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SOME PERTURBED VERSIONS OF THE GENERALIZED TRAPEZOID INEQUALITY FOR FUNCTIONS OF BOUNDED VARIATION

WENJUN LIU AND JAEKEUN PARK

Abstract. In this paper, we establish some perturbed versions of the generalized Trapezoid inequality for functions of bounded variation in terms of the cumulative variation function.

1. Introduction

In the past few years, many authors have considered various generalizations of some kinds of integral inequalities, which give explicit error bounds for some known and some new quadrature formulae. For example, in [6], Dragomir established the following generalized trapezoidal inequality for functions of bounded variation:

Theorem 1.1. *Let $f:[a,b]\to\mathbb{R}$ be a function of bounded variation. Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\frac{(x-a)f(a)+(b-x)f(b)}{b-a}\right|\le\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f), \tag{1.1}$$

where $x\in[a,b]$ and $\bigvee_a^b(f)$ denotes the total variation of f on the interval $[a,b]$. The constant $\frac{1}{2}$ cannot be replaced by a smaller one. The best inequality one can derive from (1.1) is the trapezoid inequality

$$\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\frac{f(a)+f(b)}{2}\right|\le\frac{1}{2}\bigvee_a^b(f). \tag{1.2}$$

Here the constant $\frac{1}{2}$ is also best possible.

For a function of bounded variation $v:[a,b]\to\mathbb{C}$, the Cumulative Variation Function (CVF) $V:[a,b]\to[0,\infty)$ is defined by

$$V(t):=\bigvee_a^t(v),$$

the total variation of v on the interval $[a,t]$ with $t\in[a,b]$. Recently, Dragomir [7] considered the refinement of (1.1) in terms of the cumulative variation function.

Theorem 1.2. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$. Then*

$$\begin{aligned}\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\frac{(x-a)f(a)+(b-x)f(b)}{b-a}\right| &\le\frac{1}{b-a}\left[\int_a^x\left(\bigvee_a^t(f)\right)\mathrm{d}t+\int_x^b\left(\bigvee_t^b(f)\right)\mathrm{d}t\right]\\ &\le\frac{1}{b-a}\left[(x-a)\bigvee_a^x(f)+(b-x)\bigvee_x^b(f)\right]\\ &\le\begin{cases}\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f),\\ \left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right],\end{cases}\end{aligned} \tag{1.3}$$

for any $x\in[a,b]$.

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Key words and phrases. Generalized Trapezoid inequality, Cumulative variation, Function of bounded variation, Lipschitzian function, Monotonic function.

In order to extend the classical Ostrowski's inequality for differentiable functions with bounded derivatives to the larger class of functions of bounded variation, Dragomir obtained the following result in [13]:

Theorem 1.3. *Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, b]$. Then, for all $x \in [a, b]$, we have the following inequality*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \mathrm{d}t \right| \le \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b (f). \tag{1.4}$$

The constant $\frac{1}{2}$ is the best possible. The best inequality one can obtain from (1.4) *is the midpoint inequality*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t)dt \right| \le \frac{1}{2} \bigvee_a^b (f), \tag{1.5}$$

for which the constant $\frac{1}{2}$ is also sharp.

Recently, Dragomir [8] considered the refinement of (1.4) in terms of the cumulative variation function.

Theorem 1.4. *Let $f : [a, b] \to \mathbb{C}$ be a function of bounded variation on $[a, b]$. Then*

$$\begin{aligned} \left| f(x) - \frac{1}{b-a} \int_a^b f(t) \mathrm{d}t \right| \le & \frac{1}{b-a} \left[\int_a^x \left(\bigvee_a^t (f) \right) \mathrm{d}t + \int_x^b \left(\bigvee_t^b (f) \right) \mathrm{d}t \right] \\ \le & \frac{1}{b-a} \left[(x-a) \bigvee_a^x (f) + (b-x) \bigvee_x^b (f) \right] \\ \le & \begin{cases} \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b (f), \\ \left[\frac{1}{2} \bigvee_a^b (f) + \frac{1}{2} \left| \bigvee_a^x (f) - \bigvee_x^b (f) \right| \right], \end{cases} \end{aligned} \tag{1.6}$$

for any $x \in [a, b]$.

Very recently, Dragomir [9] obtained the following perturbed Ostrowski type inequality for functions of bounded variation, in which he denoted $\ell : [a, b] \to [a, b]$ the identity function:

Theorem 1.5. *Let $f : [a, b] \to \mathbb{C}$ be a function of bounded variation on $[a, b]$, and $x \in [a, b]$. Then for any $\lambda_1(x)$ and $\lambda_2(x)$ complex numbers, we have*

$$\begin{aligned} & \left| f(x) + \frac{1}{2(b-a)} \left[(b-x)^2 \lambda_2(x) - (x-a)^2 \lambda_1(x) \right] - \frac{1}{b-a} \int_a^b f(t) \mathrm{d}t \right| \\ \le & \frac{1}{b-a} \left[\int_a^x \left(\bigvee_t^x (f - \lambda_1(x)\ell) \right) \mathrm{d}t + \int_x^b \left(\bigvee_x^t (f - \lambda_2(x)\ell) \right) \mathrm{d}t \right] \\ \le & \frac{1}{b-a} \left[(x-a) \bigvee_a^x (f - \lambda_1(x)\ell) + (b-x) \bigvee_x^b (f - \lambda_2(x)\ell) \right] \\ \le & \begin{cases} \max \left\{ \bigvee_a^x (f - \lambda_1(x)\ell), \bigvee_x^b (f - \lambda_2(x)\ell) \right\}, \\ \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \left(\bigvee_a^x (f - \lambda_1(x)\ell) + \bigvee_x^b (f - \lambda_2(x)\ell) \right), \end{cases} \end{aligned} \tag{1.7}$$

where $\bigvee_c^d (g)$ denotes the total variation of g on the interval $[c, d]$.

For related results, see [1]-[5], [11]-[12], [14]-[32].

Motivated by the above works, the purpose of this paper is to establish some perturbed versions of the generalized trapezoid inequality (1.3) for functions of bounded variation in terms of the cumulative variation function.

2. Inequalities for functions of bounded variation

As in [7], it is known that the CVF is monotonic nondecreasing on $[a,b]$ and is continuous at a point $c \in [a,b]$ if and only if the generating function v is continuous at that point. If v is Lipschitzian with the constant $L>0$, i.e.,

$$|v(t)-v(s)| \le L|t-s| \text{ for any } t,s \in [a,b],$$

then V is also Lipschitzian with the same constant.

The following lemma is of interest in itself as well, see also [10].

Lemma 2.1. *Let $f,u:[a,b]\to\mathbb{C}$. If f is continuous on $[a,b]$ and u is of bounded variation on $[a,b]$, then*

$$\left|\int_a^b f(t)\mathrm{d}u(t)\right| \le \int_a^b |f(t)|\mathrm{d}\left(\bigvee_a^t(u)\right) \le \max_{t\in[a,b]}|f(t)|\bigvee_a^b(u). \tag{2.1}$$

We have the following result:

Theorem 2.1. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$ and $x\in[a,b]$. Then for any $\lambda(x)$ complex number, we have the inequalities*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \frac{(x-a)f(a)+(b-x)f(b)}{b-a} - \lambda(x)\left(x-\frac{a+b}{2}\right)\right| \\
\le&\frac{1}{b-a}\left[\int_a^x\left(\bigvee_a^t(f-\lambda(x)\ell)\right)\mathrm{d}t + \int_x^b\left(\bigvee_t^b(f-\lambda(x)\ell)\right)\mathrm{d}t\right] \\
\le&\frac{1}{b-a}\left[(x-a)\bigvee_a^x(f-\lambda(x)\ell)+(b-x)\bigvee_x^b(f-\lambda(x)\ell)\right] \\
\le&\begin{cases}\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f-\lambda(x)\ell) \\ \frac{1}{2}\bigvee_a^b(f-\lambda(x)\ell)+\frac{1}{2}\left|\bigvee_a^x(f-\lambda(x)\ell)-\bigvee_x^b(f-\lambda(x)\ell)\right|,\end{cases}
\end{aligned} \tag{2.2}$$

where $\bigvee_c^d(g)$ denotes the total variation of g on the interval $[c,d]$ and $\ell:[a,b]\to[a,b]$ is the identity function.

Proof. We shall start with the identity obtained in [6]

$$\int_a^b f(t)\mathrm{d}t - [(x-a)f(a)+(b-x)f(b)] = \int_a^b (x-t)\mathrm{d}f(t), \tag{2.3}$$

in which the integrals in the right hand side are taken in the Riemann-Stieltjes sense. If we replace $f(t)$ with $f(t)-\lambda(x)t$ in (2.3), then we can get the following equation:

$$\int_a^b f(t)\mathrm{d}t - [(x-a)f(a)+(b-x)f(b)] - \lambda(x)(b-a)\left(x-\frac{a+b}{2}\right) = \int_a^b (x-t)\mathrm{d}\left[f(t)-\lambda(x)t\right]. \tag{2.4}$$

Taking the modulus in (2.4) and using the property (2.1), we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \frac{[(x-a)f(a)+(b-x)f(b)]}{b-a} - \lambda(x)\left(x-\frac{a+b}{2}\right)\right| \\
\le&\frac{1}{b-a}\left|\int_a^b (x-t)\mathrm{d}\left[f(t)-\lambda(x)t\right]\right| \\
\le&\frac{1}{b-a}\int_a^b |x-t|\mathrm{d}\left(\bigvee_a^t(f-\lambda(x)\ell)\right)
\end{aligned} \tag{2.5}$$

$$=\frac{1}{b-a}\left[\int_a^x (x-t)\mathrm{d}\left(\bigvee_a^t (f-\lambda(x)\ell)\right)+\int_x^b (t-x)\mathrm{d}\left(\bigvee_a^t (f-\lambda(x)\ell)\right)\right].$$

Integrating by parts in the Riemann-Stieltjes integral we have

$$\begin{aligned}\int_a^x (x-t)\mathrm{d}\left(\bigvee_a^t (f-\lambda(x)\ell)\right) &= (x-t)\bigvee_a^t (f-\lambda(x)\ell)\Big|_{t=a}^{x}+\int_a^x \left(\bigvee_a^t (f-\lambda(x)\ell)\right)\mathrm{d}t \\ &=\int_a^x \left(\bigvee_a^t (f-\lambda(x)\ell)\right)\mathrm{d}t\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}\int_x^b (t-x)\mathrm{d}\left(\bigvee_a^t (f-\lambda(x)\ell)\right) &= (t-x)\bigvee_a^t (f-\lambda(x)\ell)\Big|_{t=x}^{b}-\int_x^b \left(\bigvee_a^t (f-\lambda(x)\ell)\right)\mathrm{d}t \\ &=(b-x)\bigvee_a^b (f-\lambda(x)\ell)-\int_x^b \left(\bigvee_a^t (f-\lambda(x)\ell)\right)\mathrm{d}t \\ &=\int_x^b \left(\bigvee_t^b (f-\lambda(x)\ell)\right)\mathrm{d}t.\end{aligned} \tag{2.7}$$

Using (2.5)-(2.7), we deduce the first inequality in (2.2).

Since

$$\bigvee_a^t (f-\lambda(x)\ell)\le\bigvee_a^x (f-\lambda(x)\ell)\quad\text{for}\quad t\in[a,x]$$

and

$$\bigvee_t^b (f-\lambda(x)\ell)\le\bigvee_x^b (f-\lambda(x)\ell)\quad\text{for}\quad t\in[x,b],$$

then

$$\int_a^x \left(\bigvee_a^t (f-\lambda(x)\ell)\right)\mathrm{d}t\le(x-a)\bigvee_a^x (f-\lambda(x)\ell)$$

and

$$\int_x^b \left(\bigvee_t^b (f-\lambda(x)\ell)\right)\mathrm{d}t\le(b-x)\bigvee_x^b (f-\lambda(x)\ell),$$

which prove the second inequality in (2.2).

With the max properties we have

$$\begin{aligned}&(x-a)\bigvee_a^x (f-\lambda(x)\ell)+(b-x)\bigvee_x^b (f-\lambda(x)\ell)\\ \le&\begin{cases}\max\{x-a,b-x\}\bigvee_a^b (f-\lambda(x)\ell)\\ \max\left\{\bigvee_a^x (f-\lambda(x)\ell),\bigvee_x^b (f-\lambda(x)\ell)\right\}(b-a)\end{cases}\\ \le&\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b (f-\lambda(x)\ell)\\ \left[\frac{1}{2}\bigvee_a^b (f-\lambda(x)\ell)+\frac{1}{2}\left|\bigvee_a^x (f-\lambda(x)\ell)-\bigvee_x^b (f-\lambda(x)\ell)\right|\right](b-a),\end{cases}\end{aligned}$$

which completes the proof. □

The following trapezoid type inequality holds:

Corollary 2.1. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$. Then for any $\lambda\in\mathbb{C}$, we have the inequalities*

$$\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\frac{f(a)+f(b)}{2}\right| \le \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}}\left(\bigvee_a^t(f-\lambda\ell)\right)\mathrm{d}t+\int_{\frac{a+b}{2}}^b\left(\bigvee_t^b(f-\lambda\ell)\right)\mathrm{d}t\right] \le \frac{1}{2}\bigvee_a^b(f-\lambda\ell), \tag{2.8}$$

which is equivalent to

$$\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\frac{f(a)+f(b)}{2}\right| \le \frac{1}{b-a}\inf_{\lambda\in\mathbb{C}}\left[\int_a^{\frac{a+b}{2}}\left(\bigvee_a^t(f-\lambda\ell)\right)\mathrm{d}t+\int_{\frac{a+b}{2}}^b\left(\bigvee_t^b(f-\lambda\ell)\right)\mathrm{d}t\right] \le \frac{1}{2}\inf_{\lambda\in\mathbb{C}}\left[\bigvee_a^b(f-\lambda\ell)\right]. \tag{2.9}$$

3. Inequalities for Lipschitzian functions

We can state the following result:

Theorem 3.1. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$ and $x\in(a,b)$. If $\lambda(x)$ is a complex number and there exists the positive number $L(x)$ such that $f-\lambda(x)\ell$ is Lipschitzian with the constant $L(x)$ on the interval $[a,b]$, then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\lambda(x)\left(x-\frac{a+b}{2}\right)\right| \le \frac{L(x)}{b-a}\left[\left(x-\frac{a+b}{2}\right)^2+\frac{(b-a)^2}{4}\right]. \tag{3.1}$$

Proof. It's known that, if $g:[c,d]\to\mathbb{C}$ is Riemann integrable and $u:[c,d]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$, then the Riemann-Stieltjes integral $\int_c^d g(t)\mathrm{d}u(t)$ exists and

$$\left|\int_c^d g(t)\mathrm{d}u(t)\right| \le L\int_c^d |g(t)|\mathrm{d}t. \tag{3.2}$$

Taking the modulus in (2.4) and using the property (3.2) we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\lambda(x)\left(x-\frac{a+b}{2}\right)\right| \\
\le&\frac{1}{b-a}\left|\int_a^b (x-t)\mathrm{d}\left[f(t)-\lambda(x)t\right]\right| \\
\le&\frac{L(x)}{b-a}\left[\int_a^x (x-t)\mathrm{d}t+\int_x^b (t-x)\mathrm{d}t\right] \\
=&\frac{L(x)}{b-a}\left[\left(x-\frac{a+b}{2}\right)^2+\frac{(b-a)^2}{4}\right],
\end{aligned} \tag{3.3}$$

which proves the result. □

Corollary 3.1. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$. If λ is a complex number and there exists the positive number L such that $f-\lambda\ell$ is Lipschitzian with the constant L on the interval $[a,b]$, then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\frac{f(a)+f(b)}{2}\right| \le \frac{1}{4}L(b-a). \tag{3.4}$$

4. Inequalities for Monotonic functions

Now, the case of monotonic integrators is as follows:

Theorem 4.1. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$ and $x\in(a,b)$. If $\lambda(x)$ is a real number such that $f-\lambda(x)\ell$ is monotonic nondecreasing on the interval $[a,b]$, then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\lambda(x)\left(x-\frac{a+b}{2}\right)\right| \tag{4.1}$$
$$\leq\frac{1}{b-a}\left[(b-x)f(b)-(x-a)f(a)-\frac{1}{2}\lambda(x)[(b-x)^2+(x-a)^2]-\int_a^b sgn(t-x)f(t)\mathrm{d}t\right]$$
$$\leq\frac{1}{b-a}\left\{(x-a)[f(x)-f(a)-\lambda(x)(x-a)]+(b-x)[f(b)-f(x)-\lambda(x)(b-x)]\right\}$$

Proof. It's known that, if $g:[c,d]\to\mathbb{C}$ is continuous and $u:[c,d]\to\mathbb{C}$ is monotonic nondecreasing, then the Riemann-Stieltjes integral $\int_c^d g(t)\mathrm{d}u(t)$ exists and

$$\left|\int_c^d g(t)\mathrm{d}u(t)\right|\leq\int_c^d|g(t)|\mathrm{d}u(t). \tag{4.2}$$

Taking the modulus in (2.4) and using the property (4.2) we have

$$\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\frac{[(x-a)f(a)+(b-x)f(b)]}{b-a}-\lambda(x)\left(x-\frac{a+b}{2}\right)\right| \tag{4.3}$$
$$\leq\frac{1}{b-a}\left|\int_a^b(x-t)\mathrm{d}\left[f(t)-\lambda(x)t\right]\right|$$
$$\leq\frac{1}{b-a}\left[\int_a^x(x-t)\mathrm{d}[f(t)-\lambda(x)t]+\int_x^b(t-x)\mathrm{d}[f(t)-\lambda(x)t]\right].$$

Integrating by parts in the Riemann-Stieltjes integral we have

$$\begin{aligned}&\int_a^x(x-t)\mathrm{d}[f(t)-\lambda(x)t]\\
=&(x-t)[f(t)-\lambda(x)t]\Big|_{t=a}^{x}+\int_a^x[f(t)-\lambda(x)t]\mathrm{d}t\\
=&-(x-a)[f(a)-\lambda(x)a]+\int_a^x f(t)\mathrm{d}t-\lambda(x)\frac{x^2-a^2}{2}\\
=&-(x-a)f(a)+\lambda(x)a(x-a)+\int_a^x f(t)\mathrm{d}t-\lambda(x)\frac{x^2-a^2}{2}\\
=&-(x-a)f(a)-\frac{1}{2}\lambda(x)(x-a)^2+\int_a^x f(t)\mathrm{d}t\end{aligned}$$

and

$$\begin{aligned}&\int_x^b(t-x)\mathrm{d}[f(t)-\lambda(x)t]\\
=&(t-x)[f(t)-\lambda(x)t]\Big|_{t=x}^{b}-\int_x^b[f(t)-\lambda(x)t]\mathrm{d}t\\
=&(b-x)[f(b)-\lambda(x)b]-\int_x^b f(t)\mathrm{d}t+\lambda(x)\frac{b^2-x^2}{2}\\
=&(b-x)f(b)-\frac{1}{2}\lambda(x)(b-x)^2-\int_x^b f(t)\mathrm{d}t.\end{aligned}$$

If we add these equalities, we get

$$\int_a^x(x-t)\mathrm{d}[f(t)-\lambda(x)t]+\int_x^b(t-x)\mathrm{d}[f(t)-\lambda(x)t]$$

$$=(b-x)f(b)-(x-a)f(a)-\frac{1}{2}\lambda(x)[(b-x)^2+(x-a)^2]-\int_a^b sgn(t-x)f(t)\mathrm{d}t$$

and by (4.3) we get the first inequality in (4.1).

Now, since $f-\lambda(x)\ell$ is monotonic nondecreasing on the interval $[a,b]$, then

$$\begin{aligned}&\int_a^x (x-t)\mathrm{d}[f(t)-\lambda(x)t]\\ \leq&(x-a)[f(x)-\lambda(x)x-f(a)+\lambda(x)a]\\ =&(x-a)[f(x)-f(a)-\lambda(x)(x-a)]\end{aligned}$$

and

$$\begin{aligned}&\int_x^b (t-x)\mathrm{d}[f(t)-\lambda(x)t]\\ \leq&(b-x)[f(b)-\lambda(x)b-f(x)+\lambda(x)x]\\ =&(b-x)[f(b)-f(x)-\lambda(x)(b-x)],\end{aligned}$$

which completes the proof. □

Corollary 4.1. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$. If λ is a real number such that $f-\lambda\ell$ is monotonic nondecreasing on the interval $[a,b]$, then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\frac{f(a)+f(b)}{2}\right|\leq\frac{1}{2}[f(b)-f(a)-\lambda(b-a)]. \tag{4.4}$$

5. Conclusions

Some explicit error bounds for known or new quadrature formulae are given recently through various generalizations of some kinds of integral inequalities. In this paper, by using the ideas of Dragomir in [9], we establish some perturbed versions of the generalized trapezoid inequality for functions of bounded variation in terms of the cumulative variation function. These results can be regarded as further generalizations of [6], in which the generalized trapezoidal inequality for functions of bounded variation are established.

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A COMPANION OF OSTROWSKI LIKE INEQUALITY AND APPLICATIONS TO COMPOSITE QUADRATURE RULES

WENJUN LIU AND JAEKEUN PARK

Abstract. A companion of Ostrowski like inequality for mappings whose second derivatives belong to L^∞ spaces is established. Applications to composite quadrature rules are also given.

1. Introduction

In 1938, Ostrowski established the following interesting integral inequality (see [24]) for differentiable mappings with bounded derivatives:

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) whose derivative is bounded on (a, b) and denote $\|f'\|_\infty = \sup\limits_{t\in(a,b)} |f'(t)| < \infty$. Then for all $x \in [a, b]$ we have*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \le \left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)\|f'\|_\infty. \tag{1.1}$$

The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

This inequality has attracted considerable interest over the years, and many authors proved generalizations, modifications and applications of it. For example, the early work of Milovanović and Pečarić [21, 23] extended this inequality for differentiable mappings with bounded derivatives, to functions f that are n times differentiable with $|f^{(n)}| \le M$ and gave an application to quadrature. In [8], motivated by [12], Dragomir proved some companions of Ostrowski's inequality, as follows:

Theorem 1.2. *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function on $[a, b]$. Then the following inequalities*

$$\left|\frac{f(x)+f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\le \begin{cases} \left[\frac{1}{8} + 2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)\|f'\|_\infty, & f' \in L^\infty[a,b], \\ \frac{2^{1/q}}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{q+1} + \left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{q+1}\right]^{1/q}(b-a)^{1/q}\|f'\|_p, & \\ \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1 \quad \text{and} & f' \in L^p[a,b], \\ \left[\frac{1}{4} + \left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]\|f'\|_1, & f' \in L^1[a,b] \end{cases} \tag{1.2}$$

hold for all $x \in [a, \frac{a+b}{2}]$.

Recently, Alomari [1] introduced a companion of Dragomir's generalization of Ostrowsk's inequality for absolutely continuous mappings whose first derivatives are belong to $L^\infty([a, b])$.

Theorem 1.3. *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous mappings on (a, b) whose derivative is bounded on $[a, b]$. Then the inequality*

$$\left|\left[(1-\lambda)\frac{f(x)+f(a+b-x)}{2} + \lambda\frac{f(a)+f(b)}{2}\right] - \frac{1}{b-a}\int_a^b f(t)dt\right|$$

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$$\leq \left[\frac{1}{8}(2\lambda^2+(1-\lambda)^2)+2\frac{\left(x-\frac{(3-\lambda)a+(1+\lambda)b}{4}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty \tag{1.3}$$

holds for all $\lambda \in [0,1]$ *and* $x \in [a+\lambda\frac{b-a}{2}, \frac{a+b}{2}]$.

In (1.3), choose $\lambda = \frac{1}{2}$, one can get

$$\left|\frac{1}{2}\left[\frac{f(x)+f(a+b-x)}{2}+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq \left[\frac{3}{32}+2\frac{(x-\frac{5a+3b}{8})^2}{(b-a)^2}\right](b-a)\|f'\|_\infty. \tag{1.4}$$

And if choose $x = \frac{a+b}{2}$, then one has

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{1}{8}(b-a)\|f'\|_\infty. \tag{1.5}$$

It's shown in [1] that the constant $\frac{1}{8}$ is the best possible.

In related work, Dragomir and Sofo [10] developed the following Ostrowski like integral inequality for twice differentiable mapping.

Theorem 1.4. *Let* $f : [a,b] \to \mathbb{R}$ *be a mapping whose first derivative is absolutely continuous on* $[a,b]$ *and assume that the second derivative* $f'' \in L^\infty([a,b])$. *Then we have the inequality*

$$\left|\frac{1}{2}\left[f(x)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{2}\left(x-\frac{a+b}{2}\right)f'(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq \left[\frac{1}{48}+\frac{1}{3}\frac{|x-\frac{a+b}{2}|^3}{(b-a)^3}\right](b-a)^2\|f''\|_\infty, \tag{1.6}$$

for all $x \in [a,b]$.

In (1.6), the authors pointed out that the midpoint $x = \frac{a+b}{2}$ gives the best estimator, i.e.,

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{1}{48}(b-a)^2\|f''\|_\infty. \tag{1.7}$$

In fact, we can choose $f(t) = (t-a)^2$ in (1.7) to prove that the constant $\frac{1}{48}$ in inequality (1.7) is sharp.

For other related results, the reader may be refer to [2, 3, 4, 5, 7, 9, 11, 13, 14, 15, 16, 17, 18, 19, 20, 22, 25, 26, 27, 28, 29, 30] and the references therein. Motivated by previous works [1, 6, 8, 10], we investigate in this paper a companion of the above mentioned Ostrowski like integral inequality for twice differentiable mappings. Our result gives a smaller estimator than (1.7) (see (2.9) below). Some applications to composite quadrature rules are also given.

2. A companion of Ostrowski like inequality

The following companion of Ostrowski like inequality holds:

Theorem 2.1. *Let* $f : [a,b] \to \mathbb{R}$ *be a mapping whose first derivative is absolutely continuous on* $[a,b]$ *and assume that the second derivative* $f'' \in L^\infty([a,b])$. *Then we have the inequality*

$$\left|\frac{1}{2}\left[\frac{f(x)+f(a+b-x)}{2}+\frac{f(a)+f(b)}{2}\right]\right.$$
$$\left.-\frac{1}{2}\left(x-\frac{a+b}{2}\right)\frac{f'(x)-f'(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq \left[\frac{1}{3}\frac{(\frac{a+3b}{4}-x)(x-a)^2}{(b-a)^3}+\frac{1}{3}\frac{(\frac{a+b}{2}-x)^3}{(b-a)^3}\right](b-a)^2\|f''\|_\infty \tag{2.1}$$

for all $x \in [a, \frac{a+b}{2}]$. *The first constant* $\frac{1}{3}$ *in the right hand side of* (2.1) *is sharp in the sense that it can not be replaced by a smaller one provided that* $x \neq \frac{a+3b}{4}$ *and* $x \neq a$.

Proof. Define the kernel $K(t):[a,b]\to\mathbb{R}$ by

$$K(t):=\begin{cases} t-a, & t\in[a,x],\\ t-\frac{a+b}{2}, & t\in(x,a+b-x],\\ t-b, & t\in(a+b-x,b],\end{cases}\tag{2.2}$$

for all $x\in[a,\frac{a+b}{2}]$. Integrating by parts, we obtain (see [8])

$$\frac{1}{b-a}\int_a^b K(t)g'(t)dt=\frac{g(x)+g(a+b-x)}{2}-\frac{1}{b-a}\int_a^b g(t)dt.\tag{2.3}$$

Now choose in (2.3), $g(x)=(x-\frac{a+b}{2})f'(x)$, to get

$$\begin{aligned}&\frac{1}{b-a}\int_a^b K(t)\left[f'(t)+\left(t-\frac{a+b}{2}\right)f''(t)\right]dt\\ =&\frac{1}{2}\left(x-\frac{a+b}{2}\right)[f'(x)-f'(a+b-x)]-\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)f'(t)dt.\end{aligned}\tag{2.4}$$

Integrating by parts, we have

$$\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)f'(t)dt=\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt.\tag{2.5}$$

Also upon using (2.3), we get

$$\begin{aligned}&\frac{1}{b-a}\int_a^b K(t)\left[f'(t)+\left(t-\frac{a+b}{2}\right)f''(t)\right]dt\\ =&\frac{1}{b-a}\int_a^b K(t)f'(t)dt+\frac{1}{b-a}\int_a^b K(t)\left(t-\frac{a+b}{2}\right)f''(t)dt\\ =&\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)dt+\frac{1}{b-a}\int_a^b K(t)\left(t-\frac{a+b}{2}\right)f''(t)dt.\end{aligned}\tag{2.6}$$

It follows from (2.4)–(2.6) that

$$\begin{aligned}&\frac{1}{2(b-a)}\int_a^b K(t)\left(t-\frac{a+b}{2}\right)f''(t)dt\\ =&\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{2}\left[\frac{f(x)+f(a+b-x)}{2}+\frac{f(a)+f(b)}{2}\right]\\ &+\frac{1}{2}\left(x-\frac{a+b}{2}\right)\frac{f'(x)-f'(a+b-x)}{2}.\end{aligned}\tag{2.7}$$

Now using (2.7) we obtain

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(x)+f(a+b-x)}{2}+\frac{f(a)+f(b)}{2}\right]\right.\\ &\left.-\frac{1}{2}\left(x-\frac{a+b}{2}\right)\frac{f'(x)-f'(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|\\ \le&\frac{\|f''\|_\infty}{2(b-a)}\int_a^b|K(t)|\left|t-\frac{a+b}{2}\right|dt.\end{aligned}\tag{2.8}$$

Since $x\in[a,\frac{a+b}{2}]$, we have

$$\begin{aligned}I:=&\int_a^b|K(t)|\left|t-\frac{a+b}{2}\right|dt\\ =&\int_a^x(t-a)\left|t-\frac{a+b}{2}\right|dt+\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2dt+\int_{a+b-x}^b(b-t)\left|t-\frac{a+b}{2}\right|dt\\ =&\int_a^x(t-a)\left(\frac{a+b}{2}-t\right)dt+\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2dt+\int_{a+b-x}^b(b-t)\left(t-\frac{a+b}{2}\right)dt\\ =&\frac{(a+3b-4x)(x-a)^2}{12}+\frac{2}{3}\left(\frac{a+b}{2}-x\right)^3+\frac{(a+3b-4x)(x-a)^2}{12}\end{aligned}$$

$$=\frac{(a+3b-4x)(x-a)^2}{6}+\frac{2}{3}\left(\frac{a+b}{2}-x\right)^3,$$

and referring to (2.8), we obtain the result (2.1).

The sharpness of the constant $\frac{1}{3}$ can be proved in a special case for $x=\frac{a+b}{2}$ (see the line behind (1.7)). □

Remark 1. *If we take $x=\frac{a+b}{2}$ in (2.1), we recapture the sharp inequality (1.7). If we take $x=a$ in (2.1), we obtain the perturbed trapezoid type inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{b-a}{8}[f'(b)-f'(a)]-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\frac{(b-a)^2}{24}\|f''\|_\infty,$$

which has a smaller estimator than the sharp trapezoid inequality

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\frac{(b-a)^2}{8}\|f''\|_\infty$$

stated in [11, Proposition 2.7].

Remark 2. *Consider*

$$F(x)=\left(\frac{a+3b}{4}-x\right)(x-a)^2+\left(\frac{a+b}{2}-x\right)^3$$

for $x\in[a,\frac{a+b}{2}]$. It's easy to know that $F(x)$ obtains its minimal value at $x=\frac{3a+b}{4}$. Therefore, in (2.1), the point $x=\frac{3a+b}{4}$ gives the best estimator, i.e.,

$$\left|\frac{1}{2}\left[\frac{f(\frac{3a+b}{4})+f(\frac{a+3b}{4})}{2}+\frac{f(a)+f(b)}{2}\right]+\frac{b-a}{8}\frac{f'(\frac{3a+b}{4})-f'(\frac{a+3b}{4})}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\le\frac{1}{64}(b-a)^2\|f''\|_\infty, \tag{2.9}$$

the right hand side of which is smaller than that of (1.7).

3. Application to Composite Quadrature Rules

In [10], the authors utilized inequality (1.6) to give estimates of composite quadrature rules which was pointed out have a markedly smaller error than that which may be obtained by the classical results. In this section, we apply our previous inequality (2.1) to give us estimates of new composite quadrature rules which have a further smaller error.

Theorem 3.1. *Let $I_n: a=x_0<x_1<\cdots<x_{n-1}<x_n=b$ be a partition of the interval $[a,b]$, $h_i=x_{i+1}-x_i$, $\nu(h):=\max\{h_i: i=1,\cdots,n\}$, $\xi_i\in[x_i,\frac{x_i+x_{i+1}}{2}]$, and*

$$\begin{aligned}S(f,I_n,\xi)=&\frac{1}{4}\sum_{i=0}^{n-1}[f(x_i)+f(\xi_i)+f(x_i+x_{i+1}-\xi_i)+f(x_{i+1})]h_i\\&-\frac{1}{4}\sum_{i=0}^{n-1}h_i\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)[f'(\xi_i)-f'(x_i+x_{i+1}-\xi_i)],\end{aligned}$$

then

$$\int_a^b f(x)dx=S(f,I_n,\xi)+R(f,I_n,\xi)$$

and the remainder $R(f,I_n,\xi)$ satisfies the estimate

$$|R(f,I_n,\xi)|\le\frac{1}{3}\|f''\|_\infty\left[\sum_{i=0}^{n-1}\left(\frac{x_i+3x_{i+1}}{4}-\xi_i\right)(x_i-\xi_i)^2+\sum_{i=0}^{n-1}\left(\frac{x_i+x_{i+1}}{2}-\xi_i\right)^3\right]. \tag{3.1}$$

Proof. Inequality (2.1) can be written as

$$\begin{aligned}&\left|\int_a^b f(t)dt - \frac{1}{4}\left[f(a)+f(x)+f(a+b-x)+f(b)\right](b-a)\right.\\&\left.+\frac{b-a}{4}\left(x-\frac{a+b}{2}\right)\left[f'(x)-f'(a+b-x)\right]\right|\\&\leq\frac{1}{3}\left[\left(\frac{a+3b}{4}-x\right)(x-a)^2+\left(\frac{a+b}{2}-x\right)^3\right]\|f''\|_\infty.\end{aligned} \tag{3.2}$$

Applying (3.2) on $\xi_i \in [x_i, \frac{x_i+x_{i+1}}{2}]$, we have

$$\begin{aligned}&\left|\int_{x_i}^{x_{i+1}} f(t)dt - \frac{1}{4}\left[f(x_i)+f(\xi_i)+f(x_i+x_{i+1}-\xi_i)+f(x_{i+1})\right]h_i\right.\\&\left.+\frac{h_i}{4}\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)\left[f'(\xi_i)-f'(x_i+x_{i+1}-\xi_i)\right]\right|\\&\leq\frac{1}{3}\left[\left(\frac{x_i+3x_{i+1}}{4}-\xi_i\right)(x_i-\xi_i)^2+\left(\frac{x_i+x_{i+1}}{2}-\xi_i\right)^3\right]\|f''\|_\infty.\end{aligned}$$

Now summing over i from 0 to $n-1$ and utilizing the triangle inequality, we have

$$\begin{aligned}\left|\int_a^b f(t)dt - S(f,I_n,\xi)\right| =&\left|\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} f(t)dt - \frac{1}{4}\sum_{i=0}^{n-1}\left[f(x_i)+f(\xi_i)+f(x_i+x_{i+1}-\xi_i)+f(x_{i+1})\right]h_i\right.\\&\left.+\frac{1}{4}\sum_{i=0}^{n-1}h_i\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)\left[f'(\xi_i)-f'(x_i+x_{i+1}-\xi_i)\right]\right|\\\leq&\frac{1}{3}\|f''\|_\infty\sum_{i=0}^{n-1}\left[\left(\frac{x_i+3x_{i+1}}{4}-\xi_i\right)(x_i-\xi_i)^2+\left(\frac{x_i+x_{i+1}}{2}-\xi_i\right)^3\right]\end{aligned}$$

and therefore (3.1) holds. □

Corollary 3.1. *If we choose* $\xi_i = \frac{3x_i+x_{i+1}}{4}$, *then we have*

$$\begin{aligned}\overline{S}(f,I_n) =&\frac{1}{4}\sum_{i=0}^{n-1}\left[f(x_i)+f\left(\frac{3x_i+x_{i+1}}{4}\right)+f\left(\frac{x_i+3x_{i+1}}{4}\right)+f(x_{i+1})\right]h_i\\&+\frac{1}{16}\sum_{i=0}^{n-1}\left[f'\left(\frac{3x_i+x_{i+1}}{4}\right)-f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right]h_i^2\end{aligned}$$

and

$$|\overline{R}(f,I_n)| \leq \frac{1}{64}\|f''\|_\infty\sum_{i=0}^{n-1}h_i^3. \tag{3.3}$$

Remark 3. *It is obvious that inequality* (3.3) *is better than* [10, inequality (3.1)] *due to a smaller error.*

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A modified shift-splitting preconditioner for saddle point problems *

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Abstract

Recently, Cao, Du and Niu [Shift-splitting preconditioners for saddle point problems, *Journal of Computational and Applied Mathematics*, 272 (2014) 239-250] introduced a shift-splitting preconditioner for saddle point problems. In this paper, we establish a modified shift-splitting preconditioner for solving the large sparse augmented systems of linear equations. Furthermore, the preconditioner is based on a modified shift-splitting of the saddle point matrix, resulting in an unconditional convergent fixed-point iteration, which is a generalization of shift-splitting preconditioners. Finally, numerical examples show the spectrum of the new preconditioned matrix for the different parameters.

Key words: Saddle point problem; Shift-splitting; Krylov subspace methods; Convergence; Preconditioner.

MSC: 65F10; 65F15; 65F50

1 Introduction

For solving the large sparse augmented systems of linear equations

$$\mathcal{A}u = \begin{pmatrix} A & B^T \\ -B & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \equiv b, \tag{1}$$

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where $A \in \mathbb{R}^{n\times n}$ is a symmetric and positive definite matrix and $B \in \mathbb{R}^{m\times n}$ is a matrix of full row rank and $m < n, x, f \in \mathbb{R}^n, y, g \in \mathbb{R}^m$. It appears in many different applications of scientific computing, such as constrained optimization [32], the finite element method for solving the Navier-Stokes equation [24, 25, 26], and constrained least squares problems and generalized least squares problems [1, 29, 35, 36]. There have been several recent papers [2-24,25-29,30,31,33,37-40] for solving the augmented system (1). Santos et al. [29] studied preconditioned iterative methods for solving the singular augmented system with $A = I$. Yuan et al. [35, 36] proposed several variants of SOR method and preconditioned conjugate gradient methods for solving general augmented system (1) arising from generalized least squares problems where A can be symmetric and positive semidefinite and B can be rank deficient. The SOR-like method requires less arithmetic work per iteration step than other methods but it requires choosing an optimal iteration parameter in order to achieve a comparable rate of convergence. Golub et al. [27] presented SOR-like algorithms for solving system (1). Darvishi et al. [23] studied SSOR method for solving the augmented systems. Bai et al. [2, 3, 22, 40] presented GSOR method, parameterized Uzawa (PU) and the inexact parameterized Uzawa (PIU) methods for solving systems (1). Zhang and Lu [37] showed the generalized symmetric SOR method for augmented systems. Peng and Li [28] studied unsymmetric block overrelaxation-type methods for saddle point. Bai and Golub [4, 5, 6, 7, 11, 31] presented splitting iteration methods such as Hermitian and skew-Hermitian splitting (HSS) iteration scheme and its preconditioned variants, Krylov subspace methods such as preconditioned conjugate gradient (PCG), preconditioned MINRES (PMINRES) and restrictively preconditioned conjugate gradient (RPCG) iteration schemes, and preconditioning techniques related to Krylov subspace methods such as HSS, block-diagonal, block-triangular and constraint preconditioners and so on. Bai and Wang's 2009 LAA paper [31] and Chen and Jiang's 2008 AMC paper [22] studied some general approaches about the relaxed splitting iteration methods. Wu, Huang and Zhao [33] presented modified SSOR (MSSOR) method for augmented systems. Recently, Cao, Du and Niu [19] introduced a shift-splitting preconditioner and a local shift-splitting preconditioner for saddle point problems (1). Moreover, the authors studied some properties of the local shift-splitting preconditioned matrix and numerical experiments of a model Stokes problem are presented to show the effectiveness of the proposed preconditioners.

For large, sparse or structure matrices, iterative methods are an attractive option. In particular, Krylov subspace methods apply techniques that involve orthogonal projections onto subspaces of the form

$$\mathcal{K}(\mathcal{A}, b) \equiv \text{span}\left\{b, \mathcal{A}b, \mathcal{A}^2 b, ..., \mathcal{A}^{n-1}b, ...\right\}.$$

The conjugate gradient method (CG), minimum residual method (MINRES) and generalized minimal residual method (GMRES) are common Krylov subspace methods. The CG method is used for symmetric, positive definite matrices, MINRES for symmetric and possibly indefinite matrices and GMRES for unsymmetric matrices [30].

In this paper, based on shift-splitting preconditioners presented by Cao, Du and Niu [19], we establish a modified shift-splitting preconditioner for saddle point problems. Furthermore, the preconditioner is based on a modified shift-splitting of the saddle point matrix, resulting in an unconditional convergent fixed-point iteration, which is a generalization of shift-splitting preconditioners. Finally, numerical examples show the effectiveness of the proposed preconditioners. However, the relaxed parameters of the modified shift-splitting methods are not optimal and only lie in the convergence region of the method.

2 Modified shift-splitting preconditioner

Recently, for the coefficient matrix of the augmented system (1), Cao, Du and Niu [19] presented a shift-splitting precnditioner

$$\mathcal{P}_{SS} = \frac{1}{2}(\alpha I + \mathcal{A}),$$

where α is a positive constant and I is an identity matrix. This shift-splitting precoditioner $\mathcal{P}_{SS}$ is constructed by the shift-splitting of the matrix $\mathcal{A}$

$$\mathcal{A} = \mathcal{P}_{SS} - \mathcal{Q}_{SS} = \frac{1}{2}(\alpha I + \mathcal{A}) - \frac{1}{2}(\alpha I - \mathcal{A}),$$

which naturally leads to the shift-splitting scheme

$$u^{k+1} = (\alpha I + \mathcal{A})^{-1}(\alpha I - \mathcal{A})u^k + 2(\alpha I + \mathcal{A})^{-1}b, k = 0, 1, 2, ...$$

Based on shift-splitting preconditioners presented by Cao, Du and Niu [19], we establish a modified shift-splitting preconditione, which is as follows

$$\mathcal{A} = \frac{1}{2}(\Omega + \mathcal{A}) - \frac{1}{2}(\Omega - \mathcal{A}) = \frac{1}{2}\begin{pmatrix} \alpha I_1 + A & B^T \\ -B & \beta I_2 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \alpha I_1 - A & -B^T \\ B & \beta I_2 \end{pmatrix}, \tag{2}$$

where $\alpha \geq 0, \beta > 0$ is a constant, $\Omega = \begin{pmatrix} \alpha I_1 & 0 \\ 0 & \beta I_2 \end{pmatrix}$ and $I_1 \in \mathbb{R}^{n\times n}, I_2 \in \mathbb{R}^{m\times m}$ are the identity matrix. By this special splitting, the following shift-splitting iteration method can be defined for the saddle point problems (1).

The modified shift-splitting iteration method(MSS): Given an initial vector u^0, for $k = 0, 1, 2, ...$, until $\{u^k\}$ converges, compute

$$\frac{1}{2}\begin{pmatrix} \alpha I_1 + A & B^T \\ -B & \beta I_2 \end{pmatrix} u^{k+1} = \frac{1}{2}\begin{pmatrix} \alpha I_1 - A & -B^T \\ B & \beta I_2 \end{pmatrix} u^k + \begin{pmatrix} f \\ g \end{pmatrix}, \tag{3}$$

where $\alpha \geq 0, \beta > 0$ is a constant and $I_1 \in \mathbb{R}^{n\times n}, I_2 \in \mathbb{R}^{m\times m}$ are the identity matrix.

Remark 2.1. When the relaxed parameters $\alpha = \beta$, the modified shift-splitting iteration method (MSS) reduces to the shift-splitting iteration method (SS); When the relaxed parameters $\alpha = 0$, the modified shift-splitting iteration method (MSS) reduces to the local shift-splitting iteration method (LSS). So, MSS iteration method is the generalization of SS iteration method and LSS iteration method. Furthermore, when doing numerical examples, we may choose appropriate parameters to improve the convergence speed.

Obviously, the modified shift-splitting iteration method can naturally induce a splitting preconditioner for the Krylov subspace method. The splitting preconditioner based on iterative scheme (3) is as follows

$$\mathcal{P}_{MSS} = \frac{1}{2}(\Omega + \mathcal{A}) = \frac{1}{2}\begin{pmatrix} \alpha I_1 + A & B^T \\ -B & \beta I_2 \end{pmatrix}. \tag{4}$$

On iterative scheme (3), at each step of applying the modified shift-splitting preconditioner $\mathcal{P}_{MSS}$ within a Krylov subspace method, we need to solve a linear system with $\mathcal{P}_{MSS}$ as the coefficient matrix, which is as follows:

$$\frac{1}{2}\begin{pmatrix} \alpha I_1 + A & B^T \\ -B & \beta I_2 \end{pmatrix} z = r$$

for a given vector r at each step. Moreover, the preconditioner $\mathcal{P}_{MSS}$ can do the following matrix factorization

$$\mathcal{P}_{MSS} = \frac{1}{2}\begin{pmatrix} I_1 & \frac{1}{\beta}B^T \\ 0 & I_2 \end{pmatrix}\begin{pmatrix} \alpha I_1 + A + \frac{1}{\beta}B^T B & 0 \\ 0 & \beta I_2 \end{pmatrix}\begin{pmatrix} I_1 & 0 \\ -\frac{1}{\beta}B & I_2 \end{pmatrix}. \tag{5}$$

Let $r = [r_1^T, r_2^T]$ and $z = [z_1^T, z_2^T]$, where $r_1, z_1 \in \mathbb{R}^n$ and $r_2, z_2 \in \mathbb{R}^m$. So we can obtain

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 2\begin{pmatrix} I_1 & 0 \\ \frac{1}{\beta}B & I_2 \end{pmatrix}\begin{pmatrix} \alpha I_1 + A + \frac{1}{\beta}B^T B & 0 \\ 0 & \beta I_2 \end{pmatrix}^{-1}\begin{pmatrix} I_1 & -\frac{1}{\beta}B^T \\ 0 & I_2 \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}. \tag{6}$$

Hence, the algorithmic on the modified shift-splitting iteration method (MSS) is as follows:

Algorithm 2.1: For a given vector $r = [r_1^T, r_2^T]$, we can compute the vector $z = [z_1^T, z_2^T]$ by (6) from the following steps:

(a) $t_1 = r_1 - \frac{1}{\beta}B^T r_2$;

(b) solve $(\alpha I_1 + A + \frac{1}{\beta}B^T B)z_1 = 2t_1$;

(c) $z_2 = \frac{1}{\beta}(Bz_1 + 2r_2)$.

Remark 2.2. From Algorithm 2.1 in this paper and Algorithm 2.1 in [19], we can see that steps (a) $\sim$ (c) are different because of using different parameter β. In the second step of Algorithm 2.1, we need to solve sub-linear system with the coefficient matrix $\alpha I_1 + A + \frac{1}{\beta}B^T B$. Since the matrix $\alpha I_1 + A + \frac{1}{\beta}B^T B$ is symmetric positive definite, we may employ the CG or preconditioned CG method to solve step (b) in Algorithm 2.1.

3 Convergence of MSS method

Now, we will analyze the unconditional convergence property of the corresponding iterative method for saddle point problems. At first, similar to the proving process in [19], we can obtain the following Lemmas.

Lemma 3.1. *Let A be a symmetric positive definite matrix, and B have full row rank. If λ is an eigenvalue of $\mathcal{T}_{MSS}$, then $\lambda \neq \pm 1$, where $\mathcal{T}_{MSS}$ is the iteration matrix of the modified shift-splitting iteration, which is as follows*

$$\mathcal{T}_{MSS} = \begin{pmatrix} \alpha I_1 + A & B^T \\ -B & \beta I_2 \end{pmatrix}^{-1}\begin{pmatrix} \alpha I_1 - A & -B^T \\ B & \beta I_2 \end{pmatrix}. \tag{7}$$

Lemma 3.2. *Assume that A is symmetric positive definite, B has full row rank. Let λ be an eigenvalue of $\mathcal{T}_{MSS}$ and $[x^*, y^*]$ be the corresponding eigenvector with $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^m$. Moreover, if $y = 0$, then $|\lambda| < 1$.*

Lemma 3.3. [34] *Consider the quadratic equation $x^2 - bx + c = 0$, where b and c are real numbers. Both roots of the equation are less than one in modulus if and only if $|c| < 1$ and $|b| < 1 + c$.*

Theorem 3.4. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric and positive definite matrix, $B \in \mathbb{R}^{m\times m}$ have full row rank and let $\alpha \geq 0, \beta > 0$ be constant numbers. Let $\rho(\mathcal{T}_{MSS})$ be the spectral radius of the modified shift-splitting iteration matrix. Then it holds that*

$$\rho(\mathcal{T}_{MSS}) < 1, \forall \alpha \geq 0, \beta > 0,$$

i.e., the modified shift-splitting iteration converges to the unique solution of the saddle point problems (1).

Proof. Let λ be an eigenvalue of $\rho(\mathcal{T}_{MSS})$ and $\begin{pmatrix} x \\ y \end{pmatrix}$ be the corresponding eigenvector. Then we have

$$\begin{pmatrix} \alpha I_1 - A & -B^T \\ B & \beta I_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} \alpha I_1 + A & B^T \\ -B & \beta I_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{8}$$

Expanding out (8) we obtain

$$\begin{cases} (\lambda - 1)\xi x + (\lambda + 1)Ax + (\lambda + 1)B^T y = 0, \\ (\lambda + 1)Bx + (1 - \lambda)\beta y. \end{cases} \tag{9}$$

By Lemma 3.1, we know that $\lambda \neq 1$. So, we may get from the first equation of (9) that

$$y = \frac{\lambda + 1}{\beta(\lambda - 1)} Bx. \tag{10}$$

Substituting (10) into the first equation of (9) yields

$$\alpha(\lambda - 1)x + (\lambda + 1)Ax + \frac{(\lambda + 1)^2}{\beta(\lambda - 1)} B^T Bx = 0. \tag{11}$$

By Lemma 3.2, we know that $x \neq 0$. Multiplying $\frac{x^*}{x^*x}$ on both sides of the equation (11), we have

$$\alpha\beta(\lambda - 1)^2 + \beta(\lambda^2 - 1)\frac{x^*Ax}{x^*x} + (\lambda + 1)^2 \frac{x^*B^TBx}{x^*x} = 0. \tag{12}$$

Let

$$a = \frac{x^*Ax}{x^*x} > 0, b = \frac{x^*B^TBx}{x^*x} \geq 0.$$

Then, from (12) we know that λ satisfies the following real quadratic equation

$$\lambda^2 + \frac{2b - 2\alpha\beta}{\alpha\beta + \beta a + b}\lambda + \frac{\alpha\beta - \beta a + b}{\alpha\beta + \beta a + b}. \tag{13}$$

By Lemma 3.3, $|\lambda| < 1$ if and only if

$$\left| \frac{\alpha\beta - \beta a + b}{\alpha\beta + \beta a + b} \right| < 1 \tag{14}$$

and

$$\left|\frac{2b-2\alpha\beta}{\alpha\beta+\beta a+b}\right| < 1+\frac{\alpha\beta-\beta a+b}{\alpha\beta+\beta a+b}. \tag{15}$$

Obviously, the equations (14) and (15) hold for any $\alpha \geq 0, \beta > 0$. Hence, we have

$$\rho(\mathcal{T}_{MSS}) < 1, \forall \alpha \geq 0, \beta > 0.$$

Remark 3.1. Obviously, from Theorem 3.4, we know that the modified shift-splitting iteration method is converent unconditionally.

Remark 3.2. In actual operation, when using the Krylov subspace method like GMRES or CG method, we may choose $\mathcal{P}_{MSS}$ as the preconditioner to accelerate the convergence. Actually, the left-preconditioned linear system based on the preconditioner $\mathcal{P}_{MSS}$ is as follows

$$(I-\mathcal{T}_{MSS})u = \mathcal{P}_{MSS}^{-1}\mathcal{A}u = \mathcal{P}_{MSS}^{-1}b.$$

4 Numerical examples

In this section, to further assess the effectiveness of the modified shift-splitting preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ combined with Krylov subspace methods, we present a sample of numerical examples which are based on a two-dimensional time-harmonic Maxwell equations in mixed form in a square domain $(-1 \leq x \leq 1, -1 \leq y \leq 1)$. For the simplicity, we take the generic source: $f = 1$ and a finite element subdivision such as Figure 1 based on uniform grids of triangle elements. Three mesh sizes are considered: $h = \frac{\sqrt{2}}{8}, \frac{\sqrt{2}}{12}, \frac{\sqrt{2}}{18}$. The solutions of the preconditioned systems in each iteration are computed exactly. Information on the sparsity of relevant matrices on the different meshes is given in Table 1, where nz(A) denote the nonzero elements of matrix A.

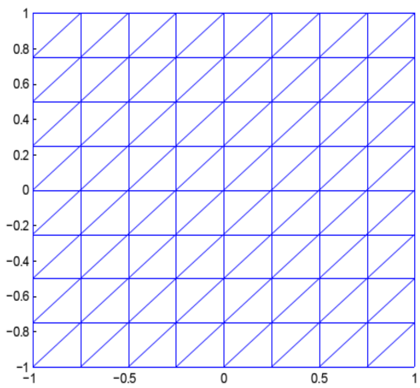

Figure 1: A uniform mesh with $h = \frac{\sqrt{2}}{4}$

Since the modified shift-splitting preconditioners have two parameters, in numerical experiments we will test different values. Numerical experiments show the spectrum of the new preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ for the different parameters.

In Figures 2, 3 and 4 we display the eigenvalues of the preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ in the case of $h = \frac{\sqrt{2}}{8}$ for different parameters. In Figures 5, 6 and 7 we display the eigenvalues of the preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ in the case of $h = \frac{\sqrt{2}}{12}$ for different parameters. In

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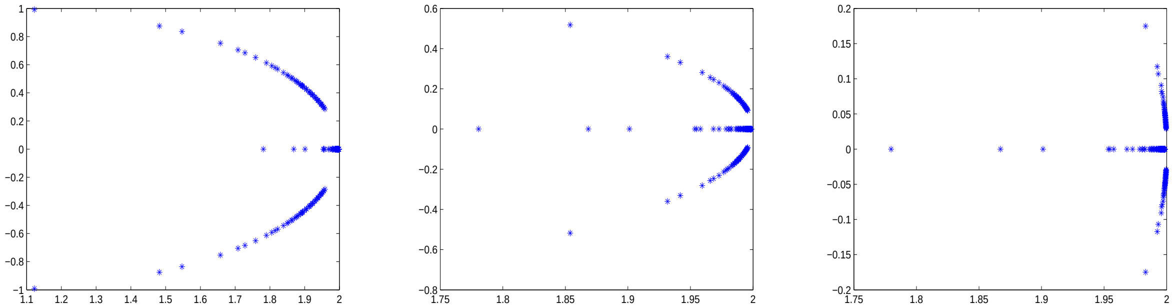

Figure 2: The eigenvalue distribution for the modified shift-splitting preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ when $\alpha = 0.01, \beta = 1$(the first), $\alpha = 0.01, \beta = 0.1$(the second) and $\alpha = 0.01, \beta = 0.01$(the third), respectively. Here, $h = \frac{\sqrt{2}}{8}$.

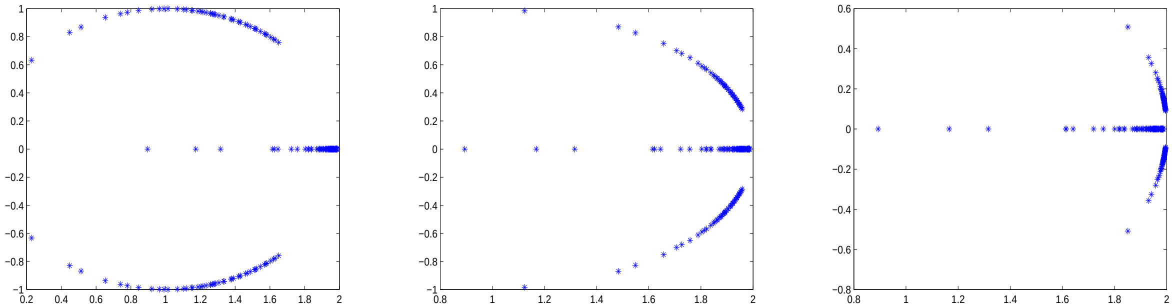

Figure 3: The eigenvalue distribution for the modified shift-splitting preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ when $\alpha = 0.1, \beta = 1$(the first), $\alpha = 0.1, \beta = 0.1$(the second) and $\alpha = 0.1, \beta = 0.01$(the third), respectively. Here, $h = \frac{\sqrt{2}}{8}$.

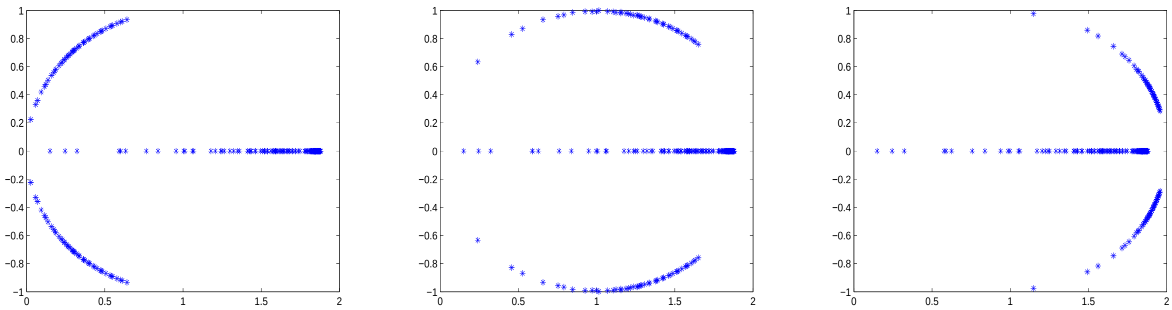

Figure 4: The eigenvalue distribution for the modified shift-splitting preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ when $\alpha = 1, \beta = 1$(the first), $\alpha = 1, \beta = 0.1$(the second) and $\alpha = 1, \beta = 0.01$(the third), respectively. Here, $h = \frac{\sqrt{2}}{8}$.

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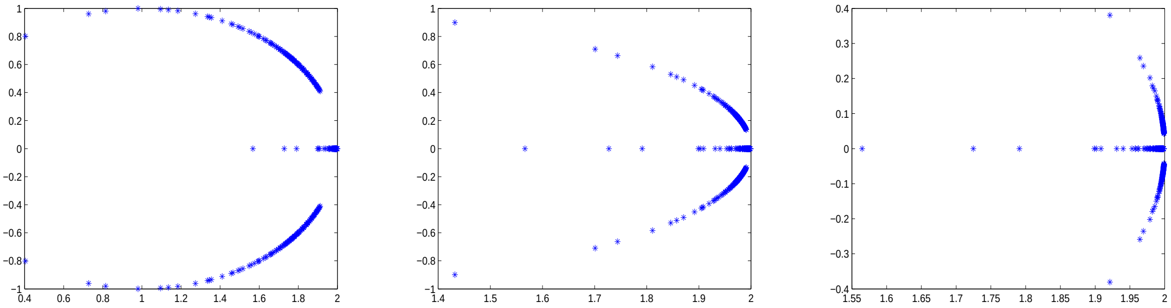

Figure 5: The eigenvalue distribution for the modified shift-splitting preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ when $\alpha = 0.01, \beta = 1$(the first), $\alpha = 0.01, \beta = 0.1$(the second) and $\alpha = 0.01, \beta = 0.01$(the third), respectively. Here, $h = \frac{\sqrt{2}}{12}$.

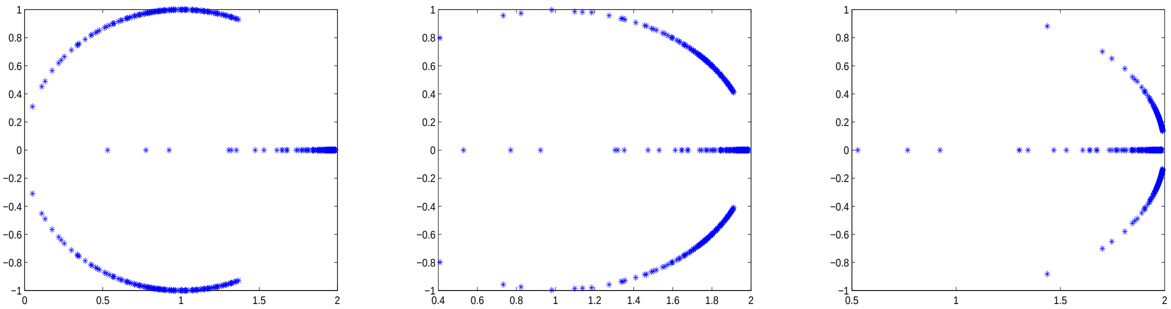

Figure 6: The eigenvalue distribution for the modified shift-splitting preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ when $\alpha = 0.1, \beta = 1$(the first), $\alpha = 0.1, \beta = 0.1$(the second) and $\alpha = 0.1, \beta = 0.01$(the third), respectively. Here, $h = \frac{\sqrt{2}}{12}$.

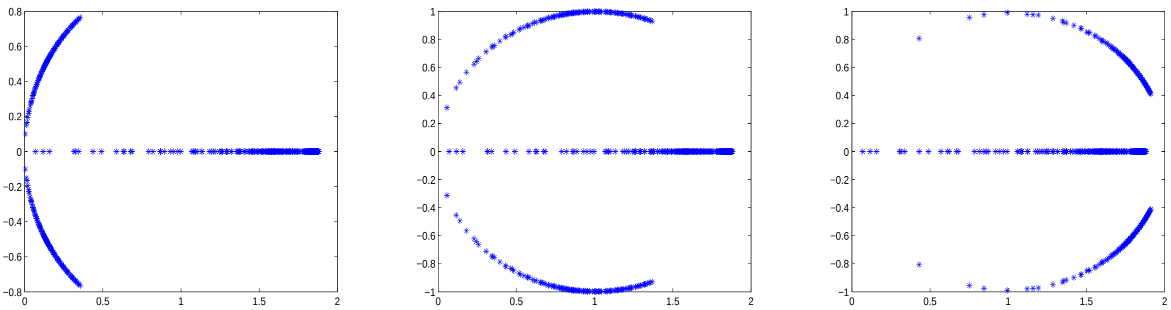

Figure 7: The eigenvalue distribution for the modified shift-splitting preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ when $\alpha = 1, \beta = 1$(the first), $\alpha = 1, \beta = 0.1$(the second) and $\alpha = 1, \beta = 0.01$(the third), respectively. Here, $h = \frac{\sqrt{2}}{12}$.

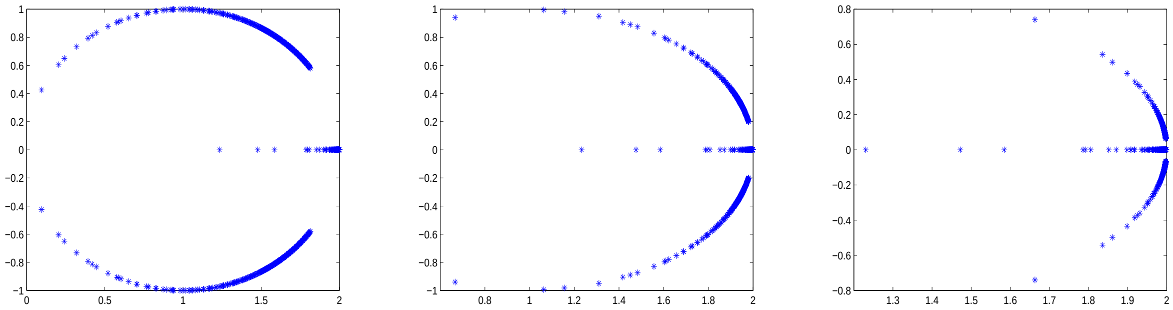

Figure 8: The eigenvalue distribution for the modified shift-splitting preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ when $\alpha = 0.01, \beta = 1$(the first), $\alpha = 0.01, \beta = 0.1$(the second) and $\alpha = 0.01, \beta = 0.01$(the third), respectively. Here, $h = \frac{\sqrt{2}}{18}$.

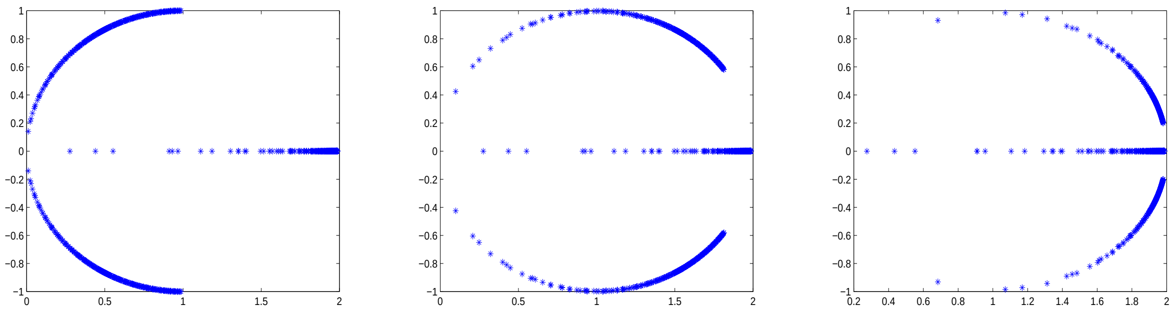

Figure 9: The eigenvalue distribution for the modified shift-splitting preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ when $\alpha = 0.1, \beta = 1$(the first), $\alpha = 0.1, \beta = 0.1$(the second) and $\alpha = 0.1, \beta = 0.01$(the third), respectively. Here, $h = \frac{\sqrt{2}}{18}$.

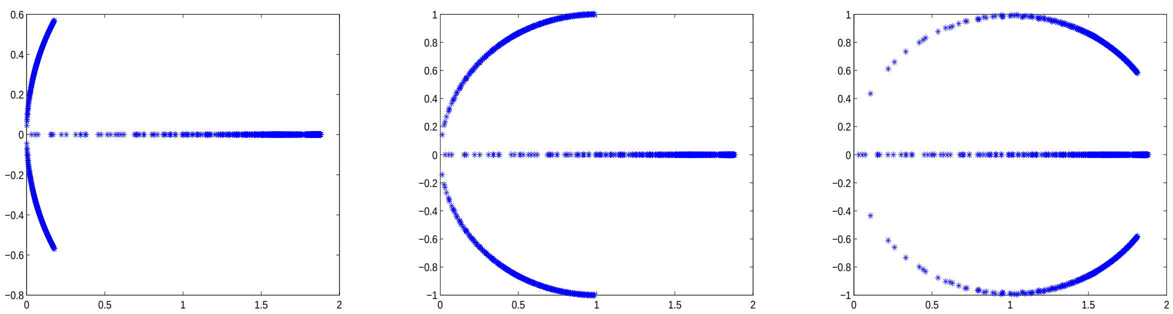

Figure 10: The eigenvalue distribution for the modified shift-splitting preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ when $\alpha = 1, \beta = 1$(the first), $\alpha = 1, \beta = 0.1$(the second) and $\alpha = 1, \beta = 0.01$(the third), respectively. Here, $h = \frac{\sqrt{2}}{18}$.

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Table 1: datasheet for different grids

Grid	m	n	nz(A)	nz(B)	nz(W)	order of $\mathcal{A}$
8×8	176	49	820	462	217	225
16×16	736	225	3556	2190	1065	961
32×32	3008	961	14788	9486	4681	3969
64×64	12160	3969	60292	39438	19593	16129

Table 2: Iteration counts and relative residual about the modified shift-splitting preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ when choosing different parameters, where the number of iterations and relative residual of unpreconditioned BICGSTAB and GMRES are $-$ and $-$, 171(1) and 7.4545×10^{-7}, respectively. Here, $h = \frac{\sqrt{2}}{8}$ denotes the size of the corresponding grid.

α	β	$It_{BICGSTAB(\mathcal{P}_{MSS}^{-1}\mathcal{A})}$	$Res_{BICGSTAB(\mathcal{P}_{MSS}^{-1}\mathcal{A})}$	$It_{GMRES(\mathcal{P}_{MSS}^{-1}\mathcal{A})}$	$Res_{GMRES(\mathcal{P}_{MSS}^{-1}\mathcal{A})}$
0.01	1	6	7.6716×10^{-7}	10(1)	7.4779×10^{-7}
0.01	0.1	3	5.4416×10^{-7}	6(1)	7.4225×10^{-7}
0.01	0.01	2	8.7718×10^{-7}	5(1)	1.8299×10^{-7}
0.1	1	21.5	5.4960×10^{-7}	24(1)	9.6647×10^{-7}
0.1	0.1	6.5	6.2392×10^{-7}	12(1)	9.3667×10^{-7}
0.1	0.01	5	3.8958×10^{-7}	8(1)	7.3712×10^{-7}
1	1	82.5	4.2920×10^{-7}	65(1)	6.5701×10^{-7}
1	0.1	31	6.0454×10^{-7}	33(1)	8.5683×10^{-7}
1	0.01	13	6.3508×10^{-7}	20(1)	5.1740×10^{-7}

Figures 8, 9 and 10 we display the eigenvalues of the preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ in the case of $h = \frac{\sqrt{2}}{18}$ for different parameters. Figures $2 \sim 10$ show that the distribution of eigenvalues of the preconditioned matrix confirms our above theoretical analysis. In Tables $2 \sim 4$ we show iteration counts and relative residual about preconditioned matrices $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ when choosing different parameters and applying to BICGSTAB and GMRES Krylov subspace iterative methods on three meshes, where $It_{BICGSTAB(\mathcal{P}_{MSS}^{-1}\mathcal{A})}$ and $Res_{BICGSTAB(\mathcal{P}_{MSS}^{-1}\mathcal{A})}$ are the iteration numbers and relative residual of the preconditioned matrices $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ when applying to BICGSTAB Krylov subspace iterative methods, respectively. $It_{GMRES(\mathcal{P}_{MSS}^{-1}\mathcal{A})}$ and $Res_{GMRES(\mathcal{P}_{MSS}^{-1}\mathcal{A})}$ are the iteration numbers and relative residual of the preconditioned matrices $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ when applying to GMRES Krylov subspace iterative methods, respectively.

Remark 4.1. From the above figures and tables, we know that the smaller the parameter β is, the gather the eigenvalues are and the fewer the iteration counts are.

Remark 4.2. From Tables 2, 3 and 4, it is very easy to see that the preconditioner $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ will improve the convergence of BICGSTAB and GMRES iteration efficiently when they are applied to the preconditioned BICGSTAB and GMRES to solove the Stokes equation and two-dimensional time-harmonic Maxwell equations by choosing different parameters.

5 Conclusions

In this paper, we establish the modified shift-splitting preconditioner for solving the large sparse augmented systems of linear equations. Furthermore, the preconditioner is based on a modified shift-splitting of the saddle point matrix, resulting in an unconditional conver-

Table 3: Iteration counts and relative residual about the modified shift-splitting preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ when choosing different parameters, where the number of iterations and relative residual of unpreconditioned BICGSTAB and GMRES are $-$ and $-$, 362(1) and 9.4148×10^{-7}, respectively. Here, $h = \frac{\sqrt{2}}{12}$ denotes the size of the corresponding grid.

α	β	$It_{BICGSTAB(\mathcal{P}_{MSS}^{-1}\mathcal{A})}$	$Res_{BICGSTAB(\mathcal{P}_{MSS}^{-1}\mathcal{A})}$	$It_{GMRES(\mathcal{P}_{MSS}^{-1}\mathcal{A})}$	$Res_{GMRES(\mathcal{P}_{MSS}^{-1}\mathcal{A})}$
0.01	1	14.5	4.1689×10^{-7}	19(1)	5.2459×10^{-7}
0.01	0.1	5.5	9.0310×10^{-7}	9(1)	7.4043×10^{-7}
0.01	0.01	3	5.2030×10^{-7}	6(1)	9.3857×10^{-7}
0.1	1	63.5	5.2347×10^{-7}	50(1)	6.7889×10^{-7}
0.1	0.1	13.5	6.1091×10^{-7}	23(1)	4.9215×10^{-7}
0.1	0.01	7.5	4.5380×10^{-7}	12(1)	8.6233×10^{-7}
1	1	216.5	4.7653×10^{-7}	123(1)	8.0138×10^{-7}
1	0.1	88	9.6032×10^{-7}	65(1)	7.5718×10^{-7}
1	0.01	27.5	1.1257×10^{-7}	34(1)	8.5489×10^{-7}

Table 4: Iteration counts and relative residual about the modified shift-splitting preconditioned matrix $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ when choosing different parameters, where the number of iterations and relative residual of unpreconditioned BICGSTAB and GMRES are 742 and 8.0810×10^{-7}, 1$-$ and $-$, respectively. Here, $h = \frac{\sqrt{2}}{18}$ denotes the size of the corresponding grid.

α	β	$It_{BICGSTAB(\mathcal{P}_{MSS}^{-1}\mathcal{A})}$	$Res_{BICGSTAB(\mathcal{P}_{MSS}^{-1}\mathcal{A})}$	$It_{GMRES(\mathcal{P}_{MSS}^{-1}\mathcal{A})}$	$Res_{GMRES(\mathcal{P}_{MSS}^{-1}\mathcal{A})}$
0.01	1	58	6.7835×10^{-7}	34(1)	8.5510×10^{-7}
0.01	0.1	7.5	7.7089×10^{-7}	16(1)	3.1469×10^{-7}
0.01	0.01	4	6.1349×10^{-7}	9(1)	2.6837×10^{-7}
0.1	1	2644.5	4.2297×10^{-7}	94(1)	9.9981×10^{-7}
0.1	0.1	34.5	8.1807×10^{-7}	43(1)	7.0956×10^{-7}
0.1	0.01	13	9.4646×10^{-7}	21(1)	5.0204×10^{-7}
1	1	8517.5	9.3710×10^{-7}	229(1)	9.1052×10^{-7}
1	0.1	116	7.8164×10^{-7}	132(1)	9.2308×10^{-7}
1	0.01	93	6.9354×10^{-7}	66(1)	8.8886×10^{-7}

gent fixed-point iteration, which is a generalization of shift-splitting preconditioners. Finally, numerical examples show the preconditioner $\mathcal{P}_{MSS}^{-1}\mathcal{A}$ will improve the convergence of BICGSTAB and GMRES iteration efficiently when they are applied to the preconditioned BICGSTAB and GMRES to solove the Stokes equation and two-dimensional time-harmonic Maxwell equations by choosing different parameters.

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CLOSED-RANGE GENERALIZED COMPOSITION OPERATORS BETWEEN BLOCH-TYPE SPACES

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ABSTRACT. Let φ denote a nonconstant analytic self-map of the open unit disk $\mathbb{D}$, g be an analytic function on $\mathbb{D}$. In this paper, we characterize the necessary or sufficient conditions for generalized composition operators

$$C_\varphi^g f(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi,$$

on the Bloch-type spaces to have a closed range. Moreover, if $g \in H^\infty$, according to relationship between α and β, we show several conclusions.

1. INTRODUCTION

Let $H(\mathbb{D})$ be the class of all holomorphic functions on $\mathbb{D}$, where $\mathbb{D}$ is the open unit disk in the complex plane $\mathbb{C}$. Denote by $H^\infty = H^\infty(\mathbb{D})$ the space of all bounded holomorphic functions on $\mathbb{D}$ with the supremum norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$.

For $0 < \alpha < \infty$, a holomorphic function f is said to be in the Bloch-type space $\mathcal{B}^\alpha$ or $\alpha-$Bloch space, if

$$\|f\|_\alpha = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)| < \infty.$$

The little Bloch-type space $\mathcal{B}_0^\alpha$, consists of all $f \in \mathcal{B}^\alpha$, such that

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha|f'(z)| = 0.$$

It is well-known that both $\mathcal{B}^\alpha$ and $\mathcal{B}_0^\alpha$ are Banach spaces under the norm

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)|.$$

Moreover, $\mathcal{B}_0^\alpha$ is the closure of polynomials in $\mathcal{B}^\alpha$. When $0 < \alpha < 1$, $\mathcal{B}^\alpha$ is the analytic Lipschitz space $Lip_{1-\alpha}$, which consists of all $f \in H(\mathbb{D})$ satisfying

$$|f(z) - f(w)| \le C|z-w|^{1-\alpha},$$

for some constant $C > 0$ and all $z, w \in \mathbb{D}$. When $\alpha = 1$, $\mathcal{B}^\alpha$ becomes the classical Bloch space $\mathcal{B}$. When $\alpha > 1$, $\mathcal{B}^\alpha$ is equivalent to the weighted Banach space $H_{\alpha-1}^\infty$. Let H_α^∞ be the weighted Banach space of holomorphic functions f on $\mathbb{D}$ satisfying

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f(z)| < \infty.$$

We refer the readers to the book [13] by K. Zhu, which is an excellent resource for the development of the theory of function spaces.

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We say that a subset H of $\mathbb{D}$ is called a *sampling set* for the Bloch-type space $\mathcal{B}^\alpha$, if there is $k>0$ such that

$$\sup\{(1-|z|^2)^\alpha|f'(z)|, z\in\mathbb{D}\}\le k\sup\{(1-|z|^2)^\alpha|f'(z)|, z\in H\}.$$

The *pseudo-hyperbolic metric* is given by

$$\rho(z,a)=|\sigma_a(z)|, \text{ where } \sigma_a(z)=\frac{a-z}{1-\bar{a}z},\ z,\ a\in\mathbb{D}.$$

$\sigma_a(z)$ is the automorphism of $\mathbb{D}$ which changes 0 and a. It is well-known that the pseudo-hyperbolic metric is Möbius-invariant. Moreover, we have that $\sigma_a'(z)=\frac{1-|a|^2}{(1-\bar{a}z)^2}$.

A subset G of $\mathbb{D}$ is an *r-net* for some $r\in(0,1)$, if for every $w\in\mathbb{D}$, there exists a $z\in G$ such that $\rho(z,w)<r$. If we define $\rho(z,E)=\inf\{\rho(z,w):w\in E\}$ for a set $E\subset D$, then a relatively closed subset E of D is an r-net if and only if $\rho(z,E)\le r$.

For every analytic self-map φ of $\mathbb{D}$ and $g\in H(\mathbb{D})$, the generalized composition operator C_φ^g is defined by

$$C_\varphi^g f(z)=\int_0^z f'(\varphi(\xi))g(\xi)d\xi,\quad z\in\mathbb{D},$$

which was firstly introduced by Li and Stević [9]. For further references and details about the generalized composition operator, we refer the readers to [10, 11] and their references. S. Li and S. Stević [9] gave the boundedness and compactness of $C_\varphi^g:\mathcal{B}^\alpha\to\mathcal{B}^\beta$, which will play a central roll in our paper, so we use the notation $\tau_{\alpha,\beta}(z)$ to state the results. For $\alpha>0$ and $\beta>0$, let

$$\tau_{\alpha,\beta}(z)=\frac{(1-|z|^2)^\beta|g(z)|}{(1-|\varphi(z)|^2)^\alpha},\ z\in\mathbb{D}.$$

Theorem A. Let $\alpha,\beta>0$, $g\in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi^g:\mathcal{B}^\alpha\to\mathcal{B}^\beta$ is bounded if and only if

$$\sup_{z\in\mathbb{D}}\tau_{\alpha,\beta}(z)<\infty.$$

Theorem B. Let $\alpha,\beta>0$, $g\in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi^g:\mathcal{B}^\alpha\to\mathcal{B}^\beta$ is compact if and only if $C_\varphi^g:\mathcal{B}^\alpha\to\mathcal{B}^\beta$ is bounded and

$$\lim_{|\varphi(z)|\to1}\tau_{\alpha,\beta}(z)=0.$$

The composition operator is defined by $C_\varphi(f)(z)=f(\varphi(z))$ on the spaces of analytic functions on $\mathbb{D}$. In 2000, Gathage, Yan and Zheng [7] characterized closed-range composition operators on Bloch spaces firstly. Chen [5] not only added a sufficient condition for [7], but also studied a sufficient and necessary condition of the boundedness from below for C_φ on the Bloch space of the unit ball. Then Gathatage, Zheng and Zorboska [8] introduced the notion of sampling sets for the bloch space and gave a necessary and sufficient condition for C_φ on the Bloch space to have closed-range. This result has been extended by Chen and Gauthire [4] to $\alpha-$Bloch spaces with $\alpha\ge1$. Soon after Zorboska [14] added new and general results on the closed-range determination of C_φ on Bloch-type spaces. There are also many articles on various other holomorphic function spaces. G. R. Chacón [3] provided a geometric characterization for those composition operators having closed-range on Dirichlet-type spaces. Recently, necessary and sufficient conditions for a closed-range composition operator on Besov spaces and more generally on Besov type spaces were given by M. Tjani [12]. Akeroyd and Fulmer [1, 2] characterized the closed range composition operators on weighted Bergman spaces.

In this paper, we give some results to determine when the generalized composition operator C_φ^g has closed-range. To some extent, our results generalize some existing results. For example, the results obtained in this paper also hold for the classical composition operator $C_\varphi:\mathcal{B}^\alpha\to\mathcal{B}^\beta$, which we get by choosing $g=\varphi'$, so some results of [14] can be got easily by this paper. In section 2, we show several necessary and sufficient conditions for the generalized composition operator C_φ^g between Bloch-type spaces to have closed-range; apart from

these, we use a set to describe when $C_\varphi^g : \mathcal{B}^\alpha/\mathbb{C} \to \mathcal{B}^\beta$ is bounded below. In section 3, if $g \in H^\infty$, according to relationship between α and β, we show several conclusions.

In order to state our main results conveniently, from now on we note $\Omega_{\varepsilon,\alpha,\beta} = \{z \in \mathbb{D}, \tau_{\alpha,\beta}(z) \geq \varepsilon\}$ and $G_{\varepsilon,\alpha,\beta} = \varphi(\Omega_{\varepsilon,\alpha,\beta})$.

Throughout the remainder of this paper, C will denote a positive constant, the exact value of which will vary from one appearance to the next. The notations $A \asymp B$, $A \preceq B$, $A \succeq B$ mean that there exist different positive constants C such that $B/C \leq A \leq CB$, $A \leq CB$, $CB \leq A$.

2. Sampling sets and r-net

A bounded generalized composition operator $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is said to be bounded below, if there exists a constant $k > 0$ such that $\|C_\varphi^g f\|_{\mathcal{B}^\beta} \geq k\|f\|_{\mathcal{B}^\alpha}$. Meanwhile, we know that C_φ^g maps any constant function to 0 function, so it is only useful to consider spaces of analytic functions modulo the constants. It follows that we can replace the norm $\|f\|_{\mathcal{B}^\alpha}$ with the seminorm $\|f\|_\alpha$ in the definition of boundedness below. Therefore, in this paper, we just show some results on $X/\mathbb{C}$, which means that a Banach space X of analytic functions on $\mathbb{D}$ modulo the constants.

Lemma 1. *Let X be Banach spaces of analytic functions. If φ is a nonconstant analytic self-map of $\mathbb{D}$, then C_φ^g is one-to-one on $X/\mathbb{C}$.*

Proof. If $C_\varphi^g f_1 = C_\varphi^g f_2$, we obtain $f_1'(\varphi(z))g(z) = f_2'(\varphi(z))g(z)$. Excluding the isolated points where g vanishes, since f_1 and f_2 are analytic, φ is a nonconstant analytic self-map of $\mathbb{D}$, the open mapping theorem for analytic functions ensures that $f_1'(z) = f_2'(z)$ for every $z \in \mathbb{D}$, and hence C_φ^g is one-to-one on $X/\mathbb{C}$. □

A basic operator theory result asserts that a one-to-one operator has a closed range if and only if it is bounded below. Therefore, Lemma 1 implies the following theorem. The detailed proof is similar to Proposition 3.30 of [6], and so we omit it.

Theorem 1. *Let $0 < \alpha, \beta < \infty$, φ be a nonconstant analytic self-map of $\mathbb{D}$. Then $C_\varphi^g : \mathcal{B}^\alpha/\mathbb{C} \to \mathcal{B}^\beta$ has a closed range if and only if it is bounded below from $\mathcal{B}^\alpha/\mathbb{C}$ to $\mathcal{B}^\beta$. This is equivalent to the condition that there exists $M > 0$ such that*

$$\|C_\varphi^g f\|_\beta \geq M\|f\|_\alpha,\ \forall f \in \mathcal{B}^\alpha/\mathbb{C}.$$

Remark 1. *Since φ is an open map, a generalized composition operator C_φ^g never has a finite rank. However, the closed subspaces of the range of a compact operator are only the finite dimensional ones, so a compact generalized composition operator can never have a closed range.*

Theorem 2. *Let $0 < \alpha, \beta < \infty$, φ be a nonconstant analytic self-map of $\mathbb{D}$. Suppose that $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded. Then $C_\varphi^g : \mathcal{B}^\alpha/\mathbb{C} \to \mathcal{B}^\beta$ has a closed range if and only if there exists $\varepsilon > 0$ such that the set $G_{\varepsilon,\alpha,\beta}$ is a sampling set on $\mathcal{B}^\alpha/\mathbb{C}$.*

Proof. Suppose that there exists $\varepsilon > 0$ such that the set $G_{\varepsilon,\alpha,\beta}$ is a sampling set on $\mathcal{B}^\alpha/\mathbb{C}$. In this case, we can find a constant $k > 0$ such that

$$\begin{aligned}
\|f\|_\alpha &\leq k\sup\{(1-|\varphi(z)|^2)^\alpha|f'(\varphi(z))|, z \in \Omega_{\varepsilon,\alpha,\beta}\}\\
&\leq k\sup\{\frac{(1-|\varphi(z)|^2)^\alpha}{(1-|z|^2)^\beta|g(z)|}(1-|z|^2)^\beta|f'(\varphi(z))g(z)|, z \in \Omega_{\varepsilon,\alpha,\beta}\}\\
&= k\sup\{\frac{1}{\tau_{\alpha,\beta}(z)}(1-|z|^2)^\beta|f'(\varphi(z))g(z)|, z \in \Omega_{\varepsilon,\alpha,\beta}\}\\
&\leq \frac{k}{\varepsilon}\sup\{(1-|z|^2)^\beta|f'(\varphi(z))g(z)|, z \in \mathbb{D}\}\\
&\leq \frac{k}{\varepsilon}\|C_\varphi^g f\|_\beta
\end{aligned}$$

and because $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded, it is bounded below. By Theorem 1, we obtain that $C_\varphi^g : \mathcal{B}^\alpha/\mathbb{C} \to \mathcal{B}^\beta$ has a closed range.

Conversely, assume that $C_\varphi^g : \mathcal{B}^\alpha/\mathbb{C} \to \mathcal{B}^\beta$ has a closed range. Then there exists $k > 0$, such that for $\forall f \in \mathcal{B}^\alpha/\mathbb{C}$, $\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|f'(\varphi(z))g(z)| \geq k\|f\|_\alpha$. Without loss of generality, we suppose that $\|f\|_\alpha = 1$. Thus, by the definition of supremum, we can choose $\omega \in \mathbb{D}$, such that $(1-|\omega|^2)^\beta|f'(\varphi(\omega))g(\omega)| \geq k/2$, that is to say,

$$\begin{aligned} (1-|\omega|^2)^\beta|f'(\varphi(\omega))g(\omega)| &= \frac{(1-|\omega|^2)^\beta|g(\omega)|}{(1-|\varphi(\omega)|^2)^\alpha}(1-|\varphi(\omega)|^2)^\alpha|f'(\varphi(\omega))| \\ &= \tau_{\alpha,\beta}(w)(1-|\varphi(\omega)|^2)^\alpha|f'(\varphi(\omega))| \\ &\geq \frac{k}{2}. \end{aligned} \tag{2.1}$$

Since $(1-|\varphi(\omega)|^2)^\alpha|f'(\varphi(\omega))| \leq 1$, $\tau_{\alpha,\beta}(w) \geq k/2$. If $\varepsilon = \frac{k}{2}$, then $\Omega_{\varepsilon,\alpha,\beta}$ contains the point ω, and so $\varphi(\omega) \in G_{\varepsilon,\alpha,\beta}$. On the other hand, $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded, Theorem A implies that there exists a constant $M > 0$, such that

$$\tau_{\alpha,\beta}(w) \quad \leq \quad M.$$

Combining the above inequality with (1), we conclude that

$$M(1-|\varphi(\omega)|^2)^\alpha|f'(\varphi(\omega))| \geq \tau_{\alpha,\beta}(w)(1-|\varphi(\omega)|^2)^\alpha|f'(\varphi(\omega))| \geq \frac{k}{2}.$$

Thus

$$(1-|\varphi(\omega)|^2)^\alpha|f'(\varphi(\omega))| \geq \frac{k}{2M}.$$

Since $\varphi(\omega) \in G_{\varepsilon,\alpha,\beta}$,

$$\sup\{(1-|z|^2)^\alpha|f'(z)|, z \in G_{\varepsilon,\alpha,\beta}\} \geq (1-|\varphi(\omega)|^2)^\alpha|f'(\varphi(\omega))| \geq \frac{k}{2M}.$$

Hence $G_{\varepsilon,\alpha,\beta}$ is a sampling set on $\mathcal{B}^\alpha/\mathbb{C}$. □

Theorem 3. *Let $0 < \alpha, \beta < \infty$, and φ be a nonconstant analytic self-map of $\mathbb{D}$. Suppose that $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded. If $C_\varphi^g : \mathcal{B}^\alpha/\mathbb{C} \to \mathcal{B}^\beta$ has a closed range, then there exist $c > 0$ and $0 < r < 1$, such that $G_{c,\alpha,\beta}$ is an r-net for $\mathbb{D}$.*

Proof. We assume that C_φ^g is bounded and has a closed-range. By Theorem A, there exists $K > 0$ such that $\sup \tau_{\alpha,\beta}(z) = K$ for $z \in \mathbb{D}$. Meanwhile, there exists $M > 0$ such that $\|C_\varphi^g f\|_\beta \geq M\|f\|_\alpha$ for all $f \in \mathcal{B}^\alpha/\mathbb{C}$.

Let $\omega \in \mathbb{D}$ and consider the function $\varphi_\omega(z)$ with $\varphi_\omega(0) = 0$ and $\varphi'_\omega(z) = (\sigma'_\omega(z))^\alpha$, where $\sigma_\omega(z) = \frac{\omega - z}{1-\overline{\omega}z}$. We have that $\varphi_\omega(z) \in \mathcal{B}^\alpha/\mathbb{C}$ and

$$\begin{aligned} \|\varphi_\omega\|_\alpha &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|\varphi'_\omega(z)| \\ &= \sup_{z\in\mathbb{D}}(1-|\sigma'_\omega(z)|^2)^\alpha \\ &= 1. \end{aligned}$$

In the above equation we use the fact that

$$1-|\sigma_\omega(z)|^2 = \frac{(1-|\omega|^2)(1-|z|^2)}{|1-\overline{\omega}z|^2} = |\sigma'_\omega(z)|(1-|z|^2).$$

Thus,

$$\begin{aligned} \|C_\varphi^g\varphi_\omega\|_\beta &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\varphi'_\omega(\varphi(z))g(z)| \\ &= \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta|g(z)|}{(1-|\varphi(z)|^2)^\alpha}(1-|\varphi(z)|^2)^\alpha|\sigma'_\omega(\varphi(z))|^\alpha \\ &= \sup_{z\in\mathbb{D}}\tau_{\alpha,\beta}(z)(1-|\sigma_\omega(\varphi(z))|^2)^\alpha. \end{aligned}$$

We shall frequently get that

$$K \geq \sup_{z\in\mathbb{D}} \tau_{\alpha,\beta}(z)(1-|\sigma_\omega(\varphi(z))|^2)^\alpha \geq M(1-|\sigma_\omega(\varphi(z))|^2)^\alpha \geq M,$$

which reveals that there exists $z_0 \in \mathbb{D}$ such that

$$\tau_{\alpha,\beta}(z_0) \geq M/2, \quad (1-|\sigma_\omega(\varphi(z_0))|^2)^\alpha \geq M/2K.$$

Thus let $\varepsilon = M/2$, $r = \sqrt{1-(M/2K)^{1/\alpha}}$, we have for all $\omega \in \mathbb{D}$, there exists $z_0 \in \Omega_{\varepsilon,\alpha,\beta}$ such that $\rho(\omega, \varphi(z_0)) < r$, and so $G_{c,\alpha,\beta}$ is an r-net for $\mathbb{D}$. □

Theorem 4. *Let $0 < \alpha, \beta < \infty$, and φ be a nonconstant analytic self-map of $\mathbb{D}$. Suppose that $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded. If there exist $\varepsilon > 0$ and $0 < r < 1$, such that $G_{\varepsilon,\alpha,\beta}$ contains the annulus $A = \{z : r < |z| < 1\}$, then $C_\varphi^g : \mathcal{B}^\alpha/\mathbb{C} \to \mathcal{B}^\beta$ has a closed range.*

Proof. Suppose that $C_\varphi^g : \mathcal{B}^\alpha/\mathbb{C} \to \mathcal{B}^\beta$ is not bounded below. Then there exists a sequence of functions $\{f_n\}$ with $\|f_n\|_\alpha = 1$ and $\|C_\varphi^g f_n\|_\beta \to 0$. It follows that for $\forall\varepsilon > 0$, there exists N_ε when $n > N_\varepsilon$, we have $\|C_\varphi^g f_n\|_\beta < \varepsilon$. Then

$$\begin{aligned}
\sup_{\omega\in G_{\varepsilon,\alpha,\beta}} (1-|\omega|^2)^\alpha |f_n'(\omega)| &= \sup_{\omega\in\Omega_{\varepsilon,\alpha,\beta}} (1-|\varphi(z)|^2)^\alpha |f_n'(\varphi(z))| \\
&= \sup_{\omega\in\Omega_{\varepsilon,\alpha,\beta}} \frac{(1-|\varphi(z)|^2)^\alpha}{(1-|z|^2)^\beta |g(z)|}(1-|z|^2)^\beta |f_n'(\varphi(z))g(z)| \\
&= \sup_{\omega\in\Omega_{\varepsilon,\alpha,\beta}} \frac{1}{\tau_{\alpha,\beta}(z)}(1-|z|^2)^\beta |f_n'(\varphi(z))g(z)| \\
&\leq \frac{1}{\varepsilon} \sup_{z\in\Omega_{\varepsilon,\alpha,\beta}} (1-|z|^2)^\beta |f_n'(\varphi(z))g(z)| \\
&= \frac{1}{\varepsilon}\|C_\varphi^g f_n\|_\beta \\
&< \varepsilon. \qquad (2.2)
\end{aligned}$$

Since $\|f_n\|_\alpha = 1$, there exists a sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{D}$, such that

$$(1-|z_n|^2)^\alpha |f_n'(z_n)| \geq 1/2 \qquad (2.3)$$

for all $n \geq 1$. If we choose $\varepsilon < 1/2$, by (2) and (3), $z_n \in \mathbb{D}/G_{\varepsilon,\alpha,\beta}$ when $n > N_\varepsilon$. Because $G_{\varepsilon,\alpha,\beta}$ contains the annulus $A = \{z : r < |z| < 1\}$, there exists $r_0 < r$ such that $|z_n| \leq r_0 \leq 1$ and $z_n \to z_0$ with $|z_0| < r_0$.

Since $\|f_n\|_\alpha = 1$, by Montel's theorem, there exists a subsequence $f_{n_k} \to f$ uniformly on every compact subsets of $\mathbb{D}$, where $f \in \mathcal{B}^\alpha/\mathbb{C}$. Cauchy's estimate gives that $f_{n_k}' \to f'$ uniformly on every compact subsets of $\mathbb{D}$. By (2), $\sup_{\omega\in G_{\varepsilon,\alpha,\beta}}(1-|\omega|^2)^\alpha|f_n'(\omega)| \to 0$ as $n \to \infty$. On the other hand, $G_{\varepsilon,\alpha,\beta}$ contains an infinite compact subset of $\mathbb{D}$, we get that $f' \equiv 0$. This contradicts the fact that $|(1-|z_0|^2)^\alpha f_n'(z_0)| \geq 1/2$. Hence, $C_\varphi^g : \mathcal{B}^\alpha/\mathbb{C} \to \mathcal{B}^\beta$ has a closed range. □

3. THE CASE OF $g \in H^\infty$

In this section we will give a special case $g \in H^\infty$. Combine α and β, we get several results.

Theorem 5. *Let φ be a nonconstant analytic self-map of $\mathbb{D}$, $\varphi(0) = 0$, $g \in H^\infty$ and $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded.*

(i) *If $0 < \alpha < \beta < \infty$ then $C_\varphi^g : \mathcal{B}^\alpha/\mathbb{C} \to \mathcal{B}^\beta$ can not have a closed range.*

(ii) *If $\alpha > \beta > 0$ and $\beta < 1$ then $C_\varphi^g : \mathcal{B}^\alpha/\mathbb{C} \to \mathcal{B}^\beta$ can not have a closed range.*

Proof. (i) Since $g \in H^\infty$, there exists a constant $k > 0$, such that $|g(z)| \leq k$, for every $z \in \mathbb{D}$. For $\varphi(0) = 0$, by Schwarz-Pick Theorem in [6], we know

$$\frac{1-|z|^2}{1-|\varphi(z)|^2} \leq 1, z \in \mathbb{D}.$$

So we have

$$\begin{aligned}\tau_{\alpha,\beta}(z) &= \frac{(1-|z|^2)^\beta|g(z)|}{(1-|\varphi(z)|^2)^\alpha}\\ &\le \frac{k(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}\\ &= \frac{k(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha}(1-|z|^2)^{\beta-\alpha}\\ &\le k(1-|\varphi(z)|^2)^{\beta-\alpha}.\end{aligned}$$

Since $0<\alpha<\beta<\infty$, as $|\varphi(z)|\to 1$, $\tau_{\alpha,\beta}(z)$ converges to 0. By Theorem B, $C_\varphi^g:\mathcal{B}^\alpha\to\mathcal{B}^\beta$ is compact. Hence $C_\varphi^g:\mathcal{B}^\alpha/\mathbb{C}\to\mathcal{B}^\beta$ can not have a closed range.

(ii) Replacing ϕ by φ, ϕ' by g in the proof of (i) of Theorem 3.6 in [14], we can get this result easily, so we omit the details here. □

Remark 2. *(i) Let φ be a nonconstant analytic self-map of $\mathbb{D}$, $\varphi(0)=0$, $g\in H^\infty$. If $\alpha=\beta$, then*

$$\begin{aligned}\tau_{\alpha,\beta}(z) &= \frac{(1-|z|^2)^\beta|g(z)|}{(1-|\varphi(z)|^2)^\alpha}\\ &\le \frac{k(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}\\ &\le k.\end{aligned}$$

By Theorem A, we obtain $C_\varphi^g:\mathcal{B}^\alpha\to\mathcal{B}^\beta$ is bounded. While apart from this, we can not get whether $C_\varphi^g:\mathcal{B}^\alpha/\mathbb{C}\to\mathcal{B}^\beta$ has a closed range or not.

(ii) Under the conditions of Theorem 5, if $\alpha>\beta\ge 1$, whether $C_\varphi^g:\mathcal{B}^\alpha/\mathbb{C}\to\mathcal{B}^\beta$ has a closed range or not is uncertain. We just give an example $\big($(ii) of Example 1$\big)$ showing that this operator sometimes do not have a closed range. While, we fail to give the concrete proof that this operator do not have a closed range always or an example to show this operator has a closed range sometimes. So this can be an open problem.

Example 1. *Let $\varphi(z)=z$, $g(z)=1$.*

(i) If $\alpha=\beta=2$, then

$$\tau_{\alpha,\beta}(z)=\frac{(1-|z|^2)^2|g(z)|}{(1-|\varphi(z)|^2)^2}=1$$

and so $\Omega_{\varepsilon,\alpha,\beta}=\mathbb{D}$ for every $0<\varepsilon<1$. In addition, $\varphi(z)=z$ is a one-to-one analytic map of the disk onto itself, therefore, $G_{\varepsilon,\alpha,\beta}=\varphi(\Omega_{\varepsilon,\alpha,\beta})=\mathbb{D}$. Then $G_{\varepsilon,\alpha,\beta}$ is a sampling set on $\mathcal{B}^\alpha/\mathbb{C}$, and by Theorem 2, $C_\varphi^g:\mathcal{B}^\alpha/\mathbb{C}\to\mathcal{B}^\beta$ has a closed range.

(ii) If $\alpha=3$, $\beta=2$, then

$$\begin{aligned}\tau_{\alpha,\beta}(z) &= \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}\\ &= (1-|z|^2)^{\beta-\alpha}\to\infty\end{aligned}$$

as $\varphi(z)\to 1$. By Theorem A, $C_\varphi^g:\mathcal{B}^\alpha\to\mathcal{B}^\beta$ is not bounded. Hence $C_\varphi^g:\mathcal{B}^\alpha/\mathbb{C}\to\mathcal{B}^\beta$ can not have a closed range.

Example 2. *Let $g(z)=z+1$, $\varphi(z)=\frac{z-1}{2}$. If $\alpha=\beta$, then*

$$\begin{aligned}\tau_{\alpha,\beta}(z) &= \frac{(1-|z|^2)^\alpha|g(z)|}{(1-|\varphi(z)|^2)^\alpha}\\ &\le \frac{4(1-|z|^2)^\alpha|z+1|}{(1-|z|)^\alpha(3+|z|)^\alpha}\\ &= \frac{4(1+|z|)^\alpha|z+1|}{(3+|z|)^\alpha}\to 0\end{aligned}$$

as $z \to -1$. By Theorem B, $C_\varphi^g : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is compact. Hence $C_\varphi^g : \mathcal{B}^\alpha/\mathbb{C} \to \mathcal{B}^\beta$ can not have a closed range.

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Approximate ternary Jordan bi-derivations on Banach Lie triple systems

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Abstract. We prove the Hyers-Ulam stability of ternary Jordan bi-derivations on Banach Lie triple systems associated to the Cauchy functional equation.

1. Introduction and preliminaries

We say that a functional equation (Q) is stable if any function g satisfying the equation (Q) approximately is near to true solution of (Q).

Ternary algebraic operations were considered in the 19th century by several mathematicians and physicists. Cayley [8] introduced the notion of cubic matrix which in turn was generalized by Kapranov, Gelfand and Zelevinskii [6]. As an application in physics, the quark model inspired a particular brand of ternary algebraic systems. The so-called Nambu mechanics which has been proposed by Nambu [11], is based on such structures. There are also some applications, although still hypothetical, in the fractional quantum Hall effect, the non-standard statistics (the anyons), supersymmetric theories, Yang-Baxter equation, etc, (cf. [15, 27]).

The comments on physical applications of ternary structures can be found in [1, 5, 10, 14, 17, 23, 24, 29].

A normed (Banach) Lie triple system is a normed (Banach) space $(A, \|\cdot\|)$ with a trilinear mapping $(x, y, z) \mapsto [x, y, z]$ from $A \times A \times A$ to A satisfying the following axioms:

$$\begin{aligned}
[x, y, z] &= -[y, x, z], \\
[x, y, z] &= -[y, z, x] - [z, x, y], \\
[u, v, [x, y, z]] &= [[u, v, x], y, z] + [x, [u, v, y], z] + [x, y, [u, v, z]], \\
\|[x, y, z]\| &\leq \|x\|\|y\|\|z\|
\end{aligned}$$

for all $u, v, x, y, z \in A$ (see [12, 16]).

Definition 1.1. Let A be a normed Lie triple system with involution $*$. A $\mathbb{C}$-bilinear mapping $D : A \times A \to A$ is called a ternary Jordan bi-derivation if it satisfies

$$\begin{aligned}
D([x, x, x], w) &= [D(x, w), x, x] + [x, D(x, w^*), x] + [x, x, D(x, w)], \\
D(x, [w, w, w]) &= [D(x, w), w, w] + [w, D(x^*, w), w] + [w, w, D(x, w)]
\end{aligned}$$

for all $x, w \in A$.

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Approximate ternary Jordan bi-derivations

The stability problem of functional equations originated from a question of Ulam [28] concerning the stability of group homomorphisms. Hyers [13] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [3] for additive mappings and by Th.M. Rassias [21] for linear mappings by considering an unbounded Cauchy difference. J.M. Rassias [20] followed the innovative approach of the Th.M. Rassias theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p\|y\|^p$ for $p, q \in \mathbb{R}$ with $p+q \neq 1$. The stability problems of various functional equations have been extensively investigated by a number of authors (see [2, 7, 9, 10, 18, 19, 22, 23, 24, 25, 26, 30, 31]).

2. Hyers-Ulam stability of ternary Jordan bi-derivations on Banach Lie triple systems

Throughout this section, assume that A is a normed Lie triple system.

For a given mapping $f : A \times A \to A$, we define

$$D_{\lambda,\mu} f(x, y, z, w) = f(\lambda x + \lambda y, \mu z + \mu w) + f(\lambda x + \lambda y, \mu z - \mu w)$$
$$+ f(\lambda x - \lambda y, \mu z + \mu w) + f(\lambda x - \lambda y, \mu z - \mu w) - 4\lambda\mu f(x, z)$$

for all $x, y, z, w \in A$ and all $\lambda, \mu \in \mathbb{T}^1 := \{\nu \in \mathbb{C} : |\nu| = 1\}$.

From now on, assume that $f(0, z) = f(x, 0) = 0$ for all $x, z \in A$.

We need the following lemma to obtain the main results.

Lemma 2.1. ([4]) *Let $f : A \times A \to B$ be a mapping satisfying $D_{\lambda,\mu} f(x, y, z, w) = 0$ for all $x, y, z, w \in A$ and all $\lambda, \mu \in \mathbb{T}^1$. Then the mapping $f : A \times A \to A$ is $\mathbb{C}$-bilinear.*

Lemma 2.2. *Let $f : A \times A \to A$ be a bi-additive mapping. Then the following assertions are equivalent:*

$$\begin{aligned} f([a, a, a], [w, w, w]) &= [f(a, w), a, a] + [a, f(a, w^*), a] + [a, a, f(a, w)], \\ f([a, a, a], [w, w, w]) &= [f(a, w), a, a] + [a, f(a^*, w), a] + [a, a, f(a, w)] \end{aligned} \tag{2.1}$$

for all $a, w \in A$, and

$$\begin{aligned} &f([a, b, c] + [b, c, a] + [c, a, b], [w, w, w]) = [f(a, w), b, c] + [a, f(b, w^*), c] + [a, b, f(c, w)] \\ &+ [f(b, w), c, a] + [b, f(c, w^*), a] + [b, c, f(a, w)] + [f(c, w), a, b] + [c, f(a, w^*), b] + [c, a, f(b, w)], \\ &f([a, a, a], [b, c, w] + [c, w, b] + [w, b, c]) = [f(a, b), c, w] + [b, f(a^*, c), w] + [b, c, f(a, w)] \\ &+ [f(a, c), w, b] + [c, f(a^*, w), b] + [c, w, f(a, b)] + [f(a, w), b, c] + [w, f(a^*, b), c] + [w, b, f(a, w)] \end{aligned} \tag{2.2}$$

for all $a, b, c, w \in A$.

Proof. Replacing a by $a + b + c$ in the first equation of (2.1), we have

$$f([a + b + c, a + b + c, a + b + c], [w, w, w]) = [f(a + b + c, w), a + b + c, a + b + c]$$
$$+ [a + b + c, f(a + b + c, w^*), a + b + c] + [a + b + c, a + b + c, f(a + b + c, w)].$$

Then we have

$$\begin{aligned} &f([a + b + c, a + b + c, a + b + c], [w, w, w]) \\ &= f([a, a, a], [w, w, w]) + f([a, b, a], [w, w, w]) + f([a, c, a], [w, w, w]) + f([b, a, a], [w, w, w]) + f([b, b, a], [w, w, w]) \\ &+ f([b, c, a], [w, w, w]) + f([c, a, a], [w, w, w]) + f([c, b, a], [w, w, w]) + f([c, c, a], [w, w, w]) + f([a, a, b], [w, w, w]) \\ &+ f([a, b, b], [w, w, w]) + f([a, c, b], [w, w, w]) + f([b, a, b], [w, w, w]) + f([b, b, b], [w, w, w]) + f([b, c, b], [w, w, w]) \end{aligned}$$

$$
\begin{aligned}
&+ f([c,a,b],[w,w,w]) + f([c,b,b],[w,w,w]) + f([c,c,b],[w,w,w]) + f([a,a,c],[w,w,w]) + f([a,b,c],[w,w,w])\\
&+ f([a,c,c],[w,w,w]) + f([b,a,c],[w,w,w]) + f([b,b,c],[w,w,w]) + f([b,c,c],[w,w,w]) + f([c,a,c],[w,w,w])\\
&+ f([c,b,c],[w,w,w]) + f([c,c,c],[w,w,w])\\
&= [f(a,w),a,a] + [a,f(a,w^*),a] + [a,a,f(a,w)] + [f(a,w),b,a] + [a,f(b,w^*),a] + [a,b,f(a,w)] + [f(a,w),c,a]\\
&+ [a,f(c,w^*),a] + [a,c,f(a,w)] + [f(b,w),a,a] + [b,f(a,w^*),a] + [b,a,f(a,w)] + [f(b,w),b,a] + [b,f(b,w^*),a]\\
&+ [b,b,f(a,w)] + [f(b,w),c,a] + [b,f(c,w^*),a] + [b,c,f(a,w)] + [f(c,w),a,a] + [c,f(a,w^*),a] + [c,a,f(a,w)]\\
&+ [f(c,w),b,a] + [c,f(b,w^*),a] + [c,b,f(a,w)] + [f(c,w),c,a] + [c,f(c,w^*),a] + [c,c,f(a,w)] + [f(a,w),a,b]\\
&+ [a,f(a,w^*),b] + [a,a,f(b,w)] + [f(a,w),b,b] + [a,f(b,w^*),b] + [a,b,f(b,w)] + [f(a,w),c,b] + [a,f(c,w^*),b]\\
&+ [a,c,f(b,w)] + [f(b,w),a,b] + [b,f(a,w^*),b] + [b,a,f(b,w)] + [f(b,w),b,b] + [b,f(b,w^*),b] + [b,b,f(b,w)]\\
&+ [f(b,w),c,b] + [b,f(c,w^*),b] + [b,c,f(b,w)] + [f(c),a,b] + [c,f(a^*),b] + [c,a,f(b)] + [f(c),b,b] + [c,f(b^*),b]\\
&+ [c,b,f(b)] + [f(c,w),c,b] + [c,f(c,w^*),b] + [c,c,f(b,w)] + [f(a,w),a,c] + [a,f(a,w^*),c] + [a,a,f(c,w)]\\
&+ [f(a,w),b,c] + [a,f(b,w^*),c] + [a,b,f(c,w)] + [f(a,w),c,c] + [a,f(c,w^*),c] + [a,c,f(c,w)] + [f(b,w),a,c]\\
&+ [b,f(a,w^*),c] + [b,a,f(c,w)] + [f(b,w),b,c] + [b,f(b,w^*),c] + [b,b,f(c,w)] + [f(b,w),c,c] + [b,f(c,w^*),c]\\
&+ [b,c,f(c,w)] + [f(c,w),a,c] + [c,f(a,w^*),c] + [c,a,f(c,w)] + [f(c,w),b,c] + [c,f(b,w^*),c] + [c,b,f(c,w)]\\
&+ [f(c,w),c,c] + [c,f(c,w^*),c] + [c,c,f(c,w)]
\end{aligned}
$$

for all $a, b, c, w \in A$.

On the other hand, for the right side of equation, we have

$$
\begin{aligned}
&[f(a+b+c,w), a+b+c, a+b+c] + [a+b+c, f(a+b+c,w^*), a+b+c] + [a+b+c, a+b+c, f(a+b+c,w)]\\
&= [f(a,w),a,a] + [f(a,w),a,b] + [f(a,w),a,c] + [f(a,w),b,a] + [f(a,w),b,b] + [f(a,w),b,c] + [f(a,w),c,a]\\
&+ [f(a,w),c,b] + [f(a,w),c,c] + [f(b,w),a,a] + [f(b,w),a,b] + [f(b,w),a,c] + [f(b,w),b,a] + [f(b,w),b,b]\\
&+ [f(b,w),b,c] + [f(b,w),c,a] + [f(b,w),c,b] + [f(b,w),c,c] + [f(c,w),a,a] + [f(c,w),a,b] + [f(c,w),a,c]\\
&+ [f(c,w),b,a] + [f(c,w),b,b] + [f(c,w),b,c] + [f(c,w),c,a] + [f(c,w),c,b] + [f(c,w),c,c] + [a,f(a,w^*),a]\\
&+ [a,f(a,w^*),b] + [a,f(a,w^*),c] + [b,f(a,w^*),a] + [b,f(a,w^*),b] + [b,f(a,w^*),c] + [c,f(a,w^*),a] + [c,f(a,w^*),b]\\
&+ [c,f(a,w^*),c] + [a,f(b,w^*),a] + [a,f(b,w^*),b] + [a,f(b,w^*),c] + [b,f(b,w^*),a] + [b,f(b,w^*),b] + [b,f(b,w^*),c]\\
&+ [c,f(b,w^*),a] + [c,f(b,w^*),b] + [c,f(b,w^*),c] + [a,f(c,w^*),a] + [a,f(c,w^*),b] + [a,f(c,w^*),c] + [b,f(c,w^*),a]\\
&+ [b,f(c,w^*),b] + [b,f(c,w^*),c] + [c,f(c,w^*),a] + [c,f(c,w^*),b] + [c,f(c,w^*),c] + [a,a,f(a,w)] + [a,b,f(a,w)]\\
&+ [a,c,f(a,w)] + [b,a,f(a,w)] + [b,b,f(a,w)] + [b,c,f(,w)] + [c,a,f(a,w)] + [c,b,f(a,w)] + [c,c,f(a,w)]\\
&+ [a,a,f(b,w)] + [a,b,f(b,w)] + [a,c,f(b,w)] + [b,a,f(b,w)] + [b,b,f(b,w)] + [b,c,f(b,w)] + [c,a,f(b,w)]\\
&+ [c,b,f(b,w)] + [c,c,f(b,w)] + [a,a,f(c,w)] + [a,b,f(c,w)] + [a,c,f(c,w)] + [b,a,f(c,w)] + [b,b,f(c,w)]\\
&+ [b,c,f(c,w)] + [c,a,f(c,w)] + [c,b,f(c,w)] + [c,c,f(c,w)]
\end{aligned}
$$

for all $a, b, c, w \in A$. It follows that

$$
\begin{aligned}
f([a,b,c]+[b,c,a]+[c,a,b],[w,w,w]) &= [f(a,w),b,c] + [a,f(b,w^*),c] + [a,b,f(c,w)]\\
&+ [f(b,w),c,a] + [b,f(c,w^*),a] + [b,c,f(a,w)] + [f(c,w),a,b] + [c,f(a,w^*),b] + [c,a,f(b,w)]
\end{aligned}
$$

for all $a, b, c, w \in A$. Hence (2.2) holds.

Similarly, we can show that

$$f([a,a,a],[b,c,w]+[c,w,b]+[w,b,c]) = [f(a,b),c,w]+[b,f(a^*,c),w]+[b,c,f(a,w)]$$
$$+[f(a,c),w,b]+[c,f(a^*,w),b]+[c,w,f(a,b)]+[f(a,w),b,c]+[w,f(a^*,b),c]+[w,b,f(a,w)]$$

for all $a, b, c, w \in A$.

For the converse, replacing b and c by a in the first equation of (2.2), we have

$$f([a,a,a]+[a,a,a]+[a,a,a],[w,w,w]) = [f(a,w),a,a]+[a,f(a,w^*),a]+[a,a,f(a,w)]+[f(a,w),a,a]$$
$$+[a,f(a,w^*),a]+[a,a,f(a,w)]+[f(a,w),a,a]+[a,f(a,w^*),a]+[a,a,f(a,w)],$$

and so

$$f\Big(([a,a,a],[w,w,w])+([a,a,a],[w,w,w])+([a,a,a],[w,w,w])\Big) = 3([f(a,w),a,a]+[a,f(a,w^*),a]+[a,a,f(a,w)]).$$

Thus

$$f\Big(3([a,a,a],[w,w,w])\Big) = 3([f(a,w),a,a]+[a,f(a,w^*),a]+[a,a,f(a,w)])$$

and so

$$f([a,a,a],[w,w,w]) = [f(a,w),a,a]+[a,f(a,w^*),a]+[a,a,f(a,w)]$$

for all $a, w \in A$.

Similarly, we can show that

$$f([a,a,a],[w,w,w]) = [f(a,w),a,a]+[a,f(a^*,w),a]+[a,a,f(a,w)]$$

for all $a, w \in A$. This completes the proof. □

Now we prove the Hyers-Ulam stability of ternary Jordan bi-derivations on Banach Lie triple systems.

Theorem 2.3. *Let p and θ be positive real numbers with $p < 2$, and let $f : A \times A \to A$ be a mapping such that*

$$\|D_{\lambda,\mu}f(x,y,z,w)\| \le \theta(\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p), \tag{2.3}$$

$$\begin{aligned}
&\|f\Big(([x,y,z]+[y,z,x]+[z,x,y]),w\Big)-[f(x,w),y,z]+[x,f(y,w^*),z]-[x,y,f(z,w)]-[f(y,w),z,x]\\
&-[y,f(z,w^*),x]-[y,z,f(x,w)]-[f(z,w),x,y]-[z,f(x,w^*),y]-[z,x,f(y,w)]\|\\
&+\|f\Big(x,([y,z,w]+[z,w,y]+[w,y,z])\Big)-[f(x,y),z,w]-[y,f(x^*,z),w]-[y,z,f(x^*,w)]-[f(x,z),w,y]\\
&-[z,f(x^*,w),y]-[z,w,f(x,y)]-[f(x,w),y,z]-[w,f(x^*,y),z]-[w,y,f(x,z)]\|\\
&\quad\le \theta(\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p)
\end{aligned} \tag{2.4}$$

for all $\lambda, \mu \in \mathbb{T}^1$ and all $x, y, z, w \in A$. Then there exists a unique ternary Jordan bi-derivations $D : A \times A \to A$ such that

$$\|f(x,y)-D(x,y)\|_B \le \frac{2\theta}{4-2^p}(\|x\|^p+\|y\|^p) \tag{2.5}$$

for all $x, y \in A$.

Proof. By the same reasoning as in the proof of [4, Theorem 2.3], there exists a unique $\mathbb{C}$-bilinear mapping $D : A \times A \to A$ satisfying (2.5). The $\mathbb{C}$-bilinear mapping $D : A \times A \to A$ is given by

$$D(x,y) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n y),$$

for all $x, y \in A$. It is easy to show that

$$D(x,y) = \lim_{n\to\infty} \frac{1}{16^n} f(8^n x, 2^n y) = \lim_{n\to\infty} \frac{1}{16^n} f(2^n x, 8^n y)$$

for all $x, y \in A$, since f is bi-additive. It follows from (2.4) that

$$\begin{aligned}
&\|D\Big(([x,y,z]+[y,z,x]+[z,x,y]),w)\Big) - [D(x,w),y,z] - [x,D(y,w^*),z] - [x,y,D(z,w)] - [D(y,w),z,x] \\
&-[y,D(z,w^*),x] - [y,z,D(x,w)] - [D(z,w),x,y] - [z,D(x,w^*),y] - [z,x,D(y,w)]\| \\
&+ \|D\Big(x,([y,z,w]+[z,w,y]+[w,y,z])\Big) - [D(x,y),z,w] - [y,D(x^*,z),w] - [y,z,D(x^*,w)] - [D(x,z),w,y] \\
&- [z,f(x^*,w),y] - [z,w,f(x,y)] - [f(x,w),y,z] - [w,f(x^*,y),z] - [w,y,f(x,z)]\| \\
&= \lim_{n\to\infty} \Big(\|\frac{1}{16^n} f\Big(2^{3n}[x,y,z]+2^{3n}[y,z,x]+2^{3n}[z,x,y],2^n w\Big) - [\frac{1}{4^n}f(2^n x,2^n w),y,z] - [x,\frac{1}{4^n}f(2^n y,2^n w^*),z] \\
&- [x,y,\frac{1}{4^n}f(2^n z,2^n w)] - [\frac{1}{4^n}f(2^n y,2^n w),z,x] - [y,\frac{1}{4^n}f(2^n z,2^n w^*),x] - [y,z,\frac{1}{4^n}f(2^n x,2^n w)] \\
&- [\frac{1}{4^n}f(2^n z,2^n w),x,y] - [z,\frac{1}{4^n}f(2^n x,2^n w^*),y] - [z,x,\frac{1}{4^n}f(2^n y,2^n w)]\| \\
&+ \Big(\|\frac{1}{16^n} f\Big(2^n x,2^{3n}[y,z,w]+2^{3n}[z,w,y]+2^{3n}[z,w,y]\Big) - [\frac{1}{4^n}f(2^n x,2^n y),z,w] + [y,\frac{1}{4^n}f(2^n x^*,2^n z),w] \\
&- [y,z,\frac{1}{4^n}f(2^n x,2^n w)] - [\frac{1}{4^n}f(2^n x,2^n z),w,y] - [z,\frac{1}{4^n}f(2^n x^*,2^n w),y] - [z,w,\frac{1}{4^n}f(2^n x,2^n y)] \\
&- [\frac{1}{4^n}f(2^n x,2^n w),y,z] - [w,\frac{1}{4^n}f(2^n x^*,2^n y),z] - [w,y,\frac{1}{4^n}f(2^n x,2^n z)]\| \\
&\le \lim_{n\to\infty} \frac{2^{np}}{16^n}\theta(\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p) = 0
\end{aligned}$$

for all $x, y, z, w \in A$. So

$$\begin{aligned}
&\|D\Big(([x,y,z]+[y,z,x]+[z,x,y]),w)\Big) - [D(x,w),y,z] - [x,D(y,w^*),z] - [x,y,D(z,w)] - [D(y,w),z,x] \\
&- [y,D(z,w^*),x] - [y,z,D(x,w)] - [D(z,w),x,y] - [z,D(x,w^*),y] - [z,x,D(y,w)]\|
\end{aligned}$$

and

$$\begin{aligned}
&+ \|D\Big(x,([y,z,w]+[z,w,y]+[w,y,z])\Big) - [D(x,y),z,w] - [y,D(x^*,z),w] - [y,z,D(x^*,w)] - [D(x,z),w,y] \\
&- [z,f(x^*,w),y] - [z,w,f(x,y)] - [f(x,w),y,z] - [w,f(x^*,y),z] - [w,y,f(x,z)]\|
\end{aligned}$$

for all $x, y, z, w \in A$. By Lemma 2.2, the mapping D is a unique ternary Jordan bi-derivation satisfying (2.5). $\square$

For the case $p > 4$, one can obtain a similar result.

Theorem 2.4. *Let p and θ be positive real numbers with $p > 4$, and let $f : A \times A \to A$ be a mapping satisfying* (2.3) *and* (2.4). *Then there exists a unique ternary Jordan bi-derivation $D : A \times A \to A$ such that*

$$\|f(x,y) - D(x,y)\| \le \frac{6\theta}{2^p - 4}(\|x\|^p + \|y\|^p)$$

for all $x, y \in A$.

Proof. The proof is similar to the proof of Theorem 2.3. $\square$

Theorem 2.5. *Let p and θ be positive real numbers with $p < \frac{1}{2}$, and let $f : A \times A \to A$ be a mapping such that*

$$\|D_{\lambda,\mu} f(x,y,z,w)\| \le \theta \cdot \|x\|^p \cdot \|y\|^p \cdot \|z\|^p \cdot \|w\|^p,$$

$$\begin{aligned}
&\|f\Big(([x,y,z]+[y,z,x]+[z,x,y]),w\Big)-[f(x,w),y,z]+[x,f(y,w^*),z]-[x,y,f(z,w)]-[f(y,w),z,x]\\
&-[y,f(z,w^*),x]-[y,z,f(x,w)]-[f(z,w),x,y]-[z,f(x,w^*),y]-[z,x,f(y,w)]\|\\
&+\|f\Big(x,([y,z,w]+[z,w,y]+[w,y,z])\Big)-[f(x,y),z,w]-[y,f(x^*,z),w]-[y,z,f(x^*,w)]-[f(x,z),w,y]\\
&-[z,f(x^*,w),y]-[z,w,f(x,y)]-[f(x,w),y,z]-[w,f(x^*,y),z]-[w,y,f(x,z)]\|\\
&\le \theta\cdot\|x\|_A^p\cdot\|y\|_A^p\cdot\|z\|_A^p\cdot\|w\|_A^p
\end{aligned}$$

for all $\lambda,\mu\in\mathbb{T}^1$ and all $x,y,z,w\in A$. Then there exists a unique ternary Jordan bi-derivations $D:A\times A\to A$ such that

$$\|f(x,y)-D(x,y)\|\le\frac{2\theta}{4-2^{4p}}\|x\|^{2p}\|y\|^{2p} \tag{2.6}$$

for all $x,y\in A$.

Proof. By the same reasoning as in the proof of [4, Theorem 2.6], there exists a unique $\mathbb{C}$-bilinear mapping $D:A\times A\to A$ satisfying (2.6). The $\mathbb{C}$-bilinear mapping $D:A\times A\to A$ is given by

$$D(x,y):=\lim_{n\to\infty}\frac{1}{4^n}f(2^nx,2^ny),$$

for all $x,y\in A$.

The rest of the proof is similar to the proof of Theorem 2.3. □

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SOME GENERALIZED DIFFERENCE SEQUENCE SPACES OF IDEAL CONVERGENCE AND ORLICZ FUNCTIONS

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Abstract. In this paper we shall introduce some generalized difference sequence spaces by using Musielak-Orlicz function, ideal convergence and an infinite matrix defined on n-normed spaces. We shall study these spaces for some linear toplogical structures and algebraic properties. We also prove some inclusion relations between these spaces

1. Introduction and Preliminaries

The notion of statistical convergence was introduced by Fast [5] and Schoenberg [31] independently. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on, it was further investigated from the sequence space point of view and linked with summability theory by Fridy [6], Connor [1], Salat [29], Isik [14], Savaş [30], Malkowsky and Savaş [19], Kolk [16], Tripathy and Sen [32] and many others. In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations are also connected with subsets of the Stone-Cech compactification of natural numbers. Moreover, statistical convergence is closely related to the concept of convergence in probability. The notion depends on the density of subsets of the set $\mathbb{N}$ of natural numbers.

A subset E of $\mathbb{N}$ is said to have the natural density $\delta(E)$ if the following limit exists: $\delta(E) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \chi_E(k)$, where χ_E is the characteristic function of E. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and $\delta(E^c) = 1 - \delta(E)$.

The notion of ideal convergence was first introduced by P.Kostyrko et.al [13] as a generalization of statistical convergence which was further studied in topological spaces by Das, Kostyrko, Wilczynski and Malik (see [2]). More applications of ideals can be seen in ([2], [3]). We continue in this direction and introduce I-convergence of generalized sequences in more general setting.

A family $\mathcal{I} \subset 2^Y$ of subsets of a non empty set Y is said to be an ideal in Yif

(1) $\phi \in \mathcal{I}$;

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(2) $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$;
(3) $A \in \mathcal{I}$, $B \subset A$ imply $B \in \mathcal{I}$, while an admissible ideal $\mathcal{I}$ of Y further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$ (see [11]).

Given $\mathcal{I} \subset 2^{\mathbb{N}}$ be a non trivial ideal in $\mathbb{N}$. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be I-convergent to $x \in X$, if for each $\epsilon > 0$ the set $A(\epsilon) = \Big\{n \in \mathbb{N} : ||x_n - x|| \geq \epsilon\Big\}$ belongs to $\mathcal{I}$ (see [10]).
The notion of difference sequence spaces was introduced by Kızmaz [15], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [4] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let w be the space of all complex or real sequences $x = (x_k)$ and let m, n be non-negative integers, then for $Z = l_\infty,\ c,\ c_0$ we have sequence spaces

$$Z(\Delta_n^m) = \{x = (x_k) \in w : (\Delta_n^m x_k) \in Z\},$$

where $\Delta_n^m x = (\Delta_n^m x_k) = (\Delta_n^{m-1} x_k - \Delta_n^{m-1} x_{k+1})$ and $\Delta_n^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_n^m x_k = \sum_{v=0}^{m} (-1)^v \begin{pmatrix} m \\ v \end{pmatrix} x_{k+nv}.$$

Taking $n = 1$, we get the spaces which were studied by Et and Çolak [4]. Taking $m = n = 1$, we get the spaces which were introduced and studied by Kızmaz [15].

The concept of 2-normed spaces was initially developed by Gähler [7] in the mid of 1960's, while that of n-normed spaces one can see in Misiak[19]. Since then, many others have studied this concept and obtained various results, see Gunawan ([8], [9]) and Gunawan and Mashadi [10] and many others. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is field of real or complex numbers of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on X^n satisfying the following four conditions:

(1) $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;
(2) $||x_1, x_2, \cdots, x_n||$ is invariant under permutation;
(3) $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha|\ \ ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{K}$, and
(4) $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called a n-norm on X, and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called a n-normed space over the field $\mathbb{K}$.
For example, we may take $X = \mathbb{R}^n$ being equipped with the n-norm $||x_1, x_2, \cdots, x_n||_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$, where script E denotes Euclidean space. Let $(X, ||\cdot, \cdots, \cdot||)$ be an n-normed space of dimension $d \geq n \geq 2$ and

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$\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $||\cdot, \cdots, \cdot||_\infty$ on X^{n-1} defined by

$$||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \ \text{ for every } \ z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to be Cauchy if

$$\lim_{k,i\to\infty} ||x_k - x_i, z_1, \cdots, z_{n-1}|| = 0 \ \text{ for every } \ z_1, \cdots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

An Orlicz function $M : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

Lindenstrauss and Tzafriri [17] used the idea of Orlicz function to define the following sequence space,

$$\ell_M = \Big\{(x_k) \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \ \text{ for some } \ \rho > 0\Big\}$$

which is called as an Orlicz sequence space. Also ℓ_M is a Banach space with the norm

$$||(x_k)|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

Also, it was shown in [17] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. An Orlicz function M satisfies Δ_2-condition if and only if for any constant $L > 1$ there exists a constant $K(L)$ such that $M(Lu) \leq K(L)M(u)$ for all values of $u \geq 0$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t)dt$$

where η is known as the kernel of M, is right differentiable for $t \geq 0, \eta(0) = 0, \eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.

A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function see ([18], [25]). A sequence $\mathcal{N} = (N_k)$ is defined by

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\}, \ k = 1, 2, \cdots$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows

$$t_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \ \text{ for some } \ c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{ x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for all } c > 0 \Big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf \Big\{ k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1 \Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf \Big\{ \frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0 \Big\}.$$

A Musielak-Orlicz function (M_k) is said to satisfy Δ_2-condition if there exist constants $a, K > 0$ and a sequence $c = (c_k)_{k=1}^{\infty} \in \ell_+^1$ (the positive cone of ℓ^1) such that the inequality

$$M_k(2u) \leq KM_k(u) + c_k$$

holds for all $k \in N$ and $u \in R_+$ whenever $M_k(u) \leq a$.

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

(1) $p(x) \geq 0$ for all $x \in X$,
(2) $p(-x) = p(x)$ for all $x \in X$,
(3) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$,
(4) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [33], Theorem 10.4.2, pp. 183). For more details about sequence spaces (see [21], [22], [23], [24], [26], [27], [28]) and reference therein.

A sequence space E is said to be solid(or normal) if $(x_k) \in E$ implies $(\alpha_k x_k) \in E$ for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ and for all $k \in \mathbb{N}$.

Let I be an admissible ideal of $\mathbb{N}$, let $p = (p_k)$ be a bounded sequence of positive real numbers for all $k \in \mathbb{N}$ and $A = (a_{nk})$ be an infinite matrix. Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $u = (u_k)$ be a sequence of strictly positive real numbers and $(X, ||., ..., .||)$ be a n-normed space. Further $w(n-x)$ denotes the space of all X-valued sequences. For every $z_1, z_2, ..., z_{n-1} \in X$, for each $\epsilon > 0$ and for some $\rho > 0$ we define the following sequence spaces:

$$W^I\big[A, \Delta_n^m, \mathcal{M}, u, p, ||., ..., .||\big] = \Big\{ x = (x_k) \in w(n-x) : \text{ for given } \epsilon > 0, \Big\{ n \in \mathbb{N} :$$

$$\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k \Delta_n^m x_k - L}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \geq \epsilon \Big\} \in I, \text{ for } L \in X \text{ and } n \in \mathbb{N} \Big\},$$

$$W_0^I\big[A,\Delta_n^m,\mathcal{M},u,p,||.,...,.||\big]=\Big\{x=(x_k)\in w(n-x):\ \text{ for given }\ \epsilon>0,\Big\{n\in\mathbb{N}:$$
$$\sum_{k=1}^{\infty}a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^m x_k}{\rho},z_1,z_2,...,z_{n-1}||\Big)\Big]^{p_k}\geq\epsilon\Big\}\in I\Big\}$$

and

$$W_\infty^I\big[A,\Delta_n^m,\mathcal{M},u,p,||.,...,.||\big]=\Big\{x=(x_k)\in w(n-x):\exists\, k>0,\Big\{n\in\mathbb{N}:$$
$$\sum_{k=1}^{\infty}a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^m x_k}{\rho},z_1,z_2,...,z_{n-1}||\Big)\Big]^{p_k}\geq K\Big\}\in I\Big\}.$$

Some special cases of the above defined sequence spaces are arises:
If $m=n=0$, then we obtain the spaces as follows

$$W^I\big[A,\mathcal{M},u,p,||.,...,.||\big]=\Big\{x=(x_k)\in w(n-x):\ \text{ for given }\ \epsilon>0,\Big\{n\in\mathbb{N}:$$
$$\sum_{k=1}^{\infty}a_{nk}\Big[M_k\Big(||\frac{u_kx_k-L}{\rho},z_1,z_2,...,z_{n-1}||\Big)\Big]^{p_k}\geq\epsilon\Big\}\in I,\ \text{ for }\ L\in X\ \text{ and }\ n\in\mathbb{N}\Big\},$$

$$W_0^I\big[A,\mathcal{M},u,p,||.,...,.||\big]=\Big\{x=(x_k)\in w(n-x):\ \text{ for given }\ \epsilon>0,\Big\{n\in\mathbb{N}:$$
$$\sum_{k=1}^{\infty}a_{nk}\Big[M_k\Big(||\frac{u_kx_k}{\rho},z_1,z_2,...,z_{n-1}||\Big)\Big]^{p_k}\geq\epsilon\Big\}\in I\Big\}$$

and

$$W_\infty^I\big[A,\mathcal{M},u,p,||.,...,.||\big]=\Big\{x=(x_k)\in w(n-x):\exists\, k>0,\Big\{n\in\mathbb{N}:$$
$$\sum_{k=1}^{\infty}a_{nk}\Big[M_k\Big(||\frac{u_kx_k}{\rho},z_1,z_2,...,z_{n-1}||\Big)\Big]^{p_k}\geq K\Big\}\in I\Big\}.$$

If $m=n=1$, then the above spaces are as follows

$$W^I\big[A,\Delta,\mathcal{M},u,p,||.,...,.||\big]=\Big\{x=(x_k)\in w(n-x):\ \text{ for given }\ \epsilon>0,\Big\{n\in\mathbb{N}:$$
$$\sum_{k=1}^{\infty}a_{nk}\Big[M_k\Big(||\frac{u_k\Delta x_k-L}{\rho},z_1,z_2,...,z_{n-1}||\Big)\Big]^{p_k}\geq\epsilon\Big\}\in I,\ \text{ for }\ L\in X\ \text{ and }\ n\in\mathbb{N}\Big\},$$

$$W_0^I\big[A,\Delta,\mathcal{M},u,p,||.,...,.||\big]=\Big\{x=(x_k)\in w(n-x):\ \text{ for given }\ \epsilon>0,\Big\{n\in\mathbb{N}:$$
$$\sum_{k=1}^{\infty}a_{nk}\Big[M_k\Big(||\frac{u_k\Delta x_k}{\rho},z_1,z_2,...,z_{n-1}||\Big)\Big]^{p_k}\geq\epsilon\Big\}\in I\Big\}$$

and

$$W_\infty^I\big[A,\Delta,\mathcal{M},u,p,||.,...,.||\big]=\Big\{x=(x_k)\in w(n-x):\exists\, k>0,\Big\{n\in\mathbb{N}:$$
$$\sum_{k=1}^{\infty}a_{nk}\Big[M_k\Big(||\frac{u_k\Delta x_k}{\rho},z_1,z_2,...,z_{n-1}||\Big)\Big]^{p_k}\geq K\Big\}\in I\Big\}.$$

If $\mathcal{M}(x) = x$ for all $x \in [0, \infty)$, then we have

$$W^I\big[A, \Delta_n^m, u, p, ||., ..., .||\big] = \Big\{x = (x_k) \in w(n-x) : \text{ for given } \epsilon > 0, \Big\{n \in \mathbb{N} :$$

$$\sum_{k=1}^{\infty} a_{nk}\Big(||\frac{u_k \Delta_n^m x_k - L}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)^{p_k} \geq \epsilon\Big\} \in I, \text{ for } L \in X \text{ and } n \in \mathbb{N}\Big\},$$

$$W_0^I\big[A, \Delta_n^m, u, p, ||., ..., .||\big] = \Big\{x = (x_k) \in w(n-x) : \text{ for given } \epsilon > 0, \Big\{n \in \mathbb{N} :$$

$$\sum_{k=1}^{\infty} a_{nk}\Big(||\frac{u_k \Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)^{p_k} \geq \epsilon\Big\} \in I\Big\}$$

and

$$W_\infty^I\big[A, \Delta_n^m, u, p, ||., ..., .||\big] = \Big\{x = (x_k) \in w(n-x) : \exists\, k > 0, \Big\{n \in \mathbb{N} :$$

$$\sum_{k=1}^{\infty} a_{nk}\Big(||\frac{u_k \Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)^{p_k} \geq K\Big\} \in I\Big\}.$$

If $p = (p_k) = 1$ for all k, then the above spaces are as follows

$$W^I\big[A, \Delta_n^m, \mathcal{M}, u, ||., ..., .||\big] = \Big\{x = (x_k) \in w(n-x) : \text{ for given } \epsilon > 0, \Big\{n \in \mathbb{N} :$$

$$\sum_{k=1}^{\infty} a_{nk} M_k\Big(||\frac{u_k \Delta_n^m x_k - L}{\rho}, z_1, z_2, ..., z_{n-1}||\Big) \geq \epsilon\Big\} \in I, \text{ for } L \in X \text{ and } n \in \mathbb{N}\Big\},$$

$$W_0^I\big[A, \Delta_n^m, \mathcal{M}, u, ||., ..., .||\big] = \Big\{x = (x_k) \in w(n-x) : \text{ for given } \epsilon > 0, \Big\{n \in \mathbb{N} :$$

$$\sum_{k=1}^{\infty} a_{nk} M_k\Big(||\frac{u_k \Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}||\Big) \geq \epsilon\Big\} \in I\Big\}$$

and

$$W_\infty^I\big[A, \Delta_n^m, \mathcal{M}, u, ||., ..., .||\big] = \Big\{x = (x_k) \in w(n-x) : \exists\, k > 0, \Big\{n \in \mathbb{N} :$$

$$\sum_{k=1}^{\infty} a_{nk} M_k\Big(||\frac{u_k \Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}||\Big) \geq K\Big\} \in I\Big\}.$$

If $A = (C, 1)$, the Cesàro matrix, then the above spaces are as follows

$$W^I\big[\Delta_n^m, \mathcal{M}, u, p, ||., ..., .||\big] = \Big\{x = (x_k) \in w(n-x) : \text{ for given } \epsilon > 0, \Big\{n \in \mathbb{N} :$$

$$\sum_{k=1}^{\infty} \Big[M_k\Big(||\frac{u_k \Delta_n^m x_k - L}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \geq \epsilon\Big\} \in I, \text{ for } L \in X \text{ and } n \in \mathbb{N}\Big\},$$

$$W_0^I\big[\Delta_n^m, \mathcal{M}, u, p, ||., ..., .||\big] = \Big\{x = (x_k) \in w(n-x) : \text{ for given } \epsilon > 0, \Big\{n \in \mathbb{N} :$$

$$\sum_{k=1}^{\infty} \Big[M_k\Big(||\frac{u_k \Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \geq \epsilon\Big\} \in I\Big\}$$

and
$W_\infty^I\big[\Delta_n^m,\mathcal{M},u,p,||.,...,.||\big]=\Big\{x=(x_k)\in w(n-x):\exists\, k>0,\Big\{n\in\mathbb{N}:$

$$\sum_{k=1}^{\infty}\Big[M_k\Big(||\frac{u_k\Delta_n^m x_k}{\rho},z_1,z_2,...,z_{n-1}||\Big)\Big]^{p_k}\geq K\Big\}\in I\Big\}.$$

If we take $A=(a_{nk})$ is a de La Valee Poussin mean i.e.

$$a_{nk}=\begin{cases}\frac{1}{\lambda_n}, & \text{if } k\in I_n=[n-\lambda_n+1,n]\\ 0, & \text{otherwise}\end{cases}$$

where (λ_n) is a non-decreasing sequence of positive numbers tending to ∞ and $\lambda_{n+1}\leq\lambda_n+1,\lambda_1=1$, then the above sequence spaces are denoted by $W^I\big[\lambda,\Delta_n^m,\mathcal{M},u,p,||.,...,.||\big]$, $W_0^I\big[\lambda,\Delta_n^m,\mathcal{M},u,p,||.,...,.||\big]$ and $W_\infty^I\big[\lambda,\Delta_n^m,\mathcal{M},u,p,||.,...,.||\big]$.
By a lacunary sequence $\theta=(k_r);r=0,1,2,...$ where $k_0=0$, we shall mean an increasing sequence of non-negative integers with $k_r-k_{r-1}\to\infty$ as $r\to\infty$. The intervals determined by θ will be denoted by $I_r=(k_{r-1},k_r]$ and $h_r=k_r-k_{r-1}$. We finally arrived, let

$$a_{nk}=\begin{cases}\frac{1}{h_r}, & \text{if } k_{r-1}<k<k_r\\ 0, & \text{otherwise.}\end{cases}$$

Then the above classes of sequences are denoted by $W^I\big[\theta,\Delta_n^m,\mathcal{M},p,||.,...,.||\big]$, $W_0^I\big[\theta,\Delta_n^m,\mathcal{M},p,||.,...,.||\big]$ and $W_\infty^I\big[\theta,\Delta_n^m,\mathcal{M},p,||.,...,.||\big]$.
The following inequality will be used throughout the paper. If $0\leq p_k\leq\sup p_k=G$, $D=\max(1,2^{G-1})$ then

$$|a_k+b_k|^{p_k}\leq D\{|a_k|^{p_k}+|b_k|^{p_k}\}\tag{1.1}$$

for all k and $a_k,b_k\in\mathbb{C}$. Also $|a|^{p_k}\leq\max(1,|a|^G)$ for all $a\in\mathbb{C}$.

The main aim of this paper is to introduce some generalized difference sequence spaces defined by ideal convergence, a Musielak-Orlicz function $\mathcal{M}=(M_k)$ and an infinite matrix $A=(a_{nk})$. I have also make an effort to study some inclusion relations and their topological properties.

2. Main Results

Theorem 2.1 *Let $\mathcal{M}=(M_k)$ be a Musielak-Orlicz function, $p=(p_k)$ be a bounded sequence of positive real numbers and $u=(u_k)$ be a sequence of strictly positive real numbers. Then $W^I\big[A,\Delta_n^m,\mathcal{M},u,p,||.,...,.||\big]$, $W_0^I\big[A,\Delta_n^m,\mathcal{M},u,p,||.,...,.||\big]$ and $W_\infty^I\big[A,\Delta_n^m,\mathcal{M},u,p,||.,...,.||\big]$ are linear spaces over the field of complex numbers $\mathbb{C}$.*

Proof. We shall prove the result for the space $W_0^I\big[A,\Delta_n^m,\mathcal{M},u,p,||.,...,.||\big]$. Let $x=(x_k)$

and $y = (y_k)$ be two elements of $W_0^I\big[A, \Delta_n^m, \mathcal{M}, u, p, ||., ..., .||\big]$. Then there exists $\rho_1 > 0$ and $\rho_2 > 0$ and for $z_1, z_2, ..., z_{n-1} \in X$ such that

$$A_{\frac{\epsilon}{2}} = \Big\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^m x_k}{\rho_1}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \geq \frac{\epsilon}{2}\Big\} \in I$$

and

$$B_{\frac{\epsilon}{2}} = \Big\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^m y_k}{\rho_2}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \geq \frac{\epsilon}{2}\Big\} \in I.$$

Let $\alpha, \beta \in \mathbb{C}$. Since $||., ..., .||$ is a n-norm, Δ_n^m is linear and the contributing of $\mathcal{M} = (M_k)$, the following inequality holds:

$$\begin{aligned}
&\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^m(\alpha x_k + \beta y_k)}{|\alpha|\rho_1 + |\beta|\rho_2}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \\
&\leq D\sum_{k=1}^{\infty} a_{nk}\Big[\frac{|\alpha|}{|\alpha|\rho_1 + |\beta|\rho_2}M_k\Big(||\frac{u_k\Delta_n^m x_k}{\rho_1}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \\
&+ D\sum_{k=1}^{\infty} a_{nk}\Big[\frac{|\beta|}{|\alpha|\rho_1 + |\beta|\rho_2}M_k\Big(||\frac{u_k\Delta_n^m y_k}{\rho_2}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \\
&\leq DK\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^m x_k}{\rho_1}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \\
&+ DK\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^m y_k}{\rho_2}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k}
\end{aligned}$$

where $K = \max\Big\{1, \frac{|\alpha|}{|\alpha|\rho_1+|\beta|\rho_2}, \frac{|\beta|}{|\alpha|\rho_1+|\beta|\rho_2}\Big\}$.
From the above relation , we get

$$\begin{aligned}
&\Big\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^m(\alpha x_k + \beta y_k)}{|\alpha|\rho_1 + |\beta|\rho_2}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \geq \epsilon\Big\} \\
&\subseteq \Big\{n \in \mathbb{N} : DK\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^m x_k}{\rho_1}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \geq \frac{\epsilon}{2}\Big\} \\
&\cup \Big\{n \in \mathbb{N} : DK\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^m y_k}{\rho_2}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \geq \frac{\epsilon}{2}\Big\}.
\end{aligned}$$

Since both the sets on the R.H.S of above relation are belongs to I, so the set on the L.H.S of the inclusion relation belongs to I. Similarly we can prove other cases. This completes the proof of the theorem.

Theorem 2.2 *Let $\mathcal{M}' = (M_k')$ and $\mathcal{M}'' = (M_k'')$ be two Musielak-orlicz functions. Then we have $W_0^I\big[A, \Delta_n^m, \mathcal{M}', u, p, ||., ..., .||\big] \cap W_0^I\big[A, \Delta_n^m, \mathcal{M}'', u, p, ||., ..., .||\big] \subseteq W_0^I\big[A, \Delta_n^m, \mathcal{M}' + \mathcal{M}'', u, p, ||., ..., .||\big]$.*

Proof. Let $x = (x_k) \in W_0^I\big[A, \Delta_n^m, \mathcal{M}', u, p, ||., ..., .||\big] \cap W_0^I\big[A, \Delta_n^m, \mathcal{M}'', u, p, ||., ..., .||\big]$.

Then we get the result by the following inequality:
$\sum_{k=1}^{\infty} a_{nk}\Big[(M'_k+M''_k)\Big(||\frac{u_k\Delta_n^m x_k}{\rho},z_1,z_2,...,z_{n-1}||\Big)\Big]^{p_k}$

$$\begin{aligned}
\leq \quad & D\sum_{k=1}^{\infty} a_{nk}\Big[M'_k\Big(||\frac{u_k\Delta_n^m x_k}{\rho},z_1,z_2,...,z_{n-1}||\Big)\Big]^{p_k}\\
+ \quad & D\sum_{k=1}^{\infty} a_{nk}\Big[M''_k\Big(||\frac{u_k\Delta_n^m x_k}{\rho},z_1,z_2,...,z_{n-1}||\Big)\Big]^{p_k}.
\end{aligned}$$

Hence
$\Big\{n\in\mathbb{N}:\sum_{k=1}^{\infty} a_{nk}\Big[(M'_k+M''_k)\Big(||\frac{u_k\Delta_n^m x_k}{\rho},z_1,z_2,...,z_{n-1}||\Big)\Big]^{p_k}\geq\epsilon\Big\}$

$$\begin{aligned}
\subseteq \quad & \Big\{n\in\mathbb{N}:D\sum_{k=1}^{\infty} a_{nk}\Big[M'_k\Big(||\frac{u_k\Delta_n^m x_k}{\rho},z_1,z_2,...,z_{n-1}||\Big)\Big]^{p_k}\geq\frac{\epsilon}{2}\Big\}\\
\cup \quad & \Big\{n\in\mathbb{N}:D\sum_{k=1}^{\infty} a_{nk}\Big[M''_k\Big(||\frac{u_k\Delta_n^m x_k}{\rho},z_1,z_2,...,z_{n-1}||\Big)\Big]^{p_k}\geq\frac{\epsilon}{2}\Big\}
\end{aligned}$$

Since both the sets on the R.H.S of above relation are belongs to I, so the set on the L.H.S of the inclusion relation belongs to I. This completes the proof of the theorem.

Theorem 2.3 *The inclusions* $Z\big[\Delta_n^{m-1},\mathcal{M},u,p,||.,...,.||\big]\subseteq Z\big[A,\Delta_n^m,\mathcal{M},u,p,||.,...,.||\big]$ *are strict for* $m\geq 1$. *In general* $Z\big[\Delta_n^{m-1},\mathcal{M},u,p,||.,...,.||\big]\subseteq Z\big[A,\Delta_n^m,\mathcal{M},u,p,||.,...,.||\big]$, *for* $m=0,1,2,...$ *where* $Z=W^I,W_0^I,W_\infty^I$.

Proof. We give the proof for $W_0^I\big[A,\Delta_n^{m-1},\mathcal{M},u,p,||.,...,.||\big]$ only. The others can be proved by similar argument. Let $x=(x_k)$ be any element in the space $W_0^I\big[A,\Delta_n^{m-1},\mathcal{M},u,p,||.,...,.||\big]$. Let $\epsilon>0$ be given. Then there exists $\rho>0$ such that the set

$$\Big\{n\in\mathbb{N}:\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^{m-1} x_k}{\rho},z_1,z_2,...,z_{n-1}||\Big)\Big]^{p_k}\geq\epsilon\Big\}\in I.$$

Since $\mathcal{M} = (M_k)$ is non-decreasing and convex for every k, it follows that

$$\begin{aligned}
&\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^m x_k}{2\rho}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \\
&= \sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^{m-1}x_{k+1} - u_k\Delta_n^{m-1}x_k}{2\rho}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \\
&\leq D\sum_{k=1}^{\infty} a_{nk}\Big[\frac{1}{2}M_k\Big(||\frac{u_k\Delta_n^{m-1}x_{k+1}}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \\
&+ D\sum_{k=1}^{\infty} a_{nk}\Big[\frac{1}{2}M_k\Big(||\frac{u_k\Delta_n^{m-1}x_k}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \\
&\leq DH\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^{m-1}x_{k+1}}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \\
&+ DH\sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^{m-1}x_k}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k},
\end{aligned}$$

where $H = \max\Big\{1, (\frac{1}{2})^G\Big\}$. Thus we have

$$\begin{aligned}
&\Big\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^m x_k}{2\rho}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \geq \epsilon\Big\} \\
&\subseteq \Big\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^{m-1}x_{k+1}}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \geq \frac{\epsilon}{2}\Big\} \\
&\cup \Big\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^{m-1}x_k}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \geq \frac{\epsilon}{2}\Big\}
\end{aligned}$$

Since both the sets in right hand side of the above relation belongs to I, therefore we get the set

$$\Big\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}||\Big)\Big]^{p_k} \geq \epsilon\Big\} \in I.$$

This inclusion is strict follows from the following example.

Example. Let $M_k(x) = x$, for all $k \in \mathbb{N}$, $u_k = p_k = 1$ for all $k \in \mathbb{N}$ and $A = (C, 1)$, the Cesaro matrix. Now consider a sequence $x = (x_k) = (k^s)$. Then for $n = 1, x = (x_k)$ belongs to $W_0^I\big[\Delta_n^m, \mathcal{M}, u, p, ||., ..., .||\big]$ but does not belongs to $W_0^I\big[\Delta_n^{m-1}, \mathcal{M}, u, p, ||., ..., .||\big]$, because $\Delta_n^m x_k = 0$ and $\Delta_n^{m-1}x_k = (-1)^{m-1}(m-1)!$.

Theorem 2.4 *For any two sequences $p = (p_k)$ and $q = (q_k)$ of positive real numbers and for any two n-norms $||., ..., .||_1$ and $||., ..., .||_2$ on X, we have the following*

$Z\big[A, \Delta_n^m, \mathcal{M}, u, p, ||., ..., .||_1\big] \cap Z\big[A, \Delta_n^m, \mathcal{M}, u, q, ||., ..., .||_2\big] \neq \phi$ *where* $Z = W^I, W_0^I$ *and* W_∞^I.

Proof. Since the zero element belongs to both the classes of sequences, so the intersection is non-empty.

Theorem 2.5 *The sequence spaces* $W_0^I\left[A,\Delta_n^m,\mathcal{M},u,p,||.,...,.||\right]$ *and* $W_\infty^I\left[A,\Delta_n^m,\mathcal{M},u,p,||.,...,.||\right]$ *are normal as well as monotone.*

Proof. We shall prove the theorem for $W_0^I\left[A,\Delta_n^m,\mathcal{M},u,p,||.,...,.||\right]$. Let $x=(x_k)\in W_0^I\left[A,\Delta_n^m,\mathcal{M},u,p,||.,...,.||\right]$ and $\alpha=(\alpha_k)$ be a sequence of scalars such that $|\alpha_k|\leq 1$ for all $k\in\mathbb{N}$. Then for given $\epsilon>0$, we have

$$\Big\{n\in\mathbb{N}:\sum_{k=1}^{\infty}a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^m(\alpha_k x_k)}{\rho},z_1,z_2,...,z_{n-1}||\Big)\Big]^{p_k}\geq\epsilon\Big\}$$

$$\subseteq\quad\Big\{n\in\mathbb{N}:\sum_{k=1}^{\infty}a_{nk}\Big[M_k\Big(||\frac{u_k\Delta_n^m(x_k)}{\rho},z_1,z_2,...,z_{n-1}||\Big)\Big]^{p_k}\geq\epsilon\Big\}\in I.$$

Hence $\alpha_k x_k\in W_0^I\left[A,\Delta_n^m,\mathcal{M},u,p,||.,...,.||\right]$. Thus the space $W_0^I\left[A,\Delta_n^m,\mathcal{M},u,p,||.,...,.||\right]$ is normal. Therefore $W_0^I\left[A,\Delta_n^m,\mathcal{M},u,p,||.,...,.||\right]$ is monotone also (see [12]). Similarly we can prove the theorem for other case. This completes the proof of the theorem.

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A general stability theorem for a class of functional equations including quadratic-additive functional equations

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Abstract. We prove a general stability theorem of an n-dimensional quadratic-additive type functional equation

$$Df(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{m} c_i f\big(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n\big) = 0$$

by using the direct method.

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Key Words: generalized Hyers-Ulam stability; functional equation; n-dimensional quadratic-additive type functional equation; quadratic-additive mapping; direct method.

1 Introduction

Let G_1 and G_2 be abelian groups. For any mapping $f : G_1 \to G_2$, let us define

$$\begin{aligned} Af(x, y) &:= f(x + y) - f(x) - f(y), \\ Qf(x, y) &:= f(x + y) + f(x - y) - 2f(x) - 2f(y) \end{aligned}$$

for all $x, y \in G_1$. A mapping $f : G_1 \to G_2$ is called an additive mapping (or a quadratic mapping) if f satisfies the functional equation $Af(x, y) = 0$ (or $Qf(x, y) = 0$) for all $x, y \in G_1$. We notice that the mappings $g, h : \mathbb{R} \to \mathbb{R}$ given by $g(x) = ax$ and $h(x) = ax^2$ are solutions of $Ag(x, y) = 0$ and $Qh(x, y) = 0$, respectively.

A mapping $f : G_1 \to G_2$ is called a quadratic-additive mapping if and only if f is represented by the sum of an additive mapping and a quadratic mapping. A functional equation is called a quadratic-additive type functional equation if and only if each of its solutions is a quadratic-additive mapping (see [9]). For example,

the mapping $f(x) = ax^2 + bx$ is a solution of the quadratic-additive type functional equation.

In the study of stability problems of quadratic-additive type functional equations, we have followed out a routine and monotonous procedure for proving the stability of the quadratic-additive type functional equations under various conditions. We can find in the books [2, 3, 7, 8] a lot of references concerning the Hyers-Ulam stability of functional equations (see also [1, 4, 5, 6, 14, 15]).

Throughout this paper, let V and W be real vector spaces, let X and Y be a real normed space resp. a real Banach space, and let $\mathbb{N}_0$ denote the set of all nonnegative integers.

In this paper, we prove a general stability theorem that can be easily applied to the (generalized) Hyers-Ulam stability of a large class of functional equations of the form $Df(x_1, x_2, \ldots, x_n) = 0$, which includes quadratic-additive type functional equations. In practice, given a mapping $f : V \to W$, $Df : V^n \to W$ is defined by

$$Df(x_1, x_2, \ldots, x_n) := \sum_{i=1}^{m} c_i f\big(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n\big) \tag{1.1}$$

for all $x_1, x_2, \ldots, x_n \in V$, where m is a positive integer and c_i, a_{ij} are real constants.

Indeed, this stability theorem can save us much trouble of proving the stability of relevant solutions repeatedly appearing in the stability problems for various functional equations (see [11, 12, 13]).

2 Preliminaries

Let V and W be real vector spaces and let X and Y be a real normed space resp. a real Banach space. For a given mapping $f : V \to W$, we use the following abbreviations

$$f_o(x) := \frac{f(x) - f(-x)}{2} \quad \text{and} \quad f_e(x) := \frac{f(x) + f(-x)}{2}$$

for all $x \in V$.

We now introduce a lemma from the paper [10, Corollary 2].

Lemma 2.1 *Let $k > 1$ be a real constant, let $\phi : V\backslash\{0\} \to [0, \infty)$ be a function satisfying either*

$$\Phi(x) := \sum_{i=0}^{\infty} \frac{1}{k^i}\phi(k^i x) < \infty \tag{2.1}$$

for all $x \in V\backslash\{0\}$ or

$$\Phi(x) := \sum_{i=0}^{\infty} k^{2i}\phi\left(\frac{x}{k^i}\right) < \infty \tag{2.2}$$

for all $x \in V\backslash\{0\}$, and let $f : V \to Y$ be an arbitrarily given mapping. If there exists a mapping $F : V \to Y$ satisfying

$$\|f(x) - F(x)\| \le \Phi(x) \tag{2.3}$$

for all $x \in V\backslash\{0\}$ *and*

$$F_e(kx) = k^2 F_e(x), \quad F_o(kx) = kF_o(x) \tag{2.4}$$

for all $x \in V$*, then* F *is a unique mapping satisfying* (2.3) *and* (2.4)*.*

We introduce a lemma that is the same as [10, Corollary 3].

Lemma 2.2 *Let* $k > 1$ *be a real number, let* $\phi, \psi : V\backslash\{0\} \to [0,\infty)$ *be functions satisfying each of the following conditions*

$$\sum_{i=0}^{\infty} k^i \psi\left(\frac{x}{k^i}\right) < \infty, \qquad \sum_{i=0}^{\infty} \frac{1}{k^{2i}} \phi(k^i x) < \infty,$$

$$\tilde{\Phi}(x) := \sum_{i=0}^{\infty} k^i \phi\left(\frac{x}{k^i}\right) < \infty, \quad \tilde{\Psi}(x) := \sum_{i=0}^{\infty} \frac{1}{k^{2i}} \psi(k^i x) < \infty$$

for all $x \in V\backslash\{0\}$*, and let* $f : V \to Y$ *be an arbitrarily given mapping. If there exists a mapping* $F : V \to Y$ *satisfying the inequality*

$$\|f(x) - F(x)\| \le \tilde{\Phi}(x) + \tilde{\Psi}(x) \tag{2.5}$$

for all $x \in V\backslash\{0\}$ *and the conditions in* (2.4) *for all* $x \in V$*, then* F *is a unique mapping satisfying* (2.4) *and* (2.5)*.*

3 Main results

In this section, let a be a real constant with $a \notin \{-1, 0, 1\}$. Lemma 2.1 plays an important role in the proofs of the following two main theorems.

Theorem 3.1 *Let* n *be a fixed integer greater than* 1*, let* $\mu : V\backslash\{0\} \to [0,\infty)$ *be a function satisfying the condition*

$$\begin{cases} \displaystyle\sum_{i=0}^{\infty} \frac{\mu(a^i x)}{a^{2i}} < \infty \quad \textit{when } |a| < 1, \\ \displaystyle\sum_{i=0}^{\infty} \frac{\mu(a^i x)}{|a|^i} < \infty \quad \textit{when } |a| > 1 \end{cases} \tag{3.1}$$

for all $x \in V\backslash\{0\}$*, and let* $\varphi : (V\backslash\{0\})^n \to [0,\infty)$ *be a function satisfying the condition*

$$\begin{cases} \displaystyle\sum_{i=0}^{\infty} \frac{\varphi(a^i x_1, a^i x_2, \ldots, a^i x_n)}{a^{2i}} < \infty \quad \textit{when } |a| < 1, \\ \displaystyle\sum_{i=0}^{\infty} \frac{\varphi(a^i x_1, a^i x_2, \ldots, a^i x_n)}{|a|^i} < \infty \quad \textit{when } |a| > 1 \end{cases} \tag{3.2}$$

for all $x_1, x_2, \ldots, x_n \in V\backslash\{0\}$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$,

$$\left\| f(ax) - \frac{a^2+a}{2} f(x) - \frac{a^2-a}{2} f(-x) \right\| \leq \mu(x) \tag{3.3}$$

for all $x \in V\backslash\{0\}$, and

$$\|Df(x_1, x_2, \ldots, x_n)\| \leq \varphi(x_1, x_2, \ldots, x_n) \tag{3.4}$$

for all $x_1, x_2, \ldots, x_n \in V\backslash\{0\}$, then there exists a unique mapping $F : V \to Y$ such that

$$DF(x_1, x_2, \ldots, x_n) = 0 \tag{3.5}$$

for all $x_1, x_2, \ldots, x_n \in V\backslash\{0\}$,

$$F_e(ax) = a^2 F_e(x) \quad \textit{and} \quad F_o(ax) = aF_o(x) \tag{3.6}$$

for all $x \in V$, and

$$\|f(x) - F(x)\| \leq \sum_{i=0}^{\infty} \left(\frac{\mu(a^i x) + \mu(-a^i x)}{2a^{2i+2}} + \frac{\mu(a^i x) + \mu(-a^i x)}{2|a|^{i+1}} \right) \tag{3.7}$$

for all $x \in V\backslash\{0\}$.

Proof. First, we define $A := \{f : V \to Y \mid f(0) = 0\}$ and a mapping $J_m : A \to A$ by

$$J_m f(x) := \frac{f(a^m x) + f(-a^m x)}{2a^{2m}} + \frac{f(a^m x) - f(-a^m x)}{2a^m}$$

for $x \in V$ and $m \in \mathbb{N}_0$. It follows from (3.3) that

$$\begin{aligned}
&\|J_m f(x) - J_{m+l} f(x)\| \\
&\leq \sum_{i=m}^{m+l-1} \|J_i f(x) - J_{i+1} f(x)\| \\
&= \sum_{i=m}^{m+l-1} \left\| \frac{f(a^i x) + f(-a^i x)}{2a^{2i}} + \frac{f(a^i x) - f(-a^i x)}{2a^i} \right. \\
&\qquad \left. - \frac{f(a^{i+1} x) + f(-a^{i+1} x)}{2a^{2i+2}} - \frac{f(a^{i+1} x) - f(-a^{i+1} x)}{2a^{i+1}} \right\| \\
&= \sum_{i=m}^{m+l-1} \left\| - \frac{1}{2a^{i+1}} \left(f(a \cdot a^i x) - \frac{a^2+a}{2} f(a^i x) - \frac{a^2-a}{2} f(-a^i x) \right) \right. \qquad (3.8) \\
&\qquad + \frac{1}{2a^{i+1}} \left(f(-a \cdot a^i x) - \frac{a^2+a}{2} f(-a^i x) - \frac{a^2-a}{2} f(a^i x) \right) \\
&\qquad - \frac{1}{2a^{2i+2}} \left(f(a \cdot a^i x) - \frac{a^2+a}{2} f(a^i x) - \frac{a^2-a}{2} f(-a^i x) \right) \\
&\qquad \left. - \frac{1}{2a^{2i+2}} \left(f(-a \cdot a^i x) - \frac{a^2+a}{2} f(-a^i x) - \frac{a^2-a}{2} f(a^i x) \right) \right\| \\
&\leq \sum_{i=m}^{m+l-1} \left(\frac{\mu(a^i x) + \mu(-a^i x)}{2a^{2i+2}} + \frac{\mu(a^i x) + \mu(-a^i x)}{2|a|^{i+1}} \right)
\end{aligned}$$

for all $x \in V\backslash\{0\}$. In view of (3.1) and (3.8), the sequence $\{J_m f(x)\}$ is a Cauchy sequence for all $x \in V\backslash\{0\}$. Since Y is complete and $f(0) = 0$, the sequence $\{J_m f(x)\}$ converges for all $x \in V$. Hence, we can define a mapping $F : V \to Y$ by

$$F(x) := \lim_{m\to\infty} J_m f(x) = \lim_{m\to\infty} \left(\frac{f(a^m x) + f(-a^m x)}{2a^{2m}} + \frac{f(a^m x) - f(-a^m x)}{2a^m} \right)$$

for all $x \in V$.

We easily obtain from the definition of F and (3.4) that

$$\begin{aligned} F_e(ax) &= \frac{F(ax) + F(-ax)}{2} \\ &= \lim_{m\to\infty} \frac{f(a^{m+1}x) + f(-a^{m+1}x)}{2a^{2m}} \\ &= a^2 \lim_{m\to\infty} \frac{f(a^{m+1}x) + f(-a^{m+1}x)}{2a^{2m+2}} \\ &= a^2 F_e(x), \\ F_o(ax) &= \frac{F(ax) - F(-ax)}{2} \\ &= \lim_{m\to\infty} \frac{f(a^{m+1}x) - f(-a^{m+1}x)}{2a^m} \\ &= a \lim_{m\to\infty} \frac{f(a^{m+1}x) - f(-a^{m+1}x)}{2a^{m+1}} \\ &= aF_o(x) \end{aligned}$$

for all $x \in V$, and by (1.1) and (3.2), we get

$$\begin{aligned} &\|DF(x_1, x_2, \dots, x_n)\| \\ &= \lim_{m\to\infty} \Bigg\| \frac{Df\big(a^m x_1, a^m x_2, \dots, a^m x_n\big) + Df\big(-a^m x_1, -a^m x_2, \dots, -a^m x_n\big)}{2a^{2m}} \\ &\qquad + \frac{Df\big(a^m x_1, a^m x_2, \dots, a^m x_n\big) - Df\big(-a^m x_1, -a^m x_2, \dots, -a^m x_n\big)}{2a^m} \Bigg\| \\ &\le \lim_{m\to\infty} \Bigg(\frac{\varphi\big(a^m x_1, a^m x_2, \dots, a^m x_n\big) + \varphi\big(-a^m x_1, -a^m x_2, \dots, -a^m x_n\big)}{2a^{2m}} \\ &\qquad + \frac{\varphi\big(a^m x_1, a^m x_2, \dots, a^m x_n\big) + \varphi\big(-a^m x_1, -a^m x_2, \dots, -a^m x_n\big)}{2|a|^m} \Bigg) \\ &= 0 \end{aligned}$$

for all $x_1, x_2, \dots, x_n \in V\backslash\{0\}$, *i.e.*, $DF(x_1, x_2, \dots, x_n) = 0$ for all $x_1, x_2, \dots, x_n \in V\backslash\{0\}$. Moreover, if we put $m = 0$ and let $l \to \infty$ in (3.8), then we obtain the inequality (3.7).

Notice that the equalities

$$F_e(|a|x) = |a|^2 F_e(x), \quad F_e\left(\frac{x}{|a|}\right) = \frac{F_e(x)}{|a|^2},$$

$$F_o(|a|x) = |a|F_o(x), \quad F_o\left(\frac{x}{|a|}\right) = \frac{F_o(x)}{|a|}$$

are true in view of (3.6).

When $|a| > 1$, in view of Lemma 2.1, there exists a unique mapping $F : V \to Y$ satisfying the equalities in (3.6) and the inequality (3.7), since the inequality

$$\begin{aligned}\|f(x) - F(x)\| &\le \sum_{i=0}^{\infty}\left(\frac{\mu(a^i x) + \mu(-a^i x)}{2a^{2i+2}} + \frac{\mu(a^i x) + \mu(-a^i x)}{2|a|^{i+1}}\right)\\ &\le \sum_{i=0}^{\infty}\frac{\phi(|a|^i x)}{|a|^i}\\ &\le \sum_{i=0}^{\infty}\frac{\phi(k^i x)}{k^i}\end{aligned}$$

holds for all $x \in V\backslash\{0\}$, where we set $k := |a|$ and $\phi(x) := \frac{\mu(x)+\mu(-x)}{2a^2} + \frac{\mu(x)+\mu(-x)}{2|a|}$.

When $|a| < 1$, in view of Lemma 2.1, there exists a unique mapping $F : V \to Y$ satisfying the equalities in (3.6) and the inequality (3.7), since the inequality

$$\begin{aligned}\|f(x) - F(x)\| &\le \sum_{i=0}^{\infty}\left(\frac{\mu(a^i x) + \mu(-a^i x)}{2a^{2i+2}} + \frac{\mu(a^i x) + \mu(-a^i x)}{2|a|^{i+1}}\right)\\ &\le \sum_{i=0}^{\infty}\frac{\phi(|a|^i x)}{|a|^{2i}}\\ &= \sum_{i=0}^{\infty}k^{2i}\phi\left(\frac{x}{k^i}\right)\end{aligned}$$

holds for all $x \in V\backslash\{0\}$, where $k := \frac{1}{|a|}$ and $\phi(x) := \frac{\mu(x)+\mu(-x)}{2a^2} + \frac{\mu(x)+\mu(-x)}{2|a|}$. □

The proof of the following theorem runs analogously to that of the previous theorem.

Theorem 3.2 *Let n be a fixed integer greater than 1, let $\mu : V\backslash\{0\} \to [0,\infty)$ be a function satisfying the condition*

$$\begin{cases}\displaystyle\sum_{i=0}^{\infty}|a|^i\mu\left(\frac{x}{a^i}\right) < \infty & \text{when } |a| < 1,\\ \displaystyle\sum_{i=0}^{\infty}a^{2i}\mu\left(\frac{x}{a^i}\right) < \infty & \text{when } |a| > 1\end{cases} \tag{3.9}$$

for all $x \in V\backslash\{0\}$, and let $\varphi : (V\backslash\{0\})^n \to [0,\infty)$ be a function satisfying the condition

$$\begin{cases}\displaystyle\sum_{i=0}^{\infty}|a|^i\varphi\left(\frac{x_1}{a^i}, \frac{x_2}{a^i}, \dots, \frac{x_n}{a^i}\right) < \infty & \text{when } |a| < 1,\\ \displaystyle\sum_{i=0}^{\infty}a^{2i}\varphi\left(\frac{x_1}{a^i}, \frac{x_2}{a^i}, \dots, \frac{x_n}{a^i}\right) < \infty & \text{when } |a| > 1\end{cases} \tag{3.10}$$

for all $x_1, x_2, \dots, x_n \in V\backslash\{0\}$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$, (3.3) for all $x \in V\backslash\{0\}$, and (3.4) for all $x_1, x_2, \dots, x_n \in V\backslash\{0\}$, then there exists a unique mapping $F : V \to Y$ satisfying (3.5) for all $x_1, x_2, \dots, x_n \in V\backslash\{0\}$ and the conditions in (3.6) for all $x \in V$, and such that

$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} \frac{a^{2i} + |a|^i}{2} \left(\mu\left(\frac{x}{a^{i+1}}\right) + \mu\left(\frac{-x}{a^{i+1}}\right)\right) \tag{3.11}$$

for all $x \in V\backslash\{0\}$.

Proof. First, we define $A := \{f : V \to Y \mid f(0) = 0\}$ and a mapping $J_m : A \to A$ by

$$J_m f(x) := \frac{a^{2m}}{2}\left(f\left(\frac{x}{a^m}\right) + f\left(\frac{-x}{a^m}\right)\right) + \frac{a^m}{2}\left(f\left(\frac{x}{a^m}\right) - f\left(\frac{-x}{a^m}\right)\right)$$

for all $x \in V$ and $m \in \mathbb{N}_0$. It follows from (3.3) that

$$\begin{aligned}
&\|J_m f(x) - J_{m+l} f(x)\| \\
&\le \sum_{i=m}^{m+l-1} \|J_i f(x) - J_{i+1} f(x)\| \\
&= \sum_{i=m}^{m+l-1} \bigg\| \frac{a^{2i}}{2}\left(f\left(\frac{x}{a^i}\right) + f\left(\frac{-x}{a^i}\right)\right) + \frac{a^i}{2}\left(f\left(\frac{x}{a^i}\right) - f\left(\frac{-x}{a^i}\right)\right) \\
&\qquad - \frac{a^{2i+2}}{2}\left(f\left(\frac{x}{a^{i+1}}\right) + f\left(\frac{-x}{a^{i+1}}\right)\right) - \frac{a^{i+1}}{2}\left(f\left(\frac{x}{a^{i+1}}\right) - f\left(\frac{-x}{a^{i+1}}\right)\right)\bigg\| \\
&= \sum_{i=m}^{m+l-1} \bigg\| \frac{a^{2i}}{2}\left(f\left(a\frac{x}{a^{i+1}}\right) - \frac{a^2+a}{2} f\left(\frac{x}{a^{i+1}}\right) - \frac{a^2-a}{2} f\left(\frac{-x}{a^{i+1}}\right)\right) \\
&\qquad + \frac{a^{2i}}{2}\left(f\left(a\frac{-x}{a^{i+1}}\right) - \frac{a^2+a}{2} f\left(\frac{-x}{a^{i+1}}\right) - \frac{a^2-a}{2} f\left(\frac{x}{a^{i+1}}\right)\right) \\
&\qquad + \frac{a^{i}}{2}\left(f\left(a\frac{x}{a^{i+1}}\right) - \frac{a^2+a}{2} f\left(\frac{x}{a^{i+1}}\right) - \frac{a^2-a}{2} f\left(\frac{-x}{a^{i+1}}\right)\right. \\
&\qquad - \frac{a^{i}}{2}\left(f\left(a\frac{-x}{a^{i+1}}\right) - \frac{a^2+a}{2} f\left(\frac{-x}{a^{i+1}}\right) - \frac{a^2-a}{2} f\left(\frac{x}{a^{i+1}}\right)\right)\bigg\| \\
&\le \sum_{i=m}^{m+l-1} \left[\frac{a^{2i}}{2}\left(\mu\left(\frac{x}{a^{i+1}}\right) + \mu\left(\frac{-x}{a^{i+1}}\right)\right) + \frac{|a|^i}{2}\left(\mu\left(\frac{x}{a^{i+1}}\right) + \mu\left(\frac{-x}{a^{i+1}}\right)\right)\right]
\end{aligned} \tag{3.12}$$

for all $x \in V\backslash\{0\}$.

On account of (3.9) and (3.12), the sequence $\{J_m f(x)\}$ is a Cauchy sequence for all $x \in V\backslash\{0\}$. Since Y is complete and $f(0) = 0$, the sequence $\{J_m f(x)\}$ converges for all $x \in V$. Hence, we can define a mapping $F : V \to Y$ by

$$F(x) := \lim_{m\to\infty} \left[\frac{a^{2m}}{2}\left(f\left(\frac{x}{a^m}\right) + f\left(\frac{-x}{a^m}\right)\right) + \frac{a^m}{2}\left(f\left(\frac{x}{a^m}\right) - f\left(\frac{-x}{a^m}\right)\right)\right]$$

for all $x \in V$. Moreover, if we put $m = 0$ and let $l \to \infty$ in (3.12), we obtain the inequality (3.11).

In view of the definition of F and (3.4), we get the equalities in (3.6) for all $x \in V$ and

$$\begin{aligned}
&\|DF(x_1, x_2, \dots, x_n)\| \\
&= \lim_{m\to\infty} \Big\| \frac{a^{2m}}{2}\Big(Df\Big(\frac{x_1}{a^m}, \frac{x_2}{a^m}, \dots, \frac{x_n}{a^m}\Big) + Df\Big(\frac{-x_1}{a^m}, \frac{-x_2}{a^m}, \dots, \frac{-x_n}{a^m}\Big)\Big) \\
&\qquad + \frac{a^m}{2}\Big(Df\Big(\frac{x_1}{a^m}, \frac{x_2}{a^m}, \dots, \frac{x_n}{a^m}\Big) - Df\Big(\frac{-x_1}{a^m}, \frac{-x_2}{a^m}, \dots, \frac{-x_n}{a^m}\Big)\Big)\Big\| \\
&\le \lim_{m\to\infty} \Big[\frac{a^{2m}}{2}\Big(\varphi\Big(\frac{x_1}{a^m}, \frac{x_2}{a^m}, \dots, \frac{x_n}{a^m}\Big) + \varphi\Big(\frac{-x_1}{a^m}, \frac{-x_2}{a^m}, \dots, \frac{-x_n}{a^m}\Big)\Big) \\
&\qquad + \frac{|a|^m}{2}\Big(\varphi\Big(\frac{x_1}{a^m}, \frac{x_2}{a^m}, \dots, \frac{x_n}{a^m}\Big) + \varphi\Big(\frac{-x_1}{a^m}, \frac{-x_2}{a^m}, \dots, \frac{-x_n}{a^m}\Big)\Big)\Big] \\
&= 0
\end{aligned}$$

for all $x_1, x_2, \dots, x_n \in V\backslash\{0\}$, *i.e.*, $DF(x_1, x_2, \dots, x_n) = 0$ for all $x_1, x_2, \dots, x_n \in V\backslash\{0\}$. We notice that the equalities

$$F_e(|a|x) = |a|^2 F_e(x), \quad F_e\Big(\frac{x}{|a|}\Big) = \frac{F_e(x)}{|a|^2},$$
$$F_o(|a|x) = |a| F_o(x), \quad F_o\Big(\frac{x}{|a|}\Big) = \frac{F_o(x)}{|a|}$$

hold in view of (3.6).

When $|a| > 1$, according to Lemma 2.1, there exists a unique mapping $F : V \to Y$ satisfying the equalities in (3.6) and the inequality (3.11), since the inequality

$$\begin{aligned}
\|f(x) - F(x)\| &\le \sum_{i=0}^{\infty} \Big[\frac{a^{2i}}{2}\Big(\mu\Big(\frac{x}{a^{i+1}}\Big) + \mu\Big(\frac{-x}{a^{i+1}}\Big)\Big) + \frac{|a|^i}{2}\Big(\mu\Big(\frac{x}{a^{i+1}}\Big) + \mu\Big(\frac{-x}{a^{i+1}}\Big)\Big)\Big] \\
&\le \sum_{i=0}^{\infty} |a|^{2i}\phi\Big(\frac{x}{|a|^i}\Big) \\
&= \sum_{i=0}^{\infty} k^{2i}\phi\Big(\frac{x}{k^i}\Big)
\end{aligned}$$

holds for all $x \in V\backslash\{0\}$, where $k := |a|$ and $\phi(x) := \mu\big(\frac{x}{a}\big) + \mu\big(\frac{-x}{a}\big)$.

When $|a| < 1$, according to Lemma 2.1, there exists a unique mapping $F : V \to Y$ satisfying the equalities in (3.6) and the inequality (3.11), since the inequality

$$\begin{aligned}
\|f(x) - F(x)\| &\le \sum_{i=0}^{\infty} \Big[\frac{a^{2i}}{2}\Big(\mu\Big(\frac{x}{a^{i+1}}\Big) + \mu\Big(\frac{-x}{a^{i+1}}\Big)\Big) + \frac{|a|^i}{2}\Big(\mu\Big(\frac{x}{a^{i+1}}\Big) + \mu\Big(\frac{-x}{a^{i+1}}\Big)\Big)\Big] \\
&\le \sum_{i=0}^{\infty} |a|^{i}\phi\Big(\frac{x}{|a|^i}\Big) \\
&\le \sum_{i=0}^{\infty} \frac{\phi(k^i x)}{k^i}
\end{aligned}$$

holds for all $x \in V\backslash\{0\}$, where $k := \frac{1}{|a|}$ and $\phi(x) := \mu\big(\frac{x}{a}\big) + \mu\big(\frac{-x}{a}\big)$. □

Lemma 2.2 is necessary for the proof of the following main theorem.

Theorem 3.3 *Let n be a fixed integer greater than 1, let $\mu : V\backslash\{0\} \to [0,\infty)$ be a function satisfying the condition*

$$\begin{cases} \sum_{i=0}^{\infty} \frac{\mu(a^i x)}{a^{2i}} < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} |a|^i \mu\Big(\frac{x}{a^i}\Big) < \infty \quad \text{when } |a| > 1, \\ \sum_{i=0}^{\infty} \frac{\mu(a^i x)}{|a|^i} < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} a^{2i} \mu\Big(\frac{x}{a^i}\Big) < \infty \quad \text{when } |a| < 1 \end{cases} \tag{3.13}$$

for all $x \in V\backslash\{0\}$, and let $\varphi : (V\backslash\{0\})^n \to [0,\infty)$ be a function satisfying the conditions

$$\begin{cases} \sum_{i=0}^{\infty} \frac{\varphi(a^i x_1, a^i x_2, \ldots, a^i x_n)}{a^{2i}} < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} |a|^i \varphi\Big(\frac{x_1}{a^i}, \frac{x_2}{a^i}, \ldots, \frac{x_n}{a^i}\Big) < \infty \\ \qquad \text{when } |a| > 1, \\ \sum_{i=0}^{\infty} \frac{\varphi(a^i x_1, a^i x_2, \ldots, a^i x_n)}{|a|^i} < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} a^{2i} \varphi\Big(\frac{x_1}{a^i}, \frac{x_2}{a^i}, \ldots, \frac{x_n}{a^i}\Big) < \infty \\ \qquad \text{when } |a| < 1 \end{cases} \tag{3.14}$$

for all $x_1, x_2, \ldots, x_n \in V\backslash\{0\}$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$ and the inequality (3.3) for all $x \in V\backslash\{0\}$ and (3.4) for all $x_1, x_2, \ldots, x_n \in V\backslash\{0\}$, then there exists a unique mapping $F : V \to Y$ satisfying the equality (3.5) for all $x_1, x_2, \ldots, x_n \in V\backslash\{0\}$, the equalities in (3.6) for all $x \in V$, and

$$\|f(x) - F(x)\| \le \begin{cases} \sum_{i=0}^{\infty} \Big[\frac{\mu(a^i x) + \mu(-a^i x)}{2a^{2i+2}} + \frac{|a|^i}{2}\Big(\mu\Big(\frac{x}{a^{i+1}}\Big) + \mu\Big(\frac{-x}{a^{i+1}}\Big)\Big)\Big] \\ \qquad \text{when } |a| > 1, \\ \sum_{i=0}^{\infty} \Big[\frac{a^{2i}}{2}\Big(\mu\Big(\frac{x}{a^{i+1}}\Big) + \mu\Big(\frac{-x}{a^{i+1}}\Big)\Big) + \frac{\mu(a^i x) + \mu(-a^i x)}{2|a|^{i+1}}\Big] \\ \qquad \text{when } |a| < 1 \end{cases} \tag{3.15}$$

for all $x \in V\backslash\{0\}$.

Proof. We will divide the proof of this theorem into two cases, one is for $|a| > 1$ and the other is for $|a| < 1$.

Case 1. Assume that $|a| > 1$. We define a set $A := \{f : V \to Y \mid f(0) = 0\}$ and a mapping $J_m : A \to A$ by

$$J_m f(x) := \frac{f(a^m x) + f(-a^m x)}{2a^{2m}} + \frac{a^m}{2}\Big(f\Big(\frac{x}{a^m}\Big) - f\Big(\frac{-x}{a^m}\Big)\Big)$$

for all $x \in V$ and $m \in \mathbb{N}_0$. It follows from (3.3) that

$$\begin{aligned}
&\|J_m f(x) - J_{m+l} f(x)\| \\
&\quad \leq \sum_{i=m}^{m+l-1} \|J_i f(x) - J_{i+1} f(x)\| \\
&\quad = \sum_{i=m}^{m+l-1} \bigg\| \frac{f(a^i x) + f(-a^i x)}{2a^{2i}} + \frac{a^i}{2}\bigg(f\Big(\frac{x}{a^i}\Big) - f\Big(\frac{-x}{a^i}\Big)\bigg) \\
&\qquad\qquad - \frac{f(a^{i+1}x) + f(-a^{i+1}x)}{2a^{2i+2}} - \frac{a^{i+1}}{2}\bigg(f\Big(\frac{x}{a^{i+1}}\Big) - f\Big(\frac{-x}{a^{i+1}}\Big)\bigg)\bigg\| \\
&\quad = \sum_{i=m}^{m+l-1} \bigg\| - \frac{1}{2a^{2i+2}}\bigg(f(a\cdot a^i x) - \frac{a^2+a}{2} f(a^i x) - \frac{a^2-a}{2} f(-a^i x)\bigg) \qquad (3.16) \\
&\qquad\qquad - \frac{1}{2a^{2i+2}}\bigg(f(-a\cdot a^i x) - \frac{a^2+a}{2} f(-a^i x) - \frac{a^2-a}{2} f(a^i x)\bigg) \\
&\qquad\qquad + \frac{a^i}{2}\bigg(f\Big(a\frac{x}{a^{i+1}}\Big) - \frac{a^2+a}{2} f\Big(\frac{x}{a^{i+1}}\Big) - \frac{a^2-a}{2} f\Big(\frac{-x}{a^{i+1}}\Big)\bigg) \\
&\qquad\qquad - \frac{a^i}{2}\bigg(f\Big(a\frac{-x}{a^{i+1}}\Big) - \frac{a^2+a}{2} f\Big(\frac{-x}{a^{i+1}}\Big) - \frac{a^2-a}{2} f\Big(\frac{x}{a^{i+1}}\Big)\bigg)\bigg\| \\
&\quad \leq \sum_{i=m}^{m+l-1} \bigg[\frac{\mu(a^i x) + \mu(-a^i x)}{2a^{2i+2}} + \frac{|a|^i}{2}\bigg(\mu\Big(\frac{x}{a^{i+1}}\Big) + \mu\Big(\frac{-x}{a^{i+1}}\Big)\bigg)\bigg]
\end{aligned}$$

for all $x \in V\backslash\{0\}$.

In view of (3.13) and (3.16), the sequence $\{J_m f(x)\}$ is a Cauchy sequence for all $x \in V\backslash\{0\}$. Since Y is complete and $f(0) = 0$, the sequence $\{J_m f(x)\}$ converges for all $x \in V$. Hence, we can define a mapping $F : V \to Y$ by

$$F(x) := \lim_{m\to\infty} \bigg[\frac{f(a^m x) + f(-a^m x)}{2a^{2m}} + \frac{a^m}{2}\bigg(f\Big(\frac{x}{a^m}\Big) - f\Big(\frac{-x}{a^m}\Big)\bigg)\bigg]$$

for all $x \in V$. Moreover, if we put $m = 0$ and let $l \to \infty$ in (3.16), we obtain the first inequality of (3.15).

Using the definition of F, (3.4), and (3.14), we get the equalities in (3.6) for all $x \in V$ and

$$\begin{aligned}
&\|DF(x_1, x_2, \ldots, x_n)\| \\
&\quad = \lim_{m\to\infty} \bigg\| \frac{Df\big(a^m x_1, a^m x_2, \ldots, a^m x_n\big) + Df\big(-a^m x_1, -a^m x_2, \ldots, -a^m x_n\big)}{2a^{2m}} \\
&\qquad\qquad + \frac{a^m}{2}\bigg(Df\Big(\frac{x_1}{a^m}, \frac{x_2}{a^m}, \ldots, \frac{x_n}{a^m}\Big) - Df\Big(\frac{-x_1}{a^m}, \frac{-x_2}{a^m}, \ldots, \frac{-x_n}{a^m}\Big)\bigg)\bigg\| \\
&\quad \leq \lim_{m\to\infty} \bigg[\frac{\varphi\big(a^m x_1, a^m x_2, \ldots, a^m x_n\big) + \varphi\big(-a^m x_1, -a^m x_2, \ldots, -a^m x_n\big)}{2a^{2m}} \\
&\qquad\qquad + \frac{|a|^m}{2}\bigg(\varphi\Big(\frac{x_1}{a^m}, \frac{x_2}{a^m}, \ldots, \frac{x_n}{a^m}\Big) + \varphi\Big(\frac{-x_1}{a^m}, \frac{-x_2}{a^m}, \ldots, \frac{-x_n}{a^m}\Big)\bigg)\bigg] \\
&\quad = 0
\end{aligned}$$

for all $x_1, x_2, \ldots, x_n \in V\backslash\{0\}$, *i.e.*, $DF(x_1, x_2, \ldots, x_n) = 0$ for all $x_1, x_2, \ldots, x_n \in V\backslash\{0\}$. We notice that the equalities

$$F_e(|a|x) = |a|^2 F_e(x) \quad \text{and} \quad F_o(|a|x) = |a|F_o(x)$$

are true in view of (3.6).

Using Lemma 2.2, we conclude that there exists a unique mapping $F : V \to Y$ satisfying the equalities in (3.6) and the first inequality in (3.15), since the inequality

$$\begin{aligned}\|f(x) - F(x)\| &\le \sum_{i=0}^{\infty} \left[\frac{\mu(a^i x) + \mu(-a^i x)}{2a^{2i+2}} + \frac{|a|^i}{2}\left(\mu\left(\frac{x}{a^{i+1}}\right) + \mu\left(\frac{-x}{a^{i+1}}\right)\right)\right] \\ &\le \sum_{i=0}^{\infty} \left(\frac{\psi(k^i x)}{k^{2i}} + k^i \phi\left(\frac{x}{k^i}\right)\right)\end{aligned}$$

holds for all $x \in V\backslash\{0\}$, where $k := |a|$, $\phi(x) := \frac{\mu\left(\frac{x}{a}\right) + \mu\left(\frac{-x}{a}\right)}{2}$, and $\psi(x) := \frac{\mu(x) + \mu(-x)}{2a^2}$.

Case 2. We now consider the case of $|a| < 1$ and define a mapping $J_m : A \to A$ by

$$J_m f(x) := \frac{a^{2m}}{2}\left(f\left(\frac{x}{a^m}\right) + f\left(\frac{-x}{a^m}\right)\right) + \frac{f(a^m x) - f(-a^m x)}{2a^m}$$

for all $x \in V$ and $n \in \mathbb{N}_0$. It follows from (3.3) that

$$\begin{aligned}
&\|J_m f(x) - J_{m+l} f(x)\| \\
&\le \sum_{i=m}^{m+l-1} \|J_i f(x) - J_{i+1} f(x)\| \\
&= \sum_{i=m}^{m+l-1} \left\| \frac{a^{2i}}{2}\left(f\left(\frac{x}{a^i}\right) + f\left(\frac{-x}{a^i}\right)\right) + \frac{f(a^i x) - f(-a^i x)}{2a^i} \right. \\
&\qquad \left. - \frac{a^{2i+2}}{2}\left(f\left(\frac{x}{a^{i+1}}\right) + f\left(\frac{-x}{a^{i+1}}\right)\right) - \frac{f(a^{i+1} x) - f(-a^{i+1} x)}{2a^{i+1}} \right\| \\
&= \sum_{i=m}^{m+l-1} \left\| \frac{a^{2i}}{2}\left(f\left(a\frac{x}{a^{i+1}}\right) - \frac{a^2 + a}{2} f\left(\frac{x}{a^{i+1}}\right) - \frac{a^2 - a}{2} f\left(\frac{-x}{a^{i+1}}\right)\right) \right. \qquad (3.17) \\
&\qquad + \frac{a^{2i}}{2}\left(f\left(a\frac{-x}{a^{i+1}}\right) - \frac{a^2 + a}{2} f\left(\frac{-x}{a^{i+1}}\right) - \frac{a^2 - a}{2} f\left(\frac{x}{a^{i+1}}\right)\right) \\
&\qquad - \frac{1}{2a^{i+1}}\left(f(a \cdot a^i x) - \frac{a^2 + a}{2} f(a^i x) - \frac{a^2 - a}{2} f(-a^i x)\right) \\
&\qquad \left. + \frac{1}{2a^{i+1}}\left(f(-a \cdot a^i x) - \frac{a^2 + a}{2} f(-a^i x) - \frac{a^2 - a}{2} f(a^i x)\right) \right\| \\
&\le \sum_{i=m}^{m+l-1} \left[\frac{a^{2i}}{2}\left(\mu\left(\frac{x}{a^{i+1}}\right) + \mu\left(\frac{-x}{a^{i+1}}\right)\right) + \frac{\mu(a^i x) + \mu(-a^i x)}{2|a|^{i+1}}\right]
\end{aligned}$$

for all $x \in V\backslash\{0\}$.

On account of (3.13) and (3.17), the sequence $\{J_m f(x)\}$ is a Cauchy sequence for all $x \in V\backslash\{0\}$. Since Y is complete and $f(0) = 0$, the sequence $\{J_m f(x)\}$ converges for all $x \in V$. Hence, we can define a mapping $F : V \to Y$ by

$$F(x) := \lim_{m\to\infty} \left[\frac{a^{2m}}{2} \left(f\left(\frac{x}{a^m}\right) + f\left(\frac{-x}{a^m}\right)\right) + \frac{f(a^m x) - f(-a^m x)}{2a^m} \right]$$

for all $x \in V$. Moreover, if we put $m = 0$ and let $l \to \infty$ in (3.17), we obtain the second inequality in (3.15).

By the definition of F, (3.4), and (3.14), we get the equalities in (3.6) for all $x \in V$ and

$$\begin{aligned}
&\|DF(x_1, x_2, \ldots, x_n)\| \\
&= \lim_{m\to\infty} \Bigg\| \frac{a^{2m}}{2} \left(Df\left(\frac{x_1}{a^m}, \frac{x_2}{a^m}, \ldots, \frac{x_n}{a^m}\right) + Df\left(\frac{-x_1}{a^m}, \frac{-x_2}{a^m}, \ldots, \frac{-x_n}{a^m}\right)\right) \\
&\qquad + \frac{Df\big(a^m x_1, a^m x_2, \ldots, a^m x_n\big) - Df\big(-a^m x_1, -a^m x_2, \ldots, -a^m x_n\big)}{2a^m} \Bigg\| \\
&\le \lim_{m\to\infty} \Bigg[\frac{a^{2m}}{2} \left(\varphi\left(\frac{x_1}{a^m}, \frac{x_2}{a^m}, \ldots, \frac{x_n}{a^m}\right) + \varphi\left(\frac{-x_1}{a^m}, \frac{-x_2}{a^m}, \ldots, \frac{-x_n}{a^m}\right)\right) \\
&\qquad + \frac{\varphi\big(a^m x_1, a^m x_2, \ldots, a^m x_n\big) + \varphi\big(-a^m x_1, -a^m x_2, \ldots, -a^m x_n\big)}{2|a|^m} \Bigg] \\
&= 0
\end{aligned}$$

for all $x_1, x_2, \ldots, x_n \in V\backslash\{0\}$, *i.e.*, $DF(x_1, x_2, \ldots, x_n) = 0$ for all $x_1, x_2, \ldots, x_n \in V\backslash\{0\}$. We remark that the equalities

$$F_e\left(\frac{x}{|a|}\right) = \frac{F_e(x)}{|a|^2} \quad \text{and} \quad F_o\left(\frac{x}{|a|}\right) = \frac{F_o(x)}{|a|}$$

hold by considering (3.6).

Using Lemma 2.2, we conclude that there exists a unique mapping $F : V \to Y$ satisfying the equalities in (3.6) and the second inequality in (3.15), since the inequality

$$\begin{aligned}
\|f(x) - F(x)\| &\le \sum_{i=0}^{\infty} \left[\frac{a^{2i}}{2} \left(\mu\left(\frac{x}{a^{i+1}}\right) + \mu\left(\frac{-x}{a^{i+1}}\right)\right) + \frac{\mu(a^i x) + \mu(-a^i x)}{2|a|^{i+1}} \right] \\
&= \sum_{i=0}^{\infty} \left[\frac{\mu(k^{i+1} x) + \mu(-k^{i+1} x)}{2k^{2i}} + \frac{k^{i+1}}{2} \left(\mu\left(\frac{x}{k^i}\right) + \mu\left(\frac{-x}{k^i}\right)\right) \right] \\
&\le \sum_{i=0}^{\infty} \left(\frac{\psi(k^i x)}{k^{2i}} + k^i \phi\left(\frac{x}{k^i}\right)\right)
\end{aligned}$$

holds for all $x \in V\backslash\{0\}$, where $k := \frac{1}{|a|}$, $\phi(x) := \frac{k}{2}\big(\mu(x) + \mu(-x)\big)$, and $\psi(x) := \frac{\mu(kx)+\mu(-kx)}{2}$. $\square$

In the following corollary, we investigate the Hyers-Ulam-Rassias stability version of Theorems 3.1, 3.2, and 3.3.

Corollary 3.4 *Let X and Y be a real normed space and a real Banach space, respectively. Let p, θ, ξ be real constants such that $p \notin \{1,2\}$, $a \notin \{-1,0,1\}$, $\xi > 0$, and $\theta > 0$. If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\left\| f(ax) - \frac{a^2+a}{2} f(x) - \frac{a^2-a}{2} f(-x) \right\| \le \xi \|x\|^p \tag{3.18}$$

for all $x \in X \backslash \{0\}$, as well as if f satisfies the inequality

$$\|Df(x_1, x_2, \ldots, x_n)\| \le \theta\big(\|x_1\|^p + \cdots + \|x_n\|^p\big) \tag{3.19}$$

for all $x_1, x_2, \ldots, x_n \in X\backslash\{0\}$, then there exists a unique mapping $F : X \to Y$ satisfying (3.5) for all $x_1, x_2, \ldots, x_n \in X\backslash\{0\}$, and the equalities in (3.6) for all $x \in X$, as well as

$$\|f(x) - F(x)\| \le \frac{\xi\|x\|^p}{|a^2 - |a|^p|} + \frac{\xi\|x\|^p}{||a| - |a|^p|} \tag{3.20}$$

for all $x \in X\backslash\{0\}$.

Proof. If we put $\varphi(x_1, x_2, \ldots, x_n) := \theta\big(\|x_1\|^p + \cdots + \|x_n\|^p\big)$ for all $x_1, x_2, \ldots, x_n \in X\backslash\{0\}$, then φ satisfies (3.2) when either $|a| > 1$ and $p < 1$ or $|a| < 1$ and $p > 2$, and φ satisfies (3.10) when either $|a| > 1$ and $p > 2$ or $|a| < 1$ and $p < 1$. Moreover, φ satisfies (3.14) when $1 < p < 2$. Therefore, by Theorems 3.1, 3.2, and 3.3, there exists a unique mapping $F : X \to Y$ such that (3.5) holds for all $x_1, x_2, \ldots, x_n \in X\backslash\{0\}$, and (3.6) holds for all $x \in X$, and such that (3.20) holds for all $x \in X\backslash\{0\}$. □

4 Quadratic-additive type functional equations

In this section, let a be a rational constant such that $a \notin \{-1,0,1\}$. Assume that the functional equation $Df(x_1, x_2, \ldots, x_n) = 0$ is a quadratic-additive type functional equation. Then $F : V \to Y$ is a solution of the functional equation $Df(x_1, x_2, \ldots, x_n) = 0$ if and only if $F : V \to Y$ is a quadratic-additive mapping. If $F : V \to Y$ is a quadratic-additive mapping, then $F_e(x)$ and $F_o(x)$ are a quadratic mapping and an additive mapping, respectively. Hence, $F_e(ax) = a^2F_e(x)$ and $F_o(ax) = aF_o(x)$ for all $x \in V$, *i.e.*, F satisfies the conditions in (3.6).

Therefore, the following theorems are direct consequences of Theorems 3.1, 3.2, and 3.3.

Theorem 4.1 *Let n be a fixed integer greater than 1, let $\mu : V \to [0,\infty)$ be a function satisfying the condition (3.1) for all $x \in V$, and let $\varphi : V^n \to [0,\infty)$ be a function satisfying the condition (3.2) for all $x_1, x_2, \ldots, x_n \in V$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$, (3.3) for all $x \in V$, and (3.4) for all $x_1, x_2, \ldots, x_n \in V$, then there exists a unique quadratic-additive mapping $F : V \to Y$ such that (3.7) holds for all $x \in V$.*

Theorem 4.2 *Let n be a fixed integer greater than 1, let $\mu : V \to [0, \infty)$ be a function satisfying the condition (3.9) for all $x \in V$, and let $\varphi : V^n \to [0, \infty)$ be a function satisfying the condition (3.10) for all $x_1, x_2, \ldots, x_n \in V$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$, (3.3) for all $x \in V$, and (3.4) for all $x_1, x_2, \ldots, x_n \in V$, then there exists a unique quadratic-additive mapping $F : V \to Y$ such that (3.11) holds for all $x \in V$.*

Theorem 4.3 *Let n be a fixed integer greater than 1, let $\mu : V \to [0, \infty)$ be a function satisfying the condition (3.13) for all $x \in V$, and let $\varphi : V^n \to [0, \infty)$ be a function satisfying the condition (3.14) for all $x_1, x_2, \ldots, x_n \in V$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$, (3.3) for all $x \in V$, and (3.4) for all $x_1, x_2, \ldots, x_n \in V$, then there exists a unique quadratic-additive mapping $F : V \to Y$ satisfying the inequality (3.15) for all $x \in V$.*

Corollary 4.4 *Let X and Y be a real normed space and a real Banach space, respectively. Let p, θ, ξ be real constants such that $p \notin \{1, 2\}$, $a \notin \{-1, 0, 1\}$, $p > 0$, $\xi > 0$, and $\theta > 0$. If a mapping $f : X \to Y$ satisfies (3.18) for all $x \in X$ and the inequality (3.19) for all $x_1, x_2, \ldots, x_n \in X$, then there exists a unique quadratic-additive mapping $F : X \to Y$ such that (3.20) holds for all $x \in X$.*

Corollary 4.5 *Let X and Y be a real normed space and a real Banach space, respectively. Let θ and ξ be real constants such that $a \notin \{-1, 0, 1\}$, $\xi > 0$, and $\theta > 0$. If a mapping $f : X \to Y$ satisfies $f(0) = 0$, and*

$$\left\| f(ax) - \frac{a^2 + a}{2} f(x) - \frac{a^2 - a}{2} f(-x) \right\| \leq \xi$$

for all $x \in X$, as well as if f satisfies the inequality

$$\| Df(x_1, x_2, \ldots, x_n) \| \leq \theta$$

for all $x_1, x_2, \ldots, x_n \in X$, then there exists a unique quadratic-additive mapping $F : X \to Y$ such that

$$\| f(x) - F(x) \| \leq \frac{\xi \|x\|^p}{|a^2 - 1|} + \frac{\xi \|x\|^p}{||a| - 1|}$$

for all $x \in X$.

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A Dynamic Programming Approach to Subsistence Consumption Constraints on Optimal Consumption and Portfolio

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We investigate an optimal consumption and portfolio selection problem of an infinitely-lived economic agent with a constant relative risk aversion (CRRA) utility function who faces subsistence consumption constraints. We provide the closed form solutions for the optimal consumption and investment policies by using the dynamic programming method and compare the solutions with those obtained by the martingale method. We show that they coincide with each other. Comparison of optimal policies with and without subsistence consumption constraints shows that the constraints have effect on the optimal consumption and portfolio policies even when the constraints do not bind.

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1 Introduction

Following the seminal contributions of Merton [6, 7] on continuous-time optimal consumption and portfolio selection problems, there have been a number of research works on the optimization problems under various economic constraints. One of the most interesting topics is optimal consumption and portfolio selection with subsistence consumption constraints (see [1, 4, 5, 8, 10, 11, 12]). Subsistence consumption constraints mean that there exists a positive minimum consumption level (that can be a constant or a deterministic/stochastic process) such that the agent can live with.

We consider the optimal consumption and investment problem with subsistence consumption constraints and a constant relative risk aversion (CRRA) utility function. We derive the optimal solutions in closed form by using the dynamic programming approach based on Karatzas *et al.* [2]. We also compare the solutions with those of Shin *et al.* [11] by using the martingale duality approach for the same optimization problem. We show that they agree with each other.

Besides the methodological contribution through the dynamic programming method, we quantitatively compare our results to those of the agent without subsistence consumption constraints. The comparison shows that the existence of the subsistence consumption constraints affects the optimal consumption and portfolio policies even when the constraints do not bind.

The prospect that the subsistence consumption constraints become binding later compels the agent to consume less and to invest in the risky asset more conservatively.

The rest of this paper is organized as follows. The financial market is introduced in Section 2. In Section 3 the optimal consumption and investment problem is considered with subsistence consumption constraints. Section 4 demonstrates the impact of the subsistence consumption constraints on the optimal policies. Section 5 summarizes the paper.

2 The Economy

In a financial market, we assume that an economic agent has investment opportunities given by a riskless asset with a constant rate of return $r > 0$ and one risky asset S_t which follows a geometric Brownian motion with a constant mean rate of return μ and a constant volatility σ, $dS_t/S_t = \mu dt + \sigma dB_t$, where B_t is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\{\mathcal{F}_t\}_{t\geq 0}$ is the $\mathbb{P}$-augmentation of the filtration generated by the standard Brownian motion $\{B_t\}_{t\geq 0}$.

A portfolio process $\boldsymbol{\pi} := \{\pi_t\}_{t\geq 0}$ meaning amounts of money invested in the risky asset at time t is a measurable process adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ and satisfies

$$\int_0^t \pi_s^2 ds < \infty, \text{ for all } t \geq 0 \text{ a.s.} \tag{1}$$

A consumption process $\mathbf{c} := \{c_t\}_{t\geq 0}$ is a measurable nonnegative process adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ and satisfies

$$\int_0^t c_s ds < \infty, \text{ for all } t \geq 0 \text{ a.s.}$$

Then, with a given initial endowment $X_0 = x > 0$, the agent's wealth process X_t at time t evolves according to

$$dX_t = [rX_t + \pi_t(\mu - r) - c_t]\,dt + \pi_t \sigma dB_t. \tag{2}$$

3 The Optimization Problem

Now we investigate the agent's optimization problem with subsistence consumption constraints. Given a positive subsistence level of consumption $R > 0$, the agent's problem is to maximize the total expected discounted utility from consumption with the constraint

$$c_t \geq R, \text{ for all } t \geq 0. \tag{3}$$

In this paper, we assume that the utility function $u(\cdot)$ is of the CRRA type

$$u(c) := \frac{c^{1-\gamma}}{1-\gamma},\ \gamma > 0\ (\gamma \neq 1),$$

where γ is the agent's coefficient of relative risk aversion. A pair $(\mathbf{c}, \boldsymbol{\pi})$ of the optimal consumption/investment processes is called *admissible* at initial capital $x > 0$, if the wealth process X_t in (2) is strictly positive and it satisfies the constraint (3). Let $\mathcal{A}(x)$ denote the set of all admissible consumption/investment pair at $x > 0$.

Then, the agent's optimization problem is given by

$$V(x) := \max_{(\mathbf{c}, \boldsymbol{\pi}) \in \mathcal{A}(x)} J(x; \mathbf{c}, \boldsymbol{\pi}), \tag{4}$$

where

$$J(x; \mathbf{c}, \boldsymbol{\pi}) := \mathbb{E}\left[\int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt\right], \tag{5}$$

subject to the budget constraint (2) and the subsistence consumption constraint (3). Here $\rho > 0$ is the subjective discount factor. In addition, we should impose a lower bound on initial wealth x as follows:

$$x > \frac{R}{r}$$

such that a pair $(\mathbf{c}, \boldsymbol{\pi})$ corresponding to the wealth dynamics (2) should be admissible (see Lemma 3.1 of Gong and Li [1]).

By the dynamic programming principle, the value function $V(x)$ in the optimization problem (4) satisfies the following Bellman equation

$$\max_{c \geq R, \pi} \left[\{rx + \pi(\mu - r) - c\} V'(x) + \frac{1}{2}\sigma^2\pi^2 V''(x) - \rho V(x) + \frac{c^{1-\gamma}}{1-\gamma} \right] = 0. \tag{6}$$

We assume that the wealth process X_t satisfies a transversality condition

$$\lim_{t \to \infty} e^{-\rho t} V(X_t) = 0, \tag{7}$$

if $V(\cdot)$ is the solution to the Bellman equation (6).

The first order conditions (FOCs) of the Bellman equation (6) for the optimal consumption/portfolio (c^*, π^*) imply

$$c^* = \left((V'(x)\right)^{-\frac{1}{\gamma}}$$

and

$$\pi^* = -\frac{\mu - r}{\sigma^2} \frac{V'(x)}{V''(x)}. \tag{8}$$

The subsistence consumption constraint (3) forces us to impose a threshold wealth level $\widetilde{x} > 0$ such that

$$c^* = \begin{cases} R, & \text{for } \ R/r < x < \widetilde{x}, \\ (V'(x))^{-\frac{1}{\gamma}}, & \text{for } \ x \geq \widetilde{x}. \end{cases} \tag{9}$$

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Substituting the FOCs (8) and (9) into the equation (6) yields

$$(rx - R)V'(x) - \frac{1}{2}\theta^2 \frac{(V'(x))^2}{V''(x)} - \rho V(x) + \frac{R^{1-\gamma}}{1-\gamma} = 0, \text{ for } \; R/r < x < \widetilde{x} \quad (10)$$

and

$$rxV'(x) - \frac{1}{2}\theta^2 \frac{(V'(x))^2}{V''(x)} - \rho V(x) + \frac{\gamma}{1-\gamma} V'(x)^{-\frac{1-\gamma}{\gamma}} = 0, \text{ for } \; x \geq \widetilde{x}, \quad (11)$$

where $\theta := (\mu - r)/\sigma$ is the market price of risk. Moreover, we define a Merton constant K such that

$$K := r + \frac{\rho - r}{\gamma} + \frac{\gamma - 1}{2\gamma^2}\theta^2 \quad (12)$$

and assume that $K > 0$ to guarantee the well-definedness of the optimization problem (4).

Lemma 3.1. *The value function V(x) in (4) is strictly concave and strictly increasing for $x > R/r$.*

Proof. The proof follows a similar line to that of Proposition 2.1 in Zariphopoulou [14]. □

Remark 3.1. *For later use, we define two quadratic algebraic equations as follows:*

$$f(m) := rm^2 - \left(\rho + r + \frac{1}{2}\theta^2\right) m + \rho = 0 \quad (13)$$

and

$$g(n) := \frac{1}{2}\theta^2 n^2 + \left(\rho - r + \frac{1}{2}\theta^2\right) n - r = 0. \quad (14)$$

$f(m) = 0$ *has two real roots* m_1 *and* m_2 *satisfying* $m_1 > 1 > m_2 > 0$ *and* $g(n) = 0$ *has two real roots* n_1 *and* n_2 *satisfying* $n_1 > 0$ *and* $n_2 < -1$. *Also*

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we have the following relationships between roots of two quadratic equations (13) and (14):

$$n_1 = \frac{1}{m_1 - 1},\ n_2 = \frac{1}{m_2 - 1}. \tag{15}$$

Theorem 3.1. *Assume that a strictly increasing and strictly concave function $v(\cdot)$ such that $v(\cdot) \in C^2(R/r, \infty)$ solves the Bellman equation (6) for $x > R/r$. Then $v(x) \geq J(x; \mathbf{c}, \boldsymbol{\pi})$ for all admissible pair $(\mathbf{c}, \boldsymbol{\pi})$. If (c_t^*, π_t^*) is the maximizer of the Bellman equation (6), then we derive*

$$v(x) = V(x) = \max_{(\mathbf{c},\boldsymbol{\pi})\in\mathcal{A}(x)} J(x; \mathbf{c}, \boldsymbol{\pi}) = J(x; \mathbf{c}^*, \boldsymbol{\pi}^*).$$

Proof. Let us define a function $U(\cdot, \cdot)$ as follows:

$$U(t, X_t) := e^{-\rho t} v(X_t). \tag{16}$$

The Itô's formula implies

$$\begin{aligned} dU(t, X_t) &= e^{-\rho t}\left[\{rX_t + \pi_t(\mu - r) - c_t\}\, v'(X_t) + \frac{1}{2}\sigma^2\pi_t^2 v''(X_t) - \rho v(X_t)\right] dt + e^{-\rho t}\sigma\pi_t v'(X_t) dB_t \\ &\leq -e^{-\rho t}\frac{c_t^{1-\gamma}}{1-\gamma} dt + e^{-\rho t}\sigma\pi_t v'(X_t) dB_t \end{aligned} \tag{17}$$

for any admissible pair (c_t, π_t) of consumption/portfolio processes. For any $t \geq 0$, we obtain

$$v(X_0) \geq \int_0^t e^{-\rho s}\frac{c_s^{1-\gamma}}{1-\gamma} ds + e^{-\rho t} v(X_t) - \int_0^t e^{-\rho s}\sigma\pi_s v'(X_s) dB_s. \tag{18}$$

From (1), the second integral of the right-hand side of (18) is a bounded local martingale and hence a martingale, so we have

$$v(x) \geq \mathbb{E}\left[\int_0^t e^{-\rho s}\frac{c_s^{1-\gamma}}{1-\gamma} ds + e^{-\rho t} v(X_t)\right]. \tag{19}$$

Letting $t \uparrow \infty$ and using the monotone convergence theorem, the Lebesgue dominated convergence theorem and the transversality condition in (7), we derive

$$v(x) \geq \mathbb{E}\left[\int_0^\infty e^{-\rho s}\frac{c_s^{1-\gamma}}{1-\gamma}ds\right] = J(x;\mathbf{c},\boldsymbol{\pi}). \tag{20}$$

If (c_t, π_t) is the maximizer of the Bellman equation (6), the inequality in (20) becomes the equality and consequently we obtain $v(x) = V(x)$. □

Theorem 3.2. *The value function $V(x)$ of the optimization problem (4) is given by*

$$V(x) = \begin{cases} C_2\left(x - \frac{R}{r}\right)^{m_2} + \frac{R^{1-\gamma}}{\rho(1-\gamma)}, & \text{for } R/r < x < \widetilde{x}, \\ \frac{r - \frac{1}{2}\theta^2 n_1}{\rho} D_1 \xi^{-\gamma(n_1+1)} + \frac{\xi^{1-\gamma}}{K(1-\gamma)}, & \text{for } x \geq \widetilde{x}, \end{cases} \tag{21}$$

where

$$D_1 = \frac{\left(\frac{m_2-1}{\gamma}+1\right)\frac{1}{K} - \frac{1}{r}}{(m_2-1)n_1 - 1} R^{\gamma n_1 + 1}, \;\; \widetilde{x} = D_1 R^{-\gamma n_1} + \frac{R}{K} \tag{22}$$

and

$$C_2 = \frac{1}{m_2}\left(\widetilde{x} - \frac{R}{r}\right)^{1-m_2} R^{-\gamma}.$$

For $x \geq \widetilde{x}$, ξ is determined from the following algebraic equation

$$x = D_1 \xi^{-\gamma n_1} + \frac{\xi}{K}.$$

Proof. For $R/r < x < \widetilde{x}$, trying a homogeneous solution of the form $\left(x - \frac{R}{r}\right)^m$ to the equation (10), then we obtain the algebraic equation $f(m) = 0$ in (13). Thus we can find the homogeneous solution $\widetilde{V}(x)$ to the equation (10) as follows:

$$\widetilde{V}(x) = C_1\left(x - \frac{R}{r}\right)^{m_1} + C_2\left(x - \frac{R}{r}\right)^{m_2},$$

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for some constants C_1 and C_2. The particular solution $\frac{R^{1-\gamma}}{\rho(1-\gamma)}$ to the equation (10) can be easily derived. Thus $V(x)$ is given by

$$V(x) = \widetilde{V}(x) + \frac{R^{1-\gamma}}{\rho(1-\gamma)} = C_1\left(x - \frac{R}{r}\right)^{m_1} + C_2\left(x - \frac{R}{r}\right)^{m_2} + \frac{R^{1-\gamma}}{\rho(1-\gamma)}.$$

If $C_1 = 0$ and $C_2 > 0$, then $V(x)$ is a concave function. Thus in order to guarantee the existence of the well-defined value function $V(x)$ we set $C_1 = 0$ and we will prove that $C_2 > 0$ in Proposition 3.1 later. Therefore $V(x)$ is given by

$$V(x) = C_2\left(x - \frac{R}{r}\right)^{m_2} + \frac{R^{1-\gamma}}{\rho(1-\gamma)}. \tag{23}$$

For $x \geq \widetilde{x}$, we set the optimal consumption $c = C(x)$ and $X(\cdot) = C^{-1}(\cdot)$, that is, $X(c) = X(C(x)) = x$. Then, from the FOCs (9), we obtain

$$V'(x) = C(x)^{-\gamma},\ V''(x) = -\gamma\frac{C(x)^{-\gamma-1}}{X'(c)}. \tag{24}$$

Plugging the conditions (24) into the equation (11), we have

$$rc^{-\gamma}X(c) + \frac{1}{2\gamma}\theta^2c^{1-\gamma}X'(c) - \rho V(X(c)) + \frac{\gamma}{1-\gamma}c^{1-\gamma} = 0. \tag{25}$$

Taking the derivative of (25) with respect to c implies

$$\frac{1}{2\gamma}\theta^2c^2X''(c) + \left(r - \rho + \frac{1-\gamma}{2\gamma}\theta^2\right)cX'(c) - r\gamma X(c) + \gamma c = 0. \tag{26}$$

Trying a homogeneous solution of the form $c^{-\gamma n}$ to the equation (26), then we obtain the algebraic equation $g(n) = 0$. Thus the homogeneous solution $\widetilde{X}(c)$ is given by

$$\widetilde{X}(c) = D_1c^{-\gamma n_1} + D_2c^{-\gamma n_2},$$

for some constants D_1 and D_2. The particular solution $\frac{c}{K}$ to the equation (26) can be easily derived. Thus $X(c)$ is given by

$$X(c) = \widetilde{X}(c) + \frac{c}{K} = D_1c^{-\gamma n_1} + D_2c^{-\gamma n_2} + \frac{c}{K}.$$

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Now we should discard the rapidly growing term by setting $D_2 = 0$. Therefore $X(c)$ is given by

$$X(c) = D_1 c^{-\gamma n_1} + \frac{c}{K}. \tag{27}$$

We will prove that $X'(c) > 0$ in Proposition 3.1 later. Thus, from (24), we obtain

$$V''(x) = -\gamma \frac{C(x)^{-\gamma-1}}{X'(c)} < 0$$

and hence $V(x)$ is a concave function for $x \geq \widetilde{x}$. From (25), we have

$$V(x) = V(X(\xi)) = \frac{r - \frac{1}{2}\theta^2 n_1}{\rho} D_1 \xi^{-\gamma(n_1+1)} + \frac{\xi^{1-\gamma}}{K(1-\gamma)},$$

where ξ is determined from the algebraic equation

$$x = D_1 \xi^{-\gamma n_1} + \frac{\xi}{K}. \tag{28}$$

From (27), we see that

$$\widetilde{x} = X(R) = D_1 R^{-\gamma n_1} + \frac{R}{K} \tag{29}$$

and

$$X'(R) = -\gamma n_1 D_1 R^{-\gamma n_1 - 1} + \frac{1}{K}. \tag{30}$$

From (23) and (24), we use C^1 and C^2 conditions at $x = \widetilde{x}$ to obtain

$$V'(\widetilde{x}) = m_2 C_2 \left(\widetilde{x} - \frac{R}{r}\right)^{m_2-1} = R^{-\gamma} \tag{31}$$

and

$$V''(\widetilde{x}) = m_2(m_2-1) C_2 \left(\widetilde{x} - \frac{R}{r}\right)^{m_2-2} = -\gamma \frac{R^{-\gamma-1}}{X'(R)}. \tag{32}$$

From (30), (31) and (32) we have

$$\widetilde{x} = -\frac{m_2 - 1}{\gamma} R X'(R) + \frac{R}{r} = (m_2-1) n_1 D_1 R^{-\gamma n_1} - \frac{m_2 - 1}{\gamma} \frac{R}{K} + \frac{R}{r}. \tag{33}$$

From (29) and (33), we derive

$$D_1 = \frac{\left(\frac{m_2-1}{\gamma}+1\right)\frac{1}{K}-\frac{1}{r}}{(m_2-1)n_1-1} R^{\gamma n_1+1} \tag{34}$$

and

$$C_2 = \frac{1}{m_2}\left(\widetilde{x}-\frac{R}{r}\right)^{1-m_2} R^{-\gamma}. \tag{35}$$

□

Proposition 3.1. *$\widetilde{x}$ is an increasing function with respect to R, $X'(c)>0$ and $\widetilde{x}>R/r$. Also $C_2>0$ as promised before.*

Proof. From (29) and (34) we have

$$\begin{aligned}\widetilde{x} &= \left[\frac{\left(\frac{m_2-1}{\gamma}+1\right)\frac{1}{K}-\frac{1}{r}}{(m_2-1)n_1-1}+\frac{1}{K}\right]R \\ &= \frac{(m_2-1)\left(\frac{1}{\gamma}+n_1\right)\frac{1}{K}-\frac{1}{r}}{(m_2-1)n_1-1}R.\end{aligned}$$

Thus $\widetilde{x}$ is a linear function of R and is an increasing function with respect to R since

$$\frac{(m_2-1)\left(\frac{1}{\gamma}+n_1\right)\frac{1}{K}-\frac{1}{r}}{(m_2-1)n_1-1} > 0,$$

because of $m_2-1<0$.

Now we use the Merton constant K in (12) and the quadratic equation (14) to obtain the inequality

$$\begin{aligned}\frac{\gamma n_1}{r}-\frac{\gamma n_1}{K}-\frac{1}{K} &= \frac{\gamma n_1 K - r\gamma n_1 - r}{rK} = \frac{n_1(\rho-r)+n_1\frac{\gamma-1}{2\gamma}\theta^2-r}{rK} \\ &= \frac{\left(\rho-r+\frac{1}{2}\theta^2\right)n_1-\frac{n_1}{2\gamma}\theta^2-r}{rK} = \frac{-\frac{1}{2}\theta^2 n_1^2-\frac{n_1}{2\gamma}\theta^2}{rK} < 0.\end{aligned}$$

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Thus we have

$$X'(R) = -\gamma n_1 D_1 R^{-\gamma n_1 - 1} + \frac{1}{K} = -\gamma n_1 \frac{\left(\frac{m_2-1}{\gamma}+1\right)\frac{1}{K}-\frac{1}{r}}{(m_2-1)n_1-1} + \frac{1}{K}$$
$$= \frac{\frac{\gamma n_1}{r} - \frac{\gamma n_1}{K} - \frac{1}{K}}{(m_2-1)n_1-1} > 0. \tag{36}$$

From the fact $c > R$, we have

$$1 > \left(\frac{R}{c}\right)^{\gamma n_1+1} \quad \text{and} \quad \frac{1}{K} > \frac{1}{K}\left(\frac{R}{c}\right)^{\gamma n_1+1}. \tag{37}$$

Thus we have

$$\begin{aligned} X'(c) &= -\gamma n_1 \frac{\left(\frac{m_2-1}{\gamma}+1\right)\frac{1}{K}-\frac{1}{r}}{(m_2-1)n_1-1}\left(\frac{R}{c}\right)^{\gamma n_1+1} + \frac{1}{K} \\ &> -\gamma n_1 \frac{\left(\frac{m_2-1}{\gamma}+1\right)\frac{1}{K}-\frac{1}{r}}{(m_2-1)n_1-1}\left(\frac{R}{c}\right)^{\gamma n_1+1} + \frac{1}{K}\left(\frac{R}{c}\right)^{\gamma n_1+1} \\ &= \left(\frac{R}{c}\right)^{\gamma n_1+1} X'(R) \\ &> 0, \end{aligned}$$

where the first inequality is obtained from (37) and the second inequality is obtained from (36). Consequently, from (33), we see that $\widetilde{x} > R/r$ and $C_2 > 0$ from (35). $\square$

Remark 3.2. *For $R/r < x < \widetilde{x}$, $V''(x)$ has a lower bound. From Proposition 3.1 and (24), $V''(x)$ has a lower bound for $\tilde{x} \le x$. From Lemma 3.1, $V'(x)$ is bounded away from zero. Hence, π^* in (8) is bounded away from zero and the Bellman equation (6) is uniformly elliptic. Therefore the solution in Theorem 3.2 is the unique solution to the Bellman equation (6) by Krylov [3]. Vila and Zariphopoulou [13] provided an alternative proof by a similar argument.*

Now we will describe the related results of Shin *et al.* [11] in the following remark. They also pay their attention to the optimal consumption and portfolio selection problem with a subsistence consumption constraint, but they use the martingale method with Lagrangian duality to derive their solutions.

Remark 3.3. *With the notations in this paper, the value function* $V^S(x)$ *and the threshold wealth level* $\widetilde{x}^S$ *based on Section 4 of Shin et al. [11] are given as follows:*

$$V^S(x) = \begin{cases} d_2\left(\frac{\frac{R}{r}-x}{d_2p_2}\right)^{\frac{p_2}{p_2-1}} + \left(x-\frac{R}{r}\right)\left(\frac{\frac{R}{r}-x}{d_2p_2}\right)^{\frac{1}{p_2-1}} + \frac{R^{1-\gamma}}{\rho(1-\gamma)}, & \text{for } R/r < x < \widetilde{x}^S, \\ c_1\left(\lambda^*\right)^{p_1} + \frac{\gamma}{K(1-\gamma)}\left(\lambda^*\right)^{-\frac{1-\gamma}{\gamma}} + \left(\lambda^*\right)x, & \text{for } x \geq \widetilde{x}^S \end{cases} \tag{38}$$

and

$$\widetilde{x}^S = -c_1p_1R^{-\gamma(p_1-1)} + \frac{R}{K},$$

where

$$c_1 = \frac{\frac{1}{K}\left(\frac{\gamma p_2}{1-\gamma}+1\right) + \frac{p_2-1}{r} - \frac{p_2}{\rho(1-\gamma)}}{p_1-p_2} R^{1-\gamma+\gamma p_1} \tag{39}$$

and

$$d_2 = \frac{\frac{1}{K}\left(\frac{\gamma p_1}{1-\gamma}+1\right) + \frac{p_1-1}{r} - \frac{p_1}{\rho(1-\gamma)}}{p_1-p_2} R^{1-\gamma+\gamma p_2}. \tag{40}$$

$p_1 > 1$ *and* $p_2 < 0$ *are two real roots of the following quadratic algebraic equation*

$$h(p) := \frac{1}{2}\theta^2p^2 + \left(\rho - r - \frac{1}{2}\theta^2\right)p - \rho = 0, \tag{41}$$

and λ^* *is determined by the following algebraic equation*

$$x = -c_1p_1\left(\lambda^*\right)^{p_1-1} + \frac{1}{K}\left(\lambda^*\right)^{-\frac{1}{\gamma}}. \tag{42}$$

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Lemma 3.2.

$$m_2 C_2 = (-d_2 p_2)^{\frac{1}{1-p_2}} \quad and \quad D_1 = -c_1 p_1. \tag{43}$$

Proof. From (29), (34) and (15), we have

$$\begin{aligned}
\widetilde{x} &= D_1 R^{-\gamma n_1} + \frac{R}{K} = \frac{\left(\frac{m_2-1}{\gamma}+1\right)\frac{1}{K}-\frac{1}{r}}{(m_2-1)n_1-1}R + \frac{R}{K} \\
&= \frac{\left(\frac{1}{\gamma}+n_2\right)\frac{1}{K}-\frac{n_2}{r}}{n_1-n_2}R + \frac{R}{K}.
\end{aligned}$$

It can be easily shown that

$$p_1 = n_1 + 1, \quad p_2 = n_2 + 1. \tag{44}$$

Thus we obtain

$$\widetilde{x} = \frac{\left(\frac{1}{\gamma}+p_2-1\right)\frac{1}{K}-\frac{p_2-1}{r}}{p_1-p_2}R + \frac{R}{K}.$$

From (35), we have

$$\begin{aligned}
m_2 C_2 &= \left(\widetilde{x} - \frac{R}{r}\right)^{1-m_2} R^{-\gamma} \\
&= \left(\widetilde{x} R^{\gamma(p_2-1)} - \frac{R^{1+\gamma(p_2-1)}}{r}\right)^{1-m_2} \\
&= \left\{\left(\frac{\left(\frac{1}{\gamma}+p_1-1\right)\frac{1}{K}+\frac{1-p_1}{r}}{p_1-p_2}\right) R^{1+\gamma(p_2-1)}\right\}^{1-m_2} \\
&= (-d_2 p_2)^{1-m_2},
\end{aligned}$$

where the last equality is obtained from the following relationships between roots and coefficients of the quadratic equation $h(p) = 0$ in (41)

$$p_1 + p_2 = \frac{\theta^2 - 2\rho + 2r}{\theta^2}, \quad p_1 p_2 = -\frac{2\rho}{\theta^2} \tag{45}$$

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and (40). Therefore we obtain

$$m_2 C_2 = (-d_2 p_2)^{\frac{1}{1-p_2}}.$$

From (34), we have

$$\begin{aligned} D_1 &= \frac{\left(\frac{m_2-1}{\gamma}+1\right)\frac{1}{K}-\frac{1}{r}}{(m_2-1)n_1-1} R^{\gamma n_1+1} \\ &= \frac{\left(\frac{1}{\gamma}+p_2-1\right)\frac{1}{K}-\frac{p_2-1}{r}}{p_1-p_2} R^{1+\gamma(p_1-1)} \qquad (46) \\ &= -c_1 p_1, \end{aligned}$$

where the last equality is also obtained from the relationships (45) and (39). □

Corollary 3.1. *D_1 in (22) is positive.*

Proof. For $p_2 < x < p_1$, we define a decreasing function $F(x)$ as follows:

$$F(x) := -\frac{h(x)}{x-p_2} = -\frac{1}{2}\theta^2(x-p_1) > 0.$$

Since $0 < F(1) < F\left(\frac{\gamma-1}{\gamma}\right)$, we have $\frac{1}{F\left(\frac{\gamma-1}{\gamma}\right)} < \frac{1}{F(1)}$ and

$$\left(\frac{1}{\gamma}+p_2-1\right)\frac{1}{K}-\frac{p_2-1}{r} > 0$$

(see also Shim and Shin [9]). From (46), we have $D_1 > 0$. □

Proposition 3.2. *The value function $V(x)$ and the threshold wealth level $\widetilde{x}$ in our optimization problem coincide with $V^S(x)$ and $\widetilde{x}^S$ of Shin et al. [11], respectively.*

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Proof. From (43) and (44), we can easily show that $\widetilde{x} = \widetilde{x}^{\mathrm{S}}$.

For $R/r < x < \widetilde{x}$, we have

$$\begin{aligned} d_2\left(\frac{\frac{R}{r}-x}{d_2p_2}\right)^{\frac{p_2}{p_2-1}} + \left(x-\frac{R}{r}\right)\left(\frac{\frac{R}{r}-x}{d_2p_2}\right)^{\frac{1}{p_2-1}} &= \frac{p_2-1}{p_2}(-d_2p_2)^{\frac{1}{1-p_2}}\left(x-\frac{R}{r}\right)^{\frac{p_2}{p_2-1}} \\ &= \frac{p_2-1}{p_2}m_2C_2\left(x-\frac{R}{r}\right)^{\frac{p_2}{p_2-1}} \\ &= C_2\left(x-\frac{R}{r}\right)^{m_2}, \end{aligned}$$

where the second equality is obtained from (43) and the third equality is obtained from (15) and (44). This equality means $V(x) = V^{\mathrm{S}}(x)$ for $R/r < x < \widetilde{x}$.

For $x \geq \widetilde{x}$, if we set $\xi = (\lambda^*)^{-1/\gamma}$, then the algebraic equation (28) coincides with the algebraic equation (42). From (38) and (42), we obtain

$$\begin{aligned} V^{\mathrm{S}}(x) &= c_1(\lambda^*)^{p_1} + \frac{\gamma}{K(1-\gamma)}(\lambda^*)^{-\frac{1-\gamma}{\gamma}} + (\lambda^*)x \\ &= \frac{p_1-1}{p_1}(-c_1p_1)(\lambda^*)^{p_1} + \frac{(\lambda^*)^{\frac{\gamma-1}{\gamma}}}{K(1-\gamma)} \\ &= \frac{n_1}{n_1+1}D_1\xi^{-\gamma(n_1+1)} + \frac{\xi^{1-\gamma}}{K(1-\gamma)} \\ &= \frac{r-\frac{1}{2}\theta^2n_1}{\rho}D_1\xi^{-\gamma(n_1+1)} + \frac{\xi^{1-\gamma}}{K(1-\gamma)} \\ &= V(x), \end{aligned}$$

where the third equality is obtained from (43) and (44) and the fourth equality is obtained from (14). $\square$

Finally we use the FOCs (8), (9) and (24) with the derived value function $V(x)$ in (21) to obtain the optimal consumption and investment strategies of this optimization problem.

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Theorem 3.3. *The optimal consumption and portfolio pair* (c^*, π^*) *is given by*

$$c_t^* = \begin{cases} R, & \text{for } \; R/r < X_t < \widetilde{x} \\ \xi_t, & \text{for } \; X_t \geq \widetilde{x} \end{cases}$$

and

$$\pi_t^* = \begin{cases} \frac{\theta}{\sigma}\frac{1}{1-m_2}\left(X_t - \frac{R}{r}\right), & \text{for } \; R/r < X_t < \widetilde{x} \\ \frac{\theta}{\sigma\gamma}\left(-\gamma n_1 D_1 \xi_t^{-\gamma n_1} + \frac{\xi_t}{K}\right), & \text{for } \; X_t \geq \widetilde{x}, \end{cases}$$

where ξ_t *is determined by the following algebraic equation*

$$X_t = D_1 \xi_t^{-\gamma n_1} + \frac{\xi_t}{K}. \tag{47}$$

Proof. The proof directly follows from the FOCs (8) and (9). □

Remark 3.4. *It is easily seen that the optimal consumption and portfolio pair* (c^*, π^*) *in our optimization problem coincides with that of Shin et al. [11].*

4 Implications

In this section, we compare the agent's optimal consumption and portfolio policies with subsistence consumption constraints to those without subsistence consumption constraints. Without subsistence consumption constraints, the optimal consumption and portfolio policies are those of the well-known Merton's problems. Let us denote by $(\mathbf{c}^M, \boldsymbol{\pi}^M)$ the optimal consumption and portfolio pair without subsistence consumption constraints.

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Then

$$c_t^M = KX_t, \tag{48}$$

$$\pi_t^M = \frac{\theta}{\sigma\gamma}X_t, \tag{49}$$

for $X_t > 0$. If we let $R \to 0$ to the consumption and portfolio pair (c^*, π^*) in Theorem 3.3, we also arrive at (c_t^M, π_t^M). Due to the subsistence consumption constraints, it is natural to consider the myopic strategies defined by

$$c_t^{myopic} := \max\{R, c_t^M\}.$$

But the myopic strategies are not optimal and the existence of the subsistence consumption constraints affect the consumption and portfolio policies even at the wealth level where the subsistence consumption constraints do not bind. This is because it is possible that the constraints will become binding later. The following proposition demonstrates quantitatively the impact of the subsistence consumption constraints on the consumption and portfolio policies when the constraints are not binding.

Proposition 4.1. *For $X_t \geq \tilde{x}$, $c_t^* < c_t^M$ and $\pi_t^* < \pi_t^M$.*

Proof. From (47) and (48), the optimal wealth process is given by

$$X_t = D_1 c_t^{*-\gamma n_1} + \frac{c_t^*}{K} = \frac{c_t^M}{K}.$$

Since $D_1 > 0$ and $X(c) := D_1 c^{-\gamma n_1} + \dfrac{c}{K}$ is an increasing function from Proposition 3.1, we obtain

$$c_t^* < c_t^M = KX_t. \tag{50}$$

Also we derive

$$\pi_t^* = \frac{\theta}{\sigma\gamma}\left(-\gamma n_1 D_1 c_t^{*-\gamma n_1} + \frac{c_t^*}{K}\right) < \frac{\theta}{\sigma\gamma}\frac{c_t^*}{K} < \frac{\theta}{\sigma\gamma}X_t = \pi_t^M,$$

where the first inequality follows from $D_1 > 0$ and the second one from (50). □

5 Concluding Remarks

In this paper we study an optimal consumption and investment problem with subsistence consumption constraints. We use the dynamic programming method to derive the closed form solutions with a CRRA utility function. We also compare our solutions with those of Shin *et al.* [11] derived by the martingale approach. We show that they coincide with each other. In addition, we point out that the optimal consumption and portfolio policies may alter even when the constraints do not bind. This is attributed to the prospect that the subsistence consumption constraints become binding later. In this case, the agent consume less and invest in the risky asset more conservatively.

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THE STABILITY OF CUBIC FUNCTIONAL EQUATION WITH INVOLUTION IN NON-ARCHIMEDEAN SPACES

CHANG IL KIM AND CHANG HYEOB SHIN*

Abstract. In this paper, using fixed point method, we prove the Hyers-Ulam stability of the following functional equation

$$f(2x+y)+f(2x+\sigma(y))-2f(x+y)-2f(x+\sigma(y))-12f(x)=0$$

with involution.

1. Introduction and Preliminaries

In 1940, Ulam [18] proposed the following problem concerning the stability of group homomorphism: *Let G_1 be a group and let G_2 a meric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \longrightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?*

Hyers [7] solved the Ulam's problem for the case of approximately additive functions in Banach spaces. Since then, the stability of several functional equations have been extensively investigated by several mathematicians [2, 3, 5, 8, 9, 13, 14, 15, 16]. Jun and Kim [11] introduced the following functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1.1}$$

and they established the general solution and generalized Hyers-Ulam-Rassias stability problem for this functional equation. It is easy to see that the function $f(x) = cx^3$ is a solution of the functional equation (1.1). Thus, it is natural that (1.1) is called *a cubic functional equation* and every solution of the cubic functional equation is said to be *a cubic function*.

Let X and Y be real vector spaces. For an additive mapping $\sigma : X \longrightarrow X$ with $\sigma(\sigma(x)) = x$ for all $x \in X$, then σ is called *an involution* of X [1, 17]. Stetkær [17] introduced the following quadratic functional equation with involution

$$f(x+y)+f(x+\sigma(y))=2f(x)+2f(\sigma(y)) \tag{1.2}$$

and solved the general solution, Belaid et al. [1] established generalized Hyers-Ulam stability in Banach space for this functional equation. Jung and Lee [12] investigated the Hyers-Ulam-Rassias stability of (1.2) in a complete β-normed space, using fixed point method.

For a given involution $\sigma : X \longrightarrow X$, the functional equation

$$f(2x+y)+f(2x+\sigma(y))=2f(x+y)+2f(x+\sigma(y))+12f(x) \tag{1.3}$$

for all $x, y \in X$ is called *the cubic functional equation with involution* and a solution of (1.3) is called *a cubic mapping with involution*.

In this paper, using fixed point method, we prove the generalized Hyers-Ulam stability of the following functional equation

$$f(2x+y)+f(2x+\sigma(y))-2f(x+y)-2f(x+\sigma(y))-12f(x)=0. \tag{1.4}$$

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A *valuation* is a function $|\cdot|$ from a field K into $[0,\infty)$ such that for any $r,s\in K$, the following conditions hold: (i) $|r|=0$ if and only if $r=0$, (ii) $|rs|=|r||s|$, and (iii) $|r+s|\le |r|+|s|$.

A field K is called *a valued field* if K carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations. If the triangle inequality is replaced by $|r+s|\le max\{|r|,|s|\}$ for all $r,s\in K$, then the valuation $|\cdot|$ is called a *non-Archimedean valuation* and the field with a non-Archimedean valuation is called *non-Archimedean field.* If $|\cdot|$ is a non-Archimedean valuation on K, then clearly, $|1|=|-1|$ and $|n|\le 1$ for all $n\in\mathbb{N}$.

Definition 1.1. Let X be a vector space over a scalar field K with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\|: X\longrightarrow\mathbb{R}$ is called *a non-Archimedean norm* if satisfies the following conditions:
(a) $\|x\|=0$ if and only if $x=0$,
(b) $\|rx\|=|r|\|x\|$, and
(c) the strong triangle inequality (ultrametric) holds, that is,

$$||x+y||\le max\{\|x\|,\|y\|\}$$

for all $x,y\in X$ and all $r\in K$.

If $\|\cdot\|$ is a non-Archimedean norm, then $(X,\|\cdot\|)$ is called *a non-Archimedean normed space.* Let $(X,\|\cdot\|)$ be a non-Archimedean normed space. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be *convergent* if there exists $x\in X$ such that $\lim_{n\longrightarrow\infty}\|x_n-x\|=0$. In that case, x is called *the limit of the sequence* $\{x_n\}$, and one denotes it by $\lim_{n\longrightarrow\infty}x_n=x$. A sequence $\{x_n\}$ is said to be *a Cauchy sequence* if $\lim_{n\longrightarrow\infty}\|x_{n+p}-x_n\|=0$ for all $p\in\mathbb{N}$. Since

$$\|x_n-x_m\|\le max\{\|x_{j+1}-x_j\|\mid m\le j\le n-1\}\quad(n>m),$$

a sequence $\{x_n\}$ is Cauchy in $(X,\|\cdot\|)$ if and only if $\{x_{n+1}-x_n\}$ converges to zero in $(X,\|\cdot\|)$. By *a complete non-Archimedean space* we mean one in which every Cauchy sequence is convergent.

In 1897, Hensel [6] discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. Fix a prime number p. For any nonzero rational number x, there exists a unique integer $n_x\in\mathbb{Z}$ such that $x=\frac{a}{b}p^{n_x}$, where a and b are integers not divisible by p. Then $|x|_p:=p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x,y)=|x-y|_p$ is denoted by $\mathbb{Q}_p$, which is called the *p-adic number field.* In fact, $\mathbb{Q}_p$ is the set of all formal series $x=\sum_{k\ge n_x}^{\infty}a_kp^k$, where $|a_k|\le p-1$ are integers. The addition and multiplication between any two elements of $\mathbb{Q}_p$ are defined naturally. The norm $\left|\sum_{k\ge n_x}^{\infty}a_kp^k\right|_p=p^{-n_x}$ is a non-Archimedean norm on $\mathbb{Q}_p$ and it makes $\mathbb{Q}_p$ a locally compact field.

Let (X,d) be a generalized metric space. An operator $T:X\longrightarrow X$ satisfies a Lipschitz condition with Lipschitz constant L if there exists a constant $L\ge 0$ such that $d(Tx,Ty)\le Ld(x,y)$ for all $x,y\in X$. If the Lipschitz constant L is less than 1, then the operator T is called a *strictly contractive operator.* Note that the distinction between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity.

Theorem 1.2. [4] *Let (X,d) be a complete generalized metric space and let $J:X\longrightarrow X$ be a strictly contractive mapping with some Lipschitz constant L with $0<L<1$. Then for each given element $x\in X$, either $d(J^nx,J^{n+1}x)=\infty$ for all nonnegative integers n or there exists a positive integer n_0 such that*
(1) *$d(J^nx,J^{n+1}x)<\infty$ for all $n\ge n_0$;*
(2) *the sequence $\{J^nx\}$ converges to a fixed point x^* of J ;*
(3) *x^* is the unique fixed point of J in the set $Y=\{y\in X\mid d(J^{n_0}x,y)<\infty\}$* and
(4) *$d(y,y^*)\le\dfrac{1}{1-L}\,d(y,Jy)$ for all $y\in Y$.*

In 1996, Issac and Rassias [10] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorem with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors.

Throughout this paper, we assume that X is a non-Archimedean normed space and Y is a complete non-Archimedean normed space.

2. The generalized Hyers-Ulam stability for (1.4)

Using the fixed point methods, we will prove the generalized Hyers-Ulam stability of the cubic functional equation (1.4) with involution σ in non-Archimedean normed spaces. For a given mapping $f : X \longrightarrow Y$, we define the difference operator $Df : X^2 \longrightarrow Y$ by

$$Df(x,y) = f(2x+y) + f(2x+\sigma(y)) - 2f(x+y) - 2f(x+\sigma(y)) - 12f(x)$$

for all $x, y \in X$.

Theorem 2.1. *Assume that $\phi : X^2 \longrightarrow [0,\infty)$ is a mapping and there exists a real number L with $0 < L < 1$ such that*

$$\phi(2x,2y) \le |8|L\phi(x,y),\ \ \phi(x+\sigma(x), y+\sigma(y)) \le |8|L\phi(x,y) \tag{2.1}$$

for all $x, y \in X$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$\|Df(x,y)\| \le \phi(x,y) \tag{2.2}$$

for all $x, y \in X$. Then there exists a unique cubic mapping $C : X \longrightarrow Y$ with involution such that

$$\|f(x) - C(x)\| \le \frac{1}{|2|^4(1-L)}\Phi(x) \tag{2.3}$$

for all $x \in X$, where $\Phi(x) = max\{\phi(x,0), \phi(0,x)\}$.

Proof. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d in S defined by $d(g,h) = inf\{c \in [0,\infty) |\ \|g(x) - h(x)\| \le c\ \Phi(x)$ for all $x \in X\}$. Then (S,d) is a complete metric space(See [12]). Define a mapping $J : S \longrightarrow S$ by

$$Jg(x) = \frac{1}{8}\{g(2x) + g(x+\sigma(x))\}$$

for all $x \in X$ and all $g \in S$. Let $g, h \in S$ and $d(g,h) \le c$ for some non-negative real number c. Then by (2.1), we have

$$\begin{aligned}\|Jg(x) - Jh(x)\| &= \frac{1}{|8|}\ \|g(2x) + g(x+\sigma(x)) - h(2x) - h(x+\sigma(x))\| \\ &\le \frac{1}{|8|}max\{\|g(2x) - h(2x)\|, \|g(x+\sigma(x)) - h(x+\sigma(x))\|\} \\ &\le cL\Phi(x)\end{aligned}$$

for all $x \in X$. Hence we have $d(Jg, Jh) \le Ld(g,h)$ for any $g, h \in S$ and so J is a strictly contractive mapping.

Next, we claim that $d(Jf, f) < \infty$. Putting $y = 0$ in (2.2), we get

$$\|f(2x) - 8f(x)\| \le \frac{1}{|2|}\phi(x,0) \tag{2.4}$$

for all $x \in X$ and putting $x = 0$ in (2.2), we get

$$\|f(y) + f(\sigma(y))\| \le \phi(0,y) \tag{2.5}$$

for all $y \in X$ and putting $y = x + \sigma(x)$ in (2.5), we get

$$\|f(x+\sigma(x))\| \le \frac{1}{|2|}\phi(0, x+\sigma(x)) \tag{2.6}$$

for all $x \in X$. By (2.4) and (2.6), we have

$$\begin{aligned}\|Jf(x) - f(x)\| &= \frac{1}{|8|}\Big\|f(2x) - 8f(x) + f(x+\sigma(x))\Big\| \\ &\leq \frac{1}{|8|}max\Big\{\|f(2x) - 8f(x)\|, \|f(x+\sigma(x))\|\Big\} \\ &\leq \frac{1}{|2|^4}\Phi(x)\end{aligned}$$

for all $x \in X$. Hence

$$d(Jf, f) \leq \frac{1}{|2|^4} < \infty. \tag{2.7}$$

By Theorem 1.2, there exists a mapping $C : X \longrightarrow Y$ which is a fixed point of J such that $d(J^n f, C) \to 0$ as $n \to \infty$. By induction, we can easily show that

$$(J^n f)(x) = \frac{1}{2^{3n}}\Big\{f(2^n x) + (2^n - 1)f\Big(2^{n-1}\big(x + \sigma(x)\big)\Big)\Big\}$$

for all $x \in X$ and $n \in \mathbb{N}$. Since $d(J^n f, C) \to 0$ as $n \to \infty$, there exists a sequence $\{c_n\}$ in $\mathbb{R}$ such that $c_n \to 0$ as $n \to \infty$ and $d(J^n f, C) \leq c_n$ for every $n \in \mathbb{N}$. Hence, it follows from the definition of d that

$$\|(J^n f)(x) - C(x)\| \leq c_n \Phi(x)$$

for all $x \in X$. Thus for each fixed $x \in X$, we have

$$\lim_{n\longrightarrow\infty} ||(J^n f)(x) - C(x)|| = 0$$

and so

$$C(x) = \lim_{n\longrightarrow\infty} \frac{1}{2^{3n}}\Big\{f(2^n x) + (2^n - 1)f\Big(2^{n-1}\big(x + \sigma(x)\big)\Big)\Big\}. \tag{2.8}$$

It follows from (2.2) and (2.8) that

$$\begin{aligned}&\|C(2x + y) + C(2x + \sigma(y)) - 2C(x + y) - 2C(x + \sigma(y)) - 12C(x)\| \\ &\leq \lim_{n\longrightarrow\infty} \frac{1}{|8|^n}\ max\{\phi(2^n x, 2^n y), |2^n - 1|\phi(2^{n-1}(x + \sigma(x)), 2^{n-1}(y + \sigma(y)))\} \\ &\leq \lim_{n\longrightarrow\infty} L^n\ max\{\phi(x, y), |2^n - 1|\phi(x, y)\} = \lim_{n\longrightarrow\infty} L^n \phi(x, y) = 0\end{aligned}$$

for all $x, y \in X$, because $|2^n - 1| \leq 1$ for all $n \in \mathbb{N}$. Hence C satisfies (1.4), C is a cubic mapping with involution. By (4) in Theorem 1.2 and (2.4), f satisfies (2.3).

Assume that $C_1 : X \longrightarrow Y$ is another solution of (1.4) satisfying (2.3). We know that C_1 is a fixed point of J. Due to (3) in Theorem 1.2, we get $C = C_1$. This proves the uniqueness of C. □

Theorem 2.2. *Assume that $\phi : X^2 \longrightarrow [0, \infty)$ is a mapping and there exists a real number L with $0 < L < 1$ such that*

$$\phi(x, y) \leq \frac{L}{|8|}\phi(2x, 2y),\ \ \phi(x + \sigma(x), y + \sigma(y)) \leq \phi(2x, 2y) \tag{2.9}$$

for all $x, y \in X$. Let $f : X \longrightarrow Y$ be a mapping satisfying (2.2) and $f(0) = 0$. Then there exists a unique cubic mapping $C : X \longrightarrow Y$ with involution such that

$$\|f(x) - C(x)\| \leq \frac{L}{|2|^4(1 - L)}\Phi(x) \tag{2.10}$$

for all $x \in X$, where $\Phi(x) = max\{\phi(x, 0), \phi(0, x)\}$.

Proof. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d in S defined by $d(g,h) = inf\{c \in [0,\infty) |\ \|g(x) - h(x)\| \leq c\ \Phi(x) \text{ for all } x \in X\}$. Then (S,d) is a complete metric space. Define a mapping $J : S \longrightarrow S$ by

$$Jg(x) = 8\Big\{g\Big(\frac{x}{2}\Big) - g\Big(\frac{x+\sigma(x)}{4}\Big)\Big\}$$

for all $x \in X$ and all $g \in S$. Let $g,h \in S$ and $d(g,h) \leq c$ for some non-negative real number c. Then by (2.9), we have

$$\begin{aligned}\|Jg(x) - Jh(x)\| &= |8|\Big\|g\Big(\frac{x}{2}\Big) - g\Big(\frac{x+\sigma(x)}{4}\Big) - h\Big(\frac{x}{2}\Big) + h\Big(\frac{x+\sigma(x)}{4}\Big)\Big\| \\ &\leq |8|max\Big\{\Big\|g\Big(\frac{x}{2}\Big) - h\Big(\frac{x}{2}\Big)\Big\|, \Big\|g\Big(\frac{x+\sigma(x)}{4}\Big) - h\Big(\frac{x+\sigma(x)}{4}\Big)\Big\|\Big\} \\ &\leq cL\Phi(x)\end{aligned}$$

for all $x \in X$. Hence $d(Jg, Jh) \leq Ld(g,h)$ for any $g,h \in S$ and so J is a strictly contractive mapping.

Next, we claim that $d(Jf, f) < \infty$. By (2.4), (2.5) and (2.6), we have

$$\|Jf(x) - f(x)\| = \Big\|8f\Big(\frac{x}{2}\Big) - 8f\Big(\frac{x+\sigma(x)}{4}\Big) - f(x)\Big\| \leq \frac{L}{|2|^4}\Phi(x)$$

for all $x \in X$ and hence

$$d(Jf, f) \leq \frac{L}{|2|^4} < \infty.$$

By Theorem 1.2, there exists a mapping $C : X \longrightarrow Y$ which is a fixed point of J such that $d(J^n f, C) \to 0$ as $n \to \infty$. By induction, we can easily show that

$$(J^n f)(x) = 2^{3n}\Big\{f\Big(\frac{x}{2^n}\Big) - f\Big(\frac{x+\sigma(x)}{2^{n+1}}\Big)\Big\}$$

for each $n \in \mathbb{N}$. Since $d(J^n f, C) \to 0$ as $n \to \infty$, there exists a sequence $\{c_n\}$ in $\mathbb{R}$ such that $c_n \to 0$ as $n \to \infty$ and $d(J^n f, C) \leq c_n$ for every $n \in \mathbb{N}$. Hence, it follows from the definition of d that

$$\|(J^n f)(x) - C(x)\| \leq c_n\Phi(x)$$

for all $x \in X$. Thus for each fixed $x \in X$, we have

$$\lim_{n \longrightarrow \infty} \|(J^n f)(x) - C(x)\| = 0$$

and

$$C(x) = 2^{3n}\Big\{f\Big(\frac{x}{2^n}\Big) - f\Big(\frac{x+\sigma(x)}{2^{n+1}}\Big)\Big\}.$$

Analogously to the proof of Theorem 2.2, we can show that C is a unique cubic mapping with involution satisfying (2.10)

□

We can use Theorem 2.1 and Theorem 2.2 to get a classical result in the framework of non-Archimedean normed spaces. Taking $\phi(x,y) = \theta(\|x\|^p + \|y\|^p)$ or $\phi(x,y) = \theta(\|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p})$, we have the following examples.

Example 2.3. Let $\theta \geq 0$ and p be a positive real number with $p \neq 3$. Let $f : X \longrightarrow Y$ be a mapping satisfying

$$\|Df(x,y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{2.11}$$

for all $x, y \in X$. Suppose that $\|x + \sigma(x)\| \leq |2|\|x\|$ for all $x \in X$. Then there exists a unique mapping $C : X \longrightarrow Y$ with involution such that the inequality

$$||f(x) - C(x)|| \le \begin{cases} \dfrac{\theta||x||^p}{|2|(|2|^3 - |2|^p)}, & \text{if } p > 3, \\ \dfrac{\theta||x||^p}{|2|(|2|^p - |2|^3)}, & \text{if } 0 < p < 3 \end{cases}$$

holds for all $x \in X$.

Proof. Let $\phi(x,y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ and $L = |2|^{p-3}$. Then $\phi(2x, 2y) = |8||2|^{p-3}\phi(x,y)$ for all $x, y \in X$. Since $\|x + \sigma(x)\| \le |2|\|x\|$ for all $x \in X$, $\phi(x+\sigma(x), y+\sigma(y)) \le |8||2|^{p-3}\phi(x,y)$ for all $x, y \in X$. Hence if $p > 3$, then we have the results of Theorem 2.1.

Suppose that $L = |2|^{3-p}$. Then $\phi(x,y) = \frac{|2|^{3-p}}{|8|}\phi(2x,2y)$ for all $x, y \in X$ and $\phi(x+\sigma(x), y+\sigma(y)) \le |2|^p\phi(x,y) = \frac{|2|^{3-p}}{|8|}\phi(x,y)$ for all $x, y \in X$. Hence if $0 < p < 3$, then we have the results of Theorem 2.2. Thus the proof is complete. □

Example 2.4. Let $\theta \ge 0$ and p be a positive real number with $p \ne \frac{3}{2}$. Let $f : X \longrightarrow Y$ be a mapping satisfying

$$\|Df(x,y)\| \le \theta(\|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p}) \tag{2.12}$$

for all $x, y \in X$. Suppose that $\|x + \sigma(x)\| \le |2|\|x\|$ for all $x \in X$. Then there exists a unique mapping $C : X \longrightarrow Y$ with involution such that C is a solution of the functional equation (1.4) and the inequality

$$||f(x) - C(x)|| \le \begin{cases} \dfrac{\theta||x||^p}{|2|(|2|^3 - |2|^{2p})}, & \text{if } p > \frac{3}{2}, \\ \dfrac{\theta||x||^p}{|2|(|2|^{2p} - |2|^3)}, & \text{if } 0 < p < \frac{3}{2} \end{cases}$$

holds for all $x \in X$.

Using Theorem 2.1 and Theorem 2.2, we obtain the following corollary concerning the stability of (1.4).

Corollary 2.5. *Let $\alpha_i : [0, \infty) \longrightarrow [0, \infty)$ $(i = 1, 2, 3)$ be increasing mappings satisfying*
(i) $0 < \alpha_i(|2|) < 1$ and $\alpha_i(0) = 0$,
(ii) $\alpha_i(|2|t) \le \alpha_i(|2|)\alpha_i(t)$ for all $t \ge 0$.
Let $f : X \longrightarrow Y$ be a mapping such that for some $\delta \ge 0$

$$\|Df(x,y)\| \le \delta[\alpha_1(\|x\|)\alpha_1(\|y\|) + \alpha_2(\|x\|) + \alpha_3(\|y\|)] \tag{2.13}$$

for all $x, y \in X$. Suppose that $\|x + \sigma(x)\| \le |2|\|x\|$ for all $x \in X$. Then there exists a unique cubic mapping $C : X \longrightarrow Y$ with involution such that

$$||f(x) - C(x)|| \le \begin{cases} \dfrac{1}{|2|(|2|^3 - M)}\widetilde{\Phi}(x), & \text{if } 0 < M < |2|^3, \\ \dfrac{1}{|2|(N - |2|^3)}\widetilde{\Phi}(x), & \text{if } N > |2|^3 \end{cases}$$

holds for all $x \in X$, where $M = \max\{(\alpha_1(|2|))^2, \alpha_2(|2|), \alpha_3(|2|)\}$, $N = \min\{(\alpha_1(|2|))^2, \alpha_2(|2|), \alpha_3(|2|)\}$ and $\widetilde{\Phi}(x) = \delta \ \max\{\alpha_2(\|x\|), \alpha_3(\|x\|)\}$.

As examle of Corollary 2.5, we can take $\alpha_1(t) = \alpha_2(t) = \alpha_3(t) = t^p$ for all $t \ge 0$. Then we have the following example.

Example 2.6. Let $\delta \ge 0$ and p be a positive real number with $p \ne \frac{3}{2}$. Let $f : X \longrightarrow Y$ be a mapping satisfying

$$\|Df(x,y)\| \le \delta(\|x\|^p\|y\|^p + \|x\|^p + \|y\|^p) \tag{2.14}$$

and $\|x + \sigma(x)\| \le |2|\|x\|$ for all $x, y \in X$. Then there exists a unique mapping $C : X \longrightarrow Y$ with involution such that the inequality

$$||f(x) - C(x)|| \leq \begin{cases} \dfrac{\delta||x||^p}{|2|(|2|^3 - |2|^p)}, & \text{if } p > 3, \\ \dfrac{\delta||x||^p}{|2|(|2|^{2p} - |2|^3)}, & \text{if } 0 < p < \frac{3}{2} \end{cases}$$

holds for all $x \in X$.

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VALUE SHARING RESULTS FOR MEROMORPHIC FUNCTIONS WITH THEIR Q-SHIFTS

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Abstract. This research is a continuation of a recent paper [16, 17]. Shared value problems related to a meromorphic function $f(z)$ and its q-shift $f(qz+c)$ are studied. Moreover, we also consider uniqueness problems on meromorphic functions $f(z)$ share sets with $f(qz+c)$.

1. Introduction

We assume that the reader is familiar with the elementary Nevanlinna Theory, see, e.g. [8, 18]. Meromorphic functions are always non-constant, unless otherwise specified. As usual, by $S(r,f)$ we denote any quantity satisfying $S(r,f) = o(T(r,f))$ for all r outside of a possible exceptional set of finite linear measure. In particular, we denote by $S_1(r,f)$ any quality satisfying $S_1(r,f) = o(T(r,f))$ for all r on a set of logarithmic density 1.

For a meromorphic function f and a set S of complex numbers, we define the set $E(S,f) = \bigcup_{a\in S}\{z|f(z)-a=0\}$, where a zero of $f-a$ with multiplicity m counts m times in $E(S,f)$. As a special case, when $S=\{a\}$ contains only one element a, if $E(a,f)=E(a,g)$, then we say $f(z)$ and $g(z)$ share a CM; if $\overline{E}(a,f)=\overline{E}(a,g)$, then we say $f(z)$ and $g(z)$ share a IM, see [18].

The classical results due to Nevanlinna [14] in the uniqueness theory of meromorphic functions are the five-point, resp. four-point, theorems:

Theorem A. *If two meromorphic functions $f(z)$ and $g(z)$ share five distinct values $a_1,\ a_2,\ a_3,\ a_4,\ a_5 \in \mathbb{C}\cup\{\infty\}$ IM, then $f(z)\equiv g(z)$.*

Theorem B. *If two meromorphic functions $f(z)$ and $g(z)$ share four distinct values $a_1,\ a_2,\ a_3,\ a_4 \in \mathbb{C}\cup\{\infty\}$ CM, then $f(z)\equiv g(z)$ or $f(z)\equiv T\circ g(z)$, where T is a Möbius transformation.*

It is well-known that 4 CM can not be improved to 4 IM, see [6]. Further, Gundersen [7, Theorem 1] has improved the assumption 4 CM to 2 CM+2 IM, while 1 CM+3 IM is still an open problem.

Heittokangas et al. [9, 10] considered the uniqueness of a finite order meromorphic function sharing values with its shift. They proved the following theorem:

Theorem C. *Let $f(z)$ be a meromorphic function of finite order, let $c\in\mathbb{C}$, and let $a_1,a_2,a_3 \in S(f)\cup\{\infty\}$ be three distinct periodic functions with period*

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c. If $f(z)$ and $f(z+c)$ share a_1, a_2 CM and a_3 IM, then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.

Here, denote by $S(f)$ the family of all meromorphic functions $a(z)$ that satisfy $T(r, a) = o(T(r, f))$, for $r \to \infty$ outside a possible exceptional set of finite logarithmic measure.

Some improvements of Theorem C can be found in [1, 11, 12, 15]. A natural question is: what is the uniqueness result in the case when $f(z)$ shares values with $f(qz + c)$ for a zero-order meromorphic function $f(z)$. Corresponding to this question, we got the following result in [16]:

Theorem D. *Let $f(z)$ be a zero-order meromorphic function, and $q \in \mathbb{C} \setminus \{0\}$, $c \in \mathbb{C}$, and let $a_1, a_2, a_3 \in \mathbb{C} \cup \{\infty\}$ be three distinct values. If $f(z)$ and $f(qz + c)$ share a_1, a_2 CM and a_3 IM, then $f(z) = f(qz + c)$ and $|q| = 1$.*

Theorem E. *Let $f(z)$ be a zero-order entire function, $q \in \mathbb{C} \setminus \{0\}$, $c \in \mathbb{C}$, and let $a_1, a_2 \in \mathbb{C}$ be two distinct values. If $f(z)$ and $f(qz + c)$ share a_1 and a_2 IM, then $f(z) = f(qz + c)$ and $|q| = 1$.*

It seems natural to ask whether the assumption "constants a_i" can be replaced by "small functions a_i" in Theorem E. We will give a positive answer in this paper. The reminder of this paper is organized as follows: Firstly, Section 2 contains some auxiliary results. We consider the value sharing problem for $f(z)$ and $f(qz + c)$ in Section 3. Section 4 is devoted to proving some uniqueness results for meromorphic functions $f(z)$ share sets with $f(qz + c)$.

2. Some Lemmas

Lemma 2.1. [13, Theorem 2.1] *Let $f(z)$ be a zero-order meromorphic function, and $q \in \mathbb{C} \setminus \{0\}$, $c \in \mathbb{C}$. Then*

$$m\left(r, \frac{f(qz+c)}{f(z)}\right) = S_1(r, f).$$

Lemma 2.2. [16, Theorem 3.2] *Let $f(z)$ be a zero-order meromorphic function, and $q \in \mathbb{C} \setminus \{0\}$, $c \in \mathbb{C}$. Then*

$$m\left(r, \frac{f(z)}{f(qz+c)}\right) = S_1(r, f) \tag{2.1}$$

and

$$T(r, f(qz+c)) = T(r, f(z)) + S_1(r, f). \tag{2.2}$$

Lemma 2.3. [13, Theorem 2.4] *Let $f(z)$ be a zero-order meromorphic solution of*

$$f(z)^n P(z, f) = Q(z, f),$$

where $P(z, f)$ and $Q(z, f)$ are q-shift difference polynomials in $f(z)$. If the degree of $Q(z, f)$ as a polynomial in $f(z)$ and its q-shifts is at most n, then

$$m(r, P(z, f)) = S_1(r, f).$$

3. Improvement of Theorem E

Next we show that "constants a_i" in Theorems E can be replaced by "small functions a_i".

Theorem 3.1. *Let $f(z)$ be a zero-order entire function, $q \in \mathbb{C} \setminus \{0\}$, $c \in \mathbb{C}$, and let $a_1, a_2 \in S(f)$. If $f(z)$ and $f(qz+c)$ share a_1 and a_2 IM, then $f(z) = f(qz+c)$ and $|q| = 1$.*

Remarks. (1). Theorem E and Theorem 3.1 seem to be so similar. However, our proof is different to the one in Theorem E.

(2). We tried to improve Theorem D, unfortunately, we cannot get any improvement in this paper.

Proof of Theorem 3.1. From the fact that a non-constant meromorphic function of zero-order can have at most one Picard exceptional value (see, e. g., [3, p. 114]), it can be concluded that $N(r, \frac{1}{f-a_1}) \neq 0$ and $N(r, \frac{1}{f-a_2}) \neq 0$. Define

$$H(z) = \frac{H_1(z)(f(z) - f(qz+c))}{(f(z)-a(z))(f(z)-b(z))}, \tag{3.1}$$

where

$$H_1(z) = (f(z)-a(z))(f'(z)-b'(z)) - (f'(z)-a'(z))(f(z)-b(z)).$$

And

$$G(z) = \frac{G_1(z)(f(z) - f(qz+c))}{(f(qz+c)-a(z))(f(qz+c)-b(z))}, \tag{3.2}$$

where

$$G_1(z) = (f(qz+c)-a(z))(f'(qz+c)-b'(z)) - (f'(qz+c)-a'(z))(f(qz+c)-b(z)).$$

Equation (3.1) can be rewritten as

$$\begin{aligned} H(z) &= \left(\frac{f'(z)-b'(z)}{f(z)-b(z)} - \frac{f'(z)-a'(z)}{f(z)-a(z)}\right)(f(z) - f(qz+c)) \\ &= \frac{H_1(z)(f(z)-a(z)+a(z))}{(f(z)-a(z))(f(z)-b(z))}\left(1 - \frac{f(qz+c)}{f(z)}\right). \end{aligned} \tag{3.3}$$

Note

$$\begin{aligned} H_1(z) &= (f(z)-a(z))(f'(z)-b'(z)) - (f'(z)-a'(z))(f(z)-b(z)) \\ &= (f(z)-b(z))(a'(z)-b'(z)) - (f'(z)-b'(z))(a(z)-b(z)), \end{aligned}$$

hence equation (3.3) can be expressed as

$$\begin{aligned} H(z) &= \left(1 - \frac{f(qz+c)}{f(z)}\right)\left(\frac{H_1(z)}{f(z)-b(z)} + a(z)\frac{H_1(z)}{(f(z)-a(z))(f(z)-b(z))}\right) \\ &= \left(1 - \frac{f(qz+c)}{f(z)}\right)\left(\frac{(f(z)-b(z))(a'(z)-b'(z)) - (f'(z)-b'(z))(a(z)-b(z))}{f(z)-b(z)}\right. \\ &\quad \left. + a(z)\frac{(f(z)-a(z))(f'(z)-b'(z)) - (f'(z)-a'(z))(f(z)-b(z))}{(f(z)-a(z))(f(z)-b(z))}\right). \end{aligned} \tag{3.4}$$

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By the assumption $f(z)$ and $f(qz+c)$ share $a(z)$, $b(z)$ IM and equation (3.3), we get

$$N(r, H(z)) \leq N(r, a(z)) + N(r, b(z)) = S(r, f). \tag{3.5}$$

From equation (3.4), Lemma 2.1 and the lemma of logarithmic derivative, we know

$$m(r, H(z)) = S_1(r, f).$$

Hence,

$$T(r, H(z)) = S_1(r, f). \tag{3.6}$$

Similarly as above, we know

$$G(z) = \left(\frac{f'(qz+c) - b'(z)}{f(qz+c) - b(z)} - \frac{f'(qz+c) - a'(z)}{f(qz+c) - a(z)} \right) (f(z) - f(qz+c)). \tag{3.7}$$

Using a similar way, we obtain that

$$T(r, G(z)) = S_1(r, f). \tag{3.8}$$

Denote

$$U(z) = mH(z) - nG(z). \tag{3.9}$$

Next, suppose on the contrary that $f(z) \not\equiv f(qz+c)$, and head for a contradiction.

Case 1. Assume that there exists two integers m, n such that $U(z) = 0$. Then from (3.3) and (3.7), we deduce that

$$m \left(\frac{f'(z) - b'(z)}{f(z) - b(z)} - \frac{f'(z) - a'(z)}{f(z) - a(z)} \right) = n \left(\frac{f'(qz+c) - b'(z)}{f(qz+c) - b(z)} - \frac{f'(qz+c) - a'(z)}{f(qz+c) - a(z)} \right),$$

which implies that

$$\left(\frac{f(z) - b(z)}{f(z) - a(z)} \right)^m = A \left(\frac{f(qz+c) - b(z)}{f(qz+c) - a(z)} \right)^n,$$

where A is a non-zero constant. If $m \neq n$, then we get from above equality and (2.2) that

$$mT(r, f(z)) = nT(r, f(qz+c)) + S_1(r, f) = nT(r, f(z)) + S_1(r, f),$$

which is a contradiction. If $m = n$, then we get

$$\frac{f(z) - b(z)}{f(z) - a(z)} = B \frac{f(qz+c) - b(z)}{f(qz+c) - a(z)}, \tag{3.10}$$

where B satisfies $B^m = A$.

If $B = 1$, then we obtain $f(z) = f(qz+c)$, which contradicts the assumption $f(z) \not\equiv f(qz+c)$. It remains to consider the case that $B \neq 1$. The equation (3.10) gives

$$f(z)((B-1)f(qz+c)+a(z)-Bb(z)) = (Ba(z)-b(z))f(qz+c)+(1-B)a(z)b(z).$$

Apply Lemma 2.3 to the above equation, resulting in

$$m(r, ((B-1)f(qz+c) + a(z) - Bb(z))) = S_1(r, f).$$

Consequently,

$$T(r, f(qz+c)) = T(r, f) + S_1(r, f) = S_1(r, f),$$

which is impossible.

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Case 2. There does not exist two positive integers m, n such that $U(z) = 0$. In what follows, we denote $S_{f\sim g(n,m)}(a)$ for the set of those points $z \in \mathbb{C}$ such that z is an a-point of f with multiplicity n and an a-point of g with multiplicity m such that $a(z) \neq \infty, b(z) \neq \infty, a(z) - b(z) \neq 0$. Let $N_{(n,m)}(r, \frac{1}{f-a})$ and $\overline{N}_{(n,m)}(r, \frac{1}{f-a})$ denote the counting function and reduced counting function of $f(z)$ with respect to the set $S_{f\sim g(n,m)}(a)$, respectively.

Take z_0 such that $z_0 \in S_{f(z)\sim f(qz+c)(n,m)}(a(z))$, we have $mn \neq 0$, since $a(z)$ is not a Picard exceptional value of $f(z)$ as we discuss above. Combining (3.3), (3.7) with (3.9), by calculating carefully, it follows that $U(z_0) = 0$. From (3.6), (3.8) and (3.9), we have

$$\overline{N}_{(n,m)}\left(r, \frac{1}{f(z)-a(z)}\right) \leq N\left(r, \frac{1}{U(z)}\right) = N\left(r, \frac{1}{mH(z)-nG(z)}\right) = S_1(r,f).$$

Using the same reason, we get

$$\overline{N}_{(n,m)}\left(r, \frac{1}{f(z)-b(z)}\right) \leq N\left(r, \frac{1}{U(z)}\right) = N\left(r, \frac{1}{nH(z)-mG(z)}\right) = S_1(r,f).$$

Consequently,

$$\overline{N}_{(n,m)}\left(r, \frac{1}{f(z)-a(z)}\right) + \overline{N}_{(n,m)}\left(r, \frac{1}{f(z)-b(z)}\right) = S_1(r,f). \tag{3.11}$$

Combining (2.2) with (3.11), it follows that

$$\begin{aligned}
T(r, f(z)) &\leq \overline{N}\Big(r, \frac{1}{f(z)-a(z)}\Big) + \overline{N}\Big(r, \frac{1}{f(z)-b(z)}\Big) + S_1(r,f) \\
&= \sum_{n,n}\left(\overline{N}_{(n,m)}(r, \frac{1}{f(z)-a(z)}) + \overline{N}_{(n,m)}(r, \frac{1}{f(z)-b(z)})\right) + S_1(r,f) \\
&= \sum_{m+n\geq 5}\left(\overline{N}_{(n,n)}(r, \frac{1}{f(z)-a(z)}) + \overline{N}_{(n,m)}(r, \frac{1}{f(z)-b(z)})\right) + S_1(r,f) \\
&\leq \frac{1}{5}\sum_{m+n\geq 5}\Big(N_{(n,m)}(r, \frac{1}{f(z)-a(z)}) + N_{(n,m)}(r, \frac{1}{f(z)-b(z)}) \\
&\quad + N_{(n,m)}(r, \frac{1}{f(qz+c)-a(z)}) + N_{(n,m)}(r, \frac{1}{f(qz+c)-b(z)})\Big) + S_1(r,f) \\
&\leq \frac{2}{5}T(r,f) + \frac{2}{5}T(r, f(qz+c)) + S_1(r,f) \\
&= \frac{4}{5}T(r,f) + S_1(r,f),
\end{aligned}$$

which is a contradiction. Therefore, we get $f(z) = f(qz+c)$.

The rest of proof consists of the conclusion that $|q| = 1$. The proof is similar as [10, Theorem 1.5]. In fact, we have given the proof in [16]. The proof is stated explicitly for the convenience of the reader. If $f(z)$ is transcendental and suppose first $|q| < 1$. It can be assumed that there exists one point z_0 such that $f(z_0) = a_1$ and that z_0 is not a fixed point of $qz + c$. By the sharing assumptions of Theorem 3.1, we get $f(qz_0 + c) = a_1$ as well. By calculation, we know $f(q^n z_0 + c(1 + \cdots + q^{n-1})) = a_1$ for all $n \in \mathbb{N}$. Letting $n \to \infty$, it is concluded that a_1-points of f accumulate to $z = \frac{c}{1-q}$, which

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is a contradiction. If $|q| > 1$, then set $g(z) = f(qz + c)$. Assume that g has at least one a_1 point, say at z_0. From the sharing assumptions, we get $g(\frac{1}{q^n}z - c(\frac{1}{q} + \cdots + \frac{1}{q^n})) = a_1$ for all all $n \in \mathbb{N}$. Using the same way above, we get a_1-point of g accumulate to $z = \frac{c}{1-q}$, which is a contradiction. Hence $|q| = 1$.

If f is a rational function, then set $f(z) = \frac{\sum_{i=1}^{m} a_i z^i}{\sum_{j=1}^{n} b_i z^j}$ and $f(qz + c) = \frac{\sum_{i=1}^{m} a_i (qz+c)^i}{\sum_{j=1}^{n} b_i (qz+c)^j}$. By simply calculations, it follow that $|q| = 1$. This completes the proof of Theorem 3.1.

4. Sharing sets results

Gross [4, Question 6] asked the following question:

Question. Can one find (even one set) finite sets S_j $(j = 1, 2)$ such that any two entire functions $f(z)$ and $g(z)$ satisfying $E(S_j, f) = E(S_j, g)$ $(j = 1, 2)$ must be identical?

Since then, many results have been obtained for this and related topics (see [2, 19, 20, 21]). Here, we just recall the following two results only.

Theorem F [5]. *Let $S_1 = \{1, -1\}$, $S_2 = \{0\}$. If $f(z)$ and $g(z)$ are entire functions of finite order such that $E(S_j, f) = E(S_j, g)$ for $j = 1, 2$, then $f(z) = \pm g(z)$ or $f(z)g(z) = 1$.*

Theorem G [22]. *Let $S_1 = \{1, \omega, \ldots \omega^{n-1}\}$ and $S_2 = \{\infty\}$, where $\omega = \cos(2\pi/n) + i\sin(2\pi/n)$ and $n \geq 6$ be a positive integer. Suppose that $f(z)$ and $g(z)$ are meromorphic functions such that $E(S_j, f) = E(S_j, g)$ for $j = 1, 2$, then $f(z) = tg(z)$ or $f(z)g(z) = t$, where $t^n = 1$.*

It is natural to ask what will happen if $g(z)$ is replaced by q-shift of $f(z)$ in Theorems F and G. In the following, we answer this problem, and get shared sets results for $f(z)$ and its q-shift $f(qz + c)$.

Theorem 4.1. *Let S_1, S_2 be given as in Theorem G, and let $f(z)$ be a zero-order meromorphic function satisfying $E(S_j, f(z)) = E(S_j, f(qz + c))$ for $j = 1, 2$, $c \in \mathbb{C}$ and $q \in \mathbb{C} \setminus \{0\}$. If $n \geq 4$, then $f(z) = tf(qz + c)$, $t^n = 1$ and $|q| = 1$.*

By the same reasoning as in the proof of Theorem 4.1, we obtain the following result. We omit the proof here.

Corollary 4.2. *Theorem 4.1 still holds if f is a zero-order entire function and $n \geq 3$.*

In the following, we give a partial answer as to what may happen if $n = 2$ in Corollary 4.2, which can be seen an analogue for q-shift of Theorem F.

Theorem 4.3. *Suppose $f(z)$ is a zero-order entire function and $q \in \mathbb{C} \setminus \{0\}$, $c \in \mathbb{C}$. If $f(z)$ and $f(qz + c)$ share the set $\{a(z), -a(z)\}$ CM, where $a(z)$ is a non-vanishing small function of $f(z)$, then one of the following statements hold:*
(1). $C^2 f(z) = f(q^2 z + qc + c)$, where C is a constant such that $C^2 \neq 1$;
(2). $f(z) = \pm f(qz + c)$, and $|q| = 1$.

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Corollary 4.4. *Suppose a is a non-zero constant in Theorem 4.3, then we get $f(z) = \pm f(qz+c)$, where $|q| = 1$.*

Corollary 4.5. *Under the assumptions of Theorem 4.3, if $f(z)$ and $f(qz+c)$ share sets $\{a(z), -a(z)\}$, $\{0\}$ CM, then $f(z) = \pm f(qz+c)$, where $|q| = 1$.*

Proof of Theorem 4.1. By the sharing assumption, we get $f(z)^n$ and $f(qz+c)^n$ share 1 and ∞ CM. This implies,

$$\frac{f(qz+c)^n - 1}{f(z)^n - 1} = \gamma, \tag{4.1}$$

where γ is a non-zero constant. This gives

$$f(qz+c)^n = \gamma(f(z)^n - 1 + \frac{1}{\gamma}). \tag{4.2}$$

Denote

$$G(z) = \frac{f(z)^n}{1 - \frac{1}{\gamma}}.$$

Suppose $\gamma \not\equiv 1$, then by the second main theorem and Lemma 2.2 to $G(z)$, it follows that

$$\begin{aligned}
nT(r,f) + S(r,f) = T(r,G) &\leq \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}(r,G) + \overline{N}\left(r, \frac{1}{G-1}\right) + S(r,G) \\
&\leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f(z)^n - 1 + \frac{1}{\gamma}}\right) + S(r,f) \\
&\leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f(qz+c)}\right) + S(r,f) \\
&\leq 2T(r,f) + T(r, f(qz+c)) + S(r,f) \leq 3T(r,f) + S_1(r,f).
\end{aligned}$$

This together with the assumption $n \geq 4$ results in a contradiction. Hence, $\gamma \equiv 1$, that is, $f(z)^n = f(qz+c)^n$. This yields $f(z) = tf(qz+c)$ for a constant t with $t^n = 1$. Let $F(z) = f(z)^n$ and $F(qz+c) = f(qz+c)^n$, then we get $F(z) = F(qz+c)$. Similarly as Theorem 3.1, we have $|q| = 1$. The conclusion follows.

Proof of Theorem 4.3. It follows by the assumptions that

$$(f(qz+c) - a(z))(f(qz+c) + a(z)) = C^2(f(z) - a(z))(f(z) + a(z)), \tag{4.3}$$

where C is a non-zero constant.

Case 1. Suppose first that $C^2 \not\equiv 1$. Denote

$$h_1(z) = f(z) - \frac{1}{C}f(qz+c), \quad h_2(z) = f(z) + \frac{1}{C}f(qz+c).$$

Then

$$f(z) = \frac{1}{2}(h_1(z) + h_2(z)), \quad f(qz+c) = \frac{C}{2}(h_2(z) - h_1(z)). \tag{4.4}$$

Moreover, we have

$$h_1(z)h_2(z) = (1 - \frac{1}{C^2})a^2(z). \tag{4.5}$$

From above equation, we get

$$N\left(r,\frac{1}{h_1}\right)=S(r,f),\quad N\left(r,\frac{1}{h_2}\right)=S(r,f). \tag{4.6}$$

By definitions of $h_1(z)$ and $h_2(z)$, Lemma 2.2 yields

$$T(r,h_i)\le 2T(r,f)+S_1(r,f),$$

which means $S_1(r,h_i)=o(T(r,f))$ for all r on a set of logarithmic density 1, $i=1,2$.
Denote

$$\alpha(z)=\frac{h_1(qz+c)}{h_1(z)},\quad \beta(z)=\frac{h_2(qz+c)}{h_2(z)}.$$

From (4.6) and Lemma 2.1, we obtain that

$$\begin{aligned}T(r,\alpha)&=m(r,\alpha)+N\left(r,\frac{1}{h_1}\right)=S_1(r,f),\\ T(r,\beta)&=m(r,\beta)+N\left(r,\frac{1}{h_2}\right)=S_1(r,f).\end{aligned} \tag{4.7}$$

From definitions of $h_1(z)$, $h_2(z)$ and equation (4.4), we conclude that

$$Ch_2(z)-Ch_1(z)=h_1(qz+c)+h_2(qz+c).$$

Dividing above equation with $h_1(z)h_2(z)$, we obtain

$$(\alpha+C)h_1(z)=(C-\beta)h_2(z). \tag{4.8}$$

By combining (4.5) and (4.8), it follows that

$$(\alpha+C)h_1^2(z)-(C-\beta)(1-\frac{1}{C^2})a^2(z)=0. \tag{4.9}$$

From (4.7) and (4.9), we get $\alpha=-C$ and $\beta=C$. Otherwise, we know $T(r,h_1)=S_1(r,f)$, which means $T(r,f)=S_1(r,f)$ from (4.4) and (4.5), a contradiction. Hence, we have

$$h_1(qz+c)=-Ch_1(z),\quad h_2(qz+c)=Ch_2(z),$$

from definitions of $\alpha(z)$ and $\beta(z)$, that is

$$\begin{cases}-C(f(z)-\frac{1}{C}f(qz))=f(qz)-\frac{1}{C}f(q(qz+c)+c),\\ C(f(z)+\frac{1}{C}f(qz))=f(qz)+\frac{1}{C}f(q(qz+c)+c).\end{cases}$$

The above equations give $C^2f(z)=f(q^2z+qc+c)$.

Case 2. $C^2\equiv 1$. The equation (4.3) implies that $f(z)=\pm f(qz+c)$. Using a similar way as Theorem 3.1, we get $|q|=1$ in Case 2.

Proof of Corollary 4.4. Similarly as Theorem 4.3, we obtain equations (4.4) and (4.5) hold as well. Equation (4.5) and the assumption that a is non-zero constant give

$$N\left(r,\frac{1}{h_1}\right)=0,\quad N\left(r,\frac{1}{h_2}\right)=0. \tag{4.10}$$

Combining (4.10) with the definitions of $h_1(z)$ and $h_2(z)$, we conclude that $h_1(z)$ and $h_2(z)$ are constants. From (4.4), we get $f(z)$ is a constant, which contradicts the assumption. Hence, only Case 2 of Theorem 4.3 holds, the conclusion follows.

Proof of Corollary 4.5. It suffices to prove the case $C^2 f(z) = f(q^2 z + qc + c)$ in Theorem 4.3 cannot hold. Suppose that $f(z_0) = 0$, then by the sharing assumption and (4.4), it follows that

$$h_1(z_0) + h_2(z_0) = 0, \quad h_1(qz_0 + c) + h_2(qz_0 + c) = 0.$$

Hence,

$$\frac{h_1(qz_0 + c)}{h_1(z_0)} \frac{h_2(z_0)}{h_2(qz_0 + c)} = 1.$$

From the proof of Theorem 4.3, we know

$$\alpha = \frac{h_1(qz_0 + c)}{h_1(z_0)} = -C, \quad \beta = \frac{h_2(qz_0 + c)}{h_2(z_0)} = C,$$

which means that

$$\frac{h_1(qz_0 + c)}{h_1(z_0)} \frac{h_2(z_0)}{h_2(qz_0 + c)} = -1.$$

which is impossible. This contradiction is only avoided when 0 is the Picard exceptional value of $f(z)$ and $f(qz + c)$. Since $f(z)$ is a zero-order entire function, we conclude that $f(z)$ must be a constant, which contradicts the assumption. Hence, $f(z) = \pm f(qz + c)$, where $|q| = 1$.

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RANDOM NORMED SPACE AND MIXED TYPE AQ–FUNCTIONAL EQUATION

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ABSTRACT. We investigate the stability problems for the following functional equation

$$f(x+ay)+f(x-ay)-2f(x)+\frac{a-a^2}{2}f(y)-\frac{a+a^2}{2}f(-y)-f(ay)=0$$

in random normed spaces.

1. Introduction and Preliminaries

We first demonstrate the usual terminology, notations and conventions of the theory of random normed spaces [7, 8]. The space of all probability distribution functions is denoted by

$$\Delta^+ := \{F : \mathbb{R}\cup\{-\infty,\infty\}\to[0,1]\,\big|\,F \text{ is left-continuous and nondecreasing on } \mathbb{R}, \\ \text{where } F(0)=0 \text{ and } F(+\infty)=1\}.$$

And let $D^+ := \{F \in \Delta^+ \,|\, l^-F(+\infty)=1\}$, where $l^-f(x)$ denotes the left limit of the function f at the point x. The space Δ^+ is partially ordered by the usual pointwise ordering of functions, i.e., $F \le G$ if and only if $F(t)\le G(t)$ for all $t\in\mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function $\varepsilon_0 : \mathbb{R}\cup\{0\}\to[0,\infty)$ given by

$$\varepsilon_0(t)=\begin{cases}0, & \text{if } t\le 0,\\ 1, & \text{if } t>0.\end{cases}$$

Definition 1.1. ([7]) A mapping $\tau : [0,1]\times[0,1]\to[0,1]$ is called a *continuous triangular norm* (briefly, a *continuous t-norm*) if τ satisfies the following conditions:

(TN1) τ is commutative and associative;
(TN2) τ is continuous;
(TN3) $\tau(a,1)=a$ for all $a\in[0,1]$;
(TN4) $\tau(a,b)\le\tau(c,d)$ whenever $a\le c$ and $b\le d$ for all $a,b,c,d\in[0,1]$.

Typical examples of continuous t-norms are $\tau_P(a,b)=ab$, $\tau_M(a,b)=\min(a,b)$ and $\tau_L(a,b)=\max(a+b-1,0)$.

Definition 1.2. ([8]) A *random normed space* (briefly, *RN-space*) is a triple (X,μ,τ), where X is a vector space, τ is a continuous t-norm and μ is a mapping from X into D^+ such that the following conditions hold:

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(RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$,
(RN2) $\mu_{\alpha x}(t) = \mu_x(t/|\alpha|)$ for all $x \in X$, $\alpha \neq 0$ and all $t \geq 0$,
(RN3) $\mu_{x+y}(t+s) \geq \tau(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and all $t, s \geq 0$.

If $(X, \|\cdot\|)$ is a normed space, we can define a mapping $\mu : X \to D^+$ by $\mu_x(t) = \frac{t}{t+\|x\|}$ for all $x \in X$ and all $t > 0$. Then (X, μ, τ_M) is a random normed space, which is called the *induced random normed space*.

Definition 1.3. Let (X, μ, τ) be an *RN*-space.

- (A_1) A sequence $\{x_n\}$ in X is said to be *convergent* to a point $x \in X$ if for every $t > 0$ and $\varepsilon > 0$, there exists a positive integer N such that $\mu_{x_n - x}(t) > 1 - \varepsilon$ whenever $n \geq N$.
- (A_2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if for every $t > 0$ and $\varepsilon > 0$, there exists a positive integer N such that $\mu_{x_n - x_m}(t) > 1 - \varepsilon$ whenever $n \geq m \geq N$.
- (A_3) An RN-space (X, μ, τ) is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X.

Theorem 1.4. ([7]) *If (X, μ, τ) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$.*

The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. The stability problem for functional equations originated from questions of Ulam [9] concerning the stability of group homomorphisms. Hyers [2] had answered affirmatively the question of Ulam for Banach spaces. A generalized version of the theorem of Hyers for additive mappings was given by Aoki [1] and for linear mappings was presented by Rassias [6]. Since then, many interesting results of the stability of various functional equation have been extensively investigated.

Now we take into account the following *mixed type additive-quadratic functional equation* (briefly, *AQ-functional equation*)

$$f(x+ay) + f(x-ay) - 2f(x) + \frac{a-a^2}{2} f(y) - \frac{a+a^2}{2} f(-y) - f(ay) = 0. \tag{1.1}$$

Here we promise that each solution of equation (1.1) is said to be an *additive-quadratic mapping.* Quite recently, the stability of functional equation (1.1) in the case when $a = 1$ was investigated in [3, 4, 5].

The main aim of this work is to establish the stability for the functional equation (1.1) in random normed spaces.

2. Main results

Let E_1 and E_2 be vector spaces. For convenience, we use the following abbreviations for a given mapping $f : E_1 \to E_2$,

$$Af(x,y) := f(x+y) - f(x) - f(y),$$
$$Qf(x,y) := f(x+y) + f(x-y) - 2f(x) - 2f(y),$$
$$Df(x,y) := f(x+ay) + f(x-ay) - 2f(x) + \frac{a-a^2}{2} f(y) - \frac{a+a^2}{2} f(-y) - f(ay)$$

for all $x, y \in E_1$, where $a > \frac{1}{2}$ is a rational number.

A solution of $Af = 0$ is said to be an *additive mapping* and a solution of $Qf = 0$ is called a *quadratic mapping.* If a mapping f is represented by sum of additive mapping and quadratic mapping, we say that f is an *additive-quadratic mapping.*

Lemma 2.1. *A mapping $f : E_1 \to E_2$ satisfies the functional equation $Df(x, y) = 0$ for all $x, y \in E_1$ if and only if there exist a quadratic mapping $g : E_1 \to E_2$ and an additive mapping $h : E_1 \to E_2$ such that $f(x) = g(x) + h(x)$ for all $x \in E_1$.*

Proof. (Necessity) We decompose f into the even part and the odd part by considering

$$g(x) = \frac{f(x) + f(-x)}{2},\ h(x) = \frac{f(x) - f(-x)}{2}$$

for all $x \in E_1$. It is note that $f(0) = \frac{-Df(0,0)}{a^2+1} = 0$. The following functional equalities

$$\begin{aligned} Qg(x, y) =& Dg(x, y/a) - Dg(0, y/a) = 0, \\ Ah(x, y) =& -Dh\Big(\frac{x+y}{2}, \frac{x-y}{2a}\Big) + Dh\Big(\frac{x+y}{2}, \frac{x+y}{2a}\Big) + Dh\Big(0, \frac{x-y}{2a}\Big) - Dh\Big(0, \frac{x+y}{2a}\Big) \\ =& 0 \end{aligned}$$

give that g is a quadratic mapping and h is an additive mapping.

(Sufficiency) Assume that there exist a quadratic mapping $g : E_1 \to E_2$ and an additive mapping $h : E_1 \to E_2$ such that $f(x) = g(x) + h(x)$ for all $x \in E_1$. Then we see that

$$\begin{aligned} Df(x, y) &= Dg(x, y) + Dh(x, y) \\ &= Qg(x, ay) + g(ay) - a^2 g(y) - Ah(x + ay, x - ay) + Ah(x, x) + ah(y) - h(ay) \\ &= 0 \end{aligned}$$

for all $x, y \in E_1$. Therefore we arrive at the desired conclusion. □

In the following theorem, we establish the stability of the functional equation (1.1) in random normed spaces.

Theorem 2.2. *Let (Y, μ, τ_M) and (Z, μ', τ_M) be a complete RN-space and an RN-space, respectively. Suppose that V is a vector space and $f : V \to Y$ is a mapping with $f(0) = 0$ for which there exists a mapping $\varphi : V^2 \to Z$ such that*

$$\mu_{Df(x,y)}(t) \geq \mu'_{\varphi(x,y)}(t) \tag{2.1}$$

for all $x, y \in V$ and all $t > 0$. If a mapping φ satisfies one of the following conditions:

(i) $\mu'_{\alpha\varphi(x,y)}(t) \leq \mu'_{\varphi(2ax,2ay)}(t)$ *for some* $0 < \alpha < 2a$,

(ii) $\mu'_{\varphi(2ax,2ay)}(t) \leq \mu'_{\alpha\varphi(x,y)}(t) \leq \mu'_{\varphi((2a)^2x,(2a)^2y)}(t)$ *for some* $2a < \alpha < (2a)^2$,

(iii) $\mu'_{\varphi((2a)^2x,(2a)^2y)}(t) \leq \mu'_{\alpha\varphi(x,y)}(t)$ *for some* $(2a)^2 < \alpha$

for all $x, y \in V$ and all $t > 0$, then there exists a unique additive-quadratic mapping $F : V \to Y$ such that

$$\mu_{f(x)-F(x)}(t) \geq \begin{cases} \sup_{t'<t}\{M\left(x, (2a - \alpha)t'\right)\} & \textit{if } \varphi \textit{ satisfies } \text{(i)}, \\ \sup_{t'<t}\{M(x, \frac{((2a)^2-\alpha)(2a-\alpha)t'}{4((2a)^2-2a)})\} & \textit{if } \varphi \textit{ satisfies } \text{(ii)}, \\ \sup_{t'<t}\{M(x, (\alpha - (2a)^2)t')\} & \textit{if } \varphi \textit{ satisfies } \text{(iii)} \end{cases} \tag{2.2}$$

for all $x \in V$ and all $t > 0$, where

$$M(x, t) := \tau_M\big\{\mu'_{\varphi(ax,x)}(t), \mu'_{\varphi(-ax,-x)}(t), \mu'_{\varphi(0,x)}(t), \mu'_{\varphi(0,-x)}(t)\big\}.$$

Proof. We will take into account three different cases for the assumption of φ.

Case 1. Let φ satisfy the condition (i) for some α with $0 < \alpha < 2a$ and let $J_n f : V \to Y$ be a mapping defined by

$$J_n f(x) := \frac{f((2a)^n x) - f(-(2a)^n x)}{2(2a)^n} + \frac{f((2a)^n x) + f(-(2a)^n x)}{2(2a)^{2n}}$$

for all $x \in V$ and all $n \in \mathbb{N}$. Then $J_0 f(x) = f(x)$, $J_j f(0) = f(0)$ and

$$\begin{aligned} J_j f(x) - J_{j+1} f(x) = &\frac{(2a)^{j+1} - 1}{2(2a)^{2j+2}} \left[Df(-(2a)^j ax, -(2a)^j x) - 3Df(0, (2a)^j x) \right] \\ &- \frac{(2a)^{j+1} + 1}{2(2a)^{2j+2}} \left[Df((2a)^j ax, (2a)^j x) - 3Df(0, -(2a)^j x) \right] \end{aligned} \tag{2.3}$$

for all $x \in V$ and all $j \geq 0$. It implies that if $n + m > n \geq 0$, then we get by (RN2), (RN3), (2.1) and (2.2)

$$\begin{aligned} &\mu_{J_n f(x) - J_{n+m} f(x)} \Big(\sum_{j=n}^{n+m-1} \frac{4\alpha^j t}{(2a)^{j+1}} \Big) \\ &\geq \mu_{\sum_{j=n}^{n+m-1} (J_j f(x) - J_{j+1} f(x))} \Big(\sum_{j=n}^{n+m-1} \frac{4\alpha^j t}{(2a)^{j+1}} \Big) \\ &\geq \tau_M{}_{j=n}^{n+m-1} \Big\{ \mu_{J_j f(x) - J_{j+1} f(x)} \Big(\frac{4\alpha^j t}{(2a)^{j+1}} \Big) \Big\} \\ &\geq \tau_M{}_{j=n}^{n+m-1} \Big\{ \tau \Big\{ \mu_{-\frac{((2a)^{j+1}+1) Df((2a)^j \cdot ax, (2a)^j x)}{2(2a)^{2j+2}}} \Big(\frac{((2a)^{j+1} + 1)\alpha^j t}{2(2a)^{2j+2}} \Big), \\ &\quad \mu_{\frac{((2a)^{j+1}-1) Df(-(2a)^j \cdot ax, -(2a)^j x)}{2(2a)^{2j+2}}} \Big(\frac{((2a)^{j+1} - 1)\alpha^j t}{2(2a)^{2j+2}} \Big), \\ &\quad \mu_{\frac{3((2a)^{j+1}+1) Df(0, -(2a)^j x)}{2(2a)^{2j+2}}} \Big(\frac{3((2a)^{j+1} + 1)\alpha^j t}{2(2a)^{2j+2}} \Big), \\ &\quad \mu_{-\frac{3((2a)^{j+1}-1) Df(0, (2a)^j x)}{2(2a)^{2j+2}}} \Big(\frac{3((2a)^{j+1} - 1)\alpha^j t}{2(2a)^{2j+2}} \Big) \Big\} \\ &\geq M(x, t) \end{aligned} \tag{2.4}$$

for all $x \in V$ and all $t > 0$. Let $c > 0$ and $\varepsilon > 0$ be given. Since $\lim_{t \to \infty} \mu'_z(t) = 1$ for all $z \in Z$, there is some $t_0 > 0$ such that $M(x, t_0) \geq 1 - \varepsilon$. Fix some $t > t_0$. Since $\alpha < 2a$, we know that the series $\sum_{j=0}^{\infty} \frac{4\alpha^j t}{(2a)^{j+1}}$ converges. It guarantees that there exists some $n_0 \geq 0$ such that $\sum_{j=n}^{n+m-1} \frac{4\alpha^j t}{(2a)^{j+1}} < c$ for all $n \geq n_0$ and all $m > 0$. Together with (RN3) and (2.4), this implies that

$$\begin{aligned} \mu_{J_n f(x) - J_{n+m} f(x)}(c) &\geq \mu_{J_n f(x) - J_{n+m} f(x)} \Big(\sum_{j=n}^{n+m-1} \frac{4\alpha^j t}{(2a)^{j+1}} \Big) \\ &\geq M(x, t) \geq M(x, t_0) \geq 1 - \varepsilon \end{aligned}$$

for all $x \in V$. Hence $\{J_n f(x)\}$ is a Cauchy sequence in the complete RN-space (Y, μ, τ_M) and so we can define a mapping $F : X \to Y$ by $F(x) := \lim_{n\to\infty} J_n f(x)$. Moreover, if we put $m = 0$ in (2.4), we have

$$\mu_{f(x)-J_nf(x)}(t) \geq M\left(x, \frac{t}{\sum_{j=0}^{n-1} \frac{4\alpha^j t}{(2a)^{j+1}}}\right) \tag{2.5}$$

for all $x \in V$.

Next we are in the position to show that F is an additive-quadratic mapping. In view of (RN3), we figure out the relation

$$\begin{aligned}\mu_{DF(x,y)}(t) \geq \tau_M\Big\{&\mu_{(F-J_nf)(x+ay)}\Big(\frac{t}{12}\Big), \mu_{(F-J_nf)(x-ay)}\Big(\frac{t}{12}\Big), \mu_{2(J_n-Ff)(x)}\Big(\frac{t}{12}\Big),\\ &\mu_{\frac{a-a^2}{2}(F-J_nf)(y)}\Big(\frac{t}{12}\Big), \mu_{-\frac{a+a^2}{2}(F-J_nf)(-y)}\Big(\frac{t}{12}\Big), \mu_{-(F-J_nf)(ay)}\Big(\frac{t}{12}\Big),\\ &\mu_{DJ_nf(x,y)}\Big(\frac{t}{2}\Big)\Big\}\end{aligned} \tag{2.6}$$

for all $x, y \in V$ and all $n \in \mathbb{N}$. The first six terms on the right hand side of the previous inequality tend to 1 as $n \to \infty$ by the definition of F. Also we consider that

$$\begin{aligned}\mu_{DJ_nf(x,y)}\Big(\frac{t}{2}\Big) &\geq \tau_M\Big\{\mu_{\frac{Df((2a)^nx,(2a)^ny)}{2\cdot(2a)^{2n}}}\Big(\frac{t}{8}\Big), \mu_{\frac{Df(-(2a)^nx,-(2a)^ny)}{2\cdot(2a)^{2n}}}\Big(\frac{t}{8}\Big),\\ &\qquad \mu_{\frac{Df((2a)^nx,(2a)^ny)}{2\cdot(2a)^{n}}}\Big(\frac{t}{8}\Big), \mu_{\frac{Df(-(2a)^nx,-(2a)^ny)}{2\cdot(2a)^{n}}}\Big(\frac{t}{8}\Big)\Big\}\\ &\geq \tau_M\Big\{\mu_{\frac{\varphi((2a)^nx,(2a)^ny)}{2\cdot(2a)^{2n}}}\Big(\frac{t}{8}\Big), \mu_{\frac{\varphi(-(2a)^nx,-(2a)^ny)}{2\cdot(2a)^{2n}}}\Big(\frac{t}{8}\Big),\\ &\qquad \mu_{\frac{\varphi((2a)^nx,(2a)^ny)}{2\cdot(2a)^{n}}}\Big(\frac{t}{8}\Big), \mu_{\frac{\varphi(-(2a)^nx,-(2a)^ny)}{2\cdot(2a)^{n}}}\Big(\frac{t}{8}\Big)\Big\}\\ &\geq \tau_M\Big\{\mu_{\varphi(x,y)}\Big(\frac{(2a)^{2n}t}{4\alpha^n}\Big), \mu_{\varphi(-x,-y)}\Big(\frac{(2a)^{2n}t}{4\alpha^n}\Big),\\ &\qquad \mu_{\varphi(x,y)}\Big(\frac{(2a)^{n}t}{4\alpha^n}\Big), \mu_{\varphi(-x,-y)}\Big(\frac{(2a)^{n}t}{4\alpha^n}\Big)\Big\},\end{aligned}$$

which tends to 1 as $n \to \infty$ by (RN3). It follows from (2.6) that $\mu_{DF(x,y)}(t) = 1$ for all $x, y \in V$ and all $t > 0$. By (RN1), this means that $DF(x, y) = 0$ for all $x, y \in V$.

We now approximate the difference between f and F. Fix $x \in V$, $t > 0$ and choose $t' < t$. For arbitrary $\varepsilon > 0$, by $F(x) := \lim_{n\to\infty} J_n f(x)$, there is a $n \in \mathbb{N}$ such that

$$\mu_{F(x)-J_nf(x)}(t - t') \geq 1 - \varepsilon.$$

It follows by (2.5) that

$$\begin{aligned}\mu_{F(x)-f(x)}(t) &\geq \tau_M\{\mu_{F(x)-J_nf(x)}(t-t'), \mu_{J_nf(x)-f(x)}(t')\}\\ &\geq \tau_M\Big\{1-\varepsilon, M\Big(x, \frac{t'}{\sum_{j=0}^{n-1} \frac{4\alpha^j t}{(2a)^{j+1}}}\Big)\Big\}\\ &\geq \tau_M\Big\{1-\varepsilon, M\Big(x, \frac{(2a-\alpha)t'}{4}\Big)\Big\}.\end{aligned}$$

Because $\varepsilon > 0$ is arbitrary, we find that

$$\mu_{F(x)-f(x)}(t) \geq M(x, (2a-\alpha)t')$$

for all $x \in V$ and $t' < t$. The first inequality in (2.2) follows from the previous inequality.

In order to prove the uniqueness of F, we assume that F' is another additive-quadratic mapping from V to Y satisfying the first inequality in (2.2) with $F'(0) = f(0)$. Note that if F' is an additive-quadratic mapping, then we have by (2.3)

$$F'(x) - J_nF'(x) = \sum_{j=0}^{n-1}(J_jF'(x) - J_{j+1}F'(x)) = 0$$

for all $x \in V$ and all $n \in \mathbb{N}$. With the help of (RN3) and the first inequality in (2.2), this result yields that for all $x \in V$ and all $n \in \mathbb{N}$,

$$\begin{aligned}
\mu_{F'(x)-J_nf(x)}(t) &= \mu_{J_nF'(x)-J_nf(x)}(t) \\
&\geq \tau_M\Big\{\mu_{\frac{(F'-f)((2a)^nx)}{2\cdot(2a)^{2n}}}\Big(\frac{t}{4}\Big), \mu_{\frac{(F'-f)(-(2a)^nx)}{2\cdot(2a)^{2n}}}\Big(\frac{t}{4}\Big), \mu_{\frac{(F'-f)((2a)^nx)}{2\cdot(2a)^{n}}}\Big(\frac{t}{4}\Big), \\
&\quad \mu_{\frac{(F'-f)(-(2a)^nx)}{2\cdot(2a)^{n}}}\Big(\frac{t}{4}\Big)\Big\} \\
&\geq \tau_M\Big\{\sup_{t'<t}\Big\{M\Big(x, \Big(\frac{2a}{\alpha}\Big)^n\frac{(2a-\alpha)t'}{4}\Big), \sup_{t'<t}\Big\{M\Big(x, \Big(\frac{4a^2}{\alpha}\Big)^n\frac{(2a-\alpha)t'}{4}\Big)\Big\}.
\end{aligned}$$

Observe that

$$\lim_{n\to\infty}\Big(\frac{2a}{\alpha}\Big)^n\frac{(2a-\alpha)t'}{4} = \infty,$$

which gives that

$$\lim_{n\to\infty}\mu_{F'(x)-J_nf(x)}(t) = 1$$

and then we have by (RN1)

$$F'(x) = \lim_{n\to\infty}J_nf(x) = F(x)$$

for all $x \in V$.

Case 2. Assume that φ satisfies the condition (ii) for some α with $2a < \alpha < 4a^2$ and $J_nf : V \to Y$ is a mapping defined by

$$J_nf(x) := \frac{f((2a)^nx) + f(-(2a)^nx)}{2\cdot(2a)^{2n}} + \frac{(2a)^n}{2}\Big[f\Big(\frac{x}{(2a)^n}\Big) - f\Big(\frac{-x}{(2a)^n}\Big)\Big]$$

for all $x \in V$. Then we have $J_0f(x) = f(x)$, $J_jf(0) = f(0)$ and

$$\begin{aligned}
J_jf(x) - J_{j+1}f(x) = &-\frac{Df(-(2a)^jax, -(2a)^jx) - 3Df(0,(2a)^jx)}{2(2a)^{2j+2}} \\
&-\frac{Df((2a)^jax, (2a)^jx) - 3Df(0,-(2a)^jx)}{2(2a)^{2j+2}} \\
&+\frac{(2a)^j}{2}\Big[Df\Big(\frac{x}{2(2a)^j}, \frac{x}{(2a)^{j+1}}\Big) - 3Df\Big(0, \frac{-x}{(2a)^{j+1}}\Big)\Big] \\
&-\frac{(2a)^j}{2}\Big[Df\Big(\frac{-x}{2(2a)^j}, \frac{-x}{(2a)^{j+1}}\Big) - 3Df\Big(0, \frac{x}{(2a)^{j+1}}\Big)\Big]
\end{aligned} \tag{2.7}$$

for all $x \in V$ and all $j \geq 0$. If $n+m > n \geq 0$, then we deduce that

$$\begin{aligned}
&\mu_{J_nf(x)-J_{n+m}f(x)}\left(\sum_{j=n}^{n+m-1}\Big(\frac{4}{(2a)^2}\Big(\frac{\alpha}{(2a)^2}\Big)^j+\frac{4}{\alpha}\Big(\frac{(2a)}{\alpha}\Big)^j\Big)t\right)\\
&=\mu_{\sum_{j=m}^{l+m-1}(J_jf(x)-J_{j+1}f(x))}\left(\sum_{j=n}^{n+m-1}\Big(\frac{4}{(2a)^2}\Big(\frac{\alpha}{(2a)^2}\Big)^j+\frac{4}{\alpha}\Big(\frac{(2a)}{\alpha}\Big)^j\Big)t\right)\\
&\geq \tau_{M}{}_{j=n}^{n+m-1}\Big\{\mu_{J_jf(x)-J_{j+1}f(x)}\Big(\Big(\frac{4}{(2a)^2}\Big(\frac{\alpha}{(2a)^2}\Big)^j+\frac{4}{\alpha}\Big(\frac{(2a)}{\alpha}\Big)^j\Big)t\Big)\Big\} \qquad (2.8)\\
&\geq \tau_{M}{}_{j=n}^{n+m-1}\Big\{\tau_M\Big\{\mu_{-\frac{Df((2a)^jax,(2a)^jx)}{2(2a)^{2j+2}}}\Big(\frac{\alpha^jt}{2(2a)^{2j+2}}\Big),\mu_{\frac{-Df(-(2a)^jax,-(2a)^jx)}{2(2a)^{2j+2}}}\Big(\frac{\alpha^jt}{2(2a)^{2j+2}}\Big),\\
&\quad\mu_{\frac{3Df(0,-(2a)^jx)}{2(2a)^{2j+2}}}\Big(\frac{3\alpha^jt}{2(2a)^{2j+2}}\Big),\mu_{\frac{3Df(0,(2a)^jx)}{2(2a)^{2j+2}}}\Big(\frac{3\alpha^jt}{2(2a)^{2j+2}}\Big),\mu_{\frac{(2a)^j}{2}Df\big(\frac{x}{2(2a)^j},\frac{x}{(2a)^{j+1}}\big)}\Big(\frac{(2a)^jt}{2\alpha^{j+1}}\Big),\\
&\quad\mu_{\frac{-3(2a)^j}{2}Df\big(0,\frac{-x}{(2a)^{j+1}}\big)}\Big(\frac{3(2a)^jt}{2\alpha^{j+1}}\Big),\mu_{\frac{-(2a)^j}{2}Df\big(\frac{-x}{2(2a)^j},\frac{-x}{(2a)^{j+1}}\big)}\Big(\frac{(2a)^jt}{2\alpha^{j+1}}\Big),\\
&\quad\mu_{\frac{3(2a)^j}{2}Df\big(0,\frac{x}{(2a)^{j+1}}\big)}\Big(\frac{3(2a)^jt}{2\alpha^{j+1}}\Big)\Big\}\Big\}\\
&\geq M(x,t)
\end{aligned}$$

for all $x \in V$ and all $t > 0$. Therefore the Cauchy sequence $\{J_nf(x)\}$ has the limit $F(x) := \lim_{n\to\infty} J_nf(x)$ for all $x \in V$ and

$$\mu_{f(x)-J_nf(x)}(t) \geq M\left(x, \frac{t}{\sum_{j=0}^{n-1}\big(\frac{4}{(2a)^2}\big(\frac{\alpha}{(2a)^2}\big)^j+\frac{4}{\alpha}\big(\frac{(2a)}{\alpha}\big)^j\big)}\right) \tag{2.9}$$

for all $x \in V$.

Now, to prove that $DF(x,y) = 0$ for all $x, y \in V$, we consider (2.6) in case 1. By virtue of (RN3) and (2.1), we see that

$$\begin{aligned}
\mu_{DJ_nf(x,y)}\Big(\frac{t}{2}\Big) &\geq \tau_M\Big\{\mu_{\frac{Df((2a)^nx,(2a)^ny)}{2\cdot(2a)^{2n}}}\Big(\frac{t}{8}\Big),\mu_{\frac{Df(-(2a)^nx,-(2a)^ny)}{2\cdot(2a)^{2n}}}\Big(\frac{t}{8}\Big),\\
&\qquad\mu_{\frac{(2a)^n}{2}Df\big(\frac{x}{(2a)^n},\frac{y}{(2a)^n}\big)}\Big(\frac{t}{8}\Big),\mu_{-\frac{(2a)^n}{2}Df\big(\frac{-x}{(2a)^n},\frac{-y}{(2a)^n}\big)}\Big(\frac{t}{8}\Big)\Big\}\\
&\geq \tau_M\Big\{\mu_{\varphi(x,y)}\Big(\frac{(2a)^{2n}t}{4\alpha^n}\Big),\mu_{\varphi(-x,-y)}\Big(\frac{(2a)^{2n}t}{4\alpha^n}\Big),\\
&\qquad\mu_{\varphi(x,y)}\Big(\frac{\alpha^nt}{4(2a)^n}\Big),\mu_{\varphi(-x,-y)}\Big(\frac{\alpha^nt}{4(2a)^n}\Big)\Big\}
\end{aligned}$$

for all $x, y \in V$ and all $t > 0$, which tends to 1 as $n \to \infty$. It implies that all the terms of (2.6) are equal to 1 as $n \to \infty$ and then we know that F is an additive-quadratic mapping.

Employing the same argument as in the proof of case 1, the second inequality in (2.2) follows from (2.9).

Finally, it remains to prove the uniqueness of F. Let us assume that $F' : V \to Y$ is another additive-quadratic mapping satisfying (2.2). Note that if F' is an additive-quadratic mapping then by (2.7)

$$F'(x) - J_nF'(x) = \sum_{j=0}^{n-1} \big(J_jF'(x) - J_{j+1}F'(x)\big) = 0$$

for all $x \in V$ and all $n \in \mathbb{N}$. This relation with (RN3) and (2.2) imply that

$$\begin{aligned}
\mu_{F'(x)-J_nf(x)}(t) &= \mu_{J_nF'(x)-J_nf(x)}(t)\\
&\geq \tau_M\Big\{\mu_{\frac{(F'-f)((2a)^nx)}{2\cdot(2a)^{2n}}}\Big(\frac{t}{4}\Big), \mu_{\frac{(F'-f)(-(2a)^nx)}{2\cdot(2a)^{2n}}}\Big(\frac{t}{4}\Big),\\
&\quad \mu_{\frac{(2a)^n}{2}(F'-f)\left(\frac{x}{(2a)^n}\right)}\Big(\frac{t}{4}\Big), \mu_{\frac{(2a)^n}{2}(F'-f)\left(\frac{-x}{(2a)^n}\right)}\Big(\frac{t}{4}\Big)\Big\}\\
&\geq \tau_M\Big\{\sup_{t'<t}\Big\{M\Big(x, \frac{((2a)^2-\alpha)(2a-\alpha)t'}{2((2a)^2-2a)}\Big(\frac{\alpha}{2a}\Big)^n\Big),\\
&\quad \sup_{t'<t}\Big\{M\Big(x, \frac{((2a)^2-\alpha)(2a-\alpha)t'}{2((2a)^2-2a)}\Big)^n\Big)\Big\}
\end{aligned}$$

for all $x \in V$ and all $n \in \mathbb{N}$. Due to the fact that

$$\lim_{n\to\infty} \frac{((2a)^2-\alpha)(2a-\alpha)t'}{2((2a)^2-2a)}\Big(\frac{(2a)^2}{\alpha}\Big)^n = \infty,\ \lim_{n\to\infty} \frac{((2a)^2-\alpha)(2a-\alpha)t'}{2((2a)^2-2a)}\Big(\frac{\alpha}{2a}\Big)^n = \infty$$

for $2a < \alpha < 4a^2$, we have

$$\lim_{n\to\infty} \mu_{F'(x)-J_nf(x)}(t) = 1.$$

Of course, by virtue of (RN1), we see that

$$F'(x) = \lim_{n\to\infty} J_nf(x) = F(x)$$

for all $x \in V$.

Case 3. Suppose that φ satisfies the condition (iii) for some α with $\alpha > (2a)^2$ and and $J_nf : V \to Y$ is a mapping defined by

$$J_nf(x) = \frac{(2a)^{2n}+(2a)^n}{2}f\Big(\frac{x}{(2a)^n}\Big) + \frac{(2a)^{2n}-(2a)^n}{2}f\Big(\frac{-x}{(2a)^n}\Big)$$

for all $x \in V$. Then we have $J_0f(x) = f(x)$ and

$$\begin{aligned}
J_jf(x) - J_{j+1}f(x) &= \frac{(2a)^{2j}+(2a)^j}{2}\Big[Df\Big(\frac{x}{2\cdot(2a)^j}, \frac{x}{(2a)^{j+1}}\Big) - 3Df\Big(0, \frac{-x}{(2a)^{j+1}}\Big)\Big]\\
&\quad + \frac{(2a)^{2j}-(2a)^j}{2}\Big[Df\Big(\frac{-x}{2\cdot(2a)^j}, \frac{-x}{(2a)^{j+1}}\Big) - 3Df\Big(0, \frac{x}{(2a)^{j+1}}\Big)\Big]
\end{aligned} \tag{2.10}$$

for all $x \in V$ and all $j \geq 0$. Moreover, if $n+m > n \geq 0$, then we get the inequality

$$\begin{aligned}
&\mu_{J_nf(x)-J_{n+m}f(x)}\Big(\sum_{j=n}^{n+m-1}\Big(\Big(\frac{(2a)^2}{\alpha}\Big)^j\frac{4t}{\alpha}\Big)\Big)\\
&\geq \mu_{\sum_{j=n}^{n+m-1}(J_jf(x)-J_{j+1}f(x))}\Big(\sum_{j=n}^{n+m-1}\Big(\Big(\frac{(2a)^2}{\alpha}\Big)^j\frac{4t}{\alpha}\Big)\Big)\\
&\geq \tau_{M\,j=n}^{\;n+m-1}\Big\{\mu_{J_jf(x)-J_{j+1}f(x)}\Big(\Big(\frac{(2a)^2}{\alpha}\Big)^j\frac{4t}{\alpha}\Big)\Big\}\\
&\geq \tau_{M\,j=n}^{\;n+m-1}\Big\{\tau_M\Big\{\mu_{\frac{(2a)^j((2a)^j+1)}{2}Df\big(\frac{x}{2(2a)^j},\frac{x}{(2a)^{j+1}}\big)}\Big(\frac{(2a)^j((2a)^j+1)t}{2\alpha^{j+1}}\Big),\\
&\quad\mu_{\frac{-3(2a)^j((2a)^j+1)}{2}Df\big(0,\frac{-x}{(2a)^{j+1}}\big)}\Big(\frac{3(2a)^j((2a)^j+1)t}{2\alpha^{j+1}}\Big),\\
&\quad\mu_{\frac{(2a)^j((2a)^j-1)}{2}Df\big(\frac{-x}{2(2a)^j},\frac{-x}{(2a)^{j+1}}\big)}\Big(\frac{(2a)^j((2a)^j-1)t}{2\alpha^{j+1}}\Big)\\
&\quad\mu_{\frac{3(2a)^j((2a)^j-1)}{2}Df\big(0,\frac{x}{(2a)^{j+1}}\big)}\Big(\frac{3(2a)^j((2a)^j-1)t}{2\alpha^{j+1}}\Big)\Big\}\Big\}\\
&\geq M(x,t)
\end{aligned}$$

for all $x \in V$ and all $t > 0$. And so we can define a mapping $F : V \to Y$ by $F(x) := \lim_{n\to\infty} J_nf(x)$ for all $x \in V$ and

$$\mu_{f(x)-J_nf(x)}(t) \geq M\left(x, \frac{t}{\sum_{j=0}^{n-1}\left(\frac{(2a)^2}{\alpha}\right)^j\frac{4}{\alpha}}\right) \tag{2.11}$$

for all $x \in V$. Note that for all $x, y \in V$ and all $t > 0$,

$$\begin{aligned}
\mu_{DJ_nf(x,y)}\Big(\frac{t}{2}\Big) &\geq \tau_M\Big\{\mu_{\frac{(2a)^{2n}}{2}Df\big(\frac{x}{(2a)^n},\frac{y}{(2a)^n}\big)}\Big(\frac{t}{8}\Big),\mu_{\frac{(2a)^{2n}}{2}Df\big(\frac{-x}{(2a)^n},\frac{-y}{(2a)^n}\big)}\Big(\frac{t}{8}\Big),\\
&\quad\mu_{\frac{(2a)^n}{2}Df\big(\frac{x}{(2a)^n},\frac{y}{(2a)^n}\big)}\Big(\frac{t}{8}\Big),\mu_{-\frac{(2a)^n}{2}Df\big(\frac{-x}{(2a)^n},\frac{-y}{(2a)^n}\big)}\Big(\frac{t}{8}\Big)\Big\}\\
&\geq \tau_M\Big\{\mu_{\varphi(x,y)}\Big(\frac{\alpha^nt}{4(2a)^{2n}}\Big),\mu_{\varphi(-x,-y)}\Big(\frac{\alpha^nt}{4(2a)^{2n}}\Big),\\
&\quad\mu_{\varphi(x,y)}\Big(\frac{\alpha^nt}{4(2a)^n}\Big),\mu_{\varphi(-x,-y)}\Big(\frac{\alpha^nt}{4(2a)^n}\Big)\Big\},
\end{aligned}$$

which tends to 1 as $n \to \infty$. Therefore we can show that F is an additive-quadratic mapping by using the similar fashion after (2.6).

By the same reasoning as in the proof of case 1, the relation (2.2) yields the third inequality in (2.11).

To complete the proof of the theorem, we are enough to show the uniqueness of F. Suppose that $F' : V \to Y$ is another mapping satisfying the third inequality in (2.2). If g is an additive-quadratic mapping, then, by (2.9), we have $g(x) = J_ng(x)$ for all $x \in V$

and all $n \in \mathbb{N}$. Observe that

$$\begin{aligned}
\mu_{F(x)-F'(x)}(t) &= \mu_{J_nF(x)-J_nF'(x)}(t)\\
&\geq \tau_M\Big\{\mu_{J_nF(x)-J_nf(x)}\Big(\frac{t}{2}\Big), \mu_{J_nf(x)-J_nF'(x)}\Big(\frac{t}{2}\Big)\Big\}\\
&\geq \tau_M\Big\{\mu_{\frac{(2a)^{2n}}{2}(F-f)\left(\frac{x}{(2a)^n}\right)}\Big(\frac{t}{8}\Big), \mu_{\frac{(2a)^{2n}}{2}(f-F')\left(\frac{x}{(2a)^n}\right)}\Big(\frac{t}{8}\Big),\\
&\quad \mu_{\frac{(2a)^{2n}}{2}(F-f)\left(\frac{-x}{(2a)^n}\right)}\Big(\frac{t}{8}\Big), \mu_{\frac{(2a)^{2n}}{2}(f-F')\left(\frac{-x}{(2a)^n}\right)}\Big(\frac{t}{8}\Big),\\
&\quad \mu_{\frac{(2a)^{n}}{2}(F-f)\left(\frac{x}{(2a)^n}\right)}\Big(\frac{t}{8}\Big), \mu_{\frac{(2a)^{n}}{2}(f-F')\left(\frac{x}{(2a)^n}\right)}\Big(\frac{t}{8}\Big),\\
&\quad \mu_{\frac{(2a)^{n}}{2}(F-f)\left(\frac{-x}{(2a)^n}\right)}\Big(\frac{t}{8}\Big), \mu_{\frac{(2a)^{n}}{2}(f-F')\left(\frac{-x}{(2a)^n}\right)}\Big(\frac{t}{8}\Big)\Big\}\\
&\geq \tau_M\Big\{\sup_{t'<t}\Big\{M\Big(x, \frac{(\alpha-n^2)t'}{4}\Big(\frac{\alpha}{n}\Big)^m\Big), \sup_{t'<t}\Big\{M\Big(x, \frac{(\alpha-n^2)t'}{4}\Big(\frac{\alpha}{(2a)^2}\Big)^n\Big)\Big\}\Big\}
\end{aligned}$$

for all $x \in V$ and all $n \in \mathbb{N}$. Since $\alpha > (2a)^2$, the last term in (2.6) tends to 1 as $n \to \infty$ by (RN3) and $F(0) = F'(0)$. Therefore $F = F'$. □

Corollary 2.3. *Let X and Y be a vector space and a complete normed space, respectively. Suppose that $f : X \to Y$ is a mapping with $f(0) = 0$ for which there is $\varphi : X^2 \to \mathbb{R}$ such that*

$$\|Df(x,y)\| \leq \varphi(x,y) \tag{2.12}$$

for all $x, y \in X$. If φ satisfies one of the following conditions:

(i) $\alpha\varphi(x,y) \geq \varphi(2ax, 2ay)$ *for some* $0 < \alpha < 2a$,
(ii) $\varphi(2ax, 2ay) \geq \alpha\varphi(x,y) \geq \varphi(4a^2x, 4a^2y)$ *for some* $2a < \alpha < 4a^2$,
(iii) $\varphi(4a^2x, 4a^2y) \geq \alpha\varphi(x,y)$ *for some* $4a^2 < \alpha$

for all $x, y \in X$, then there exists a unique additive-quadratic mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \leq \begin{cases} \frac{\Phi(x)}{2a-\alpha} & \text{if } \varphi \text{ satisfies (i)},\\ \frac{(4a^2-2a)\Phi(x)}{(4a^2-\alpha)(\alpha-2a)} & \text{if } \varphi \text{ satisfies (ii)},\\ \frac{\Phi(x)}{\alpha-4a^2} & \text{if } \varphi \text{ satisfies (iii)} \end{cases} \tag{2.13}$$

for all $x, y \in X$, where

$$\Phi(x) = \max\{\varphi(ax, x), \varphi(-ax, -x), \varphi(0, x), \varphi(0, -x)\}.$$

Proof. Let (Y, μ, τ_M) and $(\mathbb{R}, \mu', \tau_M)$ be the induced random normed RN-spaces. Then the inequality

$$\mu_{Df(x,y)}(t) \geq \mu'_{\varphi(x,y)}(t)$$

follows from the inequality (2.12) and φ satisfies one of the conditions in Theorem 2.2. So there exists a unique additive-quadratic mapping $F : X \to Y$ satisfying (2.13). □

From Corollary 2.3, we can obtain the following result.

Corollary 2.4. *Let X be a normed space and let $p \neq 1, 2$ be a positive real number. If a mapping $f : X \to Y$ satisfies the inequality*

$$\|Df(x,y)\| \leq \theta\big(\|x\|^p + \|y\|^p\big)$$

for all $x, y \in X$ and for some $\theta \geq 0$, then there exists a unique additive-quadratic mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \leq \begin{cases} \frac{2\theta\|x\|^p}{2a-(2a)^p} & \text{if } p < 1, \\ \frac{2\theta\|x\|^p}{(2a)^p-2a} + \frac{2\theta\|x\|^p}{4a^2-(2a)^p} & \text{if } 1 < p < 2, \\ \frac{2\theta\|x\|^p}{(2a)^p-4a^2} & \text{if } p > 2 \end{cases}$$

for all $x \in X$.

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Blow-up of solutions for a vibrating riser equation with dissipative term

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Abstract: In this paper we consider a vibrating riser equation with dissipative term and the homogeneous Dirichlet boundary condition. By developing the method in [9] and [16], we establish a blow-up result for certain solutions with non-positive initial energy as well as positive initial energy. Estimates of the lifespan of solutions are also given.

Keywords: Blow-up of solution, quasilinear riser problem, positive initial energy

AMS Subject Classification (2000): 35L70, 35L15

1 Introduction and main result

In this paper we consider the problem

$$\begin{cases} u_{tt} + pu_t + 2qu_{xxxx} - 2[(ax+b)u_x]_x + \dfrac{q}{3}(u_x^3)_{xxx} \\ \qquad -[(ax+b)u_x^3]_x - q(u_{xx}^2 u_x)_x = f(u), \quad (x,t) \in [0,1] \times (0,T), \\ u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, \quad t \in (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad x \in [0,1], \end{cases} \tag{1.1}$$

where a, b, p, q are nonnegative constant, $f(u)$ is a $C(R)$ function satisfying some conditions to be special later.

Problem (1.1) models the behavior of a riser vibrating due to effects of waves and current [14]. In 1997, Bayrack and Can [1] studied problem (1.1) and proved that, under suitable conditions on f and the initial data, all solutions of (1.1) blow up in finite time in the L^2 space. To establish their result, the authors used the standard concavity method due to [7]. Gmira and Guedda [4] extended the result of [1] to the multi-dimensional version of the problem (1.1) by using the modified concavity method introduced in [6].

More recently, Hao et al. [5] discussed (1.1) and showed that, under suitable conditions, the solution blows up in finite time *with a negative initial energy* while exists globally with a nonnegative initial energy for the case $p = 0$. Precisely, the following blow-up result was established.

Theorem 1 *Let $u(x,t)$ be a classical solution of the system (1.1). Assume that there exists a positive constant A such that the function $f(s)$ satisfies*

$$sf(s) \geq (4+A)\int_0^s f(v)\mathrm{d}v \quad \textit{for } s \in R, \tag{1.2}$$

and the initial values satisfy

$$E(0) = \frac{1}{2}\|u_1\|_2^2 + q\|u_{0xx}\|_2^2 + \int_0^1 (ax+b)u_{0x}^2 \mathrm{d}x + \frac{q}{2}\|u_{0x}u_{0xx}\|_2^2 + \frac{1}{4}\int_0^1 (ax+b)u_{0x}^4 \mathrm{d}x - \int_0^1 \int_0^{u_0} f(\upsilon)\mathrm{d}\upsilon \mathrm{d}x < 0 \tag{1.3}$$

and

$$\int_0^1 u_0 u_1 \mathrm{d}x > 0. \tag{1.4}$$

Then the solution $u(x,t)$ of the system (1.1) blows up in a finite time.

In the present paper, we shall improve the results of [5] and derive the blow-up properties of solutions of problem (1.1) *with non-positive initial energy as well as positive initial energy* by developing the method in [9] and [16] (see Remark 2). Estimates of the lifespan of solutions will also be given. For the convenience of our computation, we set $p = q = 1$ and $f(s) = |s|^{r-1}s$. Then the condition (1.2) holds when $r > 4$.

We define the energy function for the solution u of (1.1) by

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \|u_{xx}\|_2^2 + \int_0^1 (ax+b)u_x^2 \mathrm{d}x + \frac{1}{2}\|u_x u_{xx}\|_2^2 + \frac{1}{4}\int_0^1 (ax+b)u_x^4 \mathrm{d}x - \frac{1}{r}\|u\|_r^r. \tag{1.5}$$

Then

$$E'(t) = -\|u_t\|_2^2 \le 0, \quad \text{for } t \ge 0, \tag{1.6}$$

and

$$E(t) = E(0) - \int_0^t \|u_\tau(\tau)\|_2^2 \mathrm{d}\tau, \quad t \ge 0. \tag{1.7}$$

We also set

$$\alpha_1 = \left(\frac{2}{B^r}\right)^{\frac{1}{r-2}}, \quad E_1 = \frac{r-2}{r}\alpha_1^2 = \left(\frac{1}{2} - \frac{1}{r}\right) B^r \alpha_1^r. \tag{1.8}$$

where B is the optimal constant of the embedding inequality

$$\|u\|_r \le B\|u_{xx}\|_2, \quad u \in H^2([0,1]) \cap H_0^1([0,1]), \tag{1.9}$$

for $2 < r < +\infty$, that is

$$B^{-1} = \inf_{u \in H^2([0,1]) \cap H_0^1([0,1]), u \ne 0} \frac{\|u_{xx}\|_2}{\|u\|_r}.$$

We introduce the functionals

$$a(t) = \int_0^1 u^2 \mathrm{d}x + \int_0^t \int_0^1 u^2 \mathrm{d}x\mathrm{d}t, \ t \ge 0 \tag{1.10}$$

and

$$G(t) = [a(t) + (T_1 - t)\|u_0\|_2^2]^{-\delta}, \quad t \in [0, T_1], \tag{1.11}$$

where $\delta = \frac{r-2}{4}$ and $T_1 > 0$ is a certain constant to be specified later.

Our main result reads as follows.

Theorem 2 *Let $u(x,t)$ be a classical solution of the system (1.1). Assume that $r > 4$ and either one of the following four conditions is satisfied:*

1. $E(0) < 0$,
2. $E(0) = 0$ *and* $\int_0^1 u_0u_1\mathrm{d}x > 0$,
3. $0 < E(0) < E_1$ *and* $\|u_{0xx}\|_2 > \alpha_1$,
4. $E_1 \le E(0) < \min\left\{\frac{r+2}{2r}\left[\left(1+\sqrt{\frac{r-2}{r+2}}\right)\int_0^1 u_0u_1\mathrm{d}x - 2\|u_0\|_2^2\right],\ \frac{\left(\int_0^1 u_0u_1\mathrm{d}x\right)^2}{2(1+T_1)\|u_0\|_2^2}\right\}$.

Then the solution u of the problem (1.1) blows up in a finite time T^* in the sense of (2.25). Moreover, the upper bounds for T^* can be estimated according to the sign of $E(0)$:

For the case 1,

$$T^* \le t_0 - \frac{G(t_0)}{G'(t_0)}.$$

Furthermore, if $G(t_0) < \min\{1, \sqrt{\frac{\alpha}{-\beta}}\}$, then

$$T^* \le t_0 + \frac{1}{\sqrt{-\beta}}\ln\frac{\sqrt{\frac{\alpha}{-\beta}}}{\sqrt{\frac{\alpha}{-\beta}} - G(t_0)}.$$

For the case 2,

$$T^* \le -\frac{G(0)}{G'(0)} = \frac{2(T_1 - t + 1)\|u_0\|_2^2}{(r-2)\int_0^1 u_0u_1\mathrm{d}x} \quad or \quad T^* \le \frac{G(0)}{\sqrt{\alpha}}.$$

For the case 3,

$$T^* \le t_0 - \frac{G(t_0)}{G'(t_0)}.$$

Furthermore, if $G(t_0) < \min\{1, \sqrt{\frac{\alpha'}{-\beta'}}\}$, then

$$T^* \le t_0 + \frac{1}{\sqrt{-\beta'}}\ln\frac{\sqrt{\frac{\alpha'}{-\beta'}}}{\sqrt{\frac{\alpha'}{-\beta'}} - G(t_0)}.$$

For the case 4,

$$T^* \le 2^{(3\delta+1)/2\delta}\frac{\delta c}{\sqrt{\alpha}}\{1 - [1 + cG(0)]^{-1/2\delta}\}.$$

where $c = (\alpha/\beta)^{2+1/\delta}$. Here α, β, α' and β' are given in (2.23), (2.24), (2.27) and (2.28), respectively. And $t_0 = t^*$ is given by (2.12) for the case 1 and $t_0 = t_1^*$ is given by (2.13) for the case 3.

Remark 1 *Compared with Theorem 1, we have no the restriction $\int_0^1 u_0u_1\mathrm{d}x > 0$ in Theorem 2 when $E(0) < 0$.*

Remark 2 *E_1 defined in (1.8) is exactly the potential well depth obtained by Payne and Sattinger (see [13]). In [16], a global nonexistence theorem for abstract evolution equations with*

nonlinear damping terms was proved by combining the arguments in [3] and [8], where positive initial energy less than E_1 was demanded while we allow here a larger positive initial energy (see the case 4). In this work, we divide the case $E(0) > 0$ into two cases: the case 3 and 4. Unlike [9], we discuss cautiously the case 3 by combining the method of [16] (see Lemma 7). We also note that the case 4 is allowed here since the damping term involved in problem (1.1) is linear.

There are many related works on the existence and non-existence of global solutions to the hyperbolic equations with dissipative terms and damping terms, please see [2, 11, 12, 15] and the references therein.

2 Blow-up of the solutions

In this section, we shall prove Theorem 2. We start with a series of Lemmas.

Lemma 3 *Suppose $u(x,t)$ is a classical solution of the system (1.1). Assume that $E(0) < E_1$ and $\|u_{0xx}\|_2 > \alpha_1$. Then there exists a positive constant $\alpha_2 > \alpha_1$, such that*

$$\|u_{xx}(\cdot,t)\|_2 \geq \alpha_2, \quad \forall\ t \geq 0, \tag{2.1}$$

and

$$\|u(\cdot,t)\|_r \geq B\alpha_2, \quad \forall\ t \geq 0. \tag{2.2}$$

Proof. The idea follows from [16] where different type of equations were discussed. We first note that, by (1.5) and (1.9),

$$E(t) \geq \|u_{xx}\|_2^2 - \frac{1}{r}\|u\|_r^r \geq \|u_{xx}\|_2^2 - \frac{1}{r}B^r\|u_{xx}\|_2^r = \alpha^2 - \frac{1}{r}B^r\alpha^r := g(\alpha), \tag{2.3}$$

where $\alpha = \|u_{xx}\|_2$. It is easy to verify that g is increasing for $0 < \alpha < \alpha_1$, decreasing for $\alpha > \alpha_1$; $g(\alpha) \to -\infty$ as $\alpha \to +\infty$ and $g(\alpha_1) = E_1$, where α_1 is given in (1.8). Since $E(0) < E_1$, there exists $\alpha_2 > \alpha_1$ such that $g(\alpha_2) = E(0)$. Let $\alpha_0 = \|u_{0xx}\|_2$, then by (2.3) we have $g(\alpha_0) \leq E(0) = g(\alpha_2)$, which implies that $\alpha_0 \geq \alpha_2$.

To establish (2.1), we suppose by contradiction that $\|u_{xx}(t_0)\|_2 < \alpha_2$ for some $t_0 > 0$. By the continuity of $\|u_{xx}(\cdot,t)\|_2$ we can choose t_0 such that $\|u_{xx}(t_0)\|_2 > \alpha_1$. It follows from (2.3) that

$$E(t_0) \geq g(\|u_{xx}(t_0)\|_2) > g(\alpha_2) = E(0).$$

This is impossible since $E(t) \leq E(0)$ for all $t \geq 0$. Hence (2.1) is established.

To prove (2.2), we exploit (1.5) to see that

$$\|u_{xx}\|_2^2 \leq E(0) + \frac{1}{r}\|u\|_r^r.$$

Consequently,

$$\frac{1}{r}\|u\|_r^r \geq \|u_{xx}\|_2^2 - E(0) \geq \alpha_2^2 - E(0) \geq \alpha_2^2 - g(\alpha_2) = \frac{1}{r}B^r\alpha_2^r. \tag{2.4}$$

Therefore (2.2) is concluded.

Lemma 4 [9] *Let $\delta > 0$ and $B(t) \in C^2(0,\infty)$ be a nonnegative function satisfying*

$$B''(t) - 4(\delta+1)B'(t) + 4(\delta+1)B(t) \geq 0. \tag{2.5}$$

If

$$B'(0) > r_2 B(0) + k_0, \tag{2.6}$$

then $B'(t) > k_0$ for $t > 0$, where $r_2 = 2(\delta+1) - 2\sqrt{(\delta+1)\delta}$ is the smallest root of the equation

$$r^2 - 4(\delta+1)r + 4(\delta+1) = 0.$$

Lemma 5 [9] *If $G(t)$ is a non-increasing function on $[t_0, +\infty), t_0 \geq 0$ and satisfies the differential inequality*

$$G'(t)^2 \geq a + bG(t)^{2+\frac{1}{\delta}}, \quad \text{for } t \geq 0, \tag{2.7}$$

where $a > 0, \delta > 0$ and $b \in R$, then there exists a finite time T^* such that

$$\lim_{t \to T^{*-}} G(t) = 0$$

and the upper bound of T^* is estimated respectively by the following cases:

(i) If $b < 0$ and $G(t_0) < \min\{1, \sqrt{\frac{a}{-b}}\}$, then

$$T^* \leq t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{\frac{a}{-b}}}{\sqrt{\frac{a}{-b}} - G(t_0)}.$$

(ii) If $b = 0$, then

$$T^* \leq t_0 + \frac{G(t_0)}{\sqrt{a}}.$$

(iii) If $b > 0$, then

$$T^* \leq t_0 + 2^{(3\delta+1)/2\delta} \frac{\delta c}{\sqrt{a}} \{1 - [1 + cG(t_0)]^{-1/2\delta}\},$$

where $c = (a/b)^{2+1/\delta}$.

Lemma 6 *Assume that $r > 4$, $a(t)$ is defined by (1.10) and let u be a solution of (1.1), then we have*

$$a''(t) - 4(\delta+1)\|u_t\|_2^2 \geq Q_1(t), \tag{2.8}$$

where

$$Q_1(t) = (-4 - 8\delta)E(0) + 2r\int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau + 2(r-2)\|u_{xx}\|_2^2.$$

Proof. By the definition of $a(t)$, we have

$$a'(t) = 2\int_0^1 u u_t \mathrm{d}x + \int_0^1 u^2 \mathrm{d}x, \tag{2.9}$$

and by (1.1) and the divergence theorem we get

$$\begin{aligned}
a''(t) &= 2\int_0^1 u_t^2 \mathrm{d}x + 2\int_0^1 uu_{tt}\mathrm{d}x + 2\int_0^1 uu_t\mathrm{d}x\mathrm{d}x \\
&= 2\int_0^1 u_t^2 \mathrm{d}x + 2\int_0^1 u\left(|u|^{r-2}u + 2[(ax+b)u_x]_x + [(ax+b)u_x^3]_x \right.\\
&\qquad\qquad \left. +(u_{xx}^2u_x)_x - 2u_{xxxx} - \frac{1}{3}(u_x^3)_{xxx}\right)\mathrm{d}x \\
&= 2\|u_t\|_2^2 + 2\|u\|_r^r - 4\int_0^1 (ax+b)u_x^2\mathrm{d}x - 2\int_0^1 (ax+b)u_x^4\mathrm{d}x - 4\|u_xu_{xx}\|_2^2 - 4\|u_{xx}\|_2^2 \quad (2.10)
\end{aligned}$$

Using (1.5) and (1.7) we get

$$\begin{aligned}
& a''(t) - 4(\delta+1)\|u_t\|_2^2 \\
=\ & a''(t) - 2\|u_t\|_2^2 - \frac{1}{2}(8\delta+4)\|u_t\|_2^2 \\
=\ & 2\|u\|_r^r - 4\int_0^1 (ax+b)u_x^2\mathrm{d}x - 2\int_0^1 (ax+b)u_x^4\mathrm{d}x - 4\|u_xu_{xx}\|_2^2 - 4\|u_{xx}\|_2^2 \\
& -2r\left(E(0) - \int_0^t \|u_\tau\|_2^2\mathrm{d}\tau - \|u_{xx}\|_2^2 - \int_0^1 (ax+b)u_x^2\mathrm{d}x - \frac{1}{2}\|u_xu_{xx}\|_2^2\right.\\
& \left. -\frac{1}{4}\int_0^1 (ax+b)u_x^4\mathrm{d}x + \frac{1}{r}\|u\|_r^r\right) \\
\geq\ & (-4-8\delta)E(0) + 2r\int_0^t \|u_\tau\|_2^2\mathrm{d}\tau + 2(r-2)\|u_{xx}\|_2^2 + 2(r-2)\int_0^1 (ax+b)u_x^2\mathrm{d}x \\
& +\frac{1}{2}(r-4)\int_0^1 (ax+b)u_x^4\mathrm{d}x + (r-4)\|u_xu_{xx}\|_2^2 \\
\geq\ & (-4-8\delta)E(0) + 2r\int_0^t \|u_\tau\|_2^2\mathrm{d}\tau + 2(r-2)\|u_{xx}\|_2^2 \qquad (2.11)
\end{aligned}$$

since $r > 4$.

Lemma 7 *Assume that $r > 4$ and that either one of the following is satisfied:*

1. $E(0) < 0$,
2. $E(0) = 0$ *and* $\int_0^1 u_0u_1\mathrm{d}x > 0$,
3. $0 < E(0) < E_1$ *and* $\|u_{0xx}\|_2 > \alpha_1$,
4. $E_1 \leq E(0) < \frac{r+2}{2r}\left[\left(1+\sqrt{\frac{r-2}{r+2}}\right)\int_0^1 u_0u_1\mathrm{d}x - 2\|u_0\|_2^2\right]$.

Then $a'(t) > \|u_0\|_2^2$ for $t > t_0$, where $t_0 = t^*$ is given by (2.12) for the case 1, $t_0 = 0$ for the cases 2 and 4, and $t_0 = t_1^*$ is given by (2.14) for the case 3.

Proof. We consider different cases on the sign of the initial energy $E(0)$.

1. If $E(0) < 0$, then from (2.8), we have

$$a'(t) \geq a'(0) - 4(1+2\delta)E(0)t, \quad t \geq 0.$$

Thus $a'(t) > \|u_0\|_2^2$ for $t > t^*$, where

$$t^* = \max\left\{\frac{a'(0) - \|u_0\|_2^2}{4(1+2\delta)E(0)},\ 0\right\} = \max\left\{\frac{\int_0^1 u_0u_1\mathrm{d}x}{2(1+2\delta)E(0)},\ 0\right\}. \qquad (2.12)$$

2. If $E(0) = 0$, then $a''(t) \geq 0$ for $t \geq 0$. Furthermore, if $a'(0) > \|u_0\|_2^2$ (i.e., $\int_0^1 u_0 u_1 \mathrm{d}x > 0$), then $a'(t) > \|u_0\|_2^2$, $t \geq 0$.

3. If $0 < E(0) < E_1$, then using Lemma 3 and (1.8) we see that

$$\begin{aligned} Q_1(t) &\geq -(4+8\delta)E(0) + 2(r-2)\alpha_2^2 \\ &> (4+8\delta)(-E(0)+E_1) := C_1 > 0, \quad t > 0. \end{aligned} \tag{2.13}$$

Thus, from (2.8), we have

$$a''(t) \geq Q_1(t) > C_1 > 0, \quad t > 0.$$

Hence $a'(t) > \|u_0\|_2^2$ for $t > t_1^*$, where

$$t_1^* = \max\left\{\frac{\|u_0\|_2^2 - a'(0)}{C_1},\ 0\right\} = \max\left\{\frac{-2\int_0^1 u_0 u_1 \mathrm{d}x}{C_1},\ 0\right\}. \tag{2.14}$$

4. If $E(0) \geq E_1$, we first note

$$\int_0^1 u^2 \mathrm{d}x - \int_0^1 u_0^2 \mathrm{d}x = 2\int_0^t \int_0^1 u u_t \mathrm{d}x\mathrm{d}t. \tag{2.15}$$

By the Hölder inequality and Young inequality, we have

$$\int_0^1 u^2 \mathrm{d}x \leq \int_0^1 u_0^2 \mathrm{d}x + \int_0^t \|u\|_2^2 \mathrm{d}t + \int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau.$$

By the Hölder inequality, Young inequality again, and (2.15), it follows from (2.9) that

$$a'(t) \leq a(t) + \int_0^1 u_0^2 \mathrm{d}x + \int_0^1 u_t^2 \mathrm{d}x + \int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau. \tag{2.16}$$

In view of (2.8) and (2.16), we obtain

$$\begin{aligned} & a''(t) - 4(\delta+1)a'(t) + 4(\delta+1)a(t) + K_1 \\ \geq\ & a''(t) + 4(\delta+1)\left(-\|u_0\|_2^2 - \|u_t\|_2^2 - \int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau\right) + K_1 \\ \geq\ & (-4-8\delta)E(0) + 2r\int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau + 2(r-2)\|u_{xx}\|_2^2 - 4(\delta+1)\|u_0\|_2^2 - 4(\delta+1)\int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau + K_1 \\ \geq\ & 4\delta\int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau + 2(r-2)\|u_{xx}\|_2^2 \geq 0, \end{aligned}$$

where

$$K_1 = (4+8\delta)E(0) + 4(\delta+1)\|u_0\|_2^2.$$

Let

$$b(t) = a(t) + \frac{K_1}{4(1+\delta)}, \quad t > 0.$$

Then $b(t)$ satisfies (2.5). By (2.6), we see that if

$$a'(0) > r_2\left(a(0) + \frac{K_1}{4(1+\delta)}\right) + \|u_0\|_2^2, \tag{2.17}$$

i.e.,

$$E(0) < \frac{r+2}{2r}\left[\left(1+\sqrt{\frac{r-2}{r+2}}\right)\int_0^1 u_0u_1\mathrm{d}x - 2\|u_0\|_2^2\right],$$

then $a'(t) > \|u_0\|_2^2,\ t>0$. The proof is completed.

Hereafter, we will find an estimate for the life span of $a(t)$ and prove Theorem 2.

Proof of Theorem 2. By the definition of $G(t)$, we have

$$G'(t) = -\delta G(t)^{1+1/\delta}(a'(t) - \|u_0\|_2^2)$$

$$G''(t) = -\delta G^{1+2/\delta}(t)V(t), \tag{2.18}$$

where

$$V(t) = a''(t)[a(t) + (T_1 - t)\|u_0\|_2^2] - (1+\delta)(a'(t) - \|u_0\|_2^2)^2. \tag{2.19}$$

For simplicity of calculation, we denote

$$P = \|u\|_2^2, \quad Q = \int_0^t \|u\|_2^2\mathrm{d}t, \quad R = \|u_t\|_2^2, \quad S = \int_0^t \|u_\tau\|_2^2\mathrm{d}\tau.$$

From (2.9), (2.15) and the Hölder inequality, we get

$$a'(t) \le 2\left(\sqrt{PR} + \sqrt{QS}\right) + \int_0^1 u_0^2\mathrm{d}x. \tag{2.20}$$

For the case 1 and 2, it follows from (2.8) that

$$a''(t) \ge (-4-8\delta)E(0) + 4(1+\delta)(R+S). \tag{2.21}$$

Applying (2.20) and (2.21), it yields

$$V(t) \ge [(-4-8\delta)E(0) + 4(1+\delta)(R+S)][a(t) + (T_1-t)\|u_0\|_2^2] - 4(1+\delta)\left(\sqrt{PR}+\sqrt{QS}\right)^2.$$

Applying (1.11) and (1.10), it follows

$$\begin{aligned} V(t) &\ge (-4-8\delta)E(0)G^{-1/\delta}(t) + 4(1+\delta)(R+S)(T_1-t)\|u_0\|_2^2 \\ &\quad +4(1+\delta)\left[(R+S)(P+Q) - \left(\sqrt{PR}+\sqrt{QS}\right)^2\right] \\ &\ge (-4-8\delta)E(0)G^{-1/\delta}(t). \end{aligned}$$

In view of (2.18) we have

$$G''(t) \le \delta(4+8\delta)E(0)G^{1+1/\delta}(t), \quad t \ge t_0. \tag{2.22}$$

Note that by Lemma 7, $G'(t) < 0$ for $t > t_0$. Multiplying (2.22) by $G'(t)$ and integrating it from t_0 to t, we obtain

$$G'(t)^2 \ge \alpha + \beta G^{2+1/\delta}(t), \quad \text{for} \quad t \ge t_0,$$

where

$$\alpha = \delta^2 G(t_0)^{2+2/\delta} \left[(a'(t_0) - \|u_0\|_2^2)^2 - 8E(0)G^{-1/\delta}(t_0) \right] > 0 \tag{2.23}$$

and

$$\beta = 8\delta^2 E(0). \tag{2.24}$$

Then by Lemma 5, there exists a finite time T^* such that $\lim_{t\nearrow T^*-} G(t) = 0$. Therefore

$$\lim_{t\nearrow T^*-} \left(\int_0^1 u^2 \mathrm{d}x + \int_0^t \int_0^1 u^2 \mathrm{d}x\mathrm{d}t \right) = \infty. \tag{2.25}$$

For the case 3: $0 < E(0) < E_1$, it follows from (2.8) and (2.13) that

$$a''(t) \geq (-4 - 8\delta)E(0) + 2(r-2)\|u_{xx}\|_2^2 + 4(1+\delta)(R+S) > C_1 + 4(1+\delta)(R+S). \tag{2.26}$$

Then using the same arguments as in (1), we have

$$G''(t) \leq -\delta C_1 G^{1+1/\delta}(t),\ G'(t)^2 \geq \alpha' + \beta' G^{2+1/\delta}(t), \quad t \geq t_0,$$

where

$$\alpha' = \delta^2 G^{2+2/\delta}(t_0) \left[(a'(t_0) - \|u_0\|_2^2)^2 + \frac{2C_1}{1+2\delta} G^{-1/\delta}(t_0) \right] > 0 \tag{2.27}$$

and

$$\beta' = -\frac{2C_1\delta^2}{1+2\delta}. \tag{2.28}$$

Then by Lemma 5, there exists a finite time T^* such that (2.25) holds.

For the case 4: $E(0) \geq E_1$, applying the same discussion as in the case 1, we may get the equalities (2.23) and (2.24) under the condition

$$E(0) < \frac{(a'(t_0) - \|u_0\|_2^2)^2}{8a(t_0) + 8(T_1 - t_0)\|u_0\|_2^2} = \frac{\left(\int_0^1 u_0 u_1 \mathrm{d}x \right)^2}{2(1+T_1)\|u_0\|_2^2}.$$

Then by Lemma 5, there exists a finite time T^* such that (2.25) holds.

Remark 3 *The choice of T_1 in (1.11) is possible provided that $T_1 \geq T^*$.*

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Existence, uniqueness and asymptotic behavior of solutions for a fourth-order degenerate pseudo-parabolic equation with $p(x)$-growth conditions

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Abstract: In this paper, we consider an initial-boundary value problem for a fourth order degenerate pseudo-parabolic equation with $p(x)$-growth conditions. Under some assumptions on the initial value, we establish the existence of weak solutions by the time-discrete method. The uniqueness and asymptotic behavior of solutions are also discussed.

Keywords: Existence, asymptotic behavior, pseudo-parabolic equation

AMS Subject Classification (2000): 35G25, 35Q99, 35K55, 35K70.

1 INTRODUCTION

This paper is concerned with a fourth order degenerate pseudo-parabolic equation with $p(x)$-growth conditions

$$\frac{\partial u}{\partial t} - k\frac{\partial \Delta u}{\partial t} + \triangle(|\triangle u|^{p(x)-2}\triangle u) = 0, \quad x \in \Omega,\ t > 0, \tag{1.1}$$

with boundary condition

$$u = \triangle u = 0, \quad x \in \partial\Omega,\ t > 0, \tag{1.2}$$

and initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{1.3}$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $p(x)$ is a function defined on $\bar{\Omega}$ and $k > 0$ is the viscosity coefficient. The term $k\frac{\partial \Delta u}{\partial t}$ in (1.1) is interpreted as due to viscous relaxation effects, or viscosity.

Equation (1.1) arises as a regularization of the pseudo-parabolic equation

$$\frac{\partial u}{\partial t} - k\frac{\partial \Delta u}{\partial t} = \Delta u, \tag{1.4}$$

which arises in various physical phenomena. (1.4) can be assumed as a model for diffusion of fluids in fractured porous media [1, 5, 6], or as a model for heat conduction involving a thermodynamic temperature $\theta = u - k\Delta u$ and a conductive temperature u [4, 13]. In [2], Bernis investigates a class of higher order parabolic with degeneracy depending on both the unknown functions and its derivatives, the fourth order case of which is the equation

$$\frac{\partial}{\partial t}(|u|^{q-1}sgnu) + D^2(|D^2u|^{p-1}sgnD^2u) = f \tag{1.5}$$

where $p > 1$, $q > 1$ are constants. Some existence result of energy solutions was proved by energy method (see also [12, 17]).

Motivated by (1.4) and (1.5), we study the problem (1.1)-(1.3) in this paper. Under some assumptions on the initial value, we will establish the existence, uniqueness and asymptotic behavior of weak solutions by the time-discrete method as used in [10, 11].

Equation (1.1) is something like the p-Laplacian equation, but many methods which are useful for the p-Laplacian equation are no longer valid for this equation. Because of the degeneracy, problem (1.1)-(1.3) does not admit classical solutions in general. So, we study weak solutions in the sense of following

Definition A function u is said to be a weak solution of (1.1)-(1.3), if the following conditions are satisfied:

1. $u \in L^\infty(0,T;W_0^{2,p(x)}(\Omega)) \cap C(0,T;H^1(\Omega))$, $\frac{\partial u}{\partial t} \in L^\infty(0,T;(W^{2,p(x)})'(\Omega))$, where $(W^{2,p(x)})'(\Omega)$ is the conjugate space of $W^{2,p(x)}(\Omega)$.

2. For any $\varphi \in C_0^\infty(Q_T)$ and $Q_T = \Omega \times (0,T)$, the following integral equality holds
$$\iint_{Q_T} u\frac{\partial \varphi}{\partial t}dx\,dt + k\iint_{Q_T} \nabla u\frac{\partial \nabla\varphi}{\partial t}dx\,dt - \iint_{Q_T} |\triangle u|^{p(x)-2}\triangle u\triangle\varphi dx\,dt = 0\,.$$

3. $u(x,0) = u_0(x)$.

We need some theories on spaces $W^{m,p(x)}$ which we call generalized Lebesgue-Sobolev spaces. We refer the reader to [8] (see also [7, 9]) for some basic properties of spaces $W^{m,p(x)}$ which will be used later. For simplicity we set $k = 1$ in this paper.

This paper is arranged as following. We first discuss the existence of weak solutions in Section 2. Our method for investigating the existence of weak solutions is based on the time discrete method to construct an approximate solutions. By means of the uniform estimates on solutions of the time difference equations, we prove the existence of weak solutions of the problem (1.1)-(1.3). We also prove the uniqueness and asymptotic behavior in Section 3 and Section 4 subsequently.

2 EXISTENCE OF WEAK SOLUTIONS

In this section, we are going to prove the existence of weak solutions.

Theorem 1 *If $u_0 \in W_0^{2,p(x)}(\Omega)$, $p(x) \in C(\bar\Omega)$, $p(x)$ satisfies for some constant L*
$$-|p(x)-p(y)|\ln|x-y| \le L, \quad \textit{for any } x,y \in \bar\Omega$$
and $p_- = \min\limits_{\bar\Omega} p(x) > 2$. Then the problem (1.1)-(1.3) has at least one solution.

We use the a discrete method for constructing an approximate solution. First, divide the interval $(0,T)$ in N equal segments and set $h = \frac{T}{N}$. Then consider the problem
$$\frac{1}{h}(u_{k+1}-u_k) - \frac{1}{h}(\Delta u_{k+1} - \Delta u_k) + \triangle(|\triangle u_{k+1}|^{p(x)-2}\triangle u_{k+1}) = 0, \tag{2.1}$$
$$u_{k+1}|_{\partial\Omega} = \triangle u_{k+1}|_{\partial\Omega} = 0, \quad k = 0,1,\dots,N-1, \tag{2.2}$$

where u_0 is the initial value.

Lemma 2 *For a fixed k, if $u_k \in H_0^1(\Omega)$, problem (2.1)-(2.2) admits a weak solution $u_{k+1} \in W_0^{2,p(x)}(\Omega)$, such that for any $\varphi \in C_0^\infty(\Omega)$, have*

$$\frac{1}{h}\int_\Omega (u_{k+1}-u_k)\varphi dx + \frac{1}{h}\int_\Omega (\nabla u_{k+1}-\nabla u_k)\nabla\varphi dx + \int_\Omega |\triangle u_{k+1}|^{p(x)-2}\triangle u_{k+1}\triangle\varphi dx = 0. \quad (2.3)$$

Proof. Let us consider the following functionals on the space $W_0^{2,p(x)}(\Omega)$

$$F_1[u] = \int_\Omega \frac{1}{p(x)}|\triangle u|^{p(x)}dx, \quad F_2[u] = \frac{1}{2}\int_\Omega |u|^2 dx, \quad F_3[u] = \frac{1}{2}\int_\Omega |\nabla u|^2 dx,$$
$$H[u] = F_1[u] + \frac{1}{h}F_2[u] + \frac{1}{h}F_3[u] - \int_\Omega fudx,$$

where $f \in H^{-1}(\Omega)$ is a known function. Using Young's inequality, there exist constants $C_1 > 0$, such that

$$\begin{aligned} H[u] &= \int_\Omega \frac{1}{p(x)}|\triangle u|^{p(x)}dx + \frac{1}{2h}\int_\Omega |u|^2dx + \frac{1}{2h}\int_\Omega |\nabla u|^2 dx - \int_\Omega fu\,dx \\ &\geq \frac{1}{p_+}\int_\Omega |\triangle u|^{p(x)}dx - C_1\|f\|_{-1}. \end{aligned}$$

We need to check that $H[u]$ satisfies the coercive condition. For this purpose, we notice that by $u|_{\partial\Omega} = \triangle u|_{\partial\Omega} = 0$ and using the L^p theory for elliptic equation ([4]),

$$\|u\|_{W^{2,p(x)}} \leq C|\triangle u|_{p(x)}.$$

Therefore, we have $H[u] \to \infty$, as $\|u\|_{W^{2,p(x)}} \to +\infty$.

Since the norm is lower semi-continuous and $\int_\Omega fudx$ is a continuous functional, $H[u]$ is weakly lower semi-continuous on $W_0^{2,p(x)}(\Omega)$ and satisfying the coercive condition. From [3] we conclude that there exists $u_* \in W_0^{2,p(x)}(\Omega)$, such that

$$H[u_*] = \inf H[u],$$

and u_* is the weak solutions of the Euler equation corresponding to $H[u]$,

$$\frac{1}{h}u - \frac{1}{h}\Delta u + \triangle(|\triangle u|^{p-2}\triangle u) = f.$$

Taking $f = (u_k - \Delta u_k)/h$, we obtain a weak solutions u_{k+1} of (2.1)–(2.2). The proof is complete.

Now, we construct an approximate solution u^h of the problem (1.1)-(1.3) by defining

$$u^h(x,t) = u_k(x), \quad kh < t \leq (k+1)h,\ k = 0, 1, \ldots, N-1,$$
$$u^h(x,0) = u_0(x).$$

The desired solution of the problem (1.1)-(1.3) will be obtained as the limit of some subsequence of $\{u^h\}$. To this purpose, we need some uniform estimates on u^h.

Lemma 3 *The weak solutions u_k of (2.1)–(2.2) satisfy*

$$h\sum_{k=1}^N \int_\Omega |\triangle u_k|^{p(x)}dx \leq C, \quad (2.4)$$

$$\sup_{0<t<T}\int_\Omega |\triangle u^h(x,t)|^{p(x)}dx \leq C, \quad (2.5)$$

where C is a constant independent of h and k.

Proof. i) We take $\varphi = u_{k+1}$ in the integral equality (2.3) (we can easily prove that for $\varphi \in W_0^{2,p(x)}(\Omega)$, (2.3) also holds) and obtain

$$\begin{aligned}&\frac{1}{h}\int_\Omega |u_{k+1}|^2dx + \frac{1}{h}\int_\Omega |\nabla u_{k+1}|^2dx + \int_\Omega |\triangle u_{k+1}|^{p(x)}dx\\ &= \frac{1}{h}\int_\Omega u_k u_{k+1}dx + \frac{1}{h}\int_\Omega \nabla u_{k+1}\nabla u_k dx.\end{aligned}$$

By Young's inequality,

$$\begin{aligned}&\frac{1}{h}\int_\Omega |u_{k+1}|^2dx + \frac{1}{h}\int_\Omega |\nabla u_{k+1}|^2dx + \int_\Omega |\triangle u_{k+1}|^{p(x)}dx\\ &\le \frac{1}{2h}\int_\Omega |u_k|^2dx + \frac{1}{2h}\int_\Omega |u_{k+1}|^2dx + \frac{1}{2h}\int_\Omega |\nabla u_k|^2dx + \frac{1}{2h}\int_\Omega |\nabla u_{k+1}|^2dx;\end{aligned}$$

that is,

$$\begin{aligned}&\frac{1}{2}\int_\Omega |u_{k+1}|^2dx + \frac{1}{2}\int_\Omega |\nabla u_{k+1}|^2dx + h\int_\Omega |\triangle u_{k+1}|^{p(x)}dx\\ &\le \frac{1}{2}\int_\Omega |u_k|^2dx + \frac{1}{2}\int_\Omega |\nabla u_k|^2dx.\end{aligned} \tag{2.6}$$

Adding these inequalities for k from 0 to $N-1$, we have

$$h\sum_{k=1}^{N}\int_\Omega |\triangle u_k|^{p(x)}dx \le \frac{1}{2}\int_\Omega |u_0|^2dx + \frac{1}{2}\int_\Omega |\nabla u_0|^2dx.$$

Therefore, (2.4) holds.

ii) We take $\varphi = u_{k+1} - u_k$ in the integral equality (2.3) and integrating by parts, we have

$$\begin{aligned}&\frac{1}{h}\int_\Omega |u_{k+1} - u_k|^2dx + \frac{1}{h}\int_\Omega |\nabla u_{k+1} - \nabla u_k|^2dx\\ &\quad + \int_\Omega |\triangle u_{k+1}|^{p(x)-2}\triangle u_{k+1}\triangle(u_{k+1} - u_k)dx = 0\,.\end{aligned}$$

Since the first term and the second term of the left hand side of the above equality are nonnegative, it follows that

$$\begin{aligned}\int_\Omega |\triangle u_{k+1}|^{p(x)}dx &\le \int_\Omega |\triangle u_{k+1}|^{p(x)-2}\triangle u_{k+1}\triangle u_k\, dx\\ &\le \int_\Omega \frac{p(x)-1}{p(x)}|\triangle u_{k+1}|^{p(x)}dx + \int_\Omega \frac{1}{p(x)}|\triangle u_k|^{p(x)}dx;\end{aligned}$$

thus,

$$\int_\Omega \frac{1}{p(x)}|\triangle u_{k+1}|^{p(x)}dx \le \int_\Omega \frac{1}{p(x)}|\triangle u_k|^{p(x)}dx.$$

For any m, with $1 \le m \le N-1$, adding the above inequality for k from 0 to $m-1$, we have

$$\int_\Omega \frac{1}{p(x)}|\triangle u_m|^{p(x)}dx \le \int_\Omega \frac{1}{p(x)}|\triangle u_0|^{p(x)}dx,$$

that is

$$\frac{1}{p_+}\int_\Omega |\triangle u_m|^{p(x)}dx \le \frac{1}{p_-}\int_\Omega |\triangle u_0|^{p(x)}dx.$$

Therefore, (2.5) holds.

Lemma 4 *For a weak solutions u_{k+1} of (2.1)–(2.2), we have*

$$-Ch \leq \int_\Omega |u_{k+1}|^2 dx + \int_\Omega |\nabla u_{k+1}|^2 dx - \int_\Omega |u_k|^2 dx - \int_\Omega |\nabla u_k|^2 dx \leq 0, \tag{2.7}$$

where C is a constant independently of h.

Proof. The second inequality in (2.7) is an immediate consequence of (2.6). To prove the first inequality, we choose $\varphi = u_k$ in (2.3) and obtain

$$\begin{aligned}
&\int_\Omega |u_k|^2 dx + \int_\Omega |\nabla u_k|^2 dx - \int_\Omega u_{k+1} u_k dx - \int_\Omega \nabla u_{k+1} \nabla u_k dx \\
&= h \int_\Omega |\triangle u_{k+1}|^{p(x)-2} \triangle u_{k+1} \triangle u_k dx \\
&\leq h \int_\Omega \frac{p(x)-1}{p(x)} |\triangle u_{k+1}|^{p(x)} dx + h \int_\Omega \frac{1}{p(x)} |\triangle u_k|^{p(x)} dx.
\end{aligned}$$

Here we have used Hölder inequality. By (2.5) again, we obtain

$$\int_\Omega |u_k|^2 dx + \int_\Omega |\nabla u_k|^2 dx - \int_\Omega u_{k+1} u_k dx - \int_\Omega \nabla u_{k+1} \nabla u_k dx \leq Ch.$$

Therefore,

$$\int_\Omega |u_k|^2 dx + \int_\Omega |\nabla u_k|^2 dx - \int_\Omega |u_{k+1}|^2 dx - \int_\Omega |\nabla u_{k+1}|^2 dx \leq Ch,$$

which completes the proof.

Proof of Theorem 2.1. First, we define the operator A^t, $A^t(\triangle u^h) = |\triangle u_k|^{p(x)-2} \triangle u_k$, $\Delta^h u^h = u_{k+1} - u_k$, where $kh < t \leq (k+1)h, k = 0, 1, \ldots, N-1$. By the discrete equation (2.1) and the (2.4) in Lemma 2.2, we know that

$$\frac{1}{h} \Delta^h u^h \quad \text{in } L^\infty(0, T; (W^{2,p(x)}(\Omega))') \quad \text{is bounded.} \tag{2.9}$$

By (2.5), (2.7), (2.9) and (2.4) we known that exists a subsequence of $\{u^h\}$ (which we denote as the original sequence) such that

$$\begin{aligned}
u^h \to u \quad &\text{in } L^\infty(0, T; W^{2,p(x)}(\Omega)) \quad \text{weak-}\star, \\
\nabla u^h \to \nabla u \quad &\text{in } L^\infty(0, T; L^2(\Omega)) \quad \text{weak-}\star, \\
\frac{1}{h}(u_{k+1} - u_k) \to \frac{\partial u}{\partial t} \quad &\text{in } L^\infty(0, T; (W^{2,p(x)}(\Omega))') \quad \text{weak-}\star, \\
A^t(\triangle u^h) \to w \quad &\text{in } L^\infty(0, T; L^{p'(x)}(\Omega)) \quad \text{weak-}\star,
\end{aligned}$$

where $p'(x)$ is conjugate exponent of $p(x)$. From (2.3), we known, for any $\varphi \in C_0^\infty(Q_T)$,

$$\iint_{Q_T} \left(\frac{1}{h} \Delta^h u^h \varphi - \frac{1}{h} \Delta^h u^h \triangle \varphi + A^t(\triangle u^h) \triangle \varphi \right) dx\, dt = 0.$$

Letting $h \to 0$, we obtain, in the sense of distributions,

$$\frac{\partial u}{\partial t} - \frac{\partial \Delta u}{\partial t} + \triangle w = 0. \tag{2.10}$$

Similar as in [10], we can easily prove $w = |\triangle u|^{p(x)-2} \triangle u$ a.e. in Q_T. The strong convergence of u^h in $C(0, T; H^1(\Omega))$ and the fact that $u^h(x, 0) = u_0(x)$ completes the proof.

3 UNIQUENESS OF SOLUTIONS

In this section, we prove that the weak solution is unique. To this end we need the following lemma.

Lemma 5 *For $\varphi \in L^\infty(t_1, t_2; W_0^{2,p(x)}(\Omega))$ with $\varphi_t \in L^2(t_1, t_2; H^1(\Omega))$, the weak solutions u of the problem (1.1)-(1.3) on Q_T satisfies*

$$
\begin{aligned}
&\int_\Omega u(x,t_1)\varphi(x,t_1)dx + \int_\Omega \nabla u(x,t_1)\nabla\varphi(x,t_1)dx \\
&+ \int_{t_1}^{t_2}\int_\Omega \left(u\frac{\partial\varphi}{\partial t} + \nabla u \frac{\partial \nabla\varphi}{\partial t} + |\triangle u|^{p(x)-2}\triangle u \triangle\varphi \right) dx\, dt \\
&= \int_\Omega u(x,t_2)\varphi(x,t_2)dx + \int_\Omega \nabla u(x,t_2)\nabla\varphi(x,t_2)dx.
\end{aligned}
$$

In particular, for $\varphi \in W_0^{2,p(x)}(\Omega)$, we have

$$
\begin{aligned}
&\int_\Omega (u(x,t_1) - u(x,t_2))\varphi dx + \int_\Omega \nabla(u(x,t_1) - u(x,t_2))\nabla\varphi dx \\
&\qquad - \int_{t_1}^{t_2}\int_\Omega |\triangle u|^{p(x)-2}\triangle u\triangle\varphi dx\, dt = 0\,.
\end{aligned}
\tag{3.1}
$$

Proof. From $\varphi \in L^\infty(t_1, t_2; W_0^{2,p(x)}(\Omega))$ and $\varphi_t \in L^2(t_1, t_2; H^1(\Omega))$, it follows that there exists a sequence of functions $\{\varphi_k\}$, for fixed $t \in (t_1, t_2)$, $\varphi_k(\cdot, t) \in C_0^\infty(\Omega)$, and as $k \to \infty$

$$
\|\varphi_{kt} - \varphi_t\|_{L^2(t_1,t_2;H^1(\Omega))} \to 0, \quad \|\varphi_k - \varphi\|_{L^\infty(t_1,t_2;W_0^{2,p(x)}(\Omega))} \to 0.
$$

Choose a function $j(s) \in C_0^\infty(R)$ such that $j(s) \geq 0$, for $s \in R$; $j(s) = 0$, for $\forall |s| > 1$; $\int_R j(s)ds = 1$. For $h > 0$, define $j_h(s) = \frac{1}{h}j(\frac{s}{h})$ and

$$
\eta_h(t) = \int_{t-t_2+2h}^{t-t_1-2h} j_h(s)ds.
$$

Clearly $\eta_h(t) \in C_0^\infty(t_1, t_2)$, $\lim_{h\to 0^+}\eta_h(t) = 1$, for all $t \in (t_1, t_2)$. In the definition of weak solutions, choose $\varphi = \varphi_k(x,t)\eta_h(t)$, we have

$$
\begin{aligned}
&\int_{t_1}^{t_2}\int_\Omega u\varphi_k j_h(t-t_1-2h)dx\, dt - \int_{t_1}^{t_2}\int_\Omega u\varphi_k j_h(t-t_2+2h)dx\, dt \\
&+ \int_{t_1}^{t_2}\int_\Omega \nabla u\nabla\varphi_k j_h(t-t_1-2h)dx\, dt - \int_{t_1}^{t_2}\int_\Omega \nabla u\nabla\varphi_k j_h(t-t_2+2h)\, dx\, dt \\
&\qquad + \int_{t_1}^{t_2}\int_\Omega u\varphi_{kt}\eta_h dx\, dt + \int_{t_1}^{t_2}\int_\Omega \nabla u\nabla\varphi_{kt}\eta_h\, dx\, dt \\
&\qquad\qquad + \int_{t_1}^{t_2}\int_\Omega |\triangle u|^{p(x)-2}\triangle u\triangle\varphi_k\eta_h\, dx\, dt = 0.
\end{aligned}
$$

Observe that

$$\begin{aligned}
&\Big|\int_{t_1}^{t_2}\int_\Omega u\varphi_k j_h(t-t_1-2h)dx\,dt-\int_\Omega (u\varphi_k)|_{t=t_1}dx\Big|\\
&=\Big|\int_{t_1+h}^{t_1+3h}\int_\Omega u\varphi_k j_h(t-t_1-2h)dx\,dt-\int_{t_1+h}^{t_1+3h}\int_\Omega (u\varphi_k)|_{t=t_2} j_h(t-t_1-2h)dx\,dt\Big|\\
&\le \sup_{t_1+h<t<t_1+3h}\int_\Omega \big|(u\varphi_k)|_t-(u\varphi_k)|_{t_1}\big|dx,
\end{aligned}$$

and $u\in C(0,T;L^2(\Omega))$. We see that the right hand side tends to zero as $h\to 0$. Similarly,

$$\Big|\int_{t_1}^{t_2}\int_\Omega u\varphi_k j_h(t-t_2+2h)dx\,dt-\int_\Omega (u\varphi_k)|_{t=t_2}dx\Big|\to 0,\quad \text{as } h\to 0,$$

$$\Big|\int_{t_1}^{t_2}\int_\Omega \nabla u\nabla\varphi_k j_h(t-t_1-2h)dx\,dt-\int_\Omega (\nabla u\nabla\varphi_k)|_{t=t_1}dx\Big|\to 0,\quad \text{as } h\to 0,$$

$$\Big|\int_{t_1}^{t_2}\int_\Omega \nabla u\nabla\varphi_k j_h(t-t_2+2h)dx\,dt-\int_\Omega (\nabla u\nabla\varphi_k)|_{t=t_2}dx\Big|\to 0,\quad \text{as } h\to 0.$$

Letting $h\to 0$ and $k\to\infty$, we obtain

$$\begin{aligned}
&\int_\Omega u(x,t_1)\varphi(x,t_1)dx+\int_\Omega \nabla u(x,t_1)\nabla\varphi(x,t_1)dx\\
&+\int_{t_1}^{t_2}\int_\Omega\Big(u\frac{\partial\varphi}{\partial t}+\nabla u\frac{\partial\nabla\varphi}{\partial t}+|\triangle u|^{p(x)-2}\triangle u\triangle\varphi\Big)dx\,dt\\
&=\int_\Omega u(x,t_2)\varphi(x,t_2)dx+\int_\Omega \nabla u(x,t_2)\nabla\varphi(x,t_2)dx.
\end{aligned}$$

In particular for $\varphi\in W_0^{2,p(x)}(\Omega)$, we have

$$\begin{aligned}
\int_\Omega (u(x,t_1)-u(x,t_2))\varphi dx+\int_\Omega(\nabla u(x,t_1)-\nabla u(x,t_2))\nabla\varphi dx&\\
-\int_{t_1}^{t_2}\int_\Omega |\triangle u|^{p(x)-2}\triangle u\triangle\varphi\,dx\,dt=0&
\end{aligned}$$

which completes the proof.

For a fixed $\tau\in(0,T)$, set h satisfying $0<\tau<\tau+h<T$. Letting $t_1=\tau$, $t_2=\tau+h$, then multiply (3.1) by $\frac{1}{h}$, for $\varphi\in W_0^{2,p(x)}(\Omega)$, we obtain

$$\int_\Omega (u_h(x,\tau))_\tau\varphi(x)dx+\int_\Omega((\nabla u)_h(x,\tau))_\tau\varphi(x)dx+\int_\Omega(|\triangle u|^{p(x)-2}\triangle u)_h(x,\tau)\triangle\varphi dx=0,\quad (3.2)$$

where

$$u_h(x,t)=\begin{cases}\frac{1}{h}\int_t^{t+h}u(\cdot,\tau)d\tau, & t\in(0,T-h),\\ 0, & t>T-h.\end{cases}$$

Theorem 6 *Problem (1.1)-(1.3) admits only one weak solution.*

Proof. Suppose u_1,u_2 are two solutions of (1.1)-(1.3), then

$$\begin{aligned}
\int_\Omega (u_1(x,\tau)-u_2(x,\tau))_{h\tau}\varphi(x)dx+\int_\Omega((\nabla u_1-\nabla u_2)_h(x,\tau))_\tau\varphi(x)dx&\\
-\int_\Omega(|\triangle u_1|^{p(x)-2}\triangle u_1-|\triangle u_2|^{p(x)-2}\triangle u_2)_h(x,\tau)\triangle\varphi dx=0.&
\end{aligned}$$

For a fixed τ, we take $\varphi(x) = [u_1 - u_2]_h \in W_0^{2,p(x)}(\Omega)$, and hence

$$\begin{aligned}&\int_\Omega (u_1(x,\tau) - u_2(x,\tau))_{h\tau}(u_1 - u_2)_h dx + \int_\Omega \nabla(u_1(x,\tau) - u_2(x,\tau))_{h\tau}\nabla(u_1 - u_2)_h dx\\ &= -\int_\Omega [(|\triangle u_1|^{p(x)-2}\triangle u_1 - |\triangle u_2|^{p(x)-2}\triangle u_2)_h](x,\tau)\triangle(u_1 - u_2)_h dx.\end{aligned}$$

Integrating the above equality with respect to τ over $(0,t)$,

$$\int_\Omega |(u_1 - u_2)_h|^2(x,t)dx + \int_\Omega |\nabla(u_1 - u_2)_h|^2(x,t)dx \le 0,$$

we have $\int_\Omega |(u_1 - u_2)_h|^2 dx = 0$; therefore, $u_1 = u_2$.

4 ASYMPTOTIC BEHAVIOR

This section is devoted to the asymptotic behavior of solutions. To this purpose, we first show that:

Theorem 7 *The weak solution u obtained in Theorem 3.1, satisfies*

$$\begin{aligned}&\frac{1}{2}\int_\Omega |\nabla u(x,t)|^2 dx + \frac{1}{2}\int_\Omega |u(x,t)|^2 dx - \frac{1}{2}\int_\Omega |\nabla u_0(x)|^2 dx - \frac{1}{2}\int_\Omega |u_0(x)|^2\\ &\qquad = -\iint_{Q_t} |\triangle u|^{p(x)} dx\, d\tau,\end{aligned} \tag{4.1}$$

where $Q_t = \Omega \times (0,t)$.

Proof. In the proof of Theorem 2.1, we have

$$f(t) = \frac{1}{2}\int_\Omega |\nabla u(x,t)|^2 dx + \frac{1}{2}\int_\Omega |u(x,t)|^2 dx \in C([0,T]). \tag{4.2}$$

Consider the functional

$$K[v] = \frac{1}{2}\int_\Omega |\nabla v(x)|^2 dx + \frac{1}{2}\int_\Omega |v(x)|^2 dx.$$

It is easy to see that $K[v]$ is a convex functional on $H_0^1(\Omega)$.

For any $\tau \in (0,T)$ and $h > 0$, we have

$$K[u(\tau + h)] - K[u(\tau)] \ge \langle u(\tau + h) - u(\tau), u(x,\tau) - \triangle u(x,\tau)\rangle.$$

By $\frac{\delta K[v]}{\delta v} = v - \triangle v$, for any fixed $t_1, t_2 \in [0,T]$, $t_1 < t_2$, integrating the above inequality with respect to τ over (t_1, t_2), we have

$$\int_{t_2}^{t_2+h} K[u(\tau)]d\tau - \int_{t_1}^{t_1+h} K[u(\tau)]d\tau \ge \int_{t_1}^{t_2} \langle u(\tau + h) - u(\tau), u - \triangle u\rangle d\tau.$$

Multiplying the both side of the above inequality by $1/h$, and letting $h \to 0$, we obtain

$$K[u(t_2)] - K[u(t_1)] \ge \int_{t_1}^{t_2} \left\langle \frac{\partial u}{\partial t}, u - \triangle u \right\rangle d\tau.$$

Similarly, we have

$$K[u(\tau)] - K[u(\tau-h)] \leq \langle u(\tau) - u(\tau-h), u - \triangle u\rangle.$$

Thus

$$K[u(t_2)] - K[u(t_1)] \leq \int_{t_1}^{t_2} \left\langle \frac{\partial u}{\partial t}, u - \triangle u \right\rangle d\tau,$$

and hence

$$K[u(t_2)] - K[u(t_1)] = \int_{t_1}^{t_2} \left\langle \frac{\partial u}{\partial t}, u - \triangle u \right\rangle d\tau.$$

Taking $t_1 = 0, t_2 = t$, we get from the definition of solutions that

$$\begin{aligned} K[u(t)] - K[u(0)] &= \int_0^t \left\langle \frac{\partial u}{\partial t} - \frac{\partial \Delta u}{\partial t}, u(\tau) \right\rangle d\tau. \\ &= -\int_0^t \left\langle \triangle(|\triangle u|^{p(x)-2}\triangle u), u(\tau) \right\rangle d\tau \\ &= -\iint_{Q_t} |\triangle u|^{p(x)} dx\, d\tau. \end{aligned}$$

Theorem 8 *Let u be the weak solution of the problem (1.1)-(1.3), $p_- > 2$. Then*

$$\int_\Omega |\nabla u(x,t)|^2 dx + \int_\Omega |u(x,t)|^2 dx \leq \frac{C_3}{(C_1 t + C_2)^\alpha}, \quad C_i > 0 \ (i = 1,2,3), \quad \alpha = \frac{2}{p_- - 2}.$$

Proof. By (4.2), we have

$$f'(t) = -\int_\Omega |\triangle u|^{p(x)} dx \leq 0.$$

By $u \in W_0^{2,p(x)}(\Omega)$, we see that

$$\int_\Omega |\nabla u(x,t)|^2 dx + \int_\Omega |u(x,t)|^2 dx \leq C \int_\Omega |\triangle u|^2 dx \leq C \left(\int_\Omega |\triangle u|^{p(x)} dx \right)^{2/p_-},$$

that is $f(t) \leq C|f'(t)|^{2/p_-}$. Again by $f'(t) \leq 0$, we have $f'(t) \leq -Cf(t)^{p_-/2}$, and hence we complete the proof.

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Generalizations on some meromorphic function spaces in the unit disc

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Abstract

In this paper, we define a general spherical derivative. Making use of this general derivative, we introduce some new classes of meromorphic functions in the unit disk. Also, we introduce some new classes of meromorphic functions which are defined by means of a general chordal distance.

1 Introduction

Let Δ be the unit disk in the complex plane $\mathbb{C}$, and let $dA(z)$ be the Euclidean area element on Δ. Let $H(\Delta)$ (resp. $M(\Delta)$) denote the class of functions that are analytic (resp.meromorphic) in Δ. The Green's function in Δ with singularity at $a \in \Delta$ is given by $g(z,a) = \log \frac{1}{|\varphi_a(z)|}$, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ is the $M\ddot{o}bius$ transformation of Δ. For $0 < r < 1$, let $\Delta(a,r) = \{z \in \Delta : |\varphi_a(z)| < r\}$ be the pseudohyperbolic disk with center $a \in \Delta$ and radius r.

For $0 < p < \infty$, the spaces Q_p and M_p are defined by (see [1]):

$$Q_p = \{f \in H(\Delta) : \sup_{a\in\Delta} \int\int_{\Delta} |f'(z)|^2 (g(z,a))^p dA(z) < \infty\},$$

$$M_p = \{f \in H(\Delta) : \sup_{a\in\Delta} \int\int_{\Delta} |f'(z)|^2 (1 - |\varphi_a(z)|)^p dA(z) < \infty\}.$$

The Bloch space $\mathcal{B}$ (cf. [1] and [16]), is the space of all analytic functions belonging to $H(\Delta)$, for which

$$\mathcal{B} = \{f \in H(\Delta) : \|f\|_{\mathcal{B}} = \sup_{z\in\Delta} (1 - |z|^2)|f'(z)| < \infty\}.$$

When we study meromorphic functions in Δ, it is natural to replace $|f'(z)|$ in these expressions by the spherical derivative $f^{\#}(z) = |f'(z)|/(1 + |f(z)|^2)$ and obtain the classes $Q_p^{\#}, M_p^{\#}$ and $\mathcal{N}$, the class of normal function in Δ, respectively (see, for example, Aulaskari, Xiao and Zhao [4] and Wulan [19]).

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The meromorphic counterpart of BMOA is the set UBC of meromorphic functions of uniformly bounded characteristic introduced by Yamashita [21]. It turns out that we have $Q_p = M_p$ ([3]), $Q_p^{\#} \subsetneqq M_p^{\#}$ ([5] and [19]).
Now, let $K : [0,\infty) \to [0,\infty)$ be a right-continuous and nondecreasing function, then the spaces Q_K and $Q_K^{\#}$ are defined as follows (see [10, 20]):

Definition 1.1 *$f \in H(\Delta)$ belongs to the space Q_K if*

$$\|f\|_K^2 = \|f\|_{Q_K}^2 = \sup_{a\in\Delta} \int\int_{\Delta} |f'(z)|^2 K(g(z,a))\, dA(z) < \infty. \tag{1}$$

Definition 1.2 *$f \in M(\Delta)$ belongs to the class $Q_K^{\#}$ if*

$$\sup_{a\in\Delta} \int\int_{\Delta} (f^{\#}(z))^2 K(g(z,a))\, dA(z) < \infty. \tag{2}$$

Remark 1.1 *It should be remarked that the space $Q_K^{\#}$ is not a linear space. It is clear that Q_K and $Q_K^{\#}$ are Möbius invariant.*

Remark 1.2 *For $0 < p < \infty$, $K(t) = t^p$ gives the space Q_p and the class $Q_p^{\#}$. Choosing $K(t) = (1 - e^{-2t})^p$, we obtain M_p and $M_p^{\#}$.*

Remark 1.3 *Choosing $K(t) = 1$, we get the Dirichlet space $\mathcal{D}$ and the spherical Dirichlet class $\mathcal{D}^{\#}$. For a fixed r, $0 < r < 1$, we choose*

$$K_0(t) = \begin{cases} 1, & t \geq \log(1/r), \\ 0, & 0 < t < \log(1/r). \end{cases}$$

Then, we obtain

$$\int\int_{\Delta} |f'(z)|^2 K_0(g(z,a))\, dA(z) = \int\int_{\Delta(a,r)} |f'(z)|^2\, dA(z)$$

and

$$\int\int_{\Delta} (f^{\#}(z))^2 K_0(g(z,a))\, dA(z) = \int\int_{\Delta(a,r)} (f^{\#}(z))^2\, dA(z).$$

We conclude that $Q_{K_0} = \mathcal{B}$ (cf. Axler [6]) and $Q_{K_0}^{\#} = \mathcal{B}^{\#}$, where $\mathcal{B}^{\#}$ is the class of spherical Bloch functions (cf. Section 3 in [10]). It is easy to see that $\mathcal{N} \subset \mathcal{B}^{\#}$ (cf. Lappan [14] and the discussion after Definition 2.1 in Wulan [19]).
Now, let us introduce the following notation general spherical derivative

$$f_n^{\#}(z) = \frac{|f^{(n)}(z)|}{1 + |f(z)|^{n+1}};\ n \in \mathbb{N}.$$

This general derivative gives a plethora of new results on the meromorphic function spaces.
Note that if $n = 1$, we obtain the usual spherical derivative as defined above.
let $\omega : (0,1] \to (0,\infty)$ be a nondecreasing function. Let $\mathcal{N}_{n,\omega}^{\alpha}$ be the class of all normal functions in Δ. We recall that a function f meromorphic in Δ is said to be $\omega-$normal if and only if

$$\sup_{z\in\Delta} \frac{(1 - |z|^2)^{\alpha}}{\omega(1 - |\varphi_a(z)|)} f_n^{\#}(z) < \infty.$$

Now, we define some general meromorphic classes as follows:

Definition 1.3 *Let $K : [0,\infty) \to [0,\infty)$ be a nondecreasing function. For $n \in \mathbb{N}$, a function f meromorphic in Δ is said to belong to the class $Q^{\#}_{K,n,\omega}$ if*

$$\sup_{a\in\Delta}\int_{\Delta}(f^{\#}_n(z))^2\frac{K(g(z,a))}{\omega(1-|\varphi_a(z)|)}\,dA(z)<\infty. \tag{3}$$

Definition 1.4 *A function f meromorphic in Δ is said to be a general spherical Bloch function, denoted by $f \in \mathcal{B}^{\#}_{n,\omega}$, if there exists an r, $0<r<1$, such that*

$$\sup_{a\in\Delta}\int_{\Delta}\frac{(f^{\#}_n(z))^2}{\omega(1-|\varphi_a(z)|)}\,dA(z)<\infty. \tag{4}$$

It is easy to see that a normal function is a spherical Bloch function, that is, $\mathcal{N}_{n,\omega} \subset \mathcal{B}^{\#}_{n,\omega}$, but the converse is not true .
For more information of some related meromorphic function spaces, we refer to [1, 2, 7, 8, 9, 10, 11, 18] and others.
For a nondecreasing function $K : [0,\infty) \to [0,\infty)$, we say that the space Q_K is trivial if Q_K contains only constant functions. Whether our space Q_K is trivial or not depends on the integral

$$\int_0^{1/e} K(log(1/\rho))\rho\,d\rho=\int_1^{\infty}K(t)e^{-2t}\,dt. \tag{5}$$

The notation $A \lesssim B$ means that there exists a positive constant C such that $A \le CB$. The symbol $\gtrsim$ is understood in a similar fashion.

2 General meromorphic classes

It is necessary to know for which functions K the classes $Q^{\#}_{K,n}$ will be trivial. Here, the square of the general spherical derivative $(f^{\#}_n(z))^2$ is not necessarily subharmonic, where $f^{\#}_n(z)=\frac{|f^{(n)}(z)|}{1+|f(z)|^{n+1}}; n\in\mathbb{N}$.

Theorem 2.1 *If the integral*

$$\int_0^r\frac{K(\log(1/R))}{\omega(1-R)}R\,dR$$

is divergent, then the space $Q^{\#}_{K,n,\omega}$ contains only constant functions.

$$\begin{aligned}
\int\int_{\Delta}(f^{\#}_n(z))^2\frac{K(g(z,a))}{\omega(1-|\varphi_a(z)|)}\,dA(z) &\ge \int\int_{\Delta(a,r)}(f^{\#}_n(z))^2\frac{K(g(z,a))}{\omega(1-|\varphi_a(z)|)}\,dA(z)\\
&= \int\int_{\Delta(a,r)}\left(\frac{|f^{(n)}(z)|}{1+|f(z)|^{n+1}}\right)^2\frac{K(g(z,a))}{\omega(1-|\varphi_a(z)|)}\,dA(z)\\
&= \int\int_{|\varphi_a(z)|<r}\left(\frac{|f^{(n)}\varphi_a(z)|}{1+|f(\varphi_a(z))|^{n+1}}\right)^2|\varphi'_a(z)|^2\frac{K(\log(1/|z|))}{\omega(1-|z|)}\,dA(z)\\
&\ge \frac{\pi}{2}\left(\frac{(1-|a|^2)|f^{(n)}(a)|}{1+|f(a)|^{n+1}}\right)^2\int_0^r R\,\frac{K(\log(1/R))}{\omega(1-R)}dR=\infty.
\end{aligned}$$

This is a contradiction, and the proof is complete.

Again, we assume from now on that the functions K and ω are right-continuous and nondecreasing, and that the integral (5) is convergent.
As in [21], we can give the following result.

Theorem 2.2 *For some $r \in (0,1)$, a meromorphic function f belongs to $\mathcal{N}_{n,\omega}$ if and only if*

$$\sup_{a\in\Delta}\int\int_{\Delta(a,r)}\frac{(f_n^{\#}(z))^2}{\omega(1-|\varphi_a(z)|)}dA(z)<\pi.$$

Proof: The proof is very similar to the corresponding result in [21] with simple modifications, so it will be omitted.

Now, we consider the following question:

Question 1

Is the condition that there exists $r \in (0,1)$ such that:

$$\sup_{a\in\Delta}\int\int_{\Delta(a,r)}(f_n^{\#}(z))^2\frac{K(g(z,a))}{\omega(1-|\varphi_a(z)|)}\,dA(z)<\infty \tag{6}$$

necessary and sufficient for $f \in \mathcal{B}^{\#}_{n,\omega}$?

Answer

If (6) holds, we can conclude that, $f \in B^{\#}_{n,\omega}$. In particular, it follows that $Q^{\#}_{K,n,\omega} \subset \mathcal{B}^{\#}_{n,\omega}$. Conversely, if we assume that $f \in \mathcal{B}^{\#}_{n,\omega}$ and that K is bounded, it is easy to see that (6) will hold. If K is unbounded and $f \in \mathcal{B}^{\#}_{n,\omega} \setminus \mathcal{N}_{n,\omega}$, we claim that the supremum in (6) will be infinite for all $r \in (0,1)$. To prove the claim , we note that it follows from Theorem 2.1 that if $f \in \mathcal{B}^{\#}_{n,\omega} \setminus \mathcal{N}_{n,\omega}$, then

$$\sup_{a\in\Delta}\int\int_{\Delta(a,r)}\frac{(f_n^{\#}(z))^2dA(z)}{\omega(1-|\varphi_a(z)|)}\geq\pi\quad for\ \ all\ \ r\in(0,1).$$

if $0<\rho<r$, we see that

$$\int\int_{\Delta(a,r)}(f_n^{\#}(z))^2\frac{K(g(z,a))}{\omega(1-|\varphi_a(z)|)}\,dA(z)\geq K(log(1/\rho))\int\int_{\Delta(a,\rho)}\frac{(f_n^{\#}(z))^2}{\omega(1-|\varphi_a(z)|)}\,dA(z).$$

Using the observation above, we deduce that

$$\sup_{a\in\Delta}\int\int_{\Delta(a,r)}(f_n^{\#}(z))^2\frac{K(g(z,a))}{\omega(1-|\varphi_a(z)|)}\,dA(z)\geq\pi\ K(log(1/\rho)),\ \ 0<\rho<r$$

Letting $\rho \to 0$, we conclude that (6) cannot hold for any $r \in (0,1)$ which completes the proof.
We conclude that (6) is a sufficient condition for $f \in \mathcal{B}^{\#}_{n,\omega}$. It is also a necessary condition when K is bounded, but not when K is unbounded. Finally, if we assume that $f \in \mathcal{N}_{n,\omega}$, it is easy to prove that (7) will hold (see the proof of Theorem 2.3(ii) below).
For the weights, there are some questions, which can be stated as follows:

Question 2

Which additional conditions on K are required for the inclusion $Q^{\#}_{K,n,\omega} \subset \mathcal{N}_{n,\omega}$?
When are the classes $Q^{\#}_{K_1,n}$ and $Q^{\#}_{K_2,n,\omega}$ identical for $K_1 \neq K_2$?
Answers of the above questions can be given by the next results. First, as in [17, 18, 19], we can give the following proposition.

Proposition 2.1 *Assume that $K(r) \to \infty$ as $r \to \infty$. Then $Q^{\#}_{K,n,\omega} \subset \mathcal{N}_{n,\omega}$.*

Next, we prove the following result:

Theorem 2.3 *Assume that $K(\infty) = 1$. Then $f \in \mathcal{N}_{n,\omega}$ if and only if*

$$\sup_{a\in\Delta}\int\int_{\Delta(a,r)}(f_n^{\#}(z))^2\frac{K(g(z,a))}{\omega^2(1-|\varphi_a(z)|)}dA(z)<\pi \tag{7}$$

for some $r \in (0,1)$.

Proof: Suppose that f is a general normal function. Then for $0 < r < 1$,

$$\begin{aligned}\int\int_{\Delta(a,r)} (f_n^{\#}(z))^2 \frac{K(g(z,a))}{\omega^2(1-|\varphi_a(z)|)}\, dA(z) &\leq \|f\|^2_{\mathcal{N}_{n,\omega}} \int\int_{\Delta(a,r)} (1-|z|^2)^{-2} K(g(a,z))\, dA(z) \\ &\leq 2\pi \|f\|^2_{\mathcal{N}_{n,\omega}} (1-r^2)^{-2} \int_0^r K(log 1/\rho)\rho\, d\rho. \end{aligned} \tag{8}$$

Since

$$\int_0^r K(log 1/\rho)\rho\, d\rho \to 0, r \to 0,$$

we may choose r small enough such that the left hand member in the first inequality in (8) is less than $\pi/2$. Thus (7) holds.

Conversely, let $\lambda(< \pi)$ be the supremum in (7) assumed for some $r_0 \in (0, 1)$. Now consider $r \in (0, r_0)$. Since $\Delta(a, r) = \{z \in \Delta : g(z, a) > log(1/r)\}$,

$$\begin{aligned}&\int\int_{\Delta(a,r)} (f_n^{\#}(z))^2 dA(z) \\ &\leq \frac{\omega^2(1-r)}{K(log(1/r))} \int\int_{\Delta(a,r_0)} (f_n^{\#}(z))^2 \frac{K(g(z,a))}{\omega^2(1-|\varphi_a(z)|)}\, dA(z) \leq \lambda \frac{\omega^2(1-r)}{K(log(1/r))} < \pi\end{aligned}$$

here λ is a constant. Hereafter, λ stands for absolute constants, which may indicate different constants from one occurrence to the next. If r is small enough. Hence $f \in \mathcal{N}_{n,\omega}$ according to Theorem 2.1, the proof is established.

Corollary 2.1 *Assume that $K(\infty) = 1$. if $f \in Q^{\#}_{K,n,\omega}$ and*

$$\sup_{a\in\Delta} \int\int_{\Delta} (f_n^{\#}(z))^2 \frac{K(g(z,a))dA(z)}{\omega(1-|\varphi_a(z)|} < \pi,$$

then $f \in \mathcal{N}_{n,\omega}$.

Another important result on the weights of some meromorphic functions can be given by the following result:

Theorem 2.4 *Assume that $K(1) > 0$ and set $K_1(r) = \inf(K(r), K(1))$.*
(i) If K is bounded, then $Q^{\#}_{K,n,\omega} = Q^{\#}_{K_1,n,\omega}$.
(ii) If K is unbounded, then $Q^{\#}_{K,n,\omega} = \mathcal{N}_{n,\omega} \cap Q^{\#}_{K_1,n,\omega}$.

Proof: (i) If K is bounded , we have

$$K_1(r) \leq K(r) \leq \frac{K(\infty)}{K(1)} K_1(r)$$

and it is clear that $Q^{\#}_{K,n,\omega} = Q^{\#}_{K_1,n,\omega}$.

(ii) By Proposition 2.1, we have $Q^{\#}_{K,n,\omega} \subset \mathcal{N}_{n,\omega} \cap Q^{\#}_{K_1,n,\omega}$. Now assume that $f \in \mathcal{N}_{n,\omega} \cap Q^{\#}_{K_1,n,\omega}$. We note that $K(g(z, a)) = K_1(g(z, a))$ in $\Delta/\Delta(a, 1/e)$. (In this domain, we have $g(z, a) \leq 1$) . To compare the two suprema in the integrals defining $Q^{\#}_{K,n,\omega}$ and $Q^{\#}_{K_1,n,\omega}$, it suffices to deal with integrals over $\Delta(a, 1/e)$. Using our assumption that $f \in \mathcal{N}_{n,\omega}$, we see that

$$\begin{aligned}\int\int_{\Delta(a,1/e)} (f_n^{\#}(z))^2 \frac{K(g(z,a))}{\omega^2(1-|\varphi_a(z)|)}\, dA(z) &\leq \|f\|^2_{\mathcal{N}_{n,\omega}} \int\int_{\Delta(a,1/e)} (1-|z|^2)^{-2} K(g(z,a))\, dA(z) \\ &= \|f\|^2_{\mathcal{N}_{n,\omega}} \int\int_{\Delta(0,1/e)<r} (1-|z_1|^2)^{-2} K(\log\frac{1}{r})\, dA(z_1) \\ &= 2\pi \|f\|^2_{\mathcal{N}_{n,\omega}} \int_0^{1/e} r(1-|r|^2)^{-2} K(log(1/r))\, dr.\end{aligned}$$

the right hand member gives a bound for the supremum over $a \in \Delta$ of the first term in this chain of inequalities. Hence $f \in Q^{\#}_{K,n,\omega}$ and Theorem 2.3 is proved.

Next, we state conditions on K_1 and K_2 which imply that $Q^{\#}_{K_1,n,\omega} = Q^{\#}_{K_2,n,\omega}$.

Theorem 2.5 *Assume that K_1 and K_2 are either both bounded or both unbounded and that $K_1(r) \approx K_2(r)$ as $r \to 0$. Then $Q^{\#}_{K_1,n,\omega} = Q^{\#}_{K_2,n,\omega}$.*

Proof: We define $K_{i,1}(r) = inf(K_i(r), K_i(1))$, $i = 1, 2$. If K_1 and K_2 are bounded, it follows from our assumptions that $0 < c \le K_1(r)/K_2(r) \le c' < \infty, 0 < r < \infty$ and it is clear that we have $Q^{\#}_{K_1,n,\omega} = Q^{\#}_{K_2,n,\omega}$. If K_1 and K_2 are unbounded, we use Theorem 2.4 to deduce that

$$Q^{\#}_{K_1,n,\omega} = \mathcal{N}_{n,\omega} \cap Q^{\#}_{K_{1,1},n,\omega} = \mathcal{N}_{n,\omega} \cap Q^{\#}_{K_{2,1},n,\omega} = Q^{\#}_{K_2,n,\omega}.$$

This completes the proof of Theorem 2.5.

Theorem 2.6 *(i) If K is unbounded and (5) holds, then $Q^{\#}_{K,n,\omega} = \mathcal{N}_{n,\omega}$.*
(ii) If K is bounded and (5) holds, then $Q^{\#}_{K,n,\omega} = \mathcal{B}^{\#}_{n,\omega}$.
(iii) In (i) (resp. (ii)), (5) is a necessary condition for $Q^{\#}_{K,n,\omega} = \mathcal{N}_{n,\omega}$ (resp. $Q^{\#}_{K,n,\omega} = \mathcal{B}^{\#}_{n,\omega}$) .

Proof: (i) By Proposition 2.1 we have $Q^{\#}_{K,n,\omega} \subset \mathcal{N}_{n,\omega}$. Conversely, if $f \in N_{n,\omega}$, we know that $f^{\#}_n(z) \le \lambda(1 - |z|^2)^{-1}$ and we can use the argument in the proof of (Theorem 2.3 in [10]) to prove that $f \in Q^{\#}_{K,n,\omega}$.
(ii) By question 1, we have $Q^{\#}_{K,n,\omega} \subset \mathcal{B}^{\#}_{n,,\omega}$. It suffices to prove that $\mathcal{B}^{\#}_{n,,\omega} \subset Q^{\#}_{K,n,\omega}$. If $f \in B^{\#}_{n,\omega}$, there exists $r \in (0, 1)$ such that

$$\int\int_{\Delta(a,r)} \frac{(f^{\#}_n(z))^2}{\omega^2(1 - |\varphi_a(z)|)}\, dA(z) \le \lambda < \infty \quad \text{for all } a \in \Delta. \tag{9}$$

Let us first prove that there exists a constant C_1 depending on r and K (see below) such that

$$\int\int_{\Delta} (f^{\#}_n(z))^2 \frac{K(log(1/|z|))}{\omega^2(1 - |z|)}\, dA(z) \le \lambda \|K\|_\infty + C_1. \tag{10}$$

Our first observation in the proof of this estimate is that

$$\int\int_{|z|<r} (f^{\#}_n(z))^2 \frac{K(log(1/|z|))}{\omega^2(1 - |z|)}\, dA(z) \le B\|K\|_\infty.$$

Let $\Omega_k = \{z - (1-r)^k \le |z| \le 1-(1-r)^{k+1}\}$. We wish to cover Ω_k with disks $\Delta(a, r)$ with $|a| = 1-(1-r)^{k+1}$, it suffices to use roughly $C(r(1 - r)^{k+1})^{-1}$ such disks, where C is an absolute constant, $k = 1, 2$. Hence .

$$\begin{aligned} \int\int_{\Omega_k} (f^{\#}_n(z))^2 \frac{K(log(1/|z|))}{\omega^2(1 - |z|)}\, dA(z) &\le K(log\frac{1}{1 - (1 - r)^k})^{-1} BC(r(1 - r)^{k+1})^{-1}, \\ &\le K((1 - r)^k \gamma(r))BC(r(1 - r)^{k+1})^{-1}, \end{aligned}$$

where $\gamma(r) = (1 - r)^{-1} \frac{\log(\frac{1}{r})}{\omega(1-r)}$. It follows that

$$\begin{aligned} \int\int_{r<|z|<1} (f^{\#}_n(z))^2 \frac{K(log(1/|z|))}{\omega^2(1 - |z|)}\, dA(z) &\le \lambda\, r^{-1} \sum_1^\infty (1 - r)^{-k-1} K((1 - r)^k \gamma(r)) \\ &\le \lambda\, r^{-2}(1 - r)^{-2} \int_0^1 t^{-2} K(t\gamma(r))\, dt. \\ &= \lambda\gamma(r) r^{-2}(1 - r)^{-2} \int_0^{\gamma(r)} s^{-2} K(s)\, ds = C_1 < \infty. \end{aligned}$$

The convergence of the integral follows from (5). We have proved that (10) holds for all $f \in B^{\#}_{n,\omega}$ satisfying (10). Since for all $b \in \Delta$,

$$\sup_{a\in\Delta}\int\int_{\Delta(a,r)} \frac{((f\circ\varphi_b)^{\#}_n(z))^2}{\omega^2(1-|\varphi_a(z)|)}dA(z) = \sup_{a\in\Delta}\int\int_{\Delta(a,r)} \frac{(f^{\#}_n(z))^2}{\omega^2(1-|\varphi_a(z)|)}\,dA(z) = \lambda.$$

It follows from (9) and (10) with $f^{\#}_n$ replaced by $(f\circ\varphi_b)^{\#}_n$ that

$$\sup_{b\in\Delta}\int\int_{\Delta}(f^{\#}_n(z))^2\frac{K(log\frac{1}{|\varphi_b(z)|})}{\omega^2(1-|\varphi_b(z)|)}dA(z) = \sup_{b\in\Delta}\int\int_{\Delta}((f\circ\varphi_b)^{\#}_n(z))^2\frac{K(log(1/|z|))}{\omega^2(1-|z|)}dA(z) \le C_1+\lambda\|K\|_\infty$$

this proves Theorem 2.5(ii).
(iii) As given by Lappan and Xiao [15], there exist functions f_1 and f_2 in $\mathcal{N}_{n,\omega}$ such that

$$c_0 = \inf_{z\in\Delta}(1-|z|^2)(f^{\#}_{n,1}(z)+f^{\#}_{n,2}(z)) > 0 \tag{11}$$

If $Q^{\#}_{K,n,\omega} = \mathcal{N}_{n,\omega}$ or $Q^{\#}_{K,n,\omega} = \mathcal{B}^{\#}_{n,\omega} \supset \mathcal{N}_{n,\omega}$, we have

$$\begin{aligned}
\infty &> \sup_{a\in\Delta}\int\int_{\Delta}(f^{\#}_{n,1}(z))^2+(f^{\#}_{n,2}(z))^2\frac{K(g(z,a))}{\omega^2(1-|\varphi_a(z)|)}\,dA(z).\\
&\ge \frac{1}{2}\int\int_{\Delta}(f^{\#}_{n,1}(z)+f^{\#}_{n,2}(z))^2\frac{K(g(z,0))}{\omega^2(1-|\varphi_0(z)|)}\,dA(z).\\
&\ge (c_0^2/2)\int\int_{\Delta}(1-|z|^2)^{-2}\frac{K(g(z,0))}{\omega^2(1-|\varphi_0(z)|)}\,dA(z).\\
&= \pi c_0^2\int_0^1(1-r^2)^{-2}\frac{K(log(1/r))}{\omega^2(1-r)}r\,dr.
\end{aligned}$$

Hence (5) holds which finishes the proof of Theorem 2.5(iii).

Remark 2.1 *There is an analogue of (11) for Bloch functions with the general spherical derivatives* $f^{\#}_{n,1}$ *and* $f^{\#}_{n,2}$ *replaced by* $|f_1^{(n)}|$ *and* $|f_2^{(n)}|$.

Finally we consider the classes

$$\mathcal{B}^{\#}_{n,\omega,0} = \{f\in M(\Delta): \lim_{|a|\to1}\int\int_{\Delta(a,r)}(f^{\#}_n(z))^2\,dA(z)=0 \text{ for some } r\in(0,1)\},$$

$$Q^{\#}_{K,n,\omega,0} = \{f\in M(\Delta): \lim_{|a|\to1}\int\int_{\Delta}(f^{\#}_n(z))^2\frac{K(g(z,a))}{\omega^2(1-|\varphi_a(z)|)}\,dA(z)=0\},$$

$$\mathcal{N}_{n,\omega,0} = \{f\in M(\Delta): \frac{(1-|z|^2)}{\omega^2(1-|\varphi_a(z)|)}f^{\#}_n(z)\to 0\ ,\ |z|\to\ 1\}\ .$$

and the weighted general spherical Dirichlet class can be defined by

$$\mathcal{D}^{\#}_{n,\omega} = \{f\in M(\Delta): \int\int_{\Delta}\frac{(f^{\#}_n(z))^2}{\omega^2(1-|\varphi_a(z)|)}\,dA(z)<\infty\}$$

Arguing as in the proof of (Theorem 2.4 in [10]), we deduce:

Theorem 2.7 $Q^{\#}_{K,n,\omega,0} \subset \mathcal{B}^{\#}_{n,\omega,0} = \mathcal{N}_{n,\omega,0}$.

Theorem 2.8 *If (5) holds, then* $Q^{\#}_{K,n,\omega,0} = \mathcal{N}_{n,\omega,0}$.

Remark 2.2 *It suffices to prove that $\mathcal{N}_{n,\omega,0} \subset Q^{\#}_{K,n,\omega,0}$. We deduce this using the same argument as in the first part of the proof of (Theorem 2.5 in [10]). We note that in this argument, the growth of K at infinity is unimportant since we have $\mathcal{N}_{n,\omega,0} = \mathcal{B}^{\#}_{n,\omega,0}$.*

Theorem 2.9 .
(i) If $K(0) > 0$, then $\mathcal{D}^{\#}_{n,\omega} = Q^{\#}_{K,n,\omega}$.
(ii) $\mathcal{D}^{\#}_{n,\omega} \subset Q^{\#}_{K,n,\omega,0}$ if and only if $K(0) = 0$.
(iii) Assume that $Q^{\#}_{K,n,\omega} \neq Q^{\#}_{K,n,\omega,0}$. If $\mathcal{D}^{\#}_{n,\omega} = Q^{\#}_{K,n,\omega}$, then $K(0) > 0$.
(iv) If $\mathcal{D}^{\#}_{n,\omega} = Q^{\#}_{K,n,\omega} = Q^{\#}_{K,n,\omega,0}$, then $K(0) = 0$.

Proof:
To prove (i), we assume that $K(0) > 0$ and note that $\mathcal{D}^{\#}_{n} \subset \mathcal{B}^{\#}_{n,\omega,o} = \mathcal{N}_{n,\omega,0} \subset \mathcal{N}_{n,\omega}$. If K is bounded, it is clear that $Q^{\#}_{K,n,\omega} = \mathcal{D}^{\#}_{n,\omega}$. If K is unbounded, we use Theorem 2.3 and the fact that $Q^{\#}_{K_1,n,\omega} = \mathcal{D}^{\#}_{n,\omega}$ (we use the notation of Theorem 2.3) to obtain that $Q^{\#}_{K,n,\omega} = \mathcal{N}_{n,\omega} \cap Q^{\#}_{K_1,n,\omega} = \mathcal{N}_{n,\omega} \cap \mathcal{D}^{\#}_{n,\omega} = \mathcal{D}^{\#}_{n,\omega}$
the proof of (i) is completely established.
The proof of (ii) uses the same argument as the proof of Theorem 2.7 in [10] with some simple modifications except that we again use the fact that $\mathcal{D}^{\#}_{n,\omega} \subset \mathcal{B}^{\#}_{\omega,0} = \mathcal{N}_{n,\omega,0}$.
To prove (iii), we remark that assumptions imply that $\mathcal{D}^{\#}_{n,\omega} \not\subseteq Q^{\#}_{K,n,\omega,0}$ and use (ii).
If the assumptions of (iv) hold, we have $\mathcal{D}^{\#}_{n,\omega} \subset Q^{\#}_{K,n,\omega,0}$ and the conclusion follows from (ii).

Corollary 2.2 *$\mathcal{D}^{\#}_{n,\omega} \subset Q^{\#}_{p,n,\omega,0}$ for all p ,$0 < p < \infty$.*

3 General chordal distance

In this section, we introduce and study some certain new scales of meromorphic functions in the unit disk and solve some problems connected with a general Chordal distance in these scales of spaces.
The chordal distance between the points z and w in the extended complex plane $\widehat{C} = C \cup \{\infty\}$ is

$$\chi_n(z,w) = \begin{cases} \frac{|z-w|^n}{(1+|z|^2)^{\frac{1}{n+1}}(1+|w|^2)^{\frac{1}{n+1}}} & if \quad z, w \neq \infty;\ n \in \mathbb{N}. \\ \\ \frac{1}{(1+|z|^2)^{\frac{1}{n+1}}} & if \quad w = \infty. \end{cases}$$

Remark 3.1 *If, we put $n = 1$ in the general chordal distance, we obtain the usual chordal distance see [2].*

The meromorphic Bergman class M^P_α is defined as the set of those $f \in M(\Delta)$ for which

$$\|f\|^p_{M^p_{\alpha,\omega}} = \int_\Delta \chi_n(f(z), 0)^p \frac{(1-|z|^2)^\alpha}{\omega(1-|z|)}\, dA(z) < \infty.$$

Now, we give the following result:

Theorem 3.1 *Let $1 \le p < \infty$, and $-1 < \alpha < \infty$ and let $f \in M(\Delta)$. Suppose that*

$$\int_{|w|}^1 \frac{\left(1 - \frac{|w|}{t}\right)^\alpha}{\omega\left(1 - \frac{|w|}{t}\right)} \frac{dt}{t^3} < \infty.$$

Then there exists a positive constant C, depending only on p and α, such that

$$\int_\Delta \chi_n(f(z), f(0))^p \frac{(1-|z|^2)^\alpha}{\omega(1-|z|)}\, dA(z) \le C \int_\Delta (f^{\#}_n(z))^p \frac{(1-|z|^2)^{p+\alpha}}{\omega(1-|z|)} \frac{dA(z)}{|z|}.$$

Proof: First let $p=1$ and let $0<t<1$. Since

$$\chi_n(f(z),f(0)) \le \int_0^1 f_n^{\#}(tz)|z|\,dt,$$

Fubini's theorem and integration by parts yield

$$\begin{aligned}\int_\Delta \chi_n(f(z),f(0))\frac{(1-|z|^2)^\alpha}{\omega(1-|z|)}\,dA(z) &\lesssim \int_\Delta\int_0^1 f_n^{\#}(tz)\,dt|z|\frac{(1-|z|^2)^\alpha}{\omega(1-|z|)}\,dA(z)\\ &= \int_0^1\int_{D(0,t)} f_n^{\#}(w)|w|\frac{\big(1-\frac{|w|}{t}\big)^\alpha}{\omega\big(1-\frac{|w|}{t}\big)}\frac{dt}{t^3}\,dA(w)\\ &= \int_\Delta f_n^{\#}(w)|w|\int_{|w|}^1 \frac{\big(1-\frac{|w|}{t}\big)^\alpha}{\omega\big(1-\frac{|w|}{t}\big)}\frac{dt}{t^3}\,dA(w)\\ &\lesssim \int_\Delta (f_n^{\#}(w))|w|dA(w),\end{aligned}$$

which is the desired asymptotic inequality for $p=1$. If $p>1$, choose $q>((p-1)/p)$ such that $\alpha-pq+p>0$. By Hölder's inequality, we obtain

$$\begin{aligned}\chi_n(f(z),f(0)) &\le \int_0^1 f_n^{\#}(tz)|z|\,dt = \int_0^1 f_n^{\#}(1-t|z|)^q\frac{|z|\,dt}{(1-t|z|)^q}\\ &\le \Big(\int_0^1 f_n^{\#}(tz)^p\frac{(1-t|z|)^{pq}}{\omega^p(1-t|z|)}\,dt\Big)^{1/p}\Big(\int_0^1 \frac{|z|^{(p-1)/p}\,dt}{\omega^{\frac{-p}{p-1}}(1-t|z|)(1-t|z|)^{pq/(p-1)}}\Big)^{(p-1)/p}\\ &\lesssim \Big(\int_0^1 f_n^{\#}(tz)^p(1-t|z|)^{pq}\,dt|z|(1-|z|)^{p-1-pq}\Big)^{1/p}\end{aligned}$$

from which Fubinis theorem yields

$$\begin{aligned}\int_\Delta \chi_n(f(z),f(0))^p\frac{(1-|z|^2)^\alpha}{\omega(1-t|z|)}\,dA(z) &\lesssim \int_\Delta\int_0^1 \big(f_n^{\#}(tz)\big)^p(1-t|z|)^{pq}\,dt|z|\frac{(1-|z|)^{\alpha+p-1-pq}}{\omega(1-t|z|)}\,dA(z)\\ &= \int_0^1\int_{D(0,t)} \big(f_n^{\#}(w)\big)^p(1-|w|)^{pq}|w|\frac{\big(1-\frac{|w|}{t}\big)^{\alpha-pq+p-1}}{\omega(1-\frac{|w|}{t})}\frac{dt}{t^3}\,dA(w)\\ &= \int_\Delta \big(f_n^{\#}(w)\big)^p|w|\int_{|w|}^1 \frac{\big(1-\frac{|w|}{t}\big)^{\alpha+p-1}}{\omega(1-\frac{|w|}{t})}\frac{dt}{t^3}\,dA(w)\\ &\lesssim \int_\Delta f_n^{\#}(w)^p|w|dA(w).\end{aligned}$$

Theorem 3.2 *Let $1\le p<\infty$ and $-1<\alpha<\infty$, and let $f\in M(\Delta)$. Suppose that*

$$\int_\Delta |\varphi_w'(z)|^{\alpha+2}\frac{dA(w)}{\omega(1-|\varphi_w(z)|)|\varphi_w(z)|(1-|w|^2)^2} < C$$

where C is a positive constant. Then,

$$\int\int_\Delta \frac{(\chi_n(f(z),f(w))^p}{|1-\overline{w}z|^4}\frac{(1-|\varphi_w(z)|^2)^\alpha}{\omega(1-|\varphi_w(z)|)}\,dA(w) \le \lambda \int_\Delta |\varphi_w'(z)|^{\alpha+2}\frac{dA(w)}{\omega(1-|\varphi_w(z)|)|\varphi_w(z)|(1-|w|^2)^2}.$$

Proof: By the change of variable $z=\varphi_w(u)$, Theorem 3.1 and Fubini's theorem,

$$I(f) = \int\int_\Delta \frac{(\chi_n(f(z),f(w)))^p}{|1-\overline{w}z|^4}\frac{(1-|\varphi_w(z)|^2)^\alpha}{\omega(1-|\varphi_w(z)|)}\,dA(z)\,dA(w)$$

$$
\begin{aligned}
&= \int\int_{\Delta} (\chi_n((f\circ\varphi_w)(u),(f\circ\varphi_w)(0)))^p \frac{(1-|u|^2)^{\alpha}}{\omega(1-|u|)}\, dA(u) \frac{dA(w)}{(1-|w|^2)^2}\\
&\lesssim \int\int_{\Delta} ((f\circ\varphi_w)_n^{\#}(u))^p \frac{(1-|u|^2)^{p+\alpha}}{\omega(1-|u|)} \frac{dA(u)}{|u|} \frac{dA(w)}{(1-|w|^2)^2}\\
&= \int\int_{\Delta} (f_n^{\#}(\varphi_w(u)))^p (1-|\varphi_w(u)|^2)^p \frac{(1-|u|^2)^{\alpha}}{\omega(1-|u|)} \frac{dA(u)}{|u|} \frac{dA(w)}{(1-|w|^2)^2}\\
&= \int_{\Delta} (f_n^{\#}(z))^p (1-|z|^2)^{p+\alpha} \int_{\Delta} |\varphi_w'(z)|^{\alpha+2} \frac{dA(w)}{\omega(1-|\varphi_w(z)|)|\varphi_w(z)|(1-|w|^2)^2}\, dA(z).
\end{aligned}
$$

But since

$$\int_{\Delta} |\varphi_w'(z)|^{\alpha+2} \frac{dA(w)}{\omega(1-|\varphi_w(z)|)|\varphi_w(z)|(1-|w|^2)^2} < C.$$

Then,

$$I(f) \leq \lambda \int_{\Delta} (f_n^{\#}(z))^p (1-|z|^2)^{p+\alpha} dA(z).$$

Remark 3.2 *In Theorem 3.2, if we put $n=1$, we obtain theorem 1.2 in [2].*

Corollary 3.1 *Let $2<p<\infty$ and $f\in M(\Delta)$. Then there exists a positive constant C, depending only on p, such that*

$$\int\int_{\Delta} \frac{\chi_n(f(z)-f(w))^p}{|1-\overline{w}z|} \Big(\frac{(1-|z|^2)^{(p/2)-2}}{\omega(1-|z|)}\Big)\Big(\frac{(1-|w|^2)^{(p/2)-2}}{\omega(1-|w|)}\Big)\, dA(z)\, dA(w) \ \leq C\|f\|^p_{B^{\#}_{p,n}}.$$

An application of Theorem 3.1 with $\alpha=0$ to the function $.(fo\varphi_w)(rz)$ yields

$$\int_{\Delta(w,r)} \chi_n(f(z),f(w))^p\, dA(z) \lesssim \int_{\Delta(w,r)} (f_n^{\#}(z))^p \Big(\frac{(1-|z|^2)^p}{\omega(1-|z|)}\Big)\frac{dA(z)}{|\varphi_w(z)|}, \tag{12}$$

where $\Delta(w,r)=\{z:|\varphi_w(z)|<r\}$ is the pseudohyperbolic disc of (pseudohyperbolic) center $w\in\Delta$ and radius $r\in(0,1)$, and the constant of comparison depends only on r . This fact can be used to prove Theorem 3.3. The class $M^{\#}_{n,\omega}(p,q,s)$ consists of those $f\in M(\Delta)$ for which

$$\|f\|^p_{M^{\#}_{n,\omega}(p,q,s)} = \sup_{a\in\Delta}\int_{\Delta} (f_n^{\#}(z))^p \Big(\frac{(1-|z|^2)^q}{\omega(1-|z|)}\Big)\Big(\frac{(1-|\varphi_a(z)|^2)^s}{\omega(1-|\varphi_a(z)|)}\Big)\, dA(z) < \infty.$$

For the next result, let $|D(z,r)|$ denote the Euclidean area of $D(z,r)$, so by [[12], p. 3], we have that

$$|D(z,r)| = \pi r \frac{(1-|a|^2)^2}{(1-|a|^2r^2)^2} \tag{13}$$

Theorem 3.3 *Let $1\leq p<\infty$, $-2<q<\infty$, $0\leq s<\infty$ and $o<r<1$. Let $\alpha,\beta,\gamma,\delta\in\mathbb{R}$ such that $\alpha+\beta=q-p$, and $\gamma+\delta=s$, and let $f\in M(\Delta)$. Then*

$$
\begin{aligned}
&\sup_{a\in\Delta}\int_{\Delta}\Big(\frac{1}{|D(z,r)|}\int_{D(z,r)} \chi_n(f(z),f(w))^p \Big(\frac{(1-|z|^2)^{\alpha}}{\omega(1-|z|)}\Big)\Big(\frac{(1-|w|^2)^{\beta}}{\omega(1-|w|)}\Big)\\
&.\Big(\frac{(1-|\varphi_a(z)|^2)^{\gamma}}{\omega(1-|\varphi_a(z)|)}\Big)\Big(\frac{(1-|\varphi_a(w)|^2)^{\delta}}{\omega(1-|\varphi_a(w)|)}\Big)\, dA(w)\Big)\, dA(z) \leq \|f\|^p_{M^{\#}_{n,\omega}(p,q,s)}.
\end{aligned}
$$

Proof: Routine calculations and (15) show that for $w \in D(z,r)$ and $a \in \Delta$,

$$1-|z|^2 \simeq 1-|w|^2 \simeq 1-|\overline{w}z|^2 \simeq |D(z,r)|^{1/2}, \tag{14}$$

and

$$1-|\varphi_a(z)|^2 \simeq 1-|\varphi_a(w)|^2, \tag{15}$$

where the constants of comparison depend only on r . By (16), (17) and (14),

$$\begin{aligned} I &= \sup_{a\in\Delta}\int_\Delta \Big(\frac{1}{|D(z,r)|}\int_{D(z,r)} \big(\chi_n(f(z),f(w))\big)^p \Big(\frac{(1-|z|^2)^\alpha}{\omega(1-|z|)}\Big)\Big(\frac{(1-|w|^2)^\beta}{\omega(1-|w|)}\Big) \\ &\quad . \Big(\frac{(1-|\varphi_a(z)|^2)^\gamma}{\omega(1-|\varphi_a(z)|)}\Big)\Big(\frac{(1-|\varphi_a(w)|^2)^\delta}{\omega(1-|\varphi_a(w)|)}\Big)\, dA(w)\Big)\, dA(z) \\ &\lesssim \sup_{a\in\Delta}\int_\Delta \Big(\int_{D(z,r)} (f_n^\#(w))^p \Big(\frac{(1-|w|^2)^p}{\omega(1-|w|)}\Big)\frac{dA(w)}{|\varphi_z(w)|}\Big)\Big(\frac{(1-|z|^2)^{q-p-2}}{\omega(1-|z|)}\Big).\Big(\frac{(1-|\varphi_a(z)|^2)^s}{\omega(1-|\varphi_a(z)|)}\Big)\, dA(z) \end{aligned}$$

from which (16), (17) and Fubini's theorem yield

$$\begin{aligned} I &\lesssim \sup_{a\in\Delta}\int_\Delta \Big(\int_{D(z,r)} (f_n^\#(w))^p \Big(\frac{(1-|w|^2)^{q-2}}{\omega(1-|w|)}\Big)\Big(\frac{(1-|\varphi_a(w)|^2)^s}{\omega(1-|\varphi_a(w)|)}\Big)\frac{dA(w)}{|\varphi_z(w)|}\Big)\, dA(z) \\ &= \sup_{a\in\Delta}\int_\Delta \Big(\int_{D(z,r)} \frac{dA(z)}{|\varphi_z(w)|}\Big)(f_n^\#(w))^p \Big(\frac{(1-|w|^2)^{q-2}}{\omega(1-|w|)}\Big)\Big(\frac{(1-|\varphi_a(w)|^2)^s}{\omega(1-|\varphi_a(w)|)}\Big)\, dA(w) \\ &\simeq \sup_{a\in\Delta}\int_\Delta (f_n^\#(w))^p \Big(\frac{(1-|w|^2)^{q}}{\omega(1-|w|)}\Big)\Big(\frac{(1-|\varphi_a(w)|^2)^s}{\omega(1-|\varphi_a(w)|)}\Big)\, dA(w). \end{aligned}$$

The class $\mathcal{N}$ of normal functions consists of those $f \in M(\Delta)$ for which the family $\{fo\varphi\}$, where φ is a *Möbius* transformation of Δ, is normal in Δ in the sense of Montel. It is known that $f \in M(\Delta)$ is all normal if and only if

$$\|f\|_{\mathcal{N}_{n,\omega}} = \sup_{z\in\Delta} f_n^\#(z)\frac{(1-|z|^2)}{\omega(1-|z|)} < \infty.$$

The following result establishes a sufficient condition for the general normal meromorphic functions to belong to $M^\#_{n,\omega}(p,q,s)$.

Theorem 3.4 *Let $1 \le p < \infty$, $-2 < q < \infty$, $0 \le s < \infty$ and $0 < r < 1$, and let $f \in \mathcal{N}_{n,\omega}$. Let $\alpha,\beta,\gamma,\delta \in \mathbb{R}$ such that $\alpha+\beta = q-p$, and $\gamma+\delta = s$. Then*

$$\begin{aligned} \|f\|^P_{M^\#_{n,\omega}(p,q,s)} &\lesssim \sup_{a\in\Delta}\int_\Delta \Big(\frac{1}{|D(z,r)|}\int_{D(z,r)} \chi_n(f(z),f(w))\Big(\frac{(1-|w|^2)^{\alpha/p}}{\omega(1-|w|)}\Big)\Big(\frac{(1-|z|^2)^{\beta/p}}{\omega(1-|z|)}\Big) \\ &\quad .\Big(\frac{(1-|\varphi_a(w)|^2)^{\gamma/p}}{\omega(1-|\varphi_a(w)|)}\Big)\Big(\frac{(1-|\varphi_a(z)|^2)^{\delta/p}}{\omega(1-|\varphi_a(z)|)}\Big)\, dA(w)\Big)^p\, dA(z). \end{aligned}$$

Proof: Let $z, w \in \widehat{\mathbb{C}}$, and define

$$F_n(z,w) = \begin{cases} \frac{w-z}{1+\overline{w}z} & if \;\; w \in \mathbb{C}. \\ \frac{1}{z} & if \;\; w = \infty. \end{cases}$$

A direct calculation shows that $|F_n(z,w)|^2 = \chi_n^2(z,w)/(1-\chi_n^2(z,w))$ for all $z, w \in \widehat{\mathbb{C}}$. Denote the pseudohyperbolic distance between the points z and w in Δ by $\rho(z,w) = |\varphi_z(w)|$. By the uniform (ρ,χ)-continuity of f,

there is an $r_1 \in (0,1)$ such that $\chi_n(f(z), f(w)) < C$, for $\rho(z,w) < r_1$ [13], where C is a positive constant. Then, it follows that

$$|F_n(f(z), f(w))| = \frac{\chi_n(f(z), f(w))}{\sqrt{1-\chi_n^2(f(z), f(w))}} < C\chi_n(f(z), f(w)) \tag{16}$$

for $\rho(z,w) < r_1$. Since $f \in M(\Delta)$, there is an $r_2 \in (0,1)$ such that the function $g_z(w) = F_n((fo\varphi_z)(w), f(z))$is analytic in $D(0, r_2) = \{w : \rho(0,w) = |w| < r_2\}$ for all $z \in \Delta$, and hence its Maclaurin series is of the form $\sum\limits_{k=1}^{\infty} a_k(z)w^k$ in $D(0, r_2)$.Therefore

$$\begin{aligned} f_n^{\#}(z)(1-|z|^2) &= |a_1| = \frac{2}{r^4}\Big|\int_{D(0,r)} \overline{w} g_z(w)\, dA(w)\Big| \\ &\leq \frac{2}{r^3}\int_{D(0,r)} \big|F_n((fo\varphi_z)(w), f(z))\big|\, dA(w) \end{aligned} \tag{17}$$

for any $r \in (0, r_2)$. Now let $r < \min\{r_1, r_2\}$. Then, we obtain that

$$\begin{aligned} I(f) &= \int_\Delta (f_n^{\#}(z)\big(\frac{(1-|z|^2)^p}{\omega(1-|z|)}\big)\big(\frac{(1-|z|)^{q-p}}{\omega(1-|z|)}\big)\big(\frac{(1-|\varphi_a(z)|^2)^s}{\omega(1-|\varphi_a(z)|)}\big)\, dA(z) \\ &\leq \int_\Delta \big(\frac{2}{r^3}\int_{D(0,r)} \big|F_n((fo\varphi_z)(w), f(z))\big|\, dA(u)\big)^p \big(\frac{(1-|z|)^{q-p}}{\omega(1-|z|)}\big)\big(\frac{(1-|\varphi_a(z)|^2)^s}{\omega(1-|\varphi_a(z)|)}\big)\, dA(z) \\ &= \int_\Delta \big(\frac{2}{r^3}\int_{D(z,r)} \big|F_n((f(u), f(z))\big||\varphi_z'(u)|^2\, dA(u)\big)^p \big(\frac{(1-|z|)^{q-p}}{\omega(1-|z|)}\big)\big(\frac{(1-|\varphi_a(z)|^2)^s}{\omega(1-|\varphi_a(z)|)}\big)\, dA(z) \\ &\leq \int_\Delta \big(\frac{C}{r^3}\int_{D(z,r)} \chi_n(f(u), f(z))|\varphi_z'(u)|^2\, dA(u)\big)^p \big(\frac{(1-|z|)^{q-p}}{\omega(1-|z|)}\big)\big(\frac{(1-|\varphi_a(z)|^2)^s}{\omega(1-|\varphi_a(z)|)}\big)\, dA(z). \end{aligned} \tag{18}$$

from which the assertion for $r < \min\{r_1, r_2\}$ follows by (16) and (17). If $r \geq \min\{r_1, r_2\}$, choose $c > 1$ such that $r^* = r/c < \min\{r_1, r_2\}$. Then, we easily obtain the assertion for r^*. To obtain the assertion for r , it remains to make the set of integration larger by replacing $D(z, r^*)$ by $D(z, r)$ and note that there is a constant C, depending only on c, such that $|D(z, r^*)| \geq C|D(z, r)|$ for all $z \in \Delta$.

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Maximum Norm Superconvergence of the Trilinear Block Finite Element

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In this article we discuss a pointwise superconvergence post-processing technique for the gradient of the trilinear block finite element for the Poisson equation with homogeneous Dirichlet boundary conditions over a fully uniform mesh of the three-dimensional domain Ω. First, the supercloseness of the gradients between the piecewise trilinear finite element solution u_h and the trilinear interpolant Πu is given. Secondly, we analyze a superconvergence post-processing scheme for the gradient of the finite element solution by using the Z-Z recovery technique, which shows that the recovered gradient of u_h is superconvergent to the gradient of the true solution u in the pointwise sense of the L^∞-norm. Finally, a numerical example is given.

1 Introduction

Superconvergence of the gradient for the finite element approximation is a phenomenon whereby the convergent order of the derivatives of the finite element solutions exceeds the optimal global rate. Up to now, superconvergence is still an active research topic; see, for example, Babuška and Strouboulis [1], Chen [2], Chen and Huang [3], Lin and Yan [4], Wahlbin [5] and Zhu and Lin [6] for overviews of this field. Nevertheless, how to obtain the superconvergent numerical solution is an issue to researchers. In general, it needs to use post-processing techniques to get recovered gradients with high order accuracy from the finite element solution. Usual post-processing techniques include interpolation technique, projection technique, average technique, extrapolation technique, superconvergence patch recovery (SPR) technique introduced by Zienkiewicz and Zhu [7–9] and polynomial patch recovery (PPR) technique raised by Zhang and Naga [10]. In previous works, for the linear tetrahedral element, Chen and Wang [11] obtained the recovered gradient with $\mathcal{O}(h^2)$ order accuracy in the average sense of the L^2-norm by using the SPR technique. Using the L^2-projection technique, in the average sense of the L^2-norm, Chen [12] got the recovered gradient with $\mathcal{O}(h^{1+min(\sigma,\frac{1}{2})})$ order accuracy. Goodsell [13] derived by using the average

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technique the pointwise superconvergence estimate of the recovered gradient with $\mathcal{O}(h^{2-\varepsilon})$ order accuracy. Brandts and Křížek [14] obtained by using the interpolation technique the recovered gradient with $\mathcal{O}(h^2)$ order accuracy in the average sense of the L^2-norm. Zhang [15, 16] gave the theoretical analysis for the SPR technique for the one-dimensional two points boundary value problem and two-dimensional Laplacian equations, which proved two orders higher than the optimal convergence rate of the finite element solution at the internal nodal points over uniform meshes. Zhang and Victory [17] presented the theoretical justification for superconvergence of the SPR technique for a general second-order elliptic equation over the quadrilateral meshes. Zhang and Zhu [18, 19] also analyzed the SPR technique in details as well as its applications to a posteriori error estimation. In this article, we consider a SPR recovery scheme by using the Z-Z technique, by which the pointwise superconvergence recovered gradient from the trilinear finite element approximation can be obtained. We shall use the letter C to denote a generic constant which may not be the same in each occurrence and also use the standard notations for the Sobolev spaces and their norms.

2 Maximum Norm Supercloseness

Suppose $\Omega \subset R^3$ is a rectangular block with boundary, $\partial\Omega$, consisting of faces parallel to the x-, y-, and z-axes. Moreover, Ω is partitioned into a uniform rectangulation $\mathcal{T}^h$ with mesh size $h \in (0, 1)$ such that $\bar{\Omega} = \bigcup_{e\in\mathcal{T}^h} \bar{e}$. We consider the following Poisson equation with homogeneous Dirichlet boundary value conditions

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

The corresponding weak form is

$$a(u\,,\,v) = (f\,,\,v),\ \forall\, v \in H_0^1(\Omega), \tag{2.2}$$

where

$$a(u\,,\,v) \equiv (\nabla u\,,\,\nabla v) = \int_\Omega \nabla u \cdot \nabla v\, dxdydz.$$

We introduce a trilinear polynomial space Q_1, namely

$$q(x,y,z) = \sum_{(i,j,k)\in I} a_{ijk} x^i y^j z^k,\ q \in Q_1,$$

where the indexing set I is as follows:

$$I = \{(i,j,k) | 0 \le i,j,k \le 1\}.$$

Denote the trilinear finite element space by

$$S_0^h(\Omega) = \left\{ v \in C(\bar{\Omega}) \bigcap H_0^1(\Omega) : v|_e \in Q_1(e),\, \forall\, e \in \mathcal{T}^h \right\}. \tag{2.3}$$

Thus the finite element method is to find $u_h \in S_0^h(\Omega)$ such that

$$a(u_h\,,\,v) = (f\,,\,v),\;\forall\, v \in S_0^h(\Omega).$$

Obviously, there is the following Galerkin orthogonality relation

$$a(u - u_h\,,\,v) = 0,\;\forall\, v \in S_0^h(\Omega). \tag{2.4}$$

Let the element

$$e = (x_e - h_e, x_e + h_e) \times (y_e - k_e, y_e + k_e) \times (z_e - d_e, z_e + d_e) \equiv I_1 \times I_2 \times I_3,$$

and let $\{l_j(x)\}_{j=0}^{\infty}$, $\{\tilde{l}_j(y)\}_{j=0}^{\infty}$, $\{\bar{l}_j(z)\}_{j=0}^{\infty}$ be the normalized orthogonal Legendre polynomial systems on $L^2(I_1)$, $L^2(I_2)$, and $L^2(I_3)$, respectively. It is easy to see that $\{l_i(x)\tilde{l}_j(y)\bar{l}_k(z)\}_{i,j,k=0}^{\infty}$ is the normalized orthogonal polynomial system on $L^2(e)$. Set

$$\omega_0(x) = \tilde{\omega}_0(y) = \bar{\omega}_0(z) = 1,\; \omega_{j+1}(x) = \int_{x_e-h_e}^{x} l_j(\xi)\,d\xi,$$

$$\tilde{\omega}_{j+1}(y) = \int_{y_e-k_e}^{y} \tilde{l}_j(\xi)\,d\xi,\; \bar{\omega}_{j+1}(z) = \int_{z_e-d_e}^{z} \bar{l}_j(\xi)\,d\xi,\;\; j \geq 0.$$

Define the trilinear interpolation operator of projection type by Π^e: $H^3(e) \to Q_1(e)$ such that

$$\Pi^e u(x, y, z) = \sum_{(i,j,k)\in I} \beta_{ijk}\omega_i(x)\tilde{\omega}_j(y)\bar{\omega}_k(z). \tag{2.5}$$

where $\beta_{000} = u(x_e{-}h_e, y_e{-}k_e, z_e{-}d_e)$, $\beta_{i00} = \int_{I_1} \partial_x u(x, y_e{-}k_e, z_e{-}d_e) l_{i-1}(x)\,dx$, $\beta_{0j0} = \int_{I_2} \partial_y u(x_e{-}h_e, y, z_e{-}d_e)\tilde{l}_{j-1}(y)\,dy$, $\beta_{00k} = \int_{I_3} \partial_z u(x_e{-}h_e, y_e{-}k_e, z)\bar{l}_{k-1}(z)\,dz$, $\beta_{ij0} = \int_{I_1\times I_2} \partial_x\partial_y u(x, y, z_e - d_e) l_{i-1}(x)\tilde{l}_{j-1}(y)\,dxdy$, $\beta_{0jk} = \int_{I_2\times I_3} \partial_y\partial_z u(x_e - h_e, y, z)\tilde{l}_{j-1}(y)\bar{l}_{k-1}(z)\,dydz$, $\beta_{i0k} = \int_{I_1\times I_3} \partial_x\partial_z u(x, y_e{-}k_e, z) l_{i-1}(x)\bar{l}_{k-1}(z)\,dxdz$, $\beta_{ijk} = \int_e \partial_x\partial_y\partial_z u\, l_{i-1}(x)\tilde{l}_{j-1}(y)\bar{l}_{k-1}(z)\,dxdydz,\;\; i, j, k \geq 1.$
In addition, we define $(\Pi u)|_e = \Pi^e u$. Thus we have the global interpolation operator of projection type Π: $H^3(\Omega) \to S_0^h(\Omega)$. In [20], we obtained the following supercloseness estimate
Lemma 2.1. Let $\{\mathcal{T}^h\}$ be a regular family of rectangular partitions of Ω, and $u \in W^{3,\infty}(\Omega)\bigcap H_0^1(\Omega)$. For u_h and Πu, the trilinear block finite element approximation and the corresponding interpolant of projection type to u, respectively. Then we have the following supercloseness estimate

$$|u_h - \Pi u|_{1,\infty,\Omega} \leq Ch^2\,|\ln h|^{\frac{4}{3}}\,\|u\|_{3,\infty,\Omega}. \tag{2.6}$$

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3 Maximum Norm Superconvergence

SPR is a gradient recovery method introduced by Zienkiewicz and Zhu. This method is now widely used in engineering practices for its robustness in a posterior error estimation and its efficiency in computer implementation.

For $v \in S_0^h(\Omega)$, we denote by R_x the SPR-recovery operator (or Z-Z recovery operator) with respect to the x-derivative, and begin by defining the point values of $R_x v$ at the element nodes. After the recovered derivative values at all nodes are obtained, we construct a piecewise trilinear interpolant by using these values to obtain a global recovered derivative, namely SPR-recovery derivative $R_x v$. Obviously $R_x v \in S_0^h(\Omega)$. Similarly, we can define by R_y and R_z the recovered derivatives with respect to the y-derivative and the z-derivative, respectively. Consequently, we get a recovered gradient operator $R_h = (R_x, R_y, R_z)$. In the following, we mainly discuss the recovery operator R_x and its superconvergence properties. The superconvergence properties of R_y and R_z can be similarly derived.

Let us first assume N is an interior node of the partition $\mathcal{T}^h$, and denote by ω the element patch around N containing eight elements (see Fig.1).

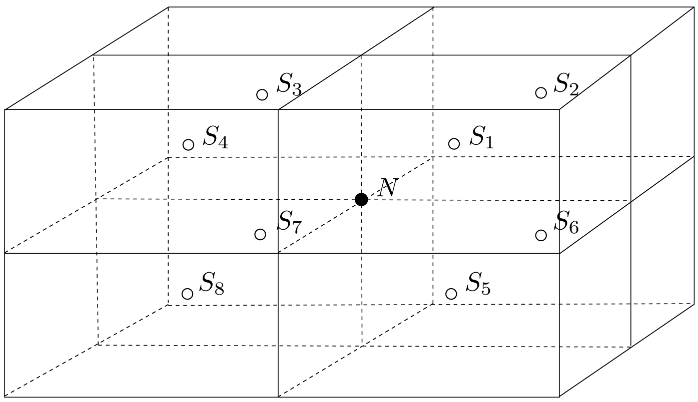

FIG. 1. Element Patch Containing Eight Elements

Under the local coordinate system centered N, we let S_j be the barycenter of an element $e_j \subset \omega$, $j = 1, 2, \cdots, 8$. SPR uses the discrete least-squares fitting to seek linear function $p \in P_1(\omega)$, such that

$$|||p - \partial_x v||| = \min_{g \in P_1(\omega)} |||g - \partial_x v|||, \tag{3.1}$$

where $|||w||| = (\sum_{j=1}^{8} |w(S_j)|^2)^{\frac{1}{2}}$. Obviously, for $w \in P_1(\omega)$, we have

$$|||w||| = 0 \Longleftrightarrow w = 0$$

. It is easy to verify that the problem (3.1) is equivalent to the following problem

$$\sum_{j=1}^{8}[p(S_j)-\partial_x v(S_j)]g(S_j)=0,\ \forall g\in P_1(\omega). \tag{3.2}$$

Then we define $R_x v(N)=p(0,0,0)$. If N is a node on the boundary, $\partial\Omega$, of Ω, we can calculate $R_x v(N)$ by the linear extrapolation from the values of $R_x v$ already obtained at two neighboring interior nodes, N_1 and N_2, namely

$$R_x v(N)=2R_x v(N_1)-R_x v(N_2). \tag{3.3}$$

Lemma 3.1. Let ω be the element patch around an interior node N, S_j the barycenter of the element $e_j\subset\omega,\ j=1.\cdots.8$, and Π the trilinear interpolation operator of projection type. For every $u\in P_2(\omega)$, we have

$$\partial_x(u-\Pi u)(S_j)=0. \tag{3.4}$$

Proof. Obviously, S_j is a Gauss point of the element $e_j\subset\omega$. From the definition of the operator Π,

$$u-\Pi u=\left(\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=2}^{\infty}+\sum_{i=0}^{1}\sum_{j=2}^{\infty}\sum_{k=0}^{\infty}+\sum_{i=2}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\right)\beta_{ijk}\omega_i(x)\tilde{\omega}_j(y)\bar{\omega}_k(z).$$

By the representation of the coefficient β_{ijk} and the orthogonality of the *Legendre* polynomial system, we obtain for $u\in P_2(\omega)$,

$$\partial_x(u-\Pi u)(S_j)=0,$$

which is the desired result (3.4).

Lemma 3.2. Let ω be the element patch around an interior node N and Π the trilinear interpolation operator of projection type. For every $u\in P_2(\omega)$, we have

$$\partial_x u-R_x\Pi u=0\ \text{ in }\omega. \tag{3.5}$$

Proof. From (3.4) and the definition (3.1) of the recovery operator R_x, we have for $u\in P_2(\omega)$,

$$R_x u=R_x\Pi u. \tag{3.6}$$

Since $u\in P_2(\omega)$, thus $\partial_x u\in P_1(\omega)$. So we obtain

$$R_x u=\partial_x u. \tag{3.7}$$

Combining (3.6) and (3.7) yields the desired result (3.5).

Lemma 3.3. For $\Pi u\in S_0^h(\Omega)$ the trilinear interpolant of projection type to u, the solution of (2.2), and R_x the x-derivative recovered operator by SPR, we have the superconvergent estimate

$$|\partial_x u-R_x\Pi u|_{0,\infty,\Omega}\le Ch^2\|u\|_{3,\infty,\Omega}. \tag{3.8}$$

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Proof. By the triangle inequality, the norms equivalence of the finite-dimensional space, and the inverse property, we have

$$\begin{aligned} & |\partial_x u - R_x\Pi u|_{0,\infty,\Omega} = |\partial_x u - R_x\Pi u|_{0,\infty,e} \le |\partial_x u|_{0,\infty,e} + |R_x\Pi u|_{0,\infty,e} \\ \le\ & C\left(|\partial_x u|_{0,\infty,e} + |||R_x\Pi u|||\right) \le C\left(|\partial_x u|_{0,\infty,e} + |||\partial_x\Pi u|||\right) \\ \le\ & C\left(|\partial_x u|_{0,\infty,\omega} + |\partial_x\Pi u|_{0,\infty,\omega}\right) \le C\left(|\partial_x u|_{0,\infty,\omega} + h^{-1}|u|_{0,\infty,\omega}\right), \end{aligned} \tag{3.9}$$

where ω is an element patch containing the element e. Let $u_I \in P_2(\omega)$ be a quadratic interpolant to u. From (3.5) and (3.9), we obtain by using the interpolation error estimate,

$$\begin{aligned} |\partial_x u - R_x\Pi u|_{0,\infty,\Omega} &= |\partial_x(u-u_I) - R_x\Pi(u-u_I)|_{0,\infty,e} \\ &\le C\left(|\partial_x(u-u_I)|_{0,\infty,\omega} + h^{-1}|u-u_I|_{0,\infty,\omega}\right), \\ &\le Ch^2\|u\|_{3,\infty,\Omega}. \end{aligned}$$

This proves the statement.

As for the y-derivative recovery operator R_y and the z-derivative recovery operator R_z, we have the following results similar to (3.8).

$$|\partial_y u - R_y\Pi u|_{0,\infty,\Omega} \le Ch^2\|u\|_{3,\infty,\Omega}. \tag{3.10}$$

$$|\partial_z u - R_z\Pi u|_{0,\infty,\Omega} \le Ch^2\|u\|_{3,\infty,\Omega}. \tag{3.11}$$

Set $R_h = (R_x, R_y, R_z)$. Combining (3.8), (3.10) and (3.11) yields

$$|\nabla u - R_h\Pi u|_{0,\infty,\Omega} \le Ch^2\|u\|_{3,\infty,\Omega}. \tag{3.12}$$

In the following, we give the main result of this article.

Theorem 3.1. For $u_h \in S_0^h(\Omega)$ the trilinear block finite element approximation to u, the solution of (2.2), and R_h the gradient recovered operator by SPR, we have the superconvergent estimate

$$|\nabla u - R_h u_h|_{0,\infty,\Omega} \le Ch^2|\ln h|^{\frac{4}{3}}\|u\|_{3,\infty,\Omega}.$$

Proof. Using the triangle inequality and the norms equivalence of the finite-dimensional space, we have

$$\begin{aligned} |\nabla u - R_h u_h|_{0,\infty,\Omega} &\le |R_h(u_h-\Pi u)|_{0,\infty,\Omega} + |\nabla u - R_h\Pi u|_{0,\infty,\Omega} \\ &= |R_h(u_h-\Pi u)|_{0,\infty,e} + |\nabla u - R_h\Pi u|_{0,\infty,\Omega} \\ &\le C\left(|||R_h(u_h-\Pi u)||| + |\nabla u - R_h\Pi u|_{0,\infty,\Omega}\right) \\ &\le C\left(|||\nabla(u_h-\Pi u)||| + |\nabla u - R_h\Pi u|_{0,\infty,\Omega}\right) \\ &\le C\left(|u_h-\Pi u|_{1,\infty,\Omega} + |\nabla u - R_h\Pi u|_{0,\infty,\Omega}\right). \end{aligned} \tag{3.13}$$

Combining (2.6), (3.12) and (3.13) yields

$$|\nabla u - R_h u_h|_{0,\infty,\Omega} \le Ch^2|\ln h|^{\frac{4}{3}}\|u\|_{3,\infty,\Omega}.$$

This proves the statement.

4 A Numerical Example

Example 1. Consider the following Poisson's equation:

$$\begin{cases} -\Delta u = f & \text{in } \Omega = [0,1] \times [0,1] \times [0,1], \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$\begin{aligned} f &= (-e^x(e^y - (e-1)y - 1) - e^y(e^x - (e-1)x - 1) \\ &\quad + \pi^2(e^x - (e-1)x - 1)(e^y - (e-1)y - 1)) \sin(\pi z). \end{aligned}$$

The exact solution is

$$u = (e^x - (e-1)x - 1)(e^y - (e-1)y - 1) \sin(\pi z).$$

Let u_h be the trilinear block finite element approximation to the exact solution u and $N_0 = (0.5, 0.5, 0.5)$. We solve Example 1 and obtain the following numerical results:

Table 4.1 Results of the derivatives post-processing at the interior vertex N_0

h	$\lvert\partial_x u(N_0) - R_x u_h(N_0)\rvert$
0.25	1.8364e-003
0.125	4.0003e-004
0.0625	9.6873e-005

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HYERS-ULAM STABILITY OF AN ADDITIVE FUNCTIONAL INEQUALITY

MING FANG AND DONGHE PEI*

Abstract. In this paper, we prove that the generalized Hyers-Ulam stability of the additive functional inequality

$$\|f(2x+y+2z)+f(2x+3y+3z)+f(4x+4y+3z)\| \leq \|8f(x+y+z)\|$$

in β-homogeneous F-spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [32] concerning the stability of group homomorphisms. Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [22] for linear mappings by considering an unbounded Cauchy difference. The paper of Rassias [22] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [9] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach. The stability problems for several functional equations or inequations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2]–[8],[10], [12]–[15], [21]–[24],[25]-[30],[34]).

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) d(x,y)=0 if and only if x=y;
(2) d(x,y)=d(y,x) for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.1 (see[6],[7]). *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for*

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each given element $x \in X$, either

$$d(J^n x, J^{n+1}x) = \infty \tag{1.1}$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty$, *for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X | d(J^{n_0}x, y) < \infty\}$;*
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all $y \in Y$.*

By the using fixed point method, the stability problems of several functional inequations have been extensively investigated by a number of authors(see[5][6][14][17]-[18]).

We recall some basic facts concerning β-homogeneous F-spaces.

Definition 1.2. Let X be a linear space. A nonnegative valued function $\|\cdot\|$ is an F-norm if it satisfies the following conditions:
(FN_1) $\|x\| = 0$ if and only if $x = 0$;
(FN_2) $\|\lambda x\| = \|x\|$ for all $x \in X$ and all λ with $|\lambda| = 1$;
(FN_3) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$;
(FN_4) $\|\lambda_n x\| \to 0$ provided $\lambda_n \to 0$;
(FN_5) $\|\lambda x_n\| \to 0$ provided $\|x_n\| \to 0$.
Then $(X, \|\cdot\|)$ is called an F^*-space. An F-space is a complete F^*-space.

A F-norm is called β-homogeneous ($\beta > 0$) if $\|tx\| = |t|^\beta \|x\|$ for all $x \in X$ and all $t \in \mathbb{R}$ (see [31]).

2. HYers-Ulam Stability In β-homogeneous F-spaces

From now on , Let $\mathcal{X}$ be a normed linear space and $\mathcal{Y}$ a β-homogeneous F-spaces.

This paper,we prove that the generalized Hyers-Ulam stability of the additive functional inequality

$$\|f(2x + y + 2z) + f(2x + 3y + 3z) + f(4x + 4y + 3z)\| \leq \|8f(x + y + z)\|$$

in β-homogeneous F-spaces.

Lemma 2.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping with $f(0) = 0$. Then it is additive if and only if it satisfies*

$$\|f(2x + y + 2z) + f(2x + 3y + 3z) + f(4x + 4y + 3z)\| \leq \|8f(x + y + z)\| \tag{2.1}$$

for all $x, y, z \in \mathcal{X}$.

Proof. If f is additive, then clearly

$$\|f(2x + y + 2z) + f(2x + 3y + 3z) + f(4x + 4y + 3z)\| = \|8f(x + y + z)\|$$

for all $x, y, z \in \mathcal{X}$.

Assume that f satisfies (2.1). Suppose that $f(0) = 0$. putting $z = 0$ and replacing y by $-x$ in (2.1), we get

$$\|f(x) + f(-x)\| \leq \|8f(0)\| = 8^\beta \|f(0)\| = 0$$

and so $f(-x) = -f(x)$ for all $x \in \mathcal{X}$. Replacing y by $-x - z$ in (2.1), we have

$$\|f(-y) + f(-x) + f(x+y)\| \leq 0$$

for all $x, y \in \mathcal{X}$. We obtain

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in \mathcal{X}$. □

Theorem 2.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping with $f(0) = 0$. If there is a function $\varphi : X^3 \to [0, \infty)$ such that*

$$\begin{aligned} &\|f(2x + y + 2z) + f(2x + 3y + 3z) + f(4x + 4y + 3z)\| \\ &\leq \|8f(x + y + z)\| + \varphi(x, y, z) \end{aligned} \tag{2.2}$$

and

$$\widetilde{\varphi}(x, y, z) := \sum_{i=0}^{\infty} \frac{1}{2^{\beta j}} \varphi\left((-2)^j x, (-2)^j y, (-2)^j z\right) < \infty \tag{2.3}$$

for all $x, y, z \in \mathcal{X}$, then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - A(x)\| \leq \widetilde{\varphi}(-x, -x, 2x) \tag{2.4}$$

for all $x \in \mathcal{X}$.

Proof. Letting $y = x$ and $z = -2x$ in (2.2), we get

$$\|2f(-x) + f(2x)\| \leq \varphi(x, x, -2x)$$

for all $x \in \mathcal{X}$. Thus

$$\left\| f(x) - \frac{f(-2x)}{-2} \right\| \leq \frac{1}{2^\beta} \varphi(-x, -x, 2x)$$

for all $x \in \mathcal{X}$.

Hence one may have the following formula for positive integers m, l with $m > l$,

$$\begin{aligned} &\left\| \frac{1}{(-2)^l} f\left((-2)^l x\right) - \frac{1}{(-2)^m} f\left((-2)^m x\right) \right\| \\ &\leq \sum_{i=l}^{m-1} \frac{1}{2^{\beta i}} \varphi\left(-(-2)^i x, -(-2)^i x, (-2)^i 2x\right) \end{aligned} \tag{2.5}$$

for all $x \in \mathcal{X}$. It follows from (2.5) that the sequence $\left\{\frac{f((-2)^k x)}{(-2)^k}\right\}$ is a Cauchy sequence for all $x \in \mathcal{X}$. Since $\mathcal{Y}$ is an F-space, the sequence $\left\{\frac{f((-2)^k x)}{(-2)^k}\right\}$ converges. So one may define the mapping $A : \mathcal{X} \to \mathcal{Y}$ by

$$A(x) := \lim_{k\to\infty} \left\{\frac{f((-2)^k x)}{(-2)^k}\right\}, \quad \forall x \in \mathcal{X}.$$

Taking $m = 0$ and letting l tend to ∞ in (2.5), we have the inequality (2.4).

It follows from (2.2) that

$$\begin{aligned}
&\|A(2x + y + 2z) + A(2x + 3y + 3z) + A(4x + 4y + 3z)\| \\
&= \lim_{k\to\infty} \left|\frac{1}{(-2)^{k\beta}}\right| \left\|f((-2)^k(2x + y + 2z)) + f((-2)^k(2x + 3y + 3z))\right. \\
&\left.+f((-2)^k(4x + 4y + 3z))\right\| \qquad (2.6)\\
&\le \lim_{k\to\infty} \left|\frac{1}{(-2)^{k\beta}}\right| \left\|8f((-2)^k(x + y + z))\right\| + \lim_{k\to\infty} \left|\frac{1}{(-2)^{k\beta}}\right| \varphi((-2)^k x, (-2)^k y, (-2)^k z) \\
&\le \|8A(x + y + z)\|
\end{aligned}$$

for all $x, y, z \in \mathcal{X}$. One see that A satisfies the inequality (2.1) and so it is additive by Lemma (2.1).

Now, we show that the uniqueness of A. Let $T : X \to Y$ be another additive mapping satisfying (2.4). Then one has

$$\begin{aligned}
\|A(x) - T(x)\| &= \left\|\frac{1}{(-2)^k} A\left((-2)^k x\right) - \frac{1}{(-2)^k} T\left((-2)^k x\right)\right\| \\
&\le \frac{1}{2^{k\beta}} \left(\left\|A\left((-2)^k x\right) - f\left((-2)^k x\right)\right\|\right. \\
&\left.+ \left\|T\left((-2)^k x\right) - f\left((-2)^k x\right)\right\|\right) \\
&\le 2\frac{1}{2^{k\beta}} \widetilde{\varphi}\left(-(-2)^k x, -(-2)^k x, (-2)^k 2x\right)
\end{aligned}$$

which tends to zero as $k \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. □

Theorem 2.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping with $f(0) = 0$. If there is a function $\varphi : X^3 \to [0, \infty)$ satisfying (2.2) such that*

$$\widetilde{\varphi}(x, y, z) := \sum_{j=1}^{\infty} 2^{\beta j} \varphi\left(\frac{x}{(-2)^j}, \frac{y}{(-2)^j}, \frac{z}{(-2)^j}\right) < \infty \qquad (2.7)$$

for all $x, y, z \in \mathcal{X}$, then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - A(x)\| \le \widetilde{\varphi}(x, x, -2x) \qquad (2.8)$$

for all $x \in \mathcal{X}$.

Proof. Letting $y = x$ and $z = -2x$ in (2.2), we get

$$\|2f(-x) + f(2x)\| \leq \varphi\left(\frac{x}{2}, \frac{x}{2}, -x\right)$$

for all $x \in \mathcal{X}$. Thus

$$\left\|f(x) - (-2)f\left(\frac{x}{-2}\right)\right\| \leq \varphi\left(\frac{x}{2}, \frac{x}{2}, -x\right)$$

for all $x \in \mathcal{X}$.

Next, we can prove that the sequence $\{(-2)^n f\left(\frac{x}{(-2)^n}\right)$ is a Cauchy sequence for all $x \in \mathcal{X}$,and define a mapping $A : \mathcal{X} \to \mathcal{Y}$ by

$$A(x) := \lim_{n\to\infty} (-2)^n f\left(\frac{x}{(-2)^n}\right)$$

for all $x \in \mathcal{X}$ that is similar to the corresponding part of the proof of Theorem (2.2). □

3. Hyers-Ulam Stability for Fixed Point Methods

Now, using fixed point theorem, we investigate the Hyers-Ulam stability of the functional inequality (2.1) in β-homogeneous F-spaces.

Theorem 3.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping for which there exists a function $\varphi : \mathcal{X}^3 \to [0, \infty)$ such that*

$$\begin{aligned} &\|f(2x + y + 2z) + f(2x + 3y + 3z) + f(4x + 4y + 3z)\| \\ &\leq \|8f(x + y + z)\| + \varphi(x, y, z) \end{aligned} \tag{3.1}$$

for all $x, y, z \in X$. If there exists $L \in (0, 1)$ such that

$$\varphi(x, y, z) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \tag{3.2}$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique additive mapping $H : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - H(x)\| \leq \frac{1}{2^\beta(1 - L)}\varphi(-x, -x, 2x) \tag{3.3}$$

for all $x \in X$.

Proof. It follows from $\varphi(x, y, z) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)$ that

$$\lim_{j\to\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y, 2^j z) = 0$$

for all $x, y, z \in \mathcal{X}$.

Consider the set

$$A := \{g : \mathcal{X} \to \mathcal{Y}\}$$

and introduce the *generalized metric* on A:

$$d(g, h) = \inf\{C \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq C\varphi(-x, -x, 2x), \forall x \in \mathcal{X}\}.$$

It is easy to show that (A, d) is complete.

Now we consider the linear mapping $J : A \to A$ such that

$$Jg(x) := \frac{1}{-2}g(-2x)$$

for all $x \in \mathcal{X}$.

By [6, Theorem 3.1]

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in A$.

Letting $y = x$ and $z = -2x$ in (3.1), we get

$$\left\| f(x) - \frac{1}{-2}f(-2x) \right\| \leq \frac{1}{2^\beta}\varphi(-x, -x, 2x)$$

for all $x \in \mathcal{X}$.

Hence $d(f, Jf) \leq \frac{1}{2^\beta}$.

By the Theorem (1.1), there exists a mapping $H : \mathcal{X} \to \mathcal{Y}$ such that

(1) H is a fixed point of J, that is

$$\frac{1}{-2}H(-2x) = H(x) \tag{3.4}$$

for all $x \in \mathcal{X}$. The mapping H is a unique fixed point of J in the set

$$B = \{g \in A : d(f, g) < \infty\}.$$

This implies that H is a unique mapping satisfying (3.4) such that there exists $C \in (0, \infty)$ satisfying

$$\|H(x) - f(x)\| \leq C\varphi(-x, -x, 2x)$$

for all $x \in \mathcal{X}$.

(2) $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the inequality

$$\lim_{n\to\infty} \frac{1}{(-2)^n} f((-2)^n x) = H(x)$$

for all $x \in \mathcal{X}$.

(3) $d(f, H) \leq \frac{1}{1-L}d(f, Jf)$, which implies the inequality

$$d(f, H) \leq \frac{1}{2^\beta(1 - L)}.$$

This implies that the inequality (3.3)holds.

Next, we show that $H(x)$ is an additive mapping.

$$\begin{aligned}
&\|H(2x+y+2z)+H(2x+3y+3z)+H(4x+4y+3z)\| \\
&=\lim_{k\to\infty}\left|\frac{1}{(-2)^{k\beta}}\right|\|f((-2)^k(2x+y+2z))+f((-2)^k(2x+3y+3z)) \\
&+f((-2)^k(4x+4y+3z))\| \qquad (3.5)\\
&\le \lim_{k\to\infty}\left|\frac{1}{(-2)^{k\beta}}\right|\|8f((-2)^k(x+y+z))\|+\lim_{k\to\infty}\left|\frac{1}{(-2)^{k\beta}}\right|\varphi((-2)^kx,(-2)^ky,(-2)^kz) \\
&\le \|8H(x+y+z)\|
\end{aligned}$$

for all $x, y, z \in \mathcal{X}$. □

Theorem 3.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping for which there exists a function $\varphi : \mathcal{X}^3 \to [0,\infty)$ satisfying (3.1) If there exists an $L \in (0,1)$ such that*

$$\varphi(x,y,z) \le \frac{1}{2}L\varphi(2x,2y,2z) \tag{3.6}$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique additive mapping $H : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x)-H(x)\| \le \frac{L}{2(1-L)}\varphi(-x,-x,2x) \tag{3.7}$$

for all $x \in X$.

Proof. It follows from $\varphi(x,y,z) \le 2L\varphi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right)$ that

$$\lim_{j\to\infty} 2^j\varphi\left(\frac{1}{2^j}x,\frac{1}{2^j}y,\frac{1}{2^j}z\right)=0$$

for all $x, y, z \in \mathcal{X}$.

Consider the set

$$A := \{g : \mathcal{X} \to \mathcal{Y}\}$$

and introduce the *generalized metric* on A:

$$d(g,h)=\inf\{C\in\mathbb{R}_+ : \|g(x)-h(x)\| \le C\varphi(x,x,-2x), \forall x\in\mathcal{X}\}.$$

It is easy to show that (A, d) is complete.

Now we consider the linear mapping $J : A \to A$ such that

$$Jg(x) := -2g\left(-\frac{x}{2}\right)$$

for all $x \in \mathcal{X}$.

By [6, Theorem 3.1]

$$d(Jg,Jh) \le Ld(g,h)$$

for all $g, h \in A$.

Letting $y = x$ and $z = x + y$ in (3.1), we get

$$\left\|f(x)-(-2)f\left(-\frac{1}{2}x\right)\right\| \le \varphi\left(\frac{x}{2},\frac{x}{2},-x\right) \le \frac{L}{2}\varphi(x,x,-2x)$$

for all $x \in \mathcal{X}$.

Hence $d(f, Jf) \leq \frac{L}{2}$. The rest of the proof is similar to the corresponding part of the proof of Theorem 3.1.

□

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Characterization of a Class of Differential Equations

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Abstract

This paper deals with a characterization of nonlinear systems of the form $\dot{x}_\gamma(t) = f\left(x_\gamma(t), u(t/\gamma)\right)$ when the parameter $\gamma \to \infty$. In particular, we are interested in the uniform convergence of the sequence of functions $x_\gamma\left(\gamma t\right)$. Necessary conditions and sufficient ones are derived for this uniform convergence to happen.

Keywords: nonlinear systems, consistent operator, uniform convergence

1 Introduction

Hysteresis is a nonlinear behavior encountered in a wide variety of processes including biology, optics, electronics, ferroelectricity, magnetism, mechanics, structures, among other areas. The detailed modeling of hysteresis systems using the laws of Physics is an arduous task, and the obtained models are often too complex to be used in applications. For this reason, alternative models of these complex systems have been proposed [15, 1, 8, 6, 9]. These models do not come, in general, from the detailed analysis of the physical behavior of the systems with hysteresis. Instead, they combine some physical understanding of the system along with some kind of black-box modeling.

This way of describing hysteresis systems led to the proliferation of hysteresis models in the last two decades. A search in the Web of Knowledge database gives more than 2000 publications. The question that arises naturally is: do these research works describe really hysteresis phenomena? In other words, does the researcher who proposes a new hysteresis model have a mathematical rule to decide whether the model they propose is indeed a hysteresis one?

Surprisingly enough, such a rule exists only for a limited number of hysteresis processes: those that possess the so-called rate-independence property. This property states that, under a time-scale change, the relationship output versus input is unchanged. Hysteresis systems that are rate-independent are listed in the survey paper [10]. However, in the last two decades, researchers have acknowledged the importance of rate-dependent processes in applications [4, 3, 2]. For this reason, a recent effort [5] proposed a mathematical framework that

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proposes a rule to decide whether or not a system may be hysteretic. The rule proposed in [5] shows that, for an input/output system with input $u(t/\gamma)$ and output $x_\gamma(t)$, the convergence of the sequence of functions $t \to x_\gamma(\gamma t)$ as $\gamma \to \infty$ is a necessary condition for the hysteresis. The previous formulation is used to study the hysteresis behavior of the generalized Duhem model [11] and the LuGre friction model [12].

In the present paper, we consider the differential equation $\dot{x} = f(x, u)$. Our objective is to derive necessary conditions and also sufficient ones for the uniform convergence of the sequence of functions $t \to x_\gamma(\gamma t)$.

This paper is organized as follows. Section 2 presents the system of study and the assumptions under which the study is performed. Sections 3 and 4 present; respectively, necessary conditions and sufficient ones for the uniform convergence of the sequence of functions $x_\gamma(\gamma t)$ as $\gamma \to \infty$. Conclusions are given in Section 5.

2 Problem Statement

The class of systems under study is

$$\dot{x}(t) = f\big(x(t), u(t)\big),\ t \geq 0, \tag{1}$$

$$x(0) = x_0, \tag{2}$$

where initial condition x_0 and state $x(t)$ take value in $\mathbb{R}^m$, and input $u \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ for some strictly positive integers n and m. The mapping $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ is a well-defined continuous function. Because of the continuity of the right-hand side of (1), the system (1)-(2) has a maximal solution which is defined on an interval of the form $[0, \omega)$, $\omega > 0$ [14, p. 67–70]. In this paper, we assume that the system (1)-(2) has a unique Carathéodory solution for all $(u, x_0) \in L^\infty(\mathbb{R}_+, \mathbb{R}^n) \times \mathbb{R}^m$.

Consider the time scale change $s_\gamma(t) = t/\gamma, \forall \gamma > 0, \forall t \geq 0$. When the input $u \circ s_\gamma$ is used instead of u, system (1)-(2) becomes

$$\dot{x}_\gamma(t) = f\big(x_\gamma(t), u \circ s_\gamma(t)\big),\ t \geq 0, \tag{3}$$

$$x_\gamma(0) = x_0, \tag{4}$$

which can be written for all $\gamma > 0$ as

$$\sigma_\gamma(t) = x_0 + \gamma \int_0^t f\big(\sigma_\gamma(\tau), u(\tau)\big) d\tau,\ \forall t \in [0, \omega_\gamma), \tag{5}$$

where $\sigma_\gamma = x_\gamma \circ s_{1/\gamma}$ and $[0, \omega_\gamma)$ is the maximal interval for the existence of solutions σ_γ.

We seek necessary conditions and also sufficient conditions for the uniform convergence of the sequence of functions σ_γ.

3 Necessary Conditions

Our aim in this section is to derive necessary conditions for the uniform covergence of the sequence of functions σ_γ.

Lemma 3.1. *Assume that the maximal solution of system (1)-(2) is defined on $\mathbb{R}_+$ for all $(u, x_0) \in L^\infty(\mathbb{R}_+, \mathbb{R}^n) \times \mathbb{R}^m$. Suppose that there exists a function $h: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$|x(t)| \leq h\big(|x_0|, \|u\|_\infty\big), \forall t \geq 0, \tag{6}$$

for each initial state $x_0 \in \mathbb{R}^n$ and each input $u \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$. Assume that there exists a function $q_u \in L^\infty(\mathbb{R}_+, \mathbb{R}^m) \cap C^0(\mathbb{R}_+, \mathbb{R}^m)$ such that $\lim_{\gamma\to\infty} \|\sigma_\gamma - q_u\|_\infty = 0$. Then, we have $f\big(x_0, u(0)\big) = 0$, $q_u(0) = x_0$, and $f\big(q_u(t), u(t)\big) = 0$, $\forall t \geq 0$.

Proof. From the fact that $\|u\|_\infty = \|u \circ s_\gamma\|, \forall \gamma > 0$ and Inequality (6) it comes that

$$\|x_\gamma\|_\infty \leq h(|x_0|, \|u\|_\infty) = a. \forall \gamma > 0,$$

Thus, we get from the continuity of σ_γ that

$$|\sigma_\gamma(t)| \leq a, \forall t \geq 0, \ \forall \gamma > 0. \tag{7}$$

Inequality (7) along with the continuity of function f and the boundedness of the input u imply that there exists a constant $r > 0$ independent of γ, such that $|f(\sigma_\gamma(\tau), u(\tau))| \leq r$, $\forall \tau \geq 0$, $\forall \gamma > 0$. This means that we can apply the Dominated Lebesgue Theorem in Equation (5) and get

$$\lim_{\gamma\to\infty} \int_0^t f(\sigma_\gamma(\tau), u(\tau))\, d\tau = \int_0^t f(q_u(\tau), u(\tau))\, d\tau, \ \forall t \geq 0, \tag{8}$$

where the continuity of f and the fact that $\lim_{\gamma\to\infty} \|\sigma_\gamma - q_u\|_\infty = 0$ are used. By Equation (7) we have $\|\sigma_\gamma - x_0\|_\infty / \gamma \to 0$ as $\gamma \to \infty$. Thus, we obtain from (5) and (8) that

$$\int_0^t f(q_u(\tau), u(\tau))\, d\tau = 0, \ \forall t \geq 0,$$

which gives $f\big(q_u(t), u(t)\big) = 0$ for almost all $t \geq 0$. From the continuity of functions f, q_u, and u it comes that

$$f\big(q_u(t), u(t)\big) = 0, \quad \text{for all } t \geq 0. \tag{9}$$

Since $\sigma_\gamma(0) = x_0$, $\forall \gamma > 0$ it comes that

$$q_u(0) = x_0. \tag{10}$$

Finally, taking $t = 0$ in (9) and using (10) provides the necessary condition

$$f\big(x_0, u(0)\big) = 0, \tag{11}$$

which completes the proof. □

Remark 1. Once chosen an input u, the term $u(0)$ is given so that any initial condition x_0 for which we have $\lim_{\gamma\to\infty} \|\sigma_\gamma - q_u\|_\infty = 0$ should satisfy (11).

4 Sufficient Conditions

In this section, we derive sufficient conditions to ensure that the sequence of functions σ_γ converges uniformly as $\gamma \to \infty$.

Definition 4.1. *[7] A continuous function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathcal{K}_\infty$ if it is strictly increasing, satisfies $\beta(0) = 0$, and $\lim_{t\to\infty} \beta(t) = \infty$.*

Lemma 4.1. *[11] Consider a function $z : [0, \omega) \subset \mathbb{R}_+ \to \mathbb{R}_+$, where ω may be infinite. Assume the following*

(i) *The function z is absolutely continuous on each compact subset of $[0, \omega)$.*

(ii) *There exist $z_1, z_2 \geq$ such that $z_1, z(0) < z_2$ and $\dot{z}(t) \leq 0$ for almost all $t \in [0, \omega)$ that satisfy $z_1 < z(t) < z_2$.*

Then, $z(t) \leq \max\big(z(0), z_1\big)$, $\forall t \in [0, \omega)$.

Corollary 4.1. *Consider a function $z : [0, \omega) \subset \mathbb{R}_+ \to \mathbb{R}_+$, where ω may be infinite. Assume the following*

(i) *The function z is absolutely continuous on each compact subset of $[0, \omega)$.*

(ii) *There exist a class $\mathcal{K}_\infty$ function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ and $z_1, z_2, z_3 \geq 0$ such that $\max\big(\beta^{-1}(z_3), z_1, z(0)\big) < z_2$, and $\dot{z}(t) \leq -\beta(z(t)) + z_3$ for almost all $t \in [0, \omega)$ that satisfy $z_1 < z(t) < z_2$.*

Then, $z(t) \leq \max\big(z(0), z_1, \beta^{-1}(z_3)\big)$, $\forall t \in [0, \omega)$.

Proof. We have $\dot{z}(t) \leq 0$ for almost all $t \in [0, \omega)$ that satisfy $\max\big(\beta^{-1}(z_3), z_1\big) < z(t) < z_2$, and hence the result follows directly from Lemma 4.1. □

Lemma 4.2. *Assume that there exists $q_u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ such that*

$$f\big(q_u(t), u(t)\big) = 0, \ \forall t \geq 0, \tag{12}$$
$$q_u(0) = x_0. \tag{13}$$

Define $y_\gamma : \mathbb{R}_+ \to \mathbb{R}^m$ as

$$y_\gamma(t) = \sigma_\gamma(t) - q_u(t) = x_\gamma(\gamma t) - q_u(t), \forall \gamma > 0, \tag{14}$$

for all $t \in [0, \omega_\gamma)$. Suppose that we can find a continuously differentiable function $V : \mathbb{R}^m \to \mathbb{R}_+$ that satisfies the following:

(i) *V is positive definite, that is $V(0) = 0$ and $V(\alpha) > 0, \forall\, 0 \neq \alpha \in \mathbb{R}^m$.*

(ii) *V is proper, that is $V(\alpha) \to \infty$ as $|\alpha| \to 0$.*

(iii) *There exist $\delta > 0$ and $\beta \in \mathcal{K}_\infty$ satisfying:*

$$\begin{cases} \left.\frac{dV(\alpha)}{d\alpha}\right|_{\alpha = y_\gamma(t)} \cdot f\big(y_\gamma(t) + q_u(t), u(t)\big) \leq -\beta\big(|y_\gamma(t)|\big), \\ \text{for all } t \in [0, \omega_\gamma) \text{ and } \forall \gamma > 0 \text{ that satisfy } |y_\gamma(t)| < \delta. \end{cases} \tag{15}$$

Then,

- *$\omega_\gamma = +\infty, \forall \gamma > 0$. Furthermore, there exist $E, \gamma^* > 0$ such that $\|x_\gamma\|_\infty \leq E, \forall \gamma > \gamma^*$, for any solution x_γ of the system (3)-(4).*

- $\lim_{\gamma\to\infty} \|\sigma_\gamma - q_u\|_\infty = 0$.

Proof. Since V is positive definite and proper, there exists $\beta_1, \beta_2 \in \mathcal{K}_\infty$ such that (see [7, p. 145])

$$\beta_1(|\alpha|) \le V(\alpha) \le \beta_2(|\alpha|), \forall \alpha \in \mathbb{R}^m. \tag{16}$$

From (5), we get for almost all $t \in [0, \omega_\gamma)$, $\forall \gamma > 0$ that

$$\begin{aligned} \dot{y}_\gamma(t) &= \gamma f\big(y_\gamma(t) + q_u(t), u(t)\big) - \dot{q}_u(t), && (17)\\ y_\gamma(0) &= 0. && (18) \end{aligned}$$

For any $\gamma > 0$, define $V_\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ as $V_\gamma(t) = V\big(y_\gamma(t)\big), \forall t \in [0, \omega_\gamma)$. Note that the function V_γ is absolutely continuous on each compact subset of $[0, \omega_\gamma)$ as a composition of a continuously differentiable function V and an absolutely continuous function y_γ. Then, we get for almost all $t \in [0, \omega_\gamma)$ and all $\gamma > 0$ that

$$\dot{V}_\gamma(t) = \frac{dV(\alpha)}{d\alpha}\bigg|_{\alpha = y_\gamma(t)} \cdot \dot{y}_\gamma(t) = \frac{dV(\alpha)}{d\alpha}\bigg|_{\alpha = y_\gamma(t)} \cdot \Big[\gamma f\big(y_\gamma(t) + q_u(t), u(t)\big) - \dot{q}_u(t)\Big]. \tag{19}$$

Let $\Omega = \big(0, \beta_1(\delta)\big)$. By (16) we have for any $\gamma > 0$, and for almost all $t \in [0, \omega_\gamma)$ that

$$V_\gamma(t) \in \Omega \Rightarrow |y_\gamma(t)| < \delta. \tag{20}$$

We conclude from (15), (19), and (20) that

$$\dot{V}_\gamma(t) \le -\gamma\,\beta(|y_\gamma(t)|) + \|\dot{q}_u\|_\infty \left| \frac{dV(\alpha)}{d\alpha}\bigg|_{\alpha = y_\gamma(t)} \right|, \text{ for almost all } t \in [0, \omega_\gamma), \forall \gamma > 0 \text{ that satisfy } V_\gamma(t) \in \Omega.$$

Thus, we deduce from the continuity of $\frac{dV(\alpha)}{d\alpha}$, the boundedness of $\dot{q}_u$, and (20) there exists some $b > 0$ independent of γ such that

$$\dot{V}_\gamma(t) \le -\gamma\,\beta(|y_\gamma(t)|) + b, \text{ for almost all } t \in [0, \omega_\gamma), \forall \gamma > 0 \text{ that satisfy } V_\gamma(t) \in \Omega.$$

Hence, (16) implies

$$\dot{V}_\gamma(t) \le -\gamma\,\beta \circ \beta_2^{-1}\big(V_\gamma(t)\big) + b, \text{ for almost all } t \in [0, \omega_\gamma), \forall \gamma > 0 \text{ that satisfy } V_\gamma(t) \in \Omega.$$

Thus, Corollary 4.1 and the fact that $V_\gamma(0) = 0, \forall \gamma > 0$, imply that $V_\gamma(t) \le \beta_2 \circ \beta^{-1}\left(\frac{b}{\gamma}\right)$, $\forall \gamma > \gamma_0, \forall t \in [0, \omega_\gamma)$ where $\gamma_0 = \frac{b}{\beta \circ \beta_2^{-1} \circ \beta_1(\delta)}$. Therefore, (16) implies that

$$|y_\gamma(t)| \le \beta_1 \circ \beta_2 \circ \beta^{-1}\left(\frac{b}{\gamma}\right), \forall \gamma > \gamma_0, \forall t \in [0, \omega_\gamma). \tag{21}$$

Thus, $\omega_\gamma = +\infty, \forall \gamma > \gamma_1$ for some $\gamma_1 > 0$, and $\lim_{\gamma\to\infty} \|y_\gamma\|_\infty = 0$, which is equivalent to $\lim_{\gamma\to\infty} \|\sigma_\gamma - q_u\|_\infty = 0$. On the other hand, (21) and the fact that $\sigma_\gamma = y_\gamma + q_u$ imply that there exists some $E, \gamma^* > 0$ such that $\|\sigma_\gamma\|_\infty \le E, \forall \gamma > \gamma^*$, and hence $\|x_\gamma\|_\infty \le E, \forall \gamma > \gamma^*$. □

Lemma 4.3. *Consider the nonlinear system [13]*

$$\begin{aligned} \dot{x} = f(x, u) &= Ax + \Phi(x) + R(u), && (22)\\ x(0) &= x_0, && (23)\\ y &= Dx, && (24) \end{aligned}$$

where $x_0 \in \mathbb{R}^m$*, A is an $m \times m$ Hurwitz matrix*[2]*, D is an $m \times m$ matrix, input $u \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$, state x, output y take values in $\mathbb{R}^m$, function $R \in C^0(\mathbb{R}^n, \mathbb{R}^m)$, and a locally Lipshitz function $\Phi \in C^0(\mathbb{R}^m, \mathbb{R}^m)$. Assume the following:*

(i) *The exists $q_u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$ such that $q_u(0) = x_0$ and*

$$Aq_u(t) + \Phi\big(q_u(t)\big) + R\big(u(t)\big) = 0, \forall t \geq 0.$$

(ii) *There exist $c_1 > 0$, $c_2 > 0$, $\xi > 0$ and $r > 2$ such that*

$$\left|\alpha \cdot \left[\Phi\big(\alpha + q_u(t)\big) - \Phi\big(q_u(t)\big)\right]\right| \leq c_1 |\alpha|^2 + c_2 |\alpha|^r, \text{ for almost all } t \geq 0, \forall \alpha \in \mathbb{R}^m \text{ that satisfy } |\alpha| < \xi.$$

(iii) *One has $c_1 < \frac{1}{2\,\lambda_{\max}}$, where $\lambda_{\max}$ is the largest eigenvalue for the $m \times m$ positive-definite symmetric matrix P that satisfies*[3]

$$PA + A^T P = -I_{m \times m}. \tag{25}$$

Let x_γ, y_γ be respectively the state and the output of (22)-(24) when we use the input $u \circ s_\gamma$ instead of u.

Then,

- *All solutions of (22)-(24) are bounded. Furthermore, there exist $E, \gamma^* > 0$ such that $\|x_\gamma\|_\infty \leq E, \forall \gamma > \gamma^*$, for any solution x_γ of the system (3)-(4).*
- $\lim_{\gamma \to \infty} \|F_\gamma - Dq_u\|_\infty = 0$*, where $F_\gamma : \mathbb{R}_+ \to \mathbb{R}^m$ is defined as $F_\gamma(t) = y_\gamma(\gamma t), \forall t \geq 0, \forall \gamma > 0$.*

Proof. Since Φ is locally Lipschitz, the right-hand side of (22) is locally Lipschitz relative to x and hence the system (22) has a unique solution. The function q_u satisfies (12)-(13) in Lemma 4.2 because of (i).

Consider the continuously differentiable quadratic Lyapunov function candidate $V : \mathbb{R}^m \to \mathbb{R}$ such that $V(\alpha) = \alpha^T P \alpha$, $\forall \alpha \in \mathbb{R}^m$. Since P is symmetric, we have $\forall \alpha \in \mathbb{R}^m$ that

$$\lambda_{\min} |\alpha|^2 \leq V(\alpha) = \alpha^T P \alpha \leq \lambda_{\max} |\alpha|^2,$$

where $\lambda_{\min}$ is the smallest eigenvalue of the matrix P. Thus V is positive definite and proper. Since P is symmetric we have

$$\left|\frac{dV(\alpha)}{d\alpha}\right| = 2\,|P\,\alpha| \leq 2\lambda_{\max} |\alpha|, \forall \alpha \in \mathbb{R}^m. \tag{26}$$

We have by (25) that

$$\frac{dV(\alpha)}{d\alpha} \cdot A\alpha = 2P\alpha \cdot A\alpha \ = \alpha^T \left(PA + A^T P\right)\alpha = -|\alpha|^2, \forall \alpha \in \mathbb{R}^m. \tag{27}$$

From Condition (i) we get for all $\gamma > 0$ that

$$\begin{aligned} \left.\frac{dV(\alpha)}{d\alpha}\right|_{\alpha = y_\gamma} \cdot f(y_\gamma + q_u, u) &= \left.\frac{dV(\alpha)}{d\alpha}\right|_{\alpha = y_\gamma} \cdot \Big[Ay_\gamma + Aq_u + \Phi(y_\gamma + q_u) + R(u)\Big] \\ &= \left.\frac{dV(\alpha)}{d\alpha}\right|_{\alpha = y_\gamma} \cdot \Big[Ay_\gamma + \Phi(y_\gamma + q_u) - \Phi(q_u)\Big]. \end{aligned} \tag{28}$$

[2]that is each eigenvalue of A has a strictly negative real part.
[3]the existence of the matrix P in (25) is guaranteed because A is Hurwitz [7, p.136].

where y_γ is defined in (14).

We get from (28), (27), (26) and Condition (ii) that

$$\left.\frac{dV(\alpha)}{d\alpha}\right|_{\alpha=y_\gamma(t)} \cdot f\big(y_\gamma(t)+q_u(t),u(t)\big) \le (-1+2c_1\,\lambda_{\max})\,|y_\gamma(t)|^2 + 2c_2\,\lambda_{\max}\,|y_\gamma(t)|^r,$$
$$\forall\gamma>0 \text{ for almost all } t\in[0,\omega_\gamma) \text{ that satisfy } |y_\gamma(t)|<\xi, \tag{29}$$

where $[0,\omega_\gamma)$ is the maximal interval of existence of σ_γ and y_γ. This leads to

$$\left.\frac{dV(\alpha)}{d\alpha}\right|_{\alpha=y_\gamma(t)} \cdot f\big(y_\gamma(t)+q_u(t),u(t)\big) \le -\frac{1-2c_1\,\lambda_{\max}}{2}\,|y_\gamma(t)|^2,$$
$$\forall\gamma>0, \text{ for almost all } t\in[0,\omega_\gamma) \text{ that satisfy } |y_\gamma(t)|<\min\left(\sqrt[r-2]{\frac{1-2c_1\,\lambda_{\max}}{4c_2\,\lambda_{\max}}},\ \xi\right). \tag{30}$$

Thus, (15) in is satisfied with $\beta(v)=\frac{1-2c_1\,\lambda_{\max}}{2}v^2, \forall v\ge 0$ and $\delta=\min\left(\sqrt[r-2]{\frac{1-2c_1\,\lambda_{\max}}{4c_2\,\lambda_{\max}}},\ \xi\right)$. Hence all conditions of Lemma 4.2 are satisfied so that the solution of (22) is bounded. Morover, there exist $E,\gamma^*>0$ such that $\|x_\gamma\|_\infty \le E, \forall\gamma>\gamma^*$. Futhermore, we have $\lim_{\gamma\to\infty}\|\sigma_\gamma-q_u\|_\infty=0$. Thus, we deduce from (24) that $\lim_{\gamma\to\infty}\|F_\gamma-Dq_u\|_\infty=0$. □

Example. Consider the system

$$\begin{aligned}\dot{x} &= -x+x^3-u, &(31)\\ x(0) &= 0. &(32)\end{aligned}$$

where state x takes values in $\mathbb{R}$ and input $u\in W^{1,\infty}(\mathbb{R}_+,\mathbb{R})$ is defined as $u(t)=0.1\sin(t), \forall t\ge 0$. The system (31)-(32) has the form (22)-(24), with $x=y$, $m=n=1$, $A=-1$, $\Phi(\alpha)=\alpha^3$, $R(\alpha)=-\alpha$, $\forall\alpha\in\mathbb{R}$, and $D=1$. Observe that P in (25) equals $1/2$ which mean that $\lambda_{\min}=\lambda_{\max}=1/2$. We have $u(0)=0$ and u is bounded with

$$u(\cdot)\in[u_{\min},u_{\max}]=[-0.1,0.1]. \tag{33}$$

Define the function $\chi:\left[-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right]\to\left[-\frac{2}{3\sqrt{3}},\frac{2}{3\sqrt{3}}\right]$ as $\chi(v)=-v+v^3, \forall v\in\left[-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right]$. The function χ is strictly decreasing, bijective and its inverse function is continuous. Hence, there exists a function $q_u\in C^0(\mathbb{R}_+,\mathbb{R})\cap L^\infty(\mathbb{R}_+,\mathbb{R})$ such that $q_u(\cdot)\in\left[-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right]$, $q_u(0)=0$ and

$$\chi\big(q_u(t)\big)=-q_u(t)+q_u^3(t)=u(t), \forall t\ge 0. \tag{34}$$

It can be checked using (33) that $\|q_u\|_\infty<0.11$ (see Figure (1b)). Thus $q_u(\cdot)\ne\frac{1}{\sqrt{3}}$. This fact and (34) implies that the function $\dot{q}_u=\dot{u}/\left(1-3q_u^2\right)$ is bounded so that $q_u\in W^{1,\infty}(\mathbb{R}_+,\mathbb{R})$. Hence Condition (i) of Lemma 4.3 is satisfied.

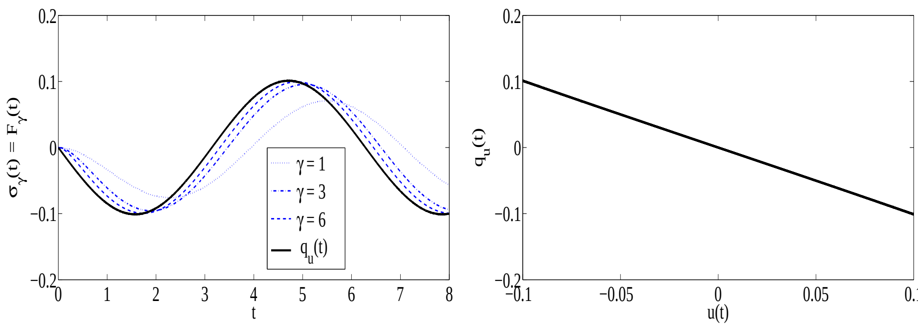
On the other hand, we have for all $\alpha\in\mathbb{R}$ that

$$\alpha\left(\Phi(\alpha+q_u)-\Phi(q_u)\right)=3q_u^2\alpha^2+3q_u\alpha^3+\alpha^4. \tag{35}$$

Since $\|q_u\|_\infty < 0.11$, one has $\left\|3q_u^2\right\|_\infty < 0.0363 = c_1$. Hence it follows from (35) that for any $\xi > 0$ we have

$$\alpha\left[\Phi\left(\alpha + q_u(t)\right) - \Phi\left(q_u(t)\right)\right] \leq c_1\alpha^2 + \left(3\|q_u\|_\infty + \xi\right)\alpha^3$$
$$\forall\alpha \in \mathbb{R}^m \text{ that satisfy } |\alpha| < \xi, \text{ for almost all } t \geq 0. \quad (36)$$

Thus, Condition (ii) in Lemma 4.3 is satisfied with $c_2 = 3\|q_u\|_\infty + \xi$. Moreover, we have $c_1 < 1 = \frac{1}{2\,\lambda_{\max}}$ which implies that Condition (ii) in Lemma 4.3 is also satisfied. Therefore, the solution of (31)-(32) is bounded, that there exist $E, \gamma^* > 0$ such that $\|x_\gamma\|_\infty \leq E, \forall\gamma > \gamma^*$, and that $\lim_{\gamma\to\infty}\|\sigma_\gamma - q_u\|_\infty = \lim_{\gamma\to\infty}\|F_\gamma - q_u\|_\infty = 0$ (observe that $\sigma_\gamma(\cdot) = F_\gamma(\cdot)$ because $x(\cdot) = y(\cdot)$). This is illustrated in Figure 1a.



(a) $F_\gamma(t)$ versus t for system (31).

(b) $q_u(t)$ versus $u(t)$ for system (31).

Figure 1: Simulations.

5 Conclusion

In [5] a rule for deciding whether a process may or may not be a hysteresis is proposed for causal operators such that a constant input leads to a constant output. That rule involves checking whether the so-called consistency and strong consistency properties hold. In this paper we derived necessary conditions and sufficient ones for the uniform convergence of the shifted solutions $\sigma_\gamma : t \to x_\gamma(\gamma t)$ of the system $\dot{x} = f(x, u \circ s_\gamma)$. This uniform convergence is related to consistency. Does this mean that the concept of consistency can be extended to study operators for which the property that a constant input leads to a constant output, that property does not hold?

This paper explores this issue for systems of the form $\dot{x} = f(x, u)$, however, no clear cut answer may be drawn for the obtained results.

Indeed, the necessary conditions alone cannot guarantee whether the uniform convergence of σ_γ when $\gamma \to \infty$ happens or not. The sufficient conditions do imply that convergence but do not guarantee that the hysteresis loop of the operator is not trivial. In the example, we have seen that q_u is a function of u so that the hysteresis loop is a curve and we cannot acertain from this whether system (31) is a hysteresis or not. This is a future research line.

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Fuzzy analytical hierarchy process based on canonical representation on fuzzy numbers

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Abstract

Fuzzy analytical hierarchy process(FAHP) is widely used in multi-criteria decision making (MCDM) under uncertain environments. Many works have been proposed. However, the existing methods are complex and time consuming. What's more, the conflict management in AHP is still an open issue. To solve these issues, a novel and simple FAHP method is proposed based on the canonical representation of multiplication operation on fuzzy numbers in this paper. We adopt the main idea of classical AHP, that is the weight of each criterion can be determined by its relative ratio. The relative ratio can be easily determined in the proposed method. In addition, the average method is adopted to handle conflicts in AHP. An example on supplier selection is used to illustrate the efficiency of our proposed method.

Keywords: Analytical Hierarchical Process, fuzzy numbers, fuzzy AHP, canonical representation of fuzzy numbers, supplier selection.

1. Introduction

Analytical Hierarchy Process(AHP) is a powerful tool for handling both qualitative and quantitative multi-criteria factors in decision-making problems, developed by Saaty [1] in the 1970s. This method has been extensively

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studied and refined since then. It provides a comprehensive and rational framework for structuring a decision problem, for representing and quantifying its elements, for relating those elements to overall goals, and for evaluating alternative solutions. With this method, a complicated problem can be converted to an ordered hierarchical structure. AHP method has been widely applied to multi-criteria decision making situations[2], such as: web sites selection[3], tools' evaluation[4], e-business [5], drugs selection[6], group decision [7, 8]and so on[9, 10, 11, 12].

Multi-Criteria analysis problems require the decision maker to make qualitative assessments regarding the performance of the decision alternatives with respect to each independent criterion and the relative importance of each independent criterion with respect to the overall objective of the problem [13, 14]. As a result, uncertain subjective data are present which make the decision making process complex. Many math tools are developed. For example, evidence theory is heavily studied since it can fuse different data which make it widely used in multi-criteria decision making [15, 16, 17]. Due to the flexibility to handle linguistic information [18], the fuzzy sets theory is also widely used in many uncertain decision makings [19, 20, 21, 22, 23]. As a result, the classical AHP is extended to fuzzy AHP (FAHP) [24] and is applied to many MCDM applications under uncertain environment,such as environmental assessment and management[25, 26, 27], supplier management[28], group decision making [29], fuzzy MCDM[30], fuzzy MADM [31], and so on [32].

Two key issues should be solved in the application of fuzzy AHP. One issue is that how to determine the weight of each criterion when the elements of comparison matrix are fuzzy numbers. Unlike the classical AHP, the eigenvector of fuzzy comparison matrix cannot be obtained directly. Hence, some other steps are inevitable to get the final weights in most existing fuzzy AHP methods[24, 33],which makes the FAHP more completed to some degrees.

The other key problem when applying the AHP is to avoid rank reversal[34]. Due to the different preference and subjective and objective factors in decision making, evidence connected from different sources are often conflicting[35, 36, 37, 38]. How to deal with conflict and dependence in AHP is still an open issue [39, 40, 41, 42]. In classical AHP, a well known coefficient, called as Consistency index(CI), is used to measure the conflicting degree in decision making. In some application systems, the AHP model should be adjusted when the CI is higher than a certain threshold value. The problem still exists in fuzzy AHP. Many methods have been proposed to handle this

problem[43, 44]. In order to construct decision matrices of pairwise comparisons based on additive transitivity, Herrera-Viedma et al. propose consistent fuzzy preference relations[45]. In [43], the distance function between two linguistic preference relations is defined, then a new CI is defined based on the distance function. In [44, 46], a method is proposed to construct fuzzy linguistic preference relations, called as fuzzy LinPreRa method. However, it should be pointed out that is difficult to give a corresponding CI in fuzzy AHP.

To handel these two issues mentioned above, we propose a novel and simple FAHP in this paper. On the one hand, we use the canonical representation of multiplication operation on fuzzy numbers, presented in [47], to obtain the weigh of each criterion in a straight and easy manner. On the other hand, we suggest to use average method to deal with conflicts in AHP decision making. The numerical example on supplier selection shows the efficiency of our proposed method. The paper is organized as follows. Section 2 begins with a brief introduction to the basic theory used in the proposed method,including AHP, fuzzy set theory and genetic algorithm. A typical fuzzy AHP is also introduced in this section. The proposed methodology is detailed in section 3. In section 4, our proposed method is applied to supplier selection. Section 5 concludes the paper.

2. Preliminaries

2.1. Analytical Hierarchy Process[1]

The first step of AHP is to establish a hierarchical structure of the problem. Then, in each hierarchical level, use a nominal scale to construct pairwise comparison judgement matrix.

Definition 2.1. *Assuming $(E_1, \cdots, E_i, \cdots, E_n)$ are n decision elements, the pairwise comparison judgement matrix is denoted as $M_{n\times n} = [m_{ij}]$, which satisfies:*

$$m_{ij} = \frac{1}{m_{ji}} \tag{1}$$

where each element m_{ij} represents the judgment concerning the relative importance of decision element E_i over E_j.

With the matrix constructed, the third step is to calculate the eigenvector of the matrix.

Definition 2.2. *Eigenvector of $n \times n$ pairwise comparison judgement matrix can be denoted as: $\vec{w} = (w_1, \cdots, w_i, \cdots, w_n)^T$, which is calculated as follows:*

$$A\vec{w} = \lambda_{\max}\vec{w}, \quad \lambda_{\max} \geq n \tag{2}$$

where $\lambda_{\max}$ is the maximum eigenvalue in the eigenvector $\vec{w}$ of matrix $M_{n\times n}$.

Before we transform the eigenvector into the weights of elements, the consistency of the matrix should be checked.

Definition 2.3. *Consistency index(CI)*[1] *is used to measure the inconsistency within each pairwise comparison judgement matrix, which is formulated as follows:*

$$CI = \frac{\lambda_{\max} - n}{n - 1} \tag{3}$$

Accordingly, the consistency ratio(CR) can be calculated by using the following equation:

$$CR = \frac{CI}{RI} \tag{4}$$

where RI is the random consistency index. The value of RI is related to the dimension of the matrix, which is listed in Table 1.

Table 1: The value of RI(random consistency index)

dimension	1	2	3	4	5	6	7	8	9	10
RI	0	0	0.52	0.89	1.12	1.26	1.36	1.41	1.46	1.49

If the result of CR is less than 0.1, the consistency of the pairwise comparison matrix M is acceptable. Moreover, the eigenvector of pairwise comparison judgement matrix can be normalized as final weights of decision elements. Otherwise, the consistency is not passed and the elements in the matrix should be revised.

2.2. Fuzzy sets

In 1965, the notion of fuzzy sets was firstly introduced by Zadeh[18], providing a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership[48].

A brief introduction of Fuzzy sets are given as follows.

Definition 2.4. *A fuzzy set A is defined on a universe X may be given as:*

$$A = \{\langle x, \mu_A(x)\rangle \,|x \in X\}$$

where $\mu_A : X \to [0,1]$ *is the membership function A. The membership value* $\mu_A(x)$ *describes the degree of belongingness of* $x \in X$ *in A.*

For a finite set $A = \{x_1, \ldots, x_i, \ldots, x_n\}$, the fuzzy set (A, m) is often denoted by $\left\{ {}^{\mu_A(x_1)}/_{x_1}, \ldots, {}^{\mu_A(x_i)}/_{x_i}, \ldots, {}^{\mu_A(x_n)}/_{x_n} \right\}$.

In real application, the domain experts may give their opinions by fuzzy numbers. For example, in a new product price estimation, one expert may give his opinion as: the lowest price is 2 dollars, the most possibility price of the product may be 3 dollars, the highest price of this product will not be in excess of 4 dollars. Hence, we can use a triangular fuzzy number (2,3,4) to represent the expert's opinion. The triangular fuzzy numbers can be defined as follows.

Definition 2.5. *A triangular fuzzy number* $\tilde{A}$ *can be defined by a triplet (a, b, c) , where the membership can be determined as follows*

A triangular fuzzy number $\tilde{A} = (a, b, c)$ can be shown in Fig.(1).

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, x < a \\ \frac{x-a}{b-a}, a \leqslant x \leqslant b \\ \frac{c-x}{c-b}, b \leqslant x \leqslant c \\ 0, x > c \end{cases} \tag{5}$$

In Fig2. $N1$, $N3$, $N5$, $N7$ and $N9$ are used to represent the pairwise comparison of decision variables from *Equal*to *Absolutely preferred*, and TFNs $N2$, $N4$, $N6$ and $N8$ represent the middle preference values between them.

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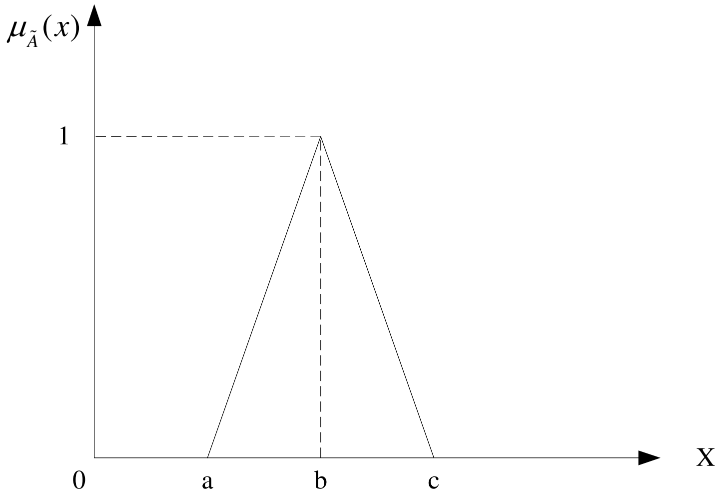

Figure 1: A triangular fuzzy number.

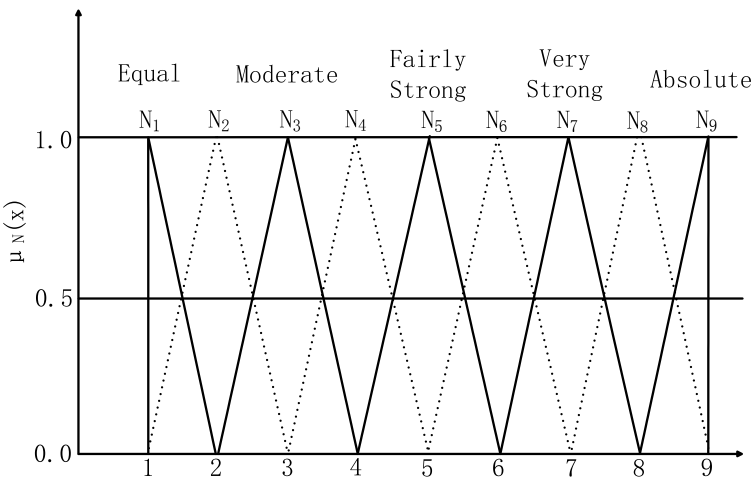

Figure 2: Nine fuzzy numbers

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2.3. Canonical representation operation on fuzzy numbers

In this section, the canonical representation of operation on triangular fuzzy numbers which are based on the graded mean integration representation method [47], is used to obtain the weight of each criterion in a simple manner. The canonical representation operation on fuzzy numbers is applied to many decision makings [49, 50].

Definition 2.6. *Given a triangular fuzzy number $\tilde{A} = (a_1, a_2, a_3)$, the graded mean integration representation of triangular fuzzy number $\tilde{A}$ is defined as:*

$$P(\tilde{A}) = \frac{1}{6}(a_1 + 4 \times a_2 + a_3) \tag{6}$$

Let $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$ be two triangular fuzzy numbers. By applying Eq.(6), the graded mean integration representation of triangular fuzzy numbers $\tilde{A}$ and $\tilde{B}$ can be obtained, respectively, as follows:

$$P(\tilde{A}) = \tfrac{1}{6}(a_1 + 4 \times a_2 + a_3)$$
$$P(\tilde{B}) = \tfrac{1}{6}(b_1 + 4 \times b_2 + b_3)$$

The representation of the addition operation $\oplus$ on triangular fuzzy numbers $\tilde{A}$ and $\tilde{B}$ can be defined as :

$$P(\tilde{A} \oplus \tilde{B}) = P(\tilde{A}) + P(\tilde{B}) = \frac{1}{6}(a_1 + 4 \times a_2 + a_3) + \frac{1}{6}(b_1 + 4 \times b_2 + b_3) \tag{7}$$

The canonical representation of the multiplication operation on triangular fuzzy numbers $\tilde{A}$ and $\tilde{B}$ is defined as :

$$P(\tilde{A} \otimes \tilde{B}) = P(\tilde{A}) \times P(\tilde{B}) = \frac{1}{6}(a_1 + 4 \times a_2 + a_3) \times \frac{1}{6}(b_1 + 4 \times b_2 + b_3) \tag{8}$$

2.4. FAHP

In this section, we briefly introduce a typical FAHP method . For detailed information, please refer [51, 52].

In the first step,triangular fuzzy numbers are used for pair-wise comparisons. Then, by using extent analysis method the synthetic extent value Si of the pair-wise comparison is introduced and by applying the principle of the comparison of fuzzy numbers, the weight vectors with respect to each element under a certain criterion is calculated. The details of the methodology are presented in the following steps:

Let $X = \{x_1, x_2, \ldots, x_n\}$ be an object set,and $U = \{u_1, u_2, \ldots, u_m\}$ be a goal set.According to the method of Changs extent analysis, each object is taken and an extent analysis for each goal, g_i, is performed.Therefore,m extent analysis values for each object can be obtained, with the following signs:
$M_{gi}^1, M_{gi}^2, \ldots, M_{gi}^m, i = 1, 2, \ldots, n$, where all the $M_{gi}^j (j = 1, 2, \ldots, m)$ are TFN's.

Step 1: The value of fuzzy synthetic extent with respect to the ith object is defined as

$$S_i = \sum_{j=1}^{m} M_{gi}^j \otimes \left\{ \sum_{i=1}^{n} \sum_{j=1}^{m} M_{gi}^j \right\}^{-1} \tag{9}$$

In order to obtain $\sum\limits_{j=1}^{m} M_{gi}^j$, perform the fuzzy addition operation of m extent analysis values for a particular matrix such that

$$\sum_{j=1}^{m} M_{gi}^j = \left(\sum_{j=1}^{m} l_j, \sum_{j=1}^{m} m_j, \sum_{j=1}^{m} u_j \right) \tag{10}$$

To obtain $\left\{ \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{m} M_{gi}^j \right\}^{-1}$, perform the fuzzy addition operation of $M_{gi}^j (j = 1, 2, \ldots, m)$ values such that

$$\sum_{i=1}^{n} \sum_{j=1}^{m} M_{gi}^j = \left(\sum_{i=1}^{n} l_i, \sum_{i=1}^{n} m_i, \sum_{i=1}^{n} u_i \right) \tag{11}$$

and then compute the inverse of the vector.

Step 2: The degree of possibility of $M_2 = (l_2, m_2, u_2) \geq M_1 = (l_1, m_1, u_1)$ is expressed as:

$$\begin{aligned} & V(M_2 \geq M_1) \\ & = hgt(M_1 \geq M_2) \\ & = \begin{cases} 1, & if\ m_2 \geq m_1 \\ \frac{(l_1 - u_2)}{((m_2 - u_2) - (m_1 - l_1))} & otherwise \\ 0, & if\ l_1 \geq u_2 \end{cases} \end{aligned} \tag{12}$$

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To compare M_1 and M_2 both $V(M_2 \geq M_1)$ and $V(M_1 \geq M_2)$ are required.

Step 3: The degree of possibility for a convex fuzzy number to be greater than k convex fuzzy numbers $M_i (i = 1, 2, \ldots, k)$ can be defined as:
$V(M \geq M_1, M_2, \ldots, M_k) = V[(M \geq M_1)$ and $(M \geq M_2)$ and ... and

$$(M \geq M_k)] = \min V(M \geq M_i), i = 1, 2, \ldots, k \tag{13}$$

Let $d'(A_i) = \min V(S_i \geq S_k)$, for $k = 1, 2, \ldots, n$; $k \neq i$. Then the weight vector is given by:

$$W' = (d'(A_1), d'(A_2), \ldots, d'(A_n))^T \tag{14}$$

Step 4: The weight vector obtained in step 3 is normalized to get the normalized weights.

3. The proposed methodology

One of the most key issue in fuzzy AHP is how to determine the weights given the fuzzy pairwise comparison judgement matrix. For example, given the linguistic data in Table 2, how can we get the weight of each criterion? In the following of this section, we solve the problem step by step.

Table 2: The Fuzzy evaluation of criteria with respect to the overall objective

	C1	C2	C3	C4	C5	W_C
C1	(1,1,1)	(3/2,2,5/2)	(3/2,2,5/2)	(5/2,3,7/2)	(5/2,3,7/2)	0.3283
C2	(2/5,1/2,2/3)	(1,1,1)	(3/2,2,5/2)	(5/2,3,7/2)	(5/2,3,7/2)	0.2839
C3	(2/5,1/2,2/3)	(2/5,1/2,2/3)	(1,1,1)	(3/2,2,5/2)	(3/2,2,5/2)	0.1798
C4	(2/7,1/3,2/5)	(2/7,1/3,2/5)	(2/5,1/2,2/3)	(1,1,1)	(3/2,2,5/2)	0.1262
C5	(2/7,1/3,2/5)	(2/7,1/3,2/5)	(2/5,1/2,2/3)	(2/5,1/2,2/3)	(1,1,1)	0.0818

3.1. Transformation of fuzzy comparison matrix

Let's consider the element in the comparison matrix classical AHP. The rating in the matrix means the relative importance of the criterion. For example, suppose only two criterion in a comparison matrix, listed as follows.

Example 3.1. *The comparison matrix is given as follows*

$$\begin{array}{c c} & \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{array}{c} C_1 \\ C_2 \end{array} & \begin{bmatrix} 1 & 3 \\ 1/3 & 1 \end{bmatrix} \end{array}$$

*From the matrix, the element $C_{12} = 3$ means that weight of the second criterion C_2 is **three times** of that of the first criterion C_1. In addition, the eigenvector of comparison matrix can be easily obtained as follows:*

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}$$

Two important points should be noticed: **First point**, *the sum of the eigenvector of comparison matrix should be* **ONE**. For example,$w_1 + w_2 = 0.75 + 0.25 = 1$. **Second point**, *the ratio of the weight should be coincide with the corresponding element in comparison matrix.* In Example 3.1, we can get $w_1/w_2 = 0.75/0.25 = 3 = C_{12}$.

This idea of AHP can be easily adopted in fuzzy AHP. For example, in the Table 2, the element $C_{12} = (3/2, 2, 5/2)$. According to the above analysis, we understand that the weight of the second criterion C_2 is **(3/2,2,5/2) times** of that of the first criterion C_1(**Notice**: for the sake of simplicity, we suppose that $(3/2, 2, 5/2)$ is not a linguistic variable N_2 shown in Fig.2, but a simple fuzzy number to model the fuzzy variable "ABOUT 2"). The only difference between this case with Example 3.1 is that one is a crisp number 3 while the other is a fuzzy number $(3/2, 2, 5/2)$. How to represent the weight of the second criterion C_2 is **(3/2,2,5/2) times** of that of the first criterion C_1 in the canonical representation of multiplication operation on fuzzy numbers? According to the Eq.(8), we obtain the follow result.

$$\begin{aligned} & P(\tilde{A} \otimes \tilde{B}) \\ & = (1, 1, 1) \otimes (3/2, 2, 5/2) \\ & = \tfrac{1}{6}(1 + 4 \times 1 + 1) \times \tfrac{1}{6}(3/2 + 4 \times 2 + 5/2) \\ & = 1 \times 2 \\ & = 2 \end{aligned} \tag{15}$$

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The Eq.(15) means that the the weight of the second criterion C_2 is **(3/2,2,5/2) times** of that of the first criterion C_1 could also be stated as "*the weight of the second criterion C_2 is **2 times** of that of the first criterion C_1 under the canonical representation of multiplication operation on fuzzy numbers*". The other element of the canonical representation of multiplication operation on fuzzy numbers can also be determined and shown in Table3.

Table 3: Evaluation of criteria with respect to the overall objective based on canonical representation of multiplication operation

	C1	C2	C3	C4	C5
C1	1	2	2	3	3
C2	46/90	1	2	3	3
C3	46/90	46/90	1	2	2
C4	212/630	212/630	46/90	1	2
C5	212/630	212/630	46/90	46/90	1

We call the matrix in Table3 the ***comparison matrix with canonical representation of multiplication operation*** (**CMCRMO**)

Let's us see the first row of Table 3. If we suppose that the relative weight of the first criterion is 1, then we get that: 1)both the the relative weight of the second and the third criterion is 2; 2)both the the relative weight of the fourth and the fifth criterion is 3. Then, a straight way to obtain the corresponding weight is with the simple normalization of these relative weights. The result can be shown as follows.

$$\begin{aligned}
w_{C_1}^1 &= \tfrac{1}{1+2+2+3+3} = \tfrac{1}{11} \\
w_{C_1}^2 &= \tfrac{2}{1+2+2+3+3} = \tfrac{2}{11} \\
w_{C_1}^3 &= \tfrac{2}{1+2+2+3+3} = \tfrac{2}{11} \\
w_{C_1}^4 &= \tfrac{3}{1+2+2+3+3} = \tfrac{3}{11} \\
w_{C_1}^5 &= \tfrac{3}{1+2+2+3+3} = \tfrac{3}{11}
\end{aligned} \tag{16}$$

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In Eq.(16), the subscript C_1 means that the weight is obtained according to the criterion C_1. The weight in Eq.(16) is not the final weight of each criterion since that there exists conflict in this situation, also called rank reversal[34, 43, 44, 46, 45]. This problem will be handled in the following part.

3.2. Conflict management with average method

It should also be mentioned that the **second point**, namely "*the ratio of the weight should be coincide with the corresponding element in comparison matrix*" can be satisfied on some ideal situations. However, the preference order will not be always keep coincided in the whole AHP process. In real application, the comparison matrix given by experts may not strictly obey the preference order as shown in Example3.2.

Example 3.2. *The comparison matrix is given as follows*

$$\begin{array}{cc} & \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{array}{c} C_1 \\ C_2 \\ C_3 \\ C_4 \end{array} & \begin{bmatrix} 1 & 3 & 5 & 7 \\ 1/3 & 1 & 1/3 & 3 \\ 1/5 & 3 & 1 & 2 \\ 1/7 & 1/3 & 1/2 & 1 \end{bmatrix} \end{array}$$

From the first row of above comparison matrix, we can see that the importance ranking corresponding to C_1 is

$$C_1 < C_2 < C_3 < C_4$$

. However, from the second row of above comparison matrix, we can see that the importance ranking corresponding to C_2 is

$$C_1 < C_3 < C_2 < C_4$$

The consistency index index defined in Definition2.3 show the conflict in preference. In classical AHP, the CI is used to determine how consistence of the comparison matrix. If the value of CI is higher than a threshold, then some adjustments to deal with rank reversal should be made. Though many methods have been proposed on this filed, it is still an open issue. In decision making with fuzzy AHP, it is also inevitable. For example, see the first line of the Table 3, we get the following preference ranking order.

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Table 4: The Fuzzy evaluation of criteria with respect to the overall objective

	Preference ranking order
C1	$C1 < C2 = C3 < C4 = C5$
C2	$C1 < C2 < C3 < C4 = C5$
C3	$C1 = C2 < C3 < C4 = C5$
C4	$C1 = C2 < C3 < C4 < C5$
C5	$C1 = C2 < C3 = C4 < C5$

As can be seen from Table4, for C_1, the weight of C_2 is equal to C_3. However, for C_2,C_3,C_4 and C_5, the weight of C_2 is less than C_3. There are many other conflicts in the ranking order. In this paper, we use average to decrease the conflict in the preference order. We average the weights of all criteria to get the final weight of each criterion. That is, if we get the the comparison matrix with canonical representation of multiplication operation (**CMCRMO**) shown in Table3, we can obtain the final weight of each criterion with the normalization of average weight of each criterion.

Example 3.3. *Suppose we get the comparison matrix with canonical representation of multiplication operation (**CMCRMO**) shown in Table3, we can get the average weight of the five criteria, respectively as follows*

$$\begin{aligned}
w_{C_1}^{av} &= \tfrac{1}{5}\sum_{i}^{5} w_{C_5}^{CR} = \tfrac{1}{5}\left(3+3+2+2+1\right)\\
w_{C_2}^{av} &= \tfrac{1}{5}\sum_{i}^{5} w_{C_4}^{CR} = \tfrac{1}{5}\left(3+3+2+1+\tfrac{46}{90}\right)\\
w_{C_3}^{av} &= \tfrac{1}{5}\sum_{i}^{5} w_{C_3}^{CR} = \tfrac{1}{5}\left(2+2+1+\tfrac{46}{90}+\tfrac{46}{90}\right)\\
w_{C_4}^{av} &= \tfrac{1}{5}\sum_{i}^{5} w_{C_2}^{CR} = \tfrac{1}{5}\left(2+1+\tfrac{46}{90}+\tfrac{212}{630}+\tfrac{212}{630}\right)\\
w_{C_5}^{av} &= \tfrac{1}{5}\sum_{i}^{5} w_{C_1}^{CR} = \tfrac{1}{5}\left(1+\tfrac{46}{90}+\tfrac{46}{90}+\tfrac{212}{630}+\tfrac{212}{630}\right)
\end{aligned} \tag{17}$$

Here, $w_{C_i}^{av}$ means the average weight of the ith's criterion, the superscript av denotes average. $w_{C_i}^{CR}$ means the canonical representation of multiplication operation of the ith's criterion, the superscript CR denotes canonical

representation. The final weight of the ith's criterion, $w^f_{C_i}$, can be obtained with the normalization of average weight of each criterion $w^{av}_{C_i}$ and listed as follows

$$
\begin{aligned}
w^f_{C_1} &= \frac{w^{av}_{C_1}}{w^{av}_{C_1}+w^{av}_{C_2}+w^{av}_{C_3}+w^{av}_{C_4}+w^{av}_{C_5}} = \frac{11}{\frac{1698}{630}+\frac{2636}{630}+\frac{3794}{630}+\frac{5992}{630}+11} = 0.3283 \\
w^f_{C_2} &= \frac{w^{av}_{C_2}}{w^{av}_{C_1}+w^{av}_{C_2}+w^{av}_{C_3}+w^{av}_{C_4}+w^{av}_{C_5}} = \frac{\frac{5992}{630}}{\frac{1698}{630}+\frac{2636}{630}+\frac{3794}{630}+\frac{5992}{630}+11} = 0.2839 \\
w^f_{C_3} &= \frac{w^{av}_{C_3}}{w^{av}_{C_1}+w^{av}_{C_2}+w^{av}_{C_3}+w^{av}_{C_4}+w^{av}_{C_5}} = \frac{\frac{3794}{630}}{\frac{1698}{630}+\frac{2636}{630}+\frac{3794}{630}+\frac{5992}{630}+11} = 0.1798 \\
w^f_{C_4} &= \frac{w^{av}_{C_4}}{w^{av}_{C_1}+w^{av}_{C_2}+w^{av}_{C_3}+w^{av}_{C_4}+w^{av}_{C_5}} = \frac{\frac{2636}{630}}{\frac{1698}{630}+\frac{2636}{630}+\frac{3794}{630}+\frac{5992}{630}+11} = 0.1262 \\
w^f_{C_5} &= \frac{w^{av}_{C_5}}{w^{av}_{C_1}+w^{av}_{C_2}+w^{av}_{C_3}+w^{av}_{C_4}+w^{av}_{C_5}} = \frac{\frac{1898}{630}}{\frac{1698}{630}+\frac{2636}{630}+\frac{3794}{630}+\frac{5992}{630}+11} = 0.0818
\end{aligned} \tag{18}
$$

Note that to detail our proposed method in a easily understood way, we suppose that the fuzzy number $C_{12} = (3/2, 2, 5/2)$ means that the the weight of the second criterion C_2 is $(3/2, 2, 5/2)times$ of that of the first criterion C_1.

However, according to the Figure2, the case is verse, where $C_{12} = (3/2, 2, 5/2)$ means that the the weight of the second criterion C_1 is $(3/2, 2, 5/2)times$ of that of the first criterion C_2. As a result, if we use the linguistic variables shown in Figure2, the final weight of each criterion are shown in the right row in Table2.

3.3. The proposed fuzzy AHP algorithm

Here we detail the proposed fuzzy AHP algorithm to determine weight vector under uncertain environment step by step.

Step1: Construct the analytical hierarchy structure by domain experts. In this step, the experts will determine the objective of decision making, the relative criteria. In addition, the rating of the comparison matrix, modelled by fuzzy numbers can be given by experts through linguistic variables(for example, shown in Figue2), listed in Table2.

Step2: For each criterion, using the canonical representation of multiplication operation on fuzzy numbers to obtain the comparison matrix with

canonical representation of multiplication operation (**CMCRMO**), shown in Table3.

Step3: Determine the average weight of the ith's criterion $w_{C_i}^{av}$, respectively by Eq.19.

$$w_{C_i}^{av} = \frac{1}{N}\sum_{i}^{N} w_{C_i}^{CR} \tag{19}$$

where $w_{C_i}^{CR}$ means the canonical representation weight of multiplication operation of the ith's criterion, the superscript CR denotes canonical representation. In this average process, the conflict in preference is handled to achieve a consensus preference.

Step4: Determine the final weight of the ith's criterion, $w_{C_i}^{f}$, with the normalization of average weight of the ith's criterion $w_{C_i}^{av}$, respectively by Eq.20.

$$w_{C_i}^{f} = \frac{w_{C_i}^{av}}{\sum\limits_{i}^{N} w_{C_i}^{av}} \tag{20}$$

4. Numerical Example

Decision making is widely used in supplier management and selection [51, 53, 54, 55, 56, 57, 58]. In this section, a numerical example originated from [51] is presented to illustrate the procedure of the proposed model.

Owing to the large number of factors affecting the supplier selection decision, an orderly sequence of steps should be required to tackle it. The problem taken here has four level of hierarchy, and the different decision criteria, attributes and the decision alternatives, will be further discussed. The main objective here is the selection of best global supplier for a manufacturing firm. Application of common criteria to all suppliers makes objective comparisons possible. The criteria which are considered here in selection of the global supplier are:
(C1)Overall cost of the product
(C2)Quality of the product
(C3)Service performance of supplier
(C4)Supplier profile
(C5)Risk factor

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The AHP model of supplier selection can be constructed as shown in Fig 3

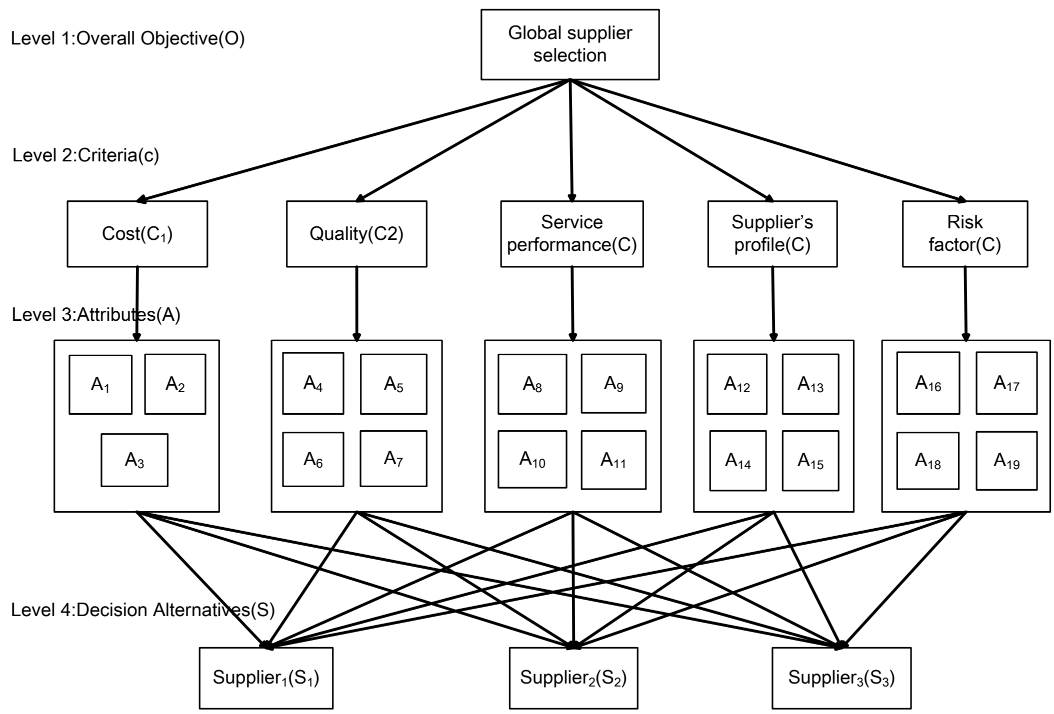

Figure 3: Hierarchy for the global supplier selection.

As been seen from Fig 3., the overall cost of the product (C1) has three factors (attributes):
(A1) Product price ,
(A2) Freight cost
(A3) Tariff and custom duties .
The quality of the product (C2) has four factors:
(A4) Rejection rate of the product ,
(A5) Increased lead time ,
(A6) Quality assessment
(A7)Remedy for quality problems.
The service performance (C3) has four attributes:
(A8) Delivery schedule ,

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(A9)Technological and R&D support ,
(A10)Response to changes
(A11) Ease of communication .
The suppliers profile (C4) has four attributes:
(A12)Financial status ,
(A13)Customer base ,
(A14)Performance history
(A15)Production facility and capacity.
The Risk factor (C5) has four attributes:
(A16)Geographical location ,
(A17)Political stability ,
(A18) Economy
(A19) Terrorism.
Refer [51] for more detailed information about the attributes mentioned above.

After the construction of the decision hierarchy of supplier selection, the fuzzy evaluation matrix of the criteria is constructed by the pairwise comparison of the different criterion relevant to the overall objective using triangular fuzzy numbers, which is shown in Table 2.

The fuzzy evaluation of criteria with respect to the overall objective can be listed in Table 2. The final weights of each criteria can be determined by the GA method. The detailed calculation process is given in Section 3. The results are listed in right side of Table 2.

In a similar way, the The fuzzy evaluation of the attributes with respect to criterion C1 to C6 can be given by domain experts and there corresponding results based on GA are listed in Table 5 to Table 9, respectively.

For the criterion C1, the summary combination of priority weights can be listed in Table 10. Also, the others summary combination of priority weights of C2 to C5 are shown in Table 11 to Table 14.

The Fuzzy evaluation of criteria with respect to the overall objective can be shown in 15. As can be seen from Table 15 and Figure 4, the best supplier is S1, which is the same to the works in [51] using the commonly used fuzzy AHP method mentioned in Section 2.4.

5. Conclusions

In this paper, a novel and simple fuzzy AHP is proposed to handle M-CDM. In our new method, the weight of each criterion can be determined by

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Table 5: The fuzzy evaluation of the attributes with respect to criterion C1

C1	A1	A2	A3	W_{C1}
A1	(1, 1, 1)	(3/2, 2, 5/2)	(3/2, 2, 5/2)	0.4747
A2	(2/5, 1/2, 2/3)	(1, 1, 1)	(3/2, 2, 5/2)	0.3333
A3	(2/5, 1/2, 2/3)	(2/5, 1/2, 2/3)	(1, 1, 1)	0.1920

Table 6: The fuzzy evaluation of the attributes with respect to criterion C2

C2	A4	A5	A6	A7	W_{C2}
A4	(1, 1, 1)	(3/2, 2, 5/2)	(2/3, 1, 3/2)	(5/2, 3, 7/2)	0.3703
A5	(2/5, 1/2, 2/3)	(1, 1, 1)	(2/3, 1, 3/2)	(3/2, 2, 5/2)	0.2391
A6	(2/3, 1, 3/2)	(2/3, 1, 3/2)	(1, 1, 1)	(3/2, 2, 5/2)	0.2663
A7	(2/7, 1/3, 2/5)	(2/5, 1/2, 2/3)	(2/5, 1/2, 2/3)	(1, 1, 1)	0.1243

Table 7: The fuzzy evaluation of the attributes with respect to criterion C3

C3	A8	A9	A10	A11	W_{C3}
A8	(1, 1, 1)	(3/2, 2, 5/2)	(5/2, 3, 7/2)	(7/2, 4, 9/2)	0.4264
A9	(2/5, 1/2, 2/3)	(1, 1, 1)	(5/2, 3, 7/2)	(5/2, 3, 7/2)	0.3274
A10	(2/7, 1/3, 2/5)	(2/7, 1/3, 2/5)	(1, 1, 1)	(3/2, 2, 5/2)	0.1566
A11	(2/9, 1/4, 2/7)	(2/7, 1/3, 2/5)	(2/5, 1/2, 2/3)	(1, 1, 1)	0.0895

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Table 8: The fuzzy evaluation of the attributes with respect to criterion C4

C4	A12	A13	A14	A15	W_{C4}
A12	(1, 1, 1)	(3/2, 2, 5/2)	(3/2, 2, 5/2)	(7/2, 4, 9/2)	0.4880
A13	(2/5, 1/2, 2/3)	(1, 1, 1)	(2/5, 1/2, 2/3)	(3/2, 2, 5/2)	0.2030
A14	(2/5, 1/2, 2/3)	(2/7, 1/3, 2/5)	(1, 1, 1)	(3/2, 2, 5/2)	0.1942
A15	(2/9, 1/4, 2/7)	(2/5, 1/2, 2/3)	(2/5, 1/2, 2/3)	(1, 1, 1)	0.1148

Table 9: The fuzzy evaluation of the attributes with respect to criterion C5

C5	A16	A17	A18	A19	W_{C5}
A16	(1, 1, 1)	(2/3,1,3/2)	(2/3, 1, 3/2)	(2/3, 1, 3/2)	0.2331
A17	(2/3, 1, 3/2)	(1, 1, 1)	(3/2, 2, 5/2)	(3/2, 2, 5/2)	0.3438
A18	(2/3, 1, 3/2)	(2/5, 1/2, 2/3)	(1, 1, 1)	(3/2, 2, 5/2)	0.2741
A19	(2/5, 1/2, 2/3)	(2/5, 1/2, 2/3)	(2/5, 1/2, 2/3)	(1, 1, 1)	0.1489

Table 10: Summary combination of priority weights: attributes of criterion C1

Weight	A1 0.4747	A2 0.3333	A3 0.1920	Alternative priority weight
Alternatives				
S1	0.71	0.44	0.69	0.6217
S2	0.13	0.36	0.08	0.1920
S3	0.16	0.20	0.23	0.1862

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Table 11: Summary combination of priority weights: attributes of criterion C2

Weight	A4 0.3703	A5 0.2391	A6 0.2663	A7 0.1243	Alternative priority weight
Alternatives					
S1	0.51	0.51	0.69	0.87	0.6027
S2	0.23	0.23	0.08	0.00	0.1615
S3	0.26	0.26	0.23	0.13	0.2359

Table 12: Summary combination of priority weights: attributes of criterion C3

Weight	A8 0.4264	A9 0.3274	A10 0.1566	A11 0.0895	Alternative priority weight
Alternatives					
S1	0.27	0.69	0.05	0.49	0.3927
S2	0.18	0.08	0.64	0.32	0.2318
S3	0.55	0.23	0.31	0.19	0.3754

Table 13: Summary combination of priority weights: attributes of criterion C4

Weight	A11 0.4880	A12 0.2030	A13 0.1942	A14 0.1148	Alternative priority weight
Alternatives					
S1	0.83	0.45	0.69	0.33	0.6683
S2	0.17	0.45	0.08	0.33	0.2277
S3	0.00	0.10	0.23	0.34	0.1040

Table 14: Summary combination of priority weights: attributes of criterion C5

	A16	A17	A18	A19	Alternative priority
Weight	0.2331	0.3438	0.2741	0.1489	weight
Alternatives					
S1	0.72	0.49	0.83	0.27	0.6040
S2	0.00	0.32	0.17	0.18	0.1834
S3	0.28	0.19	0.00	0.55	0.2125

Table 15: Summary combination of priority weights: main criteria of the overall objective

	C1	C2	C3	C4	C5	Alternative priority
Weight	0.3542	0.2696	0.1692	0.1147	0.0923	weight
Alternatives						
S1	0.6217	0.6027	0.3927	0.6683	0.6040	0.5815
S2	0.1920	0.1615	0.2318	0.2277	0.1834	0.1938
S3	0.1862	0.2359	0.3754	0.1040	0.2125	0.2246

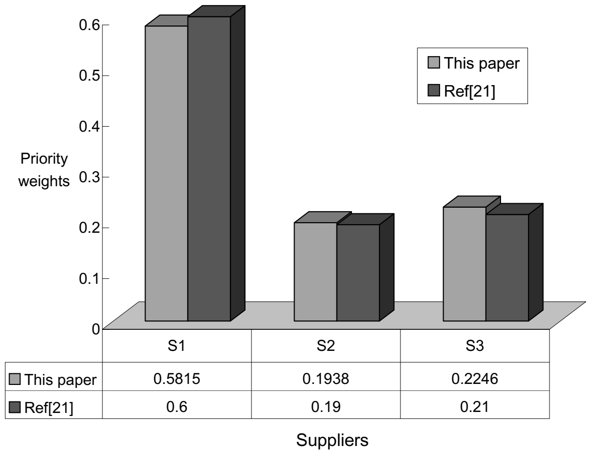

Figure 4: Comparison of proposed mthod with the previous work [21].

the the canonical representation of multiplication operation on fuzzy numbers. Instead of obtaining the eigenvector of the fuzzy comparison matrix, we get the weight simply by the ratio of each criterion. In addition, we get the final weight of each criterion by average method, which can deal with conflicts in an efficient manner. The proposed method is applied to supplier management under linguistic environment. The results show the efficiency of the proposed method. The method can be easily used in other fuzzy decision making problems.

disclosure

The author declares that there is no conflict of interests regarding the publication of this paper.

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A QUADRATURE RULE FOR THE FINITE HILBERT TRANSFORM VIA SIMPSON TYPE INEQUALITIES AND APPLICATIONS

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Abstract. In this paper, a quadrature rule on an equidistant partition of the interval $[a, b]$ for the finite Hilbert Transform of different classes of absolutely continuous functions via Simpson type inequalities is given, which may have the better error bounds than those obtained via trapezoid type inequalities. Some numerical experiments for different divisions of the interval $[a, b]$ are also presented.

1. Introduction

The finite Hilbert transform plays an important role in scientific and engineering computing. Denote by $(Tf)(a, b, \cdot)$ the finite Hilbert transform of the function $f : [a, b] \to \mathbb{R}$, i.e., we recall it

$$(Tf)(a,b;t) = \frac{1}{\pi} PV \int_a^b \frac{f(\tau)}{\tau - t} \mathrm{d}\tau := \frac{1}{\pi} \lim_{\varepsilon \to 0} \left[\int_a^{t-\varepsilon} + \int_{t+\varepsilon}^b \right] \frac{f(\tau)}{\tau - t} \mathrm{d}\tau, \tag{1.1}$$

where PV has the usual meaning of the Cauchy principle value.

There are some important approaches for evaluating finite Hilbert transforms, such as the Gaussian, Chebyshev, TANH, Iri-Moriguti-Takasawa, and double exponential quadrature methods. And for classical results on the finite Hilbert transform, see [4, 5, 6, 9, 11, 12, 13, 17].

In [5], by the use of trapezoid type rules taken on an equidistant partition of the interval $[a, b]$, Dragomir et al. proved the following inequalities for the finite Hilbert transform of different classes of absolutely continuous functions.

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function such that its derivative f' is absolutely continuous on $[a, b]$. If*

$$T_n(f;t) = \frac{f'(t)(b-a) + f(b) - f(a)}{2\pi n} + \frac{b-a}{\pi n} \sum_{i=1}^{n-1} \left[f; t - \frac{t-a}{n} \cdot i, t + \frac{b-t}{n} \cdot i \right], \tag{1.2}$$

then we have the estimate

$$\begin{aligned} &\left| (Tf)(a,b;t) - \frac{f(t)}{\pi} \ln\left(\frac{b-t}{t-a}\right) - T_n(f;t) \right| \qquad (1.3) \\ &\le \begin{cases} \dfrac{1}{4\pi n} \left[\dfrac{(b-a)^2}{4} + \left(t - \dfrac{a+b}{2}\right)^2 \right] \|f''\|_{[a,b],\infty}, & \text{if } f'' \in L_\infty[a,b]; \\ \dfrac{q}{2\pi n (q+1)^{1+\frac{1}{q}}} \left[(t-a)^{1+\frac{1}{q}} + (b-t)^{1+\frac{1}{q}} \right] \|f''\|_{[a,b],p}, & \text{if } f'' \in L_p[a,b], p > 1, \dfrac{1}{p} + \dfrac{1}{q} = 1; \\ \dfrac{1}{2\pi n} \left[\dfrac{1}{2}(b-a) + \left| t - \dfrac{a+b}{2} \right| \right] \|f''\|_{[a,b],1}, \end{cases} \\ &\le \begin{cases} \dfrac{1}{8\pi n} (b-a)^2 \|f''\|_{[a,b],\infty}, & \text{if } f'' \in L_\infty[a,b]; \\ \dfrac{q}{2\pi n (q+1)^{1+\frac{1}{q}}} (b-a)^{1+\frac{1}{q}} \|f''\|_{[a,b],p}, & \text{if } f'' \in L_p[a,b], p > 1, \dfrac{1}{p} + \dfrac{1}{q} = 1; \\ \dfrac{1}{2\pi n} (b-a) \|f''\|_{[a,b],1}, \end{cases} \end{aligned}$$

for all $t \in (a, b)$, where $[f; c, d]$ denotes the divided difference $[f; c, d] := \frac{f(c)-f(d)}{c-d}$.

Key words and phrases. Finite Hilbert transform, Simpson type inequalities.

Theorem 1.2. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable function such that the second derivative f'' is absolutely continuous on $[a,b]$. Then*

$$(1.4)\quad \left|(Tf)(a,b;t) - \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) - T_n(f;t)\right|$$
$$\leq \begin{cases} \dfrac{1}{12n^2\pi}\left[\dfrac{(b-a)^2}{12} + \left(t - \dfrac{a+b}{2}\right)^2\right](b-a)\|f'''\|_{[a,b],\infty}, & \text{if } f''' \in L_\infty[a,b]; \\ \dfrac{q[B(q+1,q+1)]^{\frac{1}{q}}}{2(2q+1)n^2\pi}\left[(t-a)^{2+\frac{1}{q}} + (b-t)^{2+\frac{1}{q}}\right]\|f'''\|_{[a,b],p}, & \text{if } f''' \in L_p[a,b], p>1, \dfrac{1}{p}+\dfrac{1}{q}=1; \\ \dfrac{1}{8\pi n^2}\left[\dfrac{(b-a)^2}{4} + \left(t - \dfrac{a+b}{2}\right)^2\right]\|f'''\|_{[a,b],1}, & \end{cases}$$
$$\leq \begin{cases} \dfrac{(b-a)^3}{36\pi n^2}\|f'''\|_{[a,b],\infty}, & \text{if } f''' \in L_\infty[a,b]; \\ \dfrac{q[B(q+1,q+1)]^{\frac{1}{q}}(b-a)^{2+\frac{1}{q}}}{2\pi(2q+1)n^2}\|f'''\|_{[a,b],p}, & \text{if } f''' \in L_p[a,b], p>1, \dfrac{1}{p}+\dfrac{1}{q}=1; \\ \dfrac{1}{16\pi n^2}(b-a)^2\|f'''\|_{[a,b],1}, & \end{cases}$$

for all $t \in (a,b)$, where $T_n(f;t)$ is defined by (1.2).

An extensive literature such as [1, 2, 3, 7, 8, 10, 14, 15, 16, 18, 19, 20, 21, 22] deal with Simpson type inequalities.

In this paper, motivated by [5], by the use of Simpson type inequalities taken on an equidistant partition of the interval $[a,b]$, a quadrature formula for the Finite Hilbert transform of different classes of absolutely continuous functions is obtained. Estimates for some error bounds and some numerical examples for the obtained approximation will also be presented.

2. THE RESULTS

Lemma 2.1. *Let $u : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. Then one has the inequalities:*

$$(2.1)\quad \left|\int_a^b u(s)\mathrm{d}s - \frac{u(a) + 4u\left(\frac{a+b}{2}\right) + u(b)}{6}(b-a)\right|$$
$$\leq \begin{cases} \dfrac{5(b-a)^2}{36}\|u'\|_{[a,b],\infty} & \text{if } u' \in L_\infty[a,b]; \\ \dfrac{2(b-a)^{1+\frac{1}{q}}\left(\frac{1}{6^{q+1}} + \frac{1}{3^{q+1}}\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\|u'\|_{[a,b],p}, & \text{if } u' \in L_p[a,b], p>1, \dfrac{1}{p}+\dfrac{1}{q}=1; \\ \dfrac{b-a}{3}\|u'\|_{[a,b],1}. & \end{cases}$$

A simple proof of this fact can be done by using the identity

$$(2.2)\quad \int_a^b u(s)\mathrm{d}s - \frac{u(a) + 4u\left(\frac{a+b}{2}\right) + u(b)}{6}(b-a) = -\left[\int_a^{\frac{a+b}{2}}\left(s - \frac{5a+b}{6}\right)u'(s)\mathrm{d}s + \int_{\frac{a+b}{2}}^b \left(s - \frac{a+5b}{6}\right)u'(s)\mathrm{d}s\right],$$

and we omit the details.

The following lemma holds.

Lemma 2.2. *Let $u : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. Then for any $t, \tau \in (a,b), t \neq \tau$ and $n \in \mathbb{N}, n \geq 1$, we have the inequality:*

$$\left|\frac{1}{\tau - t}\int_t^\tau u(s)\mathrm{d}s - \frac{1}{6n}\sum_{i=0}^{n-1}\left[u\left(t + i\cdot\frac{\tau-t}{n}\right) + 4u\left(t + \left(i+\frac{1}{2}\right)\cdot\frac{\tau-t}{n}\right) + u\left(t + (i+1)\cdot\frac{\tau-t}{n}\right)\right]\right|$$

$$\leq \begin{cases} \dfrac{5|\tau-t|}{36n}\|u'\|_{[t,\tau],\infty} & \textit{if } u'\in L_\infty[a,b];\\ \dfrac{2|\tau-t|^{\frac{1}{q}}\left(\frac{1}{6^{q+1}}+\frac{1}{3^{q+1}}\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}n}\|u'\|_{[t,\tau],p}, & \textit{if } u'\in L_p[a,b], p>1, \dfrac{1}{p}+\dfrac{1}{q}=1;\\ \dfrac{1}{3n}\|u'\|_{[t,\tau],1}, \end{cases}$$

where

$$\|u'\|_{[t,\tau],\infty} :=ess \sup_{s\in|t,\tau|}|u'(s)|, \ \textit{and}\ \|u'\|_{[t,\tau],p} := \left|\int_t^\tau |u'(s)^p|\mathrm{d}s\right|^{\frac{1}{p}}, p\geq 1.$$

Proof. Consider the equidistant division of $[t,\tau]$ (if $t<\tau$) given by

$$E_n : x_i = t + i\cdot\frac{\tau-t}{n}, i=\overline{0,n}.$$

If we apply the inequality (2.1) on the interval $[x_i,x_{i+1}]$, we may write that:

$$\left|\int_{x_i}^{x_{i+1}} u(s)\mathrm{d}s - \frac{u\Big(t+i\cdot\frac{\tau-t}{n}\Big)+4u\Big(t+\Big(i+\frac{1}{2}\Big)\cdot\frac{\tau-t}{n}\Big)+u\Big(t+(i+1)\cdot\frac{\tau-t}{n}\Big)}{6}\cdot\frac{\tau-t}{n}\right|$$

$$\leq \begin{cases} \dfrac{5(\tau-t)^2}{36n^2}\|u'\|_{[x_i,x_{i+1}],\infty}, & \text{if } u'\in L_\infty[a,b];\\ \dfrac{2|\tau-t|^{1+\frac{1}{q}}\left(\frac{1}{6^{q+1}}+\frac{1}{3^{q+1}}\right)^{\frac{1}{q}}}{n^{1+\frac{1}{q}}(q+1)^{\frac{1}{q}}}\|u'\|_{[x_i,x_{i+1}],p}, & \text{if } u'\in L_p[a,b], p>1, \dfrac{1}{p}+\dfrac{1}{q}=1;\\ \dfrac{|\tau-t|}{3n}\|u'\|_{[x_i,x_{i+1}],1}, \end{cases}$$

from which we get

$$\left|\frac{1}{\tau-t}\int_{x_i}^{x_{i+1}} u(s)\mathrm{d}s - \frac{1}{6n}\left[u\left(t+i\cdot\frac{\tau-t}{n}\right)+4u\left(t+\left(i+\frac{1}{2}\right)\cdot\frac{\tau-t}{n}\right)+u\left(t+(i+1)\cdot\frac{\tau-t}{n}\right)\right]\right|$$

$$\leq \begin{cases} \dfrac{5|\tau-t|}{36n^2}\|u'\|_{[x_i,x_{i+1}],\infty}, & \text{if } u'\in L_\infty[a,b];\\ \dfrac{2|\tau-t|^{\frac{1}{q}}\left(\frac{1}{6^{q+1}}+\frac{1}{3^{q+1}}\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}n^{1+\frac{1}{q}}}\|u'\|_{[x_i,x_{i+1}],p}, & \text{if } u'\in L_p[a,b], p>1, \dfrac{1}{p}+\dfrac{1}{q}=1;\\ \dfrac{1}{3n}\|u'\|_{[x_i,x_{i+1}],1}. \end{cases}$$

Summing over i from 0 to $n-1$ and using the generalised triangle inequality, we may write

$$\left|\frac{1}{\tau-t}\int_{t}^{\tau} u(s)\mathrm{d}s - \frac{1}{6n}\sum_{i=0}^{n-1}\left[u\left(t+i\cdot\frac{\tau-t}{n}\right)+4u\left(t+\left(i+\frac{1}{2}\right)\cdot\frac{\tau-t}{n}\right)+u\left(t+(i+1)\cdot\frac{\tau-t}{n}\right)\right]\right|$$

$$\leq \begin{cases} \dfrac{5|\tau-t|}{36n^2}\displaystyle\sum_{i=0}^{n-1}\|u'\|_{[x_i,x_{i+1}],\infty}, & \text{if } u'\in L_\infty[a,b];\\ \dfrac{2|\tau-t|^{\frac{1}{q}}\left(\frac{1}{6^{q+1}}+\frac{1}{3^{q+1}}\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}n^{1+\frac{1}{q}}}\displaystyle\sum_{i=0}^{n-1}\|u'\|_{[x_i,x_{i+1}],p}, & \text{if } u'\in L_p[a,b], p>1, \dfrac{1}{p}+\dfrac{1}{q}=1;\\ \dfrac{1}{3n}\displaystyle\sum_{i=0}^{n-1}\|u'\|_{[x_i,x_{i+1}],1}. \end{cases}$$

However,

$$\sum_{i=0}^{n-1}\|u'\|_{[x_i,x_{i+1}],\infty}\leq n\|u'\|_{[t,\tau],\infty},$$

$$\sum_{i=0}^{n-1}\|u'\|_{[x_i,x_{i+1}],p}=\sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}}|u'(s)|^p\mathrm{d}s\right|^{\frac{1}{p}}\le n^{\frac{1}{q}}\left[\left(\sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}}|u'(s)|^p\mathrm{d}s\right|^{\frac{1}{p}}\right)^p\right]^{\frac{1}{p}}=n^{\frac{1}{q}}\|u'\|_{[t,\tau],p},$$

and

$$\sum_{i=0}^{n-1}\|u'\|_{[x_i,x_{i+1}],1}\le\left|\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}|u'(s)|\mathrm{d}s\right|=\left|\int_t^\tau|u'(s)|\mathrm{d}s\right|=\|u'\|_{[t,\tau],1},$$

and the lemma is proved. □

The following theorem in approximating the Hilbert transform of a differentiable function whose derivative is absolutely continuous holds.

Theorem 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function such that its derivative f' is absolutely continuous on $[a,b]$. If*

$$\begin{aligned}T_n(f;t)=&\frac{f'(t)(b-a)+f(b)-f(a)}{6\pi n}+\frac{b-a}{3\pi n}\sum_{i=1}^{n-1}\left[f;t-\frac{t-a}{n}\cdot i,t+\frac{b-t}{n}\cdot i\right]\\&+\frac{2(b-a)}{3\pi n}\sum_{i=0}^{n-1}\left[f;t-\frac{t-a}{n}\cdot\left(i+\frac{1}{2}\right),t+\frac{b-t}{n}\cdot\left(i+\frac{1}{2}\right)\right],\end{aligned}\tag{2.3}$$

then we have the estimate

$$\left|(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-T_n(f;t)\right|\tag{2.4}$$

$$\le\begin{cases}\frac{5}{36\pi n}\left[\frac{(b-a)^2}{4}+\left(t-\frac{a+b}{2}\right)^2\right]\|f''\|_{[a,b],\infty}, & \text{if } f''\in L_\infty[a,b];\\ \frac{2q\left(\frac{1}{6^{q+1}}+\frac{1}{3^{q+1}}\right)^{\frac{1}{q}}}{\pi(q+1)^{\frac{1}{q}}}\left[(t-a)^{1+\frac{1}{q}}+(b-t)^{1+\frac{1}{q}}\right]\|f''\|_{[a,b],p}, & \text{if } f''\in L_p[a,b], p>1, \frac{1}{p}+\frac{1}{q}=1;\\ \frac{1}{3\pi n}\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]\|f''\|_{[a,b],1},\end{cases}$$

$$\le\begin{cases}\frac{5}{72\pi n}(b-a)^2\|f''\|_{[a,b],\infty}, & \text{if } f''\in L_\infty[a,b];\\ \frac{2q\left(\frac{1}{6^{q+1}}+\frac{1}{3^{q+1}}\right)^{\frac{1}{q}}}{\pi(q+1)^{\frac{1}{q}}}(b-a)^{1+\frac{1}{q}}\|f''\|_{[a,b],p}, & \text{if } f''\in L_p[a,b], p>1, \frac{1}{p}+\frac{1}{q}=1;\\ \frac{1}{3\pi n}(b-a)\|f''\|_{[a,b],1},\end{cases}$$

for all $t\in(a,b)$.

Proof. Applying Lemma 2.2 for the function f', we may write that

$$\begin{aligned}&\left|\frac{f(\tau)-f(t)}{\tau-t}-\frac{1}{6n}\left[f'(t)+\sum_{i=1}^{n-1}f'\left(t+i\cdot\frac{\tau-t}{n}\right)+4\sum_{i=0}^{n-1}f'\left(t+\left(i+\frac{1}{2}\right)\cdot\frac{\tau-t}{n}\right)\right.\right.\\&\left.\left.+f'(\tau)+\sum_{i=0}^{n-2}f'\left(t+(i+1)\cdot\frac{\tau-t}{n}\right)\right]\right|\end{aligned}\tag{2.5}$$

$$\le\begin{cases}\frac{5|\tau-t|}{36n}\|f''\|_{[t,\tau],\infty}, & \text{if } f''\in L_\infty[a,b];\\ \frac{2|\tau-t|^{\frac{1}{q}}\left(\frac{1}{6^{q+1}}+\frac{1}{3^{q+1}}\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}n}\|f''\|_{[t,\tau],p}, & \text{if } f''\in L_p[a,b], p>1, \frac{1}{p}+\frac{1}{q}=1;\\ \frac{1}{3n}\|f''\|_{[t,\tau],1}.\end{cases}$$

However,

$$\sum_{i=1}^{n-1} f'\left(t+i\cdot\frac{\tau-t}{n}\right)=\sum_{i=0}^{n-2} f'\left(t+(i+1)\cdot\frac{\tau-t}{n}\right)$$

and then by (2.5), we may write:

$$\left|\frac{f(\tau)-f(t)}{\tau-t}-\left[\frac{f'(t)+f'(\tau)}{6n}+\frac{1}{3n}\sum_{i=1}^{n-1} f'\left(t+i\cdot\frac{\tau-t}{n}\right)+\frac{2}{3n}\sum_{i=0}^{n-1} f'\left(t+\left(i+\frac{1}{2}\right)\cdot\frac{\tau-t}{n}\right)\right]\right| \tag{2.6}$$
$$\leq\begin{cases}\dfrac{5|\tau-t|}{36n}\|f''\|_{[t,\tau],\infty},\\ \dfrac{2|\tau-t|^{\frac{1}{q}}\left(\frac{1}{6^{q+1}}+\frac{1}{3^{q+1}}\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}n}\|f''\|_{[t,\tau],p},\\ \dfrac{1}{3n}\|f''\|_{[t,\tau],1},\end{cases}$$

for any $t,\tau\in[a,b], t\neq\tau$.

Consequently, we have

$$\left|\frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}\mathrm{d}\tau-\frac{1}{\pi}PV\int_a^b\left[\frac{f'(t)+f'(\tau)}{6n}+\frac{1}{3n}\sum_{i=1}^{n-1} f'\left(t+i\cdot\frac{\tau-t}{n}\right)\right.\right. \tag{2.7}$$
$$\left.\left.+\frac{2}{3n}\sum_{i=0}^{n-1} f'\left(t+\left(i+\frac{1}{2}\right)\cdot\frac{\tau-t}{n}\right)\right]\mathrm{d}\tau\right|$$
$$\leq\begin{cases}\dfrac{5}{36\pi n}PV\displaystyle\int_a^b|\tau-t|\|f''\|_{[t,\tau],\infty}\mathrm{d}\tau,\\ \dfrac{2\left(\frac{1}{6^{q+1}}+\frac{1}{3^{q+1}}\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}\pi n}PV\displaystyle\int_a^b|\tau-t|^{\frac{1}{q}}\|f''\|_{[t,\tau],p}\mathrm{d}\tau,\\ \dfrac{1}{3\pi n}PV\displaystyle\int_a^b\|f''\|_{[t,\tau],1}\mathrm{d}\tau.\end{cases}$$

Since

$$PV\int_a^b\left[\frac{f'(t)+f'(\tau)}{6n}+\frac{1}{3n}\sum_{i=1}^{n-1} f'\left(t+i\cdot\frac{\tau-t}{n}\right)+\frac{2}{3n}\sum_{i=0}^{n-1} f'\left(t+\left(i+\frac{1}{2}\right)\cdot\frac{\tau-t}{n}\right)\right]\mathrm{d}\tau$$
$$=\lim_{\varepsilon\to0^+}\left(\int_a^{t-\varepsilon}+\int_{t+\varepsilon}^b\right)\left(\frac{f'(t)+f'(\tau)}{6n}+\frac{1}{3n}\sum_{i=1}^{n-1} f'\left(t+i\cdot\frac{\tau-t}{n}\right)\right.$$
$$\left.+\frac{2}{3n}\sum_{i=0}^{n-1} f'\left(t+\left(i+\frac{1}{2}\right)\cdot\frac{\tau-t}{n}\right)\right)\mathrm{d}\tau$$
$$=\frac{f'(t)(b-a)+f(b)-f(a)}{6n}+\frac{1}{3n}\sum_{i=1}^{n-1}\left[\lim_{\varepsilon\to0^+}\left(\int_a^{t-\varepsilon}+\int_{t+\varepsilon}^b\right)\left(f'\left(t+i\cdot\frac{\tau-t}{n}\right)\right)\right]\mathrm{d}\tau$$
$$+\frac{2}{3n}\sum_{i=0}^{n-1}\left[\lim_{\varepsilon\to0^+}\left(\int_a^{t-\varepsilon}+\int_{t+\varepsilon}^b\right)\left(f'\left(t+\left(i+\frac{1}{2}\right)\cdot\frac{\tau-t}{n}\right)\right)\right]\mathrm{d}\tau$$
$$=\frac{f'(t)(b-a)+f(b)-f(a)}{6n}+\frac{1}{3n}\sum_{i=1}^{n-1}\left[\lim_{\varepsilon\to0^+}\left[\frac{n}{i}\cdot f\left(t+i\cdot\frac{\tau-t}{n}\right)\left(\Big|_a^{t-\varepsilon}+\Big|_{t+\varepsilon}^b\right)\right]\right]$$
$$+\frac{2}{3n}\sum_{i=0}^{n-1}\left[\lim_{\varepsilon\to0^+}\left[\frac{2n}{2i+1}\cdot f\left(t+\left(i+\frac{1}{2}\right)\cdot\frac{\tau-t}{n}\right)\left(\Big|_a^{t-\varepsilon}+\Big|_{t+\varepsilon}^b\right)\right]\right]$$
$$=\frac{f'(t)(b-a)+f(b)-f(a)}{6n}+\frac{1}{3n}\sum_{i=1}^{n-1}\frac{n}{i}\left[f\left(t+i\cdot\frac{b-t}{n}\right)-f\left(t+i\cdot\frac{a-t}{n}\right)\right]$$

$$+\frac{2}{3n}\sum_{i=0}^{n-1}\frac{2n}{2i+1}\left[f\left(t+\left(i+\frac{1}{2}\right)\cdot\frac{b-t}{n}\right)-f\left(t+\left(i+\frac{1}{2}\right)\cdot\frac{a-t}{n}\right)\right]$$

$$=\frac{f'(t)(b-a)+f(b)-f(a)}{6n}+\frac{b-a}{3n}\sum_{i=1}^{n-1}\left[f;t+i\cdot\frac{b-t}{n},t+i\cdot\frac{a-t}{n}\right]$$

$$+\frac{2(b-a)}{3n}\sum_{i=0}^{n-1}\left[f;t+\left(i+\frac{1}{2}\right)\cdot\frac{b-t}{n},t+\left(i+\frac{1}{2}\right)\cdot\frac{a-t}{n}\right],$$

and

$$PV\int_a^b|\tau-t|\|f''\|_{[t,\tau],\infty}\mathrm{d}\tau\le\|f''\|_{[a,b],\infty}PV\int_a^b|\tau-t|\mathrm{d}\tau=\|f''\|_{[a,b],\infty}\left[\frac{1}{4}(b-a)^2+\left(t-\frac{a+b}{2}\right)\right],$$

$$PV\int_a^b|\tau-t|^{\frac{1}{q}}\|f''\|_{[t,\tau],p}\mathrm{d}\tau\le\|f''\|_{[a,b],p}PV\int_a^b|\tau-t|^{\frac{1}{q}}\mathrm{d}\tau=\frac{q\|f''\|_{[a,b],p}}{q+1}\left[(t-a)^{1+\frac{1}{q}}+(b-t)^{1+\frac{1}{q}}\right],$$

$$PV\int_a^b\|f''\|_{[t,\tau],1}\mathrm{d}\tau=PV\left[\int_a^t\|f''\|_{[\tau,t],1}\mathrm{d}\tau+\int_t^b\|f''\|_{[t,\tau],1}\mathrm{d}\tau\right]$$

$$\le\|f''\|_{[a,t],1}(t-a)+\|f''\|_{[t,b],1}(b-t)\le\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]\|f''\|_{[a,b],1},$$

then, by (2.7) we get

$$\begin{aligned}(2.8)\quad&\left|\frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}\mathrm{d}\tau-\frac{f'(t)(b-a)+f(b)-f(a)}{6\pi n}-\frac{b-a}{3\pi n}\sum_{i=1}^{n-1}\left[f;t-\frac{t-a}{n}\cdot i,t+\frac{b-t}{n}\cdot i\right]\right.\\&\left.-\frac{2(b-a)}{3\pi n}\sum_{i=0}^{n-1}\left[f;t-\frac{t-a}{n}\cdot\left(i+\frac{1}{2}\right),t+\frac{b-t}{n}\cdot\left(i+\frac{1}{2}\right)\right]\right|\\&\le\begin{cases}\dfrac{5\|f''\|_{[a,b],\infty}}{36\pi n}\left[\dfrac{1}{4}(b-a)^2+\left(t-\dfrac{a+b}{2}\right)\right], & \text{if } f''\in L_\infty[a,b];\\ \dfrac{2q\left(\frac{1}{6^{q+1}}+\frac{1}{3^{q+1}}\right)^{\frac{1}{q}}\|f''\|_{[a,b],p}}{\pi(q+1)^{1+\frac{1}{q}}n}\left[(t-a)^{1+\frac{1}{q}}+(b-t)^{1+\frac{1}{q}}\right], & \text{if } f''\in L_p[a,b],p>1,\dfrac{1}{p}+\dfrac{1}{q}=1;\\ \dfrac{\|f''\|_{[a,b],1}}{3\pi n}\left[\dfrac{1}{2}(b-a)+\left|t-\dfrac{a+b}{2}\right|\right].\end{cases}\end{aligned}$$

On the other hand, as for the function $f_0:(a,b)\to\mathbb{R}, f_0(t)=1$, we have

$$(T,f_0)(a,b;t)=\frac{1}{\pi}\ln\left(\frac{b-a}{t-a}\right),t\in(a,b),$$

then obviously

$$(Tf)(a,b;t)=\frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)+f(t)}{\tau-t}\mathrm{d}\tau=\frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}\mathrm{d}\tau+\frac{f(t)}{\pi}PV\int_a^b\frac{\mathrm{d}\tau}{\tau-t},$$

from which we get the equality:

$$(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)=\frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}\mathrm{d}\tau.\tag{2.9}$$

Finally, using (2.8) and (2.9), we deduce (2.4). □

Before we proceed with another estimate of the remainder in approximating the Hilbert Transform for functions whose second derivatives are absolutely continuous, we need the following lemma.

Lemma 2.3. *Let $u:[a,b]\to\mathbb{R}$ be a function such that its derivative is absolutely continuous on $[a,b]$. Then one has the inequalities:*

$$\left|\int_a^b u(s)\mathrm{d}s-\frac{u(a)+4u\left(\frac{a+b}{2}\right)+u(b)}{6}(b-a)\right|\tag{2.10}$$

$$\leq \begin{cases} \dfrac{(b-a)^3}{81}\|u''\|_{[a,b],\infty}, & \textit{if } u'' \in L_\infty[a,b]; \\ \dfrac{(b-a)^{2+\frac{1}{q}}}{2}\Lambda\|u''\|_{[a,b],p}, & \textit{if } u'' \in L_p[a,b], p>1, \dfrac{1}{p}+\dfrac{1}{q}=1; \\ \dfrac{(b-a)^2}{24}\|u''\|_{[a,b],1}, \end{cases}$$

where

$$\Lambda = \left[\left(\int_0^{\frac{1}{3}} s^q\left(\frac{1}{3}-s\right)^q \mathrm{d}s + \int_{\frac{1}{3}}^{\frac{1}{2}} s^q\left(s-\frac{1}{3}\right)^q \mathrm{d}s\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^{\frac{2}{3}} (1-s)^q\left(\frac{2}{3}-s\right)^q \mathrm{d}s + \int_{\frac{2}{3}}^{1} (1-s)^q\left(s-\frac{2}{3}\right)^q \mathrm{d}s\right)^{\frac{1}{q}}\right]. \tag{2.11}$$

A simple proof of the fact can be done by the use of the following identity:

$$\int_a^b u(s)\mathrm{d}s - \frac{u(a)+4u\left(\frac{a+b}{2}\right)+u(b)}{6}(b-a) = -\frac{1}{2}\int_a^{\frac{a+b}{2}}(s-a)\left(s-\frac{2a+b}{3}\right)u''(s)\mathrm{d}s - \frac{1}{2}\int_{\frac{a+b}{2}}^b (s-b)\left(s-\frac{a+2b}{3}\right)u''(s)\mathrm{d}s, \tag{2.12}$$

and we omit the details.

The following lemma also holds.

Lemma 2.4. *Let $u:[a,b]\to\mathbb{R}$ be a differentiable function such that $u':[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$. Then for any $t,\tau\in(a,b), t\neq\tau$ and $n\in\mathbb{N}, n\geq 1$, we have the inequality:*

$$\left|\frac{1}{\tau-t}\int_t^\tau u(s)\mathrm{d}s - \frac{1}{6n}\sum_{i=0}^{n-1}\left[u\left(t+i\cdot\frac{\tau-t}{n}\right)+4u\left(t+\left(i+\frac{1}{2}\right)\cdot\frac{\tau-t}{n}\right)+u\left(t+(i+1)\cdot\frac{\tau-t}{n}\right)\right]\right|$$

$$\leq \begin{cases} \dfrac{|\tau-t|^2}{81n^2}\|u''\|_{[t,\tau],\infty}, & \textit{if } u'' \in L_\infty[a,b]; \\ \dfrac{|\tau-t|^{1+\frac{1}{q}}}{2n^2}\Lambda\|u''\|_{[t,\tau],p}, & \textit{if } u'' \in L_p[a,b], p>1, \dfrac{1}{p}+\dfrac{1}{q}=1; \\ \dfrac{|\tau-t|}{24n^2}\|u''\|_{[t,\tau],1}. \end{cases}$$

Proof. Consider the equidistant division of $[t,\tau]$ (if $t<\tau$)

$$E_n : x_i = t + i\cdot\frac{\tau-t}{n}, i=\overline{0,n}.$$

If we apply the inequality (2.10), we may state that

$$\left|\int_{x_i}^{x_{i+1}} u(s)\mathrm{d}s - \frac{u\left(t+i\cdot\frac{\tau-t}{n}\right)+4u\left(t+\left(i+\frac{1}{2}\right)\cdot\frac{\tau-t}{n}\right)+u\left(t+(i+1)\cdot\frac{\tau-t}{n}\right)}{6}\cdot\frac{\tau-t}{n}\right|$$

$$\leq \begin{cases} \dfrac{|\tau-t|^3}{81n^3}\|u''\|_{[x_i,x_{i+1}],\infty}, & \text{if } u'' \in L_\infty[a,b]; \\ \dfrac{|\tau-t|^{2+\frac{1}{q}}}{2n^{2+\frac{1}{q}}}\Lambda\|u''\|_{[x_i,x_{i+1}],p}, & \text{if } u'' \in L_p[a,b], p>1, \dfrac{1}{p}+\dfrac{1}{q}=1; \\ \dfrac{|\tau-t|^2}{24n^2}\|u''\|_{[x_i,x_{i+1}],1}. \end{cases}$$

Dividing by $|\tau-t|>0$ and using a similar argument to the one in Lemma 2.2, we conclude that the desired inequality holds. □

The following theorem in approximating the Hilbert transform of a twice differentiable function whose second derivative f'' is absolutely continuous also holds.

Theorem 2.2. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable function such that the second derivative f'' is absolutely continuous on $[a,b]$. Then*

$$
\left|(Tf)(a,b;t) - \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) - T_n(f;t)\right| \tag{2.13}
$$

$$
\leq \begin{cases} \dfrac{1}{81n^2\pi}\left[\dfrac{(b-a)^2}{12} + \left(t - \dfrac{a+b}{2}\right)^2\right](b-a)\|f'''\|_{[a,b],\infty}, & \text{if } f''' \in L_\infty[a,b]; \\ \dfrac{q[(t-a)^{2+\frac{1}{q}} + (b-t)^{2+\frac{1}{q}}]\Lambda}{2(2q+1)n^2\pi}\|f'''\|_{[a,b],p}, & \text{if } f''' \in L_p[a,b], p>1, \dfrac{1}{p}+\dfrac{1}{q}=1; \\ \dfrac{1}{24n^2\pi}\left[\dfrac{(b-a)^2}{4} + \left(t - \dfrac{a+b}{2}\right)^2\right]\|f'''\|_{[a,b],1}, & \end{cases}
$$

$$
\leq \begin{cases} \dfrac{(b-a)^3}{243n^2\pi}\|f'''\|_{[a,b],\infty}, & \text{if } f''' \in L_\infty[a,b]; \\ \dfrac{q(b-a)^{2+\frac{1}{q}}\Lambda}{2\pi(2q+1)n^2}\|f'''\|_{[a,b],p}, & \text{if } f''' \in L_p[a,b], p>1, \dfrac{1}{p}+\dfrac{1}{q}=1; \\ \dfrac{(b-a)^2}{48n^2\pi}\|f'''\|_{[a,b],1}, & \end{cases}
$$

for all $t \in (a,b)$, where $T_n(f;t)$ is defined by (2.3) *and Λ is defined by* (2.11).

Proof. Applying Lemma 2.4 for the function f', we may write that (see also Theorem 2.1)

$$
\left|\frac{f(\tau)-f(t)}{\tau - t} - \left[\frac{f'(t)+f'(\tau)}{6n} + \frac{1}{3n}\sum_{i=1}^{n-1} f'\left(t + i\cdot\frac{\tau-t}{n}\right) + \frac{2}{3n}\sum_{i=0}^{n-1} f'\left(t + \left(i+\frac{1}{2}\right)\cdot\frac{\tau-t}{n}\right)\right]\right| \tag{2.14}
$$

$$
\leq \begin{cases} \dfrac{|\tau-t|^2}{81n^2}\|f'''\|_{[t,\tau],\infty}, & \text{if } f''' \in L_\infty[a,b]; \\ \dfrac{|\tau-t|^{1+\frac{1}{q}}}{2n^2}\Lambda\|f'''\|_{[t,\tau],p}, & \text{if } f''' \in L_p[a,b], p>1, \dfrac{1}{p}+\dfrac{1}{q}=1; \\ \dfrac{|\tau-t|}{24n^2}\|f'''\|_{[t,\tau],1}, & \end{cases}
$$

for any $t, \tau \in [a,b], t \neq \tau$. Consequently, we may write:

$$
\begin{aligned} &\left|\frac{1}{\pi}PV\int_a^b \frac{f(\tau)-f(t)}{\tau-t}\mathrm{d}\tau - \frac{1}{\pi}PV\int_a^b\left[\frac{f'(t)+f'(\tau)}{6n} + \frac{1}{3n}\sum_{i=1}^{n-1} f'\left(t+i\cdot\frac{\tau-t}{n}\right)\right.\right. \\ &\left.\left. + \frac{2}{3n}\sum_{i=0}^{n-1} f'\left(t+\left(i+\frac{1}{2}\right)\cdot\frac{\tau-t}{n}\right)\right]\mathrm{d}\tau\right| \end{aligned} \tag{2.15}
$$

$$
\leq \begin{cases} \dfrac{1}{81n^2\pi}PV\displaystyle\int_a^b |\tau-t|^2\|f'''\|_{[t,\tau],\infty}\mathrm{d}\tau, \\ \dfrac{\Lambda}{2n^2\pi}PV\displaystyle\int_a^b |\tau-t|^{1+\frac{1}{q}}\|f'''\|_{[t,\tau],p}\mathrm{d}\tau, \\ \dfrac{1}{24n^2\pi}PV\displaystyle\int_a^b |\tau-t|\|f'''\|_{[t,\tau],1}\mathrm{d}\tau. \end{cases}
$$

since

$$
\begin{aligned} &PV\int_a^b |\tau-t|^2\|f'''\|_{[t,\tau],\infty}\mathrm{d}\tau \leq \|f'''\|_{[a,b],\infty}PV\int_a^b|\tau-t|^2\mathrm{d}\tau \\ =&\|f'''\|_{[a,b],\infty}\left[\frac{(t-a)^3+(b-t)^3}{3}\right] = \|f'''\|_{[a,b],\infty}\left[\frac{(b-a)^2}{12} + \left(t-\frac{a+b}{2}\right)^2\right](b-a), \end{aligned}
$$

$$
PV\int_a^b |\tau-t|^{1+\frac{1}{q}}\|f'''\|_{[t,\tau],p}\mathrm{d}\tau \leq \|f'''\|_{[a,b],p}PV\int_a^b|\tau-t|^{1+\frac{1}{q}}\mathrm{d}\tau
$$

$$=\|f'''\|_{[a,b],p}\frac{(b-t)^{2+\frac{1}{q}}+(t-a)^{2+\frac{1}{q}}}{2+\frac{1}{q}}=\frac{q\|f'''\|_{[a,b],p}}{2q+1}\left[(b-t)^{2+\frac{1}{q}}+(t-a)^{2+\frac{1}{q}}\right]$$

and

$$PV\int_a^b|\tau-t|\|f'''\|_{[t,\tau],1}\mathrm{d}\tau\le\left[\frac{(b-a)^2}{4}+\left(t-\frac{a+b}{2}\right)^2\right]\|f'''\|_{[a,b],1}.$$

Then by (2.15), we deduce the first part of (2.13). □

3. NUMERICAL EXPERIMENTS

For a function $f:[a,b]\to\mathbb{R}$, we may consider the quadrature formula

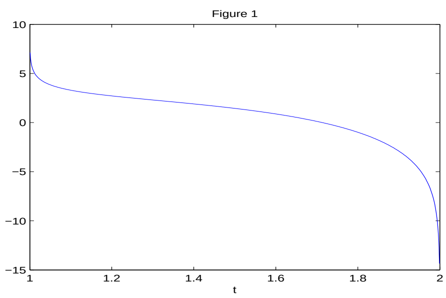
$$E_n(f;a,b,t):=\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)+T_n(f;t),t\in[a,b].$$

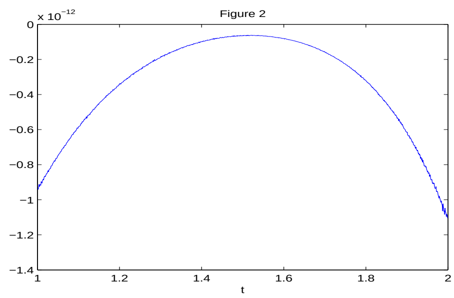
As shown in the above section, $E_n(f;a,b,t)$ provides an approximation for the Finite Hilbert Transform $(Tf)(a,b;t)$ and the error estimate fulfils the bounds described in (2.4) and (2.15).

If we consider the function $f:[1,2]\to\mathbb{R}$, $f(x)=\exp(x)$, the exact value of the Hilbert transform is

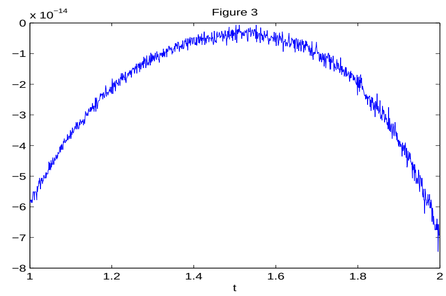
$$(Tf)(a,b;t)=\frac{\exp(t)Ei(2-t)-\exp(t)Ei(1-t)}{\pi},t\in[1,2].$$

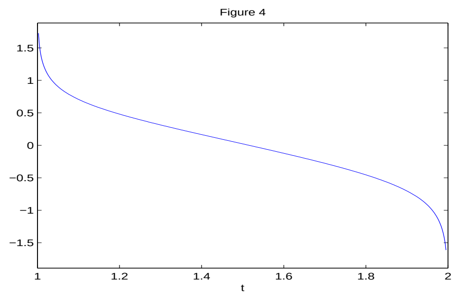
and the plot of this function is embodied in Figure 1.





If we implement the quadrature formula provided by $E_n(f;a,b,t)$ using Matlab and chose the value of $n=100$, the error $E_r(f;a,b,t):=(Tf)(a,b;t)-E_n(f;a,b,t)$ has the variation described in Figure 2.





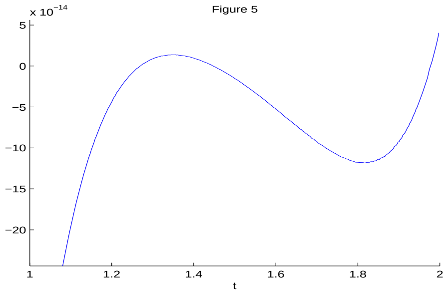
For $n=200$, the plot of $E_r(f;a,b,t)$ is embodied in the following Figure 3, from which we can see that the precision of the error gets higher when n gets bigger.

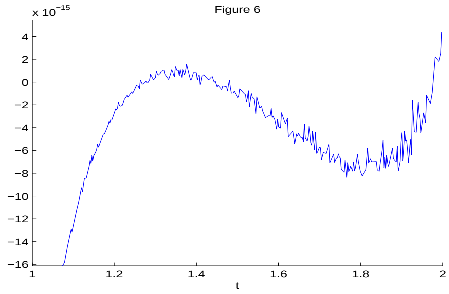
Now, if we consider another function $f:[1,2]\to\mathbb{R}$, $f(x)=\sin(x)$, then the exact value of Hilbert transform is

$$(Tf)(a,b;t)=\frac{-S_i(-2+t)\cos(t)+C_i(2-t)\sin(t)+S_i(t-1)\cos(t)-\sin(t)C_i(t-1)}{\pi},t\in[1,2]$$

having the plot embodied in Figure 4.

If we choose the value of $n=50$, then the error $E_r(f;a,b,t):=(Tf)(a,b;t)-E_n(f;a,b,t)$ for the function $f(x)=\sin x$, $x\in[a,b]$ has the variation described in Figure 5. Moreover, for $n=100$, the behaviour of $E_r(f;a,b,t)$ is plotted in Figure 6.





Remark 1. *When $n = 100$, for function $f(x) = \exp(x)$, the precision of the error is 10^{-06} in [5], while the precision obtained here is 10^{-12}. When $n = 200$, we also have the higher precision. For function $f(x) = \sin(x)$, it's the same situation. Therefore, our results may have the better error bounds.*

Acknowledgments

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A QUADRATURE FORMULA IN APPROXIMATING THE FINITE HILBERT TRANSFORM VIA PERTURBED TRAPEZOID TYPE INEQUALITIES

SHUNFENG WANG, XINGYUE GAO AND NA LU

Abstract. In this paper, we obtain the error estimation of a quadrature formula in approximating the finite Hilbert transform on an equidistant partition of the interval $[a, b]$. Some numerical examples for the obtained approximation are also presented.

1. Introduction

In the recent year, many authors tried to consider error inequalities for some known and some new quadrature rules. For example, the well-known trapezoid and midpoint quadrature rules were considered (see [1], [4], [6], [9], [11], [12], [14], [15], [18], [19] and [20]). In [5], the authors proved the following theorem:

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a mapping such the derivative $f^{(n-1)}$ $(n \geq 1)$ is absolutely continuous on $[a, b]$. Then*

$$\int_a^b f(t)\mathrm{d}t = \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left[(x-a)^{k+1}f^{(k)}(a) + (-1)^k(b-x)^{k+1}f^{(k)}(b)\right] + \frac{1}{n!}\int_a^b (x-t)^n f^{(n)}(t)\mathrm{d}t \tag{1.1}$$

for all $x \in [a, b]$.

Specially, we can obtain the following identity from (1.1) with $x = \frac{a+b}{2}$:

$$\int_a^b f(t)\mathrm{d}t = \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{2^{k+1}(k+1)!}\left[f^{(k)}(a) + (-1)^k f^{(k)}(b)\right] + \frac{(-1)^n}{n!}\int_a^b \left(t - \frac{a+b}{2}\right)^n f^{(n)}(t)\mathrm{d}t. \tag{1.2}$$

In (1.2), for $n = 1$, we obtain the trapezoid rule

$$\int_a^b f(t)\mathrm{d}t = \frac{f(b)+f(a)}{2}(b-a) + \int_a^b \left(\frac{a+b}{2} - t\right) f'(t)\mathrm{d}t. \tag{1.3}$$

The finite Hilbert transform of the function $f : (a, b) \to \mathbb{R}$ is defined as

$$T(f)(a,b;t) = \frac{1}{\pi} PV \int_a^b \frac{f(\tau)}{\tau - t}\mathrm{d}\tau = \lim_{\varepsilon \to 0}\left[\int_a^{t-\varepsilon} + \int_{t+\varepsilon}^b\right] \frac{f(\tau)}{\pi(\tau - t)}\mathrm{d}\tau$$

where PV has the usual meaning of the Cauthy principle value (see [3]).

In [7], the authors used the inequality (1.3) to approximate the finite Hilbert transform and obtain the following theorem:

Theorem 1.2. *Let $f : [a, b] \to \mathbb{R}$ be such that $f' : [a, b] \to \mathbb{R}$ is absolutely continuous on $[a, b]$. Then we have the bounds*

$$\left|T(f)(a,b;t) - \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) - \frac{1}{2\pi}\left[f(b) - f(a) + f'(t)(b-a)\right]\right|$$

$$\leq \begin{cases} \dfrac{\|f''\|_\infty}{4\pi}\left[\dfrac{(b-a)^2}{4} + \left(t - \dfrac{b+a}{2}\right)\right]^2, & f'' \in L^\infty[a,b]; \\ \dfrac{q\,\|f''\|_p}{2\pi(q+1)^{1+\frac{1}{q}}}\left[(t-a)^{1+\frac{1}{q}} + (b-t)^{1+\frac{1}{q}}\right], & p > 1, \dfrac{1}{p} + \dfrac{1}{q} = 1 \quad and \quad f'' \in L^p[a,b]; \\ \dfrac{\|f''\|_1}{2\pi}(b-a), & f'' \in L^1[a,b], \end{cases} \tag{1.4}$$

for all $t \in (a, b)$, where $\|\cdot\|_p$ are the usual Lebesgue norms in $L^p[a, b]$ $(1 \leq p \leq \infty)$.

Key words and phrases. perturbed trapezoid type inequality; numerical integration; finite Hilbert transform.

In [8], by the use of trapezoid type rules taken on an equidistant partition of the interval $[a,b]$, Dragomir et al. proved the following inequalities for the finite Hilbert transform of different classes of absolutely continuous functions.

Theorem 1.3. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function such that its derivative f' is absolutely continuous on $[a,b]$. If*

$$T_n(f;t)=\frac{f'(t)(b-a)+f(b)-f(a)}{2\pi n}+\frac{b-a}{\pi n}\sum_{i=1}^{n-1}\left[f;t-\frac{t-a}{n}\cdot i,t+\frac{b-t}{n}\cdot i\right],\tag{1.5}$$

then we have the estimate

$$\left|(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-T_n(f;t)\right|\tag{1.6}$$
$$\le\begin{cases}\dfrac{1}{4\pi n}\left[\dfrac{(b-a)^2}{4}+\left(t-\dfrac{a+b}{2}\right)^2\right]\|f''\|_{[a,b],\infty}, & \text{if } f''\in L_\infty[a,b];\\ \dfrac{q}{2\pi n(q+1)^{1+\frac1q}}\left[(t-a)^{1+\frac1q}+(b-t)^{1+\frac1q}\right]\|f''\|_{[a,b],p}, & \text{if } f''\in L_p[a,b],p>1,\dfrac1p+\dfrac1q=1;\\ \dfrac{1}{2\pi n}\left[\dfrac12(b-a)+\left|t-\dfrac{a+b}{2}\right|\right]\|f''\|_{[a,b],1}, & \end{cases}$$
$$\le\begin{cases}\dfrac{1}{8\pi n}(b-a)^2\|f''\|_{[a,b],\infty}, & \text{if } f''\in L_\infty[a,b];\\ \dfrac{q}{2\pi n(q+1)^{1+\frac1q}}(b-a)^{1+\frac1q}\|f''\|_{[a,b],p}, & \text{if } f''\in L_p[a,b],p>1,\dfrac1p+\dfrac1q=1;\\ \dfrac{1}{2\pi n}(b-a)\|f''\|_{[a,b],1}, & \end{cases}$$

for all $t\in(a,b)$, where $[f;c,d]$ denotes the divided difference $[f;c,d]:=\frac{f(c)-f(d)}{c-d}$.

If we put $n=2$ in (1.2), we can get the perturbed trapezoid rule

$$\int_a^b f(t)\mathrm{d}t=\frac{f(b)+f(a)}{2}(b-a)-\frac{(b-a)^2}{8}\left[f'(b)-f'(a)\right]+\frac12\int_a^b\left(t-\frac{a+b}{2}\right)^2f''(t)\mathrm{d}t,\tag{1.7}$$

Recently, Liu and Pan [16] proved the following inequalities for the finite Hilbert transform of different classes of absolutely continuous functions via the above rule (1.7) (see also [13] for other related results).

Theorem 1.4. *Let $f:[a,b]\to\mathbb{R}$ be such that $f'':[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$. Then we have the bounds*

$$\Bigg|T(f)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-\frac{f'(t)}{2\pi}(b-a)-\frac{5}{8\pi}\left[f(b)-f(a)\right]\tag{1.8}$$
$$+\frac{f'(b)}{8\pi}(b-t)+\frac{f''(t)}{16\pi}(a-b)(a+b-2t)-\frac{f'(a)}{8\pi}(t-a)\Bigg|$$
$$\le\begin{cases}\dfrac{\|f'''\|_\infty}{24\pi}\left[(b-a)\left(t-\dfrac{b+a}{2}\right)^2+\dfrac{(b-a)^3}{12}\right], & f'''\in L^\infty[a,b];\\ \dfrac{q\,\|f'''\|_p}{8\pi(2q+1)^{1+\frac1q}}\left[(t-a)^{2+\frac1q}+(b-t)^{2+\frac1q}\right], & p>1,\dfrac1p+\dfrac1q=1 \ \text{ and } \ f'''\in L^p[a,b];\\ \dfrac{\|f'''\|_1}{8\pi}\left[\left(t-\dfrac{a+b}{2}\right)^2+\dfrac{(b-a)^2}{4}\right], & f'''\in L^1[a,b];\end{cases}$$

for all $t\in(a,b)$, where $\|\cdot\|_p$ $(1\le p\le\infty)$ are the usual Lebesgue norms in $L^p[a,b]$.

In this paper, inspired by [8], we shall derive a quadrature formula in approximating the finite Hilbert transform of different classes of absolutely continuous functions. Some numerical examples for the obtained approximation will be presented in Section 3.

2. A quadrature formula for equidistant divisions

Lemma 2.1. *Let $u : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. Then one has the inequalities:*

$$\left| \int_a^b u(s)\mathrm{d}s - \frac{u(a)+u(b)}{2}(b-a) + \frac{(b-a)^2}{8}[u'(b)-u'(a)] \right| \tag{2.1}$$

$$\leq \begin{cases} \dfrac{(b-a)^3}{24}\|u''\|_{[a,b],\infty} & \text{if } u'' \in L_\infty[a,b]; \\ \dfrac{(b-a)^{2+\frac{1}{q}}}{8(2q+1)^{\frac{1}{q}}}\|u''\|_{[a,b],p}, & \text{if } u'' \in L_p[a,b], p>1, \dfrac{1}{p}+\dfrac{1}{q}=1; \\ \dfrac{(b-a)^2}{8}\|u''\|_{[a,b],1}. \end{cases}$$

A simple proof of this fact can be done by using the identity

$$\int_a^b u(s)\mathrm{d}s - \frac{u(a)+u(b)}{2}(b-a) + \frac{(b-a)^2}{8}[u'(b)-u'(a)] = \frac{1}{2}\int_a^b \left(s - \frac{a+b}{2}\right) u''(s)\mathrm{d}s \tag{2.2}$$

and we omit the details.

The following lemma holds.

Lemma 2.2. *Let $u : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. Then for any $t,\tau \in (a,b), t \neq \tau$ and $n \in \mathbb{N}, n \geq 1$, we have the inequality:*

$$\left| \frac{1}{\tau - t}\int_t^\tau u(s)\mathrm{d}s - \frac{1}{2n}\sum_{i=0}^{n-1}\left[u\left(t + i\cdot\frac{\tau-t}{n}\right) + u\left(t+(i+1)\cdot\frac{\tau-t}{n}\right)\right] \right. \tag{2.3}$$

$$\left. + \frac{\tau-t}{8n^2}\sum_{i=0}^{n-1}\left[u'\left(t+(i+1)\cdot\frac{\tau-t}{n}\right) - u'\left(t+i\cdot\frac{\tau-t}{n}\right)\right]\right|$$

$$\leq \begin{cases} \dfrac{|\tau-t|^2}{24n^2}\|u''\|_{[t,\tau],\infty} & \text{if } u'' \in L_\infty[a,b]; \\ \dfrac{|\tau-t|^{1+\frac{1}{q}}}{8n^2(2q+1)^{\frac{1}{q}}}\|u''\|_{[t,\tau],p}, & \text{if } u'' \in L_p[a,b], p>1, \dfrac{1}{p}+\dfrac{1}{q}=1; \\ \dfrac{|\tau-t|}{8n^2}\|u''\|_{[t,\tau],1}, \end{cases}$$

where

$$\|u''\|_{[t,\tau],\infty} := ess \sup_{s\in[t,\tau]} |u''(s)|, \text{ and } \|u''\|_{[t,\tau],p} := \left| \int_t^\tau |u''(s)^p|\mathrm{d}s \right|^{\frac{1}{p}}, p \geq 1.$$

Proof. Consider the equidistant division of $[t,\tau]$ (if $t<\tau$) given by

$$E_n : x_i = t + i\cdot\frac{\tau-t}{n}, i = \overline{0,n}.$$

If we apply the inequality (2.1) on the interval $[x_i, x_{i+1}]$, we may write that:

$$\left| \int_{x_i}^{x_{i+1}} u(s)\mathrm{d}s - \frac{u\left(t+i\cdot\frac{\tau-t}{n}\right) + u\left(t+(i+1)\cdot\frac{\tau-t}{n}\right)}{2}\cdot\frac{\tau-t}{n} \right.$$

$$\left. + \frac{(\tau-t)^2}{8n^2}\left[u'\left(t+(i+1)\cdot\frac{\tau-t}{n}\right) - u'\left(t+i\cdot\frac{\tau-t}{n}\right)\right]\right|$$

$$\leq \begin{cases} \dfrac{|\tau-t|^3}{24n^3}\|u''\|_{[x_i,x_{i+1}],\infty}, & \text{if } u'' \in L_\infty[a,b]; \\ \dfrac{|\tau-t|^{2+\frac{1}{q}}}{8n^{2+\frac{1}{q}}(2q+1)^{\frac{1}{q}}}\|u''\|_{[x_i,x_{i+1}],p}, & \text{if } u'' \in L_p[a,b], p>1, \dfrac{1}{p}+\dfrac{1}{q}=1; \\ \dfrac{|\tau-t|^2}{8n^2}\|u''\|_{[x_i,x_{i+1}],1}, \end{cases}$$

from which we get

$$\begin{aligned}&\left|\frac{1}{\tau-t}\int_{x_i}^{x_{i+1}}u(s)\mathrm{d}s-\frac{1}{2n}\left[u\Big(t+i\cdot\frac{\tau-t}{n}\Big)+u\Big(t+(i+1)\cdot\frac{\tau-t}{n}\Big)\right]\right.\\&\left.+\frac{(\tau-t)}{8n^2}\left[u'\left(t+(i+1)\cdot\frac{\tau-t}{n}\right)-u'\left(t+i\cdot\frac{\tau-t}{n}\right)\right]\right|\\&\le\begin{cases}\dfrac{|\tau-t|^2}{24n^3}\|u''\|_{[x_i,x_{i+1}],\infty}, & \text{if } u''\in L_\infty[a,b];\\ \dfrac{|\tau-t|^{1+\frac{1}{q}}}{8n^{2+\frac{1}{q}}(2q+1)^{\frac{1}{q}}}\|u''\|_{[x_i,x_{i+1}],p}, & \text{if } u''\in L_p[a,b], p>1, \dfrac{1}{p}+\dfrac{1}{q}=1;\\ \dfrac{|\tau-t|}{8n^2}\|u''\|_{[x_i,x_{i+1}],1},\end{cases}\end{aligned}$$

Summing over i from 0 to $n-1$ and using the generalised triangle inequality, we may write

$$\begin{aligned}&\left|\frac{1}{\tau-t}\int_{t}^{\tau}u(s)\mathrm{d}s-\frac{1}{2n}\sum_{i=0}^{n-1}\left[u\Big(t+i\cdot\frac{\tau-t}{n}\Big)+u\Big(t+(i+1)\cdot\frac{\tau-t}{n}\Big)\right]\right.\\&\left.+\frac{(\tau-t)}{8n^2}\sum_{i=0}^{n-1}\left[u'\left(t+(i+1)\cdot\frac{\tau-t}{n}\right)-u'\left(t+i\cdot\frac{\tau-t}{n}\right)\right]\right|\\&\le\sum_{i=0}^{n-1}\left|\frac{1}{\tau-t}\int_{x_i}^{x_{i+1}}u(s)\mathrm{d}s-\frac{1}{2n}\left[u\left(t+i\cdot\frac{\tau-t}{n}\right)+u\left(t+(i+1)\cdot\frac{\tau-t}{n}\right)\right]\right.\\&\left.+\frac{(\tau-t)}{8n^2}\left[u'\left(t+(i+1)\cdot\frac{\tau-t}{n}\right)-u'\left(t+i\cdot\frac{\tau-t}{n}\right)\right]\right|\\&\le\begin{cases}\dfrac{|\tau-t|^2}{24n^3}\displaystyle\sum_{i=0}^{n-1}\|u''\|_{[x_i,x_{i+1}],\infty}, & \text{if } u''\in L_\infty[a,b];\\ \dfrac{|\tau-t|^{1+\frac{1}{q}}}{8n^{2+\frac{1}{q}}(2q+1)^{\frac{1}{q}}}\displaystyle\sum_{i=0}^{n-1}\|u''\|_{[x_i,x_{i+1}],p}, & \text{if } u''\in L_p[a,b], p>1, \dfrac{1}{p}+\dfrac{1}{q}=1;\\ \dfrac{|\tau-t|}{8n^2}\displaystyle\sum_{i=0}^{n-1}\|u''\|_{[x_i,x_{i+1}],1},\end{cases}\end{aligned}$$

However,

$$\begin{aligned}&\sum_{i=0}^{n-1}\|u''\|_{[x_i,x_{i+1}],\infty}\le n\|u''\|_{[t,\tau],\infty},\\&\sum_{i=0}^{n-1}\|u''\|_{[x_i,x_{i+1}],p}=\sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}}|u''(s)|^p\mathrm{d}s\right|^{\frac{1}{p}}\le n^{\frac{1}{q}}\left[\left(\sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}}|u''(s)|^p\mathrm{d}s\right|^{\frac{1}{p}}\right)^p\right]^{\frac{1}{p}}=n^{\frac{1}{q}}\|u''\|_{[t,\tau],p},\end{aligned}$$

and

$$\sum_{i=0}^{n-1}\|u''\|_{[x_i,x_{i+1}],1}\le\left|\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}|u''(s)|\mathrm{d}s\right|=\left|\int_t^\tau|u''(s)|\mathrm{d}s\right|=\|u''\|_{[t,\tau],1},$$

and the lemma is proved. □

The following theorem in approximating the Hilbert transform of a differentiable function whose second derivative is absolutely continuous holds.

Theorem 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function such that its derivative f' is absolutely continuous on $[a,b]$. If*

$$T_n(f;t)=\frac{f'(t)(b-a)+f(b)-f(a)}{2\pi n}+\frac{b-a}{\pi n}\sum_{i=1}^{n-1}\left[f;t-\frac{t-a}{n}\cdot i,t+\frac{b-t}{n}\cdot i\right]\tag{2.4}$$

$$-\frac{1}{8n^2}\left[\frac{1}{2}f''(t)(a-b)(a+b-2t)-f(b)+f(a)+f'(b)(b-t)-f'(a)(a-t)\right],$$

then we have the estimate

$$\left|(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-T_n(f;t)\right| \tag{2.5}$$

$$\leq\begin{cases}\frac{1}{24\pi n^2}\left[(b-a)\left(t-\frac{a+b}{2}\right)^2+\frac{4}{3}(b-a)^3\right]\|f'''\|_{[a,b],\infty}, & \text{if } f'''\in L_\infty[a,b];\\ \frac{q}{8\pi n^2(2q+1)^{1+\frac{1}{q}}}\left[(t-a)^{2+\frac{1}{q}}+(b-t)^{2+\frac{1}{q}}\right]\|f'''\|_{[a,b],p}, & \text{if } f'''\in L_p[a,b], p>1, \frac{1}{p}+\frac{1}{q}=1;\\ \frac{1}{8\pi n^2}\left[\frac{1}{4}(b-a)^2+\left(t-\frac{a+b}{2}\right)\right]\|f'''\|_{[a,b],1}, & \end{cases}$$

$$\leq\begin{cases}\frac{1}{18\pi n^2}(b-a)^3\|f'''\|_{[a,b],\infty}, & \text{if } f'''\in L_\infty[a,b];\\ \frac{q}{4\pi n^2(2q+1)^{1+\frac{1}{q}}}\left(\frac{b-a}{2}\right)^{2+\frac{1}{q}}\|f'''\|_{[a,b],p}, & \text{if } f'''\in L_p[a,b], p>1, \frac{1}{p}+\frac{1}{q}=1;\\ \frac{1}{32\pi n^2}(b-a)^2\|f'''\|_{[a,b],1}, & \end{cases}$$

for all $t\in(a,b)$.

Proof. Applying Lemma 2.2 for the function f', we may write that

$$\left|\frac{f(\tau)-f(t)}{\tau-t}-\frac{1}{2n}\left[f'(t)+\sum_{i=1}^{n-1}f'\left(t+i\cdot\frac{\tau-t}{n}\right)+\sum_{i=0}^{n-2}f'\left(t+(i+1)\cdot\frac{\tau-t}{n}\right)+f'(\tau)\right]\right. \tag{2.6}$$

$$\left.+\frac{\tau-t}{8n^2}\left[f''(\tau)+\sum_{i=0}^{n-2}f''\left(t+(i+1)\cdot\frac{\tau-t}{n}\right)-\sum_{i=1}^{n-1}f''\left(t+i\cdot\frac{\tau-t}{n}\right)-f''(t)\right]\right|$$

$$\leq\begin{cases}\frac{|\tau-t|^2}{24n^2}\|f'''\|_{[t,\tau],\infty} & \text{if } f'''\in L_\infty[a,b];\\ \frac{|\tau-t|^{1+\frac{1}{q}}}{8n^2(2q+1)^{\frac{1}{q}}}\|f'''\|_{[t,\tau],p}, & \text{if } f'''\in L_p[a,b], p>1, \frac{1}{p}+\frac{1}{q}=1;\\ \frac{|\tau-t|}{8n^2}\|f'''\|_{[t,\tau],1}, & \end{cases}$$

However,

$$\sum_{i=1}^{n-1}f'\left(t+i\cdot\frac{\tau-t}{n}\right)=\sum_{i=0}^{n-2}f'\left(t+(i+1)\cdot\frac{\tau-t}{n}\right),\sum_{i=1}^{n-1}f''\left(t+i\cdot\frac{\tau-t}{n}\right)=\sum_{i=0}^{n-2}f''\left(t+(i+1)\cdot\frac{\tau-t}{n}\right)$$

and then by (2.6), we may write:

$$\left|\frac{f(\tau)-f(t)}{\tau-t}-\left[\frac{f'(t)+f'(\tau)}{2n}+\frac{1}{n}\sum_{i=1}^{n-1}f'\left(t+i\cdot\frac{\tau-t}{n}\right)\right]+\frac{\tau-t}{8n^2}\left[f''(\tau)-f''(t)\right]\right| \tag{2.7}$$

$$\leq\begin{cases}\frac{|\tau-t|^2}{24n^2}\|f'''\|_{[t,\tau],\infty}\\ \frac{|\tau-t|^{1+\frac{1}{q}}}{8n^2(2q+1)^{\frac{1}{q}}}\|f'''\|_{[t,\tau],p},\\ \frac{|\tau-t|}{8n^2}\|f'''\|_{[t,\tau],1},\end{cases}$$

for any $t,\tau\in[a,b], t\neq\tau$. Consequently, we have

$$\left|\frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}d\tau-\frac{1}{\pi}PV\int_a^b\left[\frac{f'(t)+f'(\tau)}{2n}+\frac{1}{n}\sum_{i=1}^{n-1}f'\left(t+i\cdot\frac{\tau-t}{n}\right)\right]d\tau\right. \tag{2.8}$$

$$
\left. + \frac{1}{\pi} PV \int_a^b \frac{\tau - t}{8n^2} [f''(\tau) - f''(t)] \mathrm{d}\tau \right|
$$

$$
\leq \begin{cases} \dfrac{1}{24\pi n^2} PV \displaystyle\int_a^b |\tau - t|^2 \|f'''\|_{[t,\tau],\infty} \mathrm{d}\tau, \\ \dfrac{1}{8\pi n^2 (2q+1)^{\frac{1}{q}}} PV \displaystyle\int_a^b |\tau - t|^{1+\frac{1}{q}} \|f'''\|_{[t,\tau],p} \mathrm{d}\tau, \\ \dfrac{1}{8\pi n^2} PV \displaystyle\int_a^b |\tau - t| \|f'''\|_{[t,\tau],1} \mathrm{d}\tau, \end{cases}
$$

Since

$$
\begin{aligned}
&PV \int_a^b \left[\frac{f'(t) + f'(\tau)}{2n} + \frac{1}{n} \sum_{i=1}^{n-1} f'\left(t + i \cdot \frac{\tau - t}{n} \right) \right] \mathrm{d}\tau \\
=& \frac{f'(t)(b-a) + f(b) - f(a)}{n} + \frac{b-a}{n} \sum_{i=1}^{n-1} \left[f; t + i \cdot \frac{b-t}{n}, t + i \cdot \frac{a-t}{n} \right],
\end{aligned}
$$

$$
PV \int_a^b \frac{\tau - t}{8n^2} [f''(\tau) - f''(t)] \mathrm{d}\tau = \frac{1}{8n^2} \left[\frac{1}{2} f''(t)(a-b)(a+b-2t) - f(b) + f(a) + f'(b)(b-t) - f'(a)(a-t) \right],
$$

and

$$
\begin{aligned}
&PV \int_a^b |\tau - t|^2 \|f'''\|_{[t,\tau],\infty} \mathrm{d}\tau \leq \|f'''\|_{[a,b],\infty} PV \int_a^b |\tau - t|^2 \mathrm{d}\tau \\
=& \|f'''\|_{[a,b],\infty} \frac{(b-t)^3 - (a-t)^3}{3} = \|f'''\|_{[a,b],\infty} \left[(b-a) \left(t - \frac{a+b}{2} \right)^2 + \frac{4}{3}(b-a)^3 \right],
\end{aligned}
$$

$$
\begin{aligned}
&PV \int_a^b |\tau - t|^{1+\frac{1}{q}} \|f'''\|_{[t,\tau],p} \mathrm{d}\tau \leq \|f'''\|_{[a,b],p} PV \int_a^b |\tau - t|^{1+\frac{1}{q}} \mathrm{d}\tau \\
=& \|f'''\|_{[a,b],p} \left[\frac{(t-a)^{2+\frac{1}{q}} + (b-t)^{2+\frac{1}{q}}}{2 + \frac{1}{q}} \right] = \frac{q \|f'''\|_{[a,b],p}}{2q+1} \left[(t-a)^{2+\frac{1}{q}} + (b-t)^{2+\frac{1}{q}} \right],
\end{aligned}
$$

and

$$
PV \int_a^b |\tau - t| \|f'''\|_{[t,\tau],1} \mathrm{d}\tau \leq \|f'''\|_{[t,\tau],1} \left[\frac{1}{4}(b-a)^2 + \left(t - \frac{a+b}{2} \right)^2 \right],
$$

then, by (2.8) we get

$$
\begin{aligned}
(2.9) \quad &\left| \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} \mathrm{d}\tau - \frac{f'(t)(b-a) + f(b) - f(a)}{2\pi n} - \frac{b-a}{\pi n} \sum_{i=1}^{n-1} \left[f; t - \frac{t-a}{n} \cdot i, t + \frac{b-t}{n} \cdot i \right] \right. \\
&\left. + \frac{1}{8n^2} \left[\frac{1}{2} f''(t)(a-b)(a+b-2t) - f(b) + f(a) + f'(b)(b-t) - f'(a)(a-t) \right] \right| \\
\leq& \begin{cases} \dfrac{\|f'''\|_{[a,b],\infty}}{24\pi n^2} \left[(b-a) \left(t - \dfrac{a+b}{2} \right)^2 + \dfrac{4}{3}(b-a)^3 \right], & \text{if } f''' \in L_\infty[a,b]; \\ \dfrac{q \|f'''\|_{[a,b],p}}{8\pi n^2 (2q+1)^{1+\frac{1}{q}}} \left[(t-a)^{2+\frac{1}{q}} + (b-t)^{2+\frac{1}{q}} \right], & \text{if } f''' \in L_p[a,b], p > 1, \dfrac{1}{p} + \dfrac{1}{q} = 1; \\ \dfrac{\|f'''\|_{[a,b],1}}{8\pi n^2} \left[\dfrac{1}{4}(b-a)^2 + \left(t - \dfrac{a+b}{2} \right)^2 \right], \end{cases}
\end{aligned}
$$

On the other hand, as for the function $f_0 : (a,b) \to \mathbb{R}, f_0(t) = 1$, we have

$$
(T, f_0)(a, b; t) = \frac{1}{\pi} \ln \left(\frac{b-a}{t-a} \right), t \in (a, b),
$$

then obviously

$$(Tf)(a,b;t) = \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t) + f(t)}{\tau - t} \mathrm{d}\tau = \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} \mathrm{d}\tau + \frac{f(t)}{\pi} PV \int_a^b \frac{\mathrm{d}\tau}{\tau - t},$$

from which we get

$$(Tf)(a,b;t) - \frac{f(t)}{\pi} \ln\left(\frac{b-t}{t-a}\right) = \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} \mathrm{d}\tau. \tag{2.10}$$

Finally, using (2.9) and (2.10), we deduce (2.5). □

3. Some numerical examples

For a function $f : [a,b] \to \mathbb{R}$, we may consider the quadrature formula

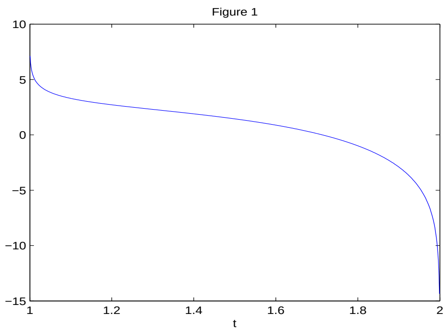
$$E_n(f;a,b,t) := \frac{f(t)}{\pi} \ln\left(\frac{b-t}{t-a}\right) + T_n(f;t), t \in [a,b].$$

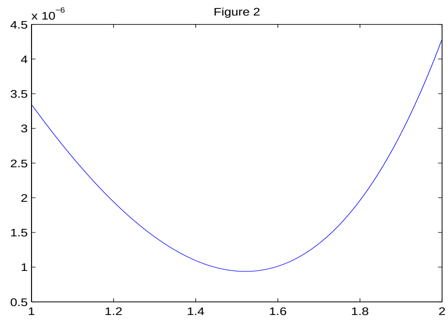
As shown in the above section, $E_n(f;a,b,t)$ provides an approximation for the Finite Hilbert Transform $(Tf)(a,b;t)$ and the error estimate fulfils the bounds described in (2.5).

If we consider the function $f : [1,2] \to \mathbb{R}$, $f(x) = \exp(x)$, the exact value of the Hilbert transform is

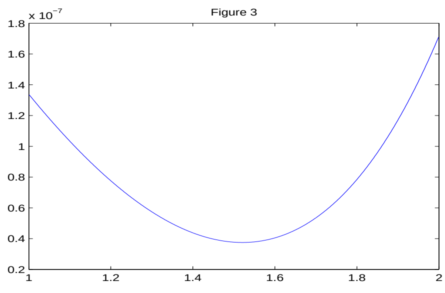
$$(Tf)(a,b;t) = \frac{\exp(t)Ei(2-t) - \exp(t)Ei(1-t)}{\pi}, t \in [1,2].$$

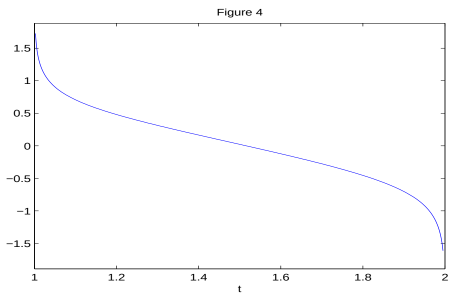
and the plot of this function is embodied in Figure 1.





If we implement the quadrature formula provided by $E_n(f;a,b,t)$ using Matlab and chose the value of $n = 200$, the error $E_r(f;a,b,t) := (Tf)(a,b;t) - E_n(f;a,b,t)$ has the variation described in Figure 2.





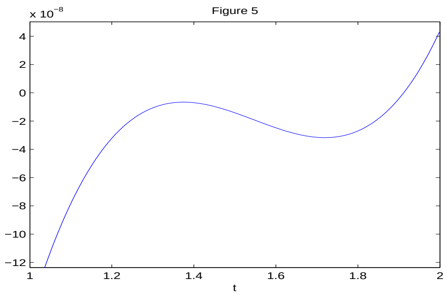
For $n = 1000$, the plot of $E_r(f;a,b,t)$ is embodied in Figure 3, from which we can see that the precision of the error gets higher when n gets bigger.

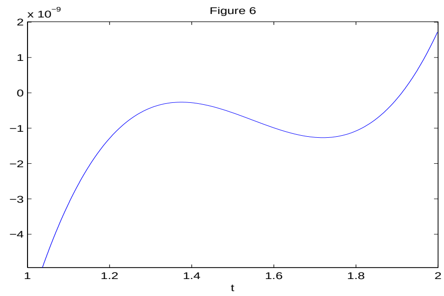
Now, if we consider another function $f : [1,2] \to \mathbb{R}$, $f(x) = \sin(x)$, then

$$(Tf)(a,b;t) = \frac{-S_i(-2+t)\cos(t) + C_i(2-t)\sin(t) + S_i(t-1)\cos(t) - \sin(t)C_i(t-1)}{\pi}, t \in [1,2]$$

having the plot embodied in Figure 4.

If we choose the value of $n = 200$, then the error $E_r(f;a,b,t) := (Tf)(a,b;t) - E_n(f;a,b,t)$ for the function $f(x) = \sin x$, $x \in [a,b]$ has the variation described in Figure 5. Moreover, for $n = 1000$, the behaviour of $E_r(f;a,b,t)$ is plotted in Figure 6.





Acknowledgments

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Pointwise Superconvergence of the Displacement of the Six-Dimensional Finite Element

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In this article we first introduce definitions of the regularized Green's function, the discrete Green's function, the discrete δ function, and the L^2-projection operator in six dimensions. Then the $W^{2,1}$-seminorm estimates for the regularized Green's function and the discrete Green's function are derived. Finally, pointwise superconvergence of the displacement of the six-dimensional finite element is obtained.

1 Introduction

There have been many studies concerned with superconvergence of the finite element method for partial differential equations. Books and survey papers have been published. For the literature, we refer to [1–17] and references therein. It is well known that estimates for the Green's function play very important roles in the study of the superconvergence (especially, pointwise superconvergence) of the finite element method (see [4, 5, 8, 12, 13, 14, 17]). For one- and two-dimensional elliptic problems, one have obtained many optimal estimates for the Green's function (see [17]). Recently, for three-dimensional elliptic problems, the $W^{2,1}$-seminorm optimal estimate with order $O(|\ln h|^{\frac{2}{3}})$ for the discrete Green's function was derived (see [12]).

In this article, we will discuss estimate for the the discrete Green's function based on the 6D Poisson equation.

we shall use the symbol C to denote a generic constant, which is independent from the discretization parameter h and which may not be the same in each occurrence and also use the standard notations for the Sobolev spaces and their norms.

We consider the following Poisson equation:

$$\mathcal{L}u \equiv -\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where $\Omega \subset \mathcal{R}^6$ is a bounded polytopic domain. The weak formulation of (1.1)

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reads,

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ satisfying} \\ a(u\,,\,v) = (f\,,\,v) \text{ for all } v \in H_0^1(\Omega). \end{cases}$$

where

$$a(u\,,\,v) \equiv \int_\Omega \nabla u \cdot \nabla v \, dX,$$

and

$$(f\,,\,v) \equiv \int_\Omega f v \, dX.$$

Let $\{\mathcal{T}^h\}$ be a regular family of partitions of $\bar{\Omega}$. Denote by $S^h(\Omega)$ a continuous piecewise $m(m \geq 2)$-degree (or tensor-product m-degree) polynomials space regarding this kind of partitions and let $S_0^h(\Omega) = S^h(\Omega) \bigcap H_0^1(\Omega)$. Discretizing the above weak formulation using $S_0^h(\Omega)$ as approximating space means,

$$\begin{cases} \text{Find } u_h \in S_0^h(\Omega) \text{ satisfying} \\ a(u_h\,,\,v) = (f\,,\,v) \text{ for all } v \in S_0^h(\Omega). \end{cases}$$

Thus, the following Galerkin orthogonality relation holds.

$$a(u - u_h\,,\,v) = 0 \;\; \forall\, v \in S_0^h(\Omega). \tag{1.2}$$

For every $Z \in \Omega$, we define the discrete δ function $\delta_Z^h \in S_0^h(\Omega)$, the regularized Green's function $G_Z^* \in H^2(\Omega) \bigcap H_0^1(\Omega)$, the discrete Green's function $G_Z^h \in S_0^h(\Omega)$ and the L^2-projection $P_h u \in S_0^h(\Omega)$ such that (see [17])

$$(v,\,\delta_Z^h) = v(Z) \;\; \forall\, v \in S_0^h(\Omega), \tag{1.3}$$

$$a(G_Z^*,\,v) = (\delta_Z^h,\,v) \;\; \forall\, v \in H_0^1(\Omega), \tag{1.4}$$

$$a(G_Z^* - G_Z^h,\,v) = 0 \;\; \forall\, v \in S_0^h(\Omega), \tag{1.5}$$

$$(u - P_h u,\,v) = 0 \;\; \forall\, v \in S_0^h(\Omega), \tag{1.6}$$

In this article, we will bound the terms $|G_Z^*|_{2,\,1}$ and $\left|G_Z^h\right|_{2,\,1}^h$. Here $\left|G_Z^h\right|_{2,\,1}^h = \sum_{e \in \mathcal{T}^h} \left|G_Z^h\right|_{2,\,1,\,e}$.

2 Estimates for the Regularized Green's Function

We first introduce the weight function defined by

$$\phi \equiv \phi(X) = \left(|X - \bar{X}|^2 + \theta^2\right)^{-3} \;\; \forall\, X \in \bar{\Omega}, \tag{2.1}$$

where $\bar{X} \in \bar{\Omega}$ is a fixed point, $\theta = \gamma h$, and $\gamma \in [6, +\infty)$ is a suitable real number.

For every $\alpha \in \mathcal{R}$, we give the following notations:

$$|\nabla^n v|^2 = \sum_{|\beta|=n} \left|D^\beta v\right|^2, |\nabla^n v|_{\phi^\alpha} = \left(\int_\Omega \phi^\alpha \, |\nabla^n v|^2 \, dX\right)^{\frac{1}{2}}, \|v\|_{m,\,\phi^\alpha}^2 = \sum_{n=0}^{m} |\nabla^n v|_{\phi^\alpha}^2 \,,$$

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where $\beta = (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6)$, $|\beta| = \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6$, and $\beta_i \geq 0$, $i = 1, \cdots, 6$ are integers. In particular, for the case of $m = 0$, we write

$$\|v\|_{\phi^\alpha} = \left(\int_\Omega \phi^\alpha |v|^2 \, dX \right)^{\frac{1}{2}}.$$

We assume there exists a real number q_0 $(1 < q_0 \leq \infty)$ such that

$$\|v\|_{2,\,q} \leq C(q) \|\mathcal{L}v\|_{0,\,q} \ \ \forall\, v \in W^{2,\,q}(\Omega) \bigcap W_0^{1,\,q}(\Omega), 1 < q < q_0, \tag{2.2}$$

which is the so-called a priori estimate (see [17]). As in the two-dimensional case (see [17]), we can obtain the following Lemma 2.1.

Lemma 2.1. *For ϕ the weight function defined by (2.1), we have the following estimates*:

$$|\nabla^n \phi^\alpha| \leq C(\alpha, n) \phi^{\alpha + \frac{n}{6}}, \ \alpha \in \mathcal{R}, \ n = 1, 2, \tag{2.3}$$

$$\int_\Omega \phi \, dX \leq C(k) |\ln \theta|, \ \theta \leq k < 1, \tag{2.4}$$

$$\int_\Omega \phi^\alpha \, dX \leq C(\alpha - 1)^{-1} \theta^{-6(\alpha - 1)} \ \forall\, \alpha > 1. \tag{2.5}$$

For the L^2-projection operator P_h and the discrete δ function δ_Z^h, similar to the arguments in the two-dimensional setting (see [17]), we have the following results (2.6)–(2.8).

Lemma 2.2. *For $P_h w$ the L^2-projection of w, we have the following stability estimate*:

$$\|P_h w\|_{0,\,q} \leq C \|w\|_{0,\,q}, \ 1 \leq q \leq \infty, \tag{2.6}$$

$$\|P_h w\|_{1,\,q} \leq C \|w\|_{1,\,q}, \ 6 < q \leq \infty. \tag{2.7}$$

Lemma 2.3. *For δ_Z^h the discrete δ function defined by (1.3), we have the following estimate*:

$$|\delta_Z^h(X)| \leq C h^{-6} \mathrm{e}^{-Ch^{-1}|X - Z|}, \tag{2.8}$$

where $X, Z \in \bar{\Omega}$, and C is independent of h, X, and Z.

In additon, we have the following weighted-norm estimate for δ_Z^h.

Lemma 2.4. *For δ_Z^h the discrete δ function defined by (1.3) and ϕ defined by (2.1), we have the following estimate*:

$$\left\|\delta_Z^h\right\|_{\phi^{-1}} \leq C. \tag{2.9}$$

Proof. From (2.1) and (2.8),

$$\begin{aligned} \left\|\delta_Z^h\right\|_{\phi^{-1}}^2 &\leq C \int_\Omega \left(|X - Z|^2 + \theta^2\right)^3 h^{-12} e^{-Ch^{-1}|X - Z|} dX \\ &\leq C \int_0^\infty \left(r^2 + \theta^2\right)^3 h^{-12} e^{-Ch^{-1} r} r^5 dr. \end{aligned}$$

Set $h^{-1}r = t$, then

$$\left\|\delta_Z^h\right\|_{\phi^{-1}}^2 \le C\int_0^\infty \left(t^2+\gamma^2\right)^3 e^{-Ct}t^5dt \le C,$$

which is the result (2.9).

Lemma 2.5. *For G_Z^* the regularized Green's function defined by (1.4) and ϕ defined by (2.1), we have the following weighted-norm estimate*:

$$\|G_Z^*\|_{\phi^{-\frac{1}{3}}} \le C\,|\ln h|^{\frac{5}{6}}\,. \tag{2.10}$$

Proof. From (1.3), (1.4), (1.6), (2.2), (2.6), the inverse estimate, the Sobolev Embedding Theorem (see [18]), and the Poincaré inequality, we have

$$\begin{aligned}\|G_Z^*\|_{\phi^{-\frac{1}{3}}}^2 &= \left(G_Z^*, \phi^{-\frac{1}{3}}G_Z^*\right) = a(G_Z^*, w) = (\delta_Z^h, w) = P_h w(Z) \le |P_h w|_{0,\,\infty}\\ &\le Ch^{-\frac{6}{q}}\,|P_h w|_{0,\,q} \le Ch^{-\frac{6}{q}}\,|w|_{0,\,q} \le Ch^{-\frac{6}{q}}q^{\frac{5}{6}}\,\|w\|_{1,\,6}\\ &\le Ch^{-\frac{6}{q}}q^{\frac{5}{6}}\,\|w\|_{2,\,3} \le Ch^{-\frac{6}{q}}q^{\frac{5}{6}}\left\|\phi^{-\frac{1}{3}}G_Z^*\right\|_{0,\,3}\\ &\le Ch^{-\frac{6}{q}}q^{\frac{5}{6}}\left|\phi^{-\frac{1}{3}}G_Z^*\right|_1,\end{aligned} \tag{2.11}$$

where $w \in W^{2,\,3}(\Omega)\bigcap W_0^{1,\,6}(\Omega)$ and $\mathcal{L}w = \phi^{-\frac{1}{3}}G_Z^*$. Set $q = |\ln h|$ in (2.11), and by the *Young* inequality, we get

$$\|G_Z^*\|_{\phi^{-\frac{1}{3}}}^2 \le C\,|\ln h|^{\frac{5}{6}}\left|\phi^{-\frac{1}{3}}G_Z^*\right|_1 \le C(\varepsilon)\,|\ln h|^{\frac{5}{3}} + \varepsilon\left|\phi^{-\frac{1}{3}}G_Z^*\right|_1^2 \tag{2.12}$$

In addition, from (1.4) and (2.3),

$$\begin{aligned}\left|\phi^{-\frac{1}{3}}G_Z^*\right|_1^2 &\le Ca\left(\phi^{-\frac{1}{3}}G_Z^*\,,\,\phi^{-\frac{1}{3}}G_Z^*\right)\\ &\le C\left|a(G_Z^*\,,\,\phi^{-\frac{2}{3}}G_Z^*)\right| + C\left|\left((G_Z^*)^2\,,\,|\nabla(\phi^{-\frac{1}{3}})|^2\right)\right|\\ &\le C\left|a(G_Z^*\,,\,\phi^{-\frac{2}{3}}G_Z^*)\right| + C\,\|G_Z^*\|_{\phi^{-\frac{1}{3}}}^2\\ &\le \tilde{C}\left\|\delta_Z^h\right\|_{\phi^{-1}}^2 + \tilde{C}\,\|G_Z^*\|_{\phi^{-\frac{1}{3}}}^2\,.\end{aligned} \tag{2.13}$$

Combining (2.9), (2.12), (2.13), and choosing ε such that $\varepsilon\tilde{C} = \frac{1}{2}$, we immediately obtain the result (2.10).

Theorem 2.1. *For G_Z^* the regularized Green's function defined by (1.4), we have the following $W^{2,\,1}$-seminorm estimate*:

$$|G_Z^*|_{2,\,1} \le C\,|\ln h|^{\frac{4}{3}}\,. \tag{2.14}$$

Proof. Obviously,

$$|G_Z^*|_{2,\,1}^2 \le \int_\Omega \phi\,dX\cdot\left\|\nabla^2 G_Z^*\right\|_{\phi^{-1}}^2\,. \tag{2.15}$$

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Furthermore,

$$\begin{aligned}
&\left\|\nabla^2 G_Z^*\right\|_{\phi^{-1}}^2 = \int_\Omega \phi^{-1}\left|\nabla^2 G_Z^*\right|^2 dX = \int_\Omega \left(\phi^{-\frac{1}{2}}\left|\nabla^2 G_Z^*\right|\right)^2 dX \\
\leq\ & C\left(\int_\Omega \left|\nabla^2\left(\phi^{-\frac{1}{2}} G_Z^*\right)\right|^2 dX + \int_\Omega \left|\nabla^2 \phi^{-\frac{1}{2}} G_Z^*\right|^2 dX \right.\\
&\left. + \int_\Omega \left|\nabla \phi^{-\frac{1}{2}}\right|^2 \left|\nabla G_Z^*\right|^2 dx\right) \\
\leq\ & C\left(\left\|\nabla^2\left(\phi^{-\frac{1}{2}} G_Z^*\right)\right\|_0^2 + \|G_Z^*\|_{\phi^{-\frac{1}{3}}}^2 + |G_Z^*|_{1,\,\phi^{-\frac{2}{3}}}^2\right) \\
\leq\ & C\left(\left\|\mathcal{L}\left(\phi^{-\frac{1}{2}} G_Z^*\right)\right\|_0^2 + \|G_Z^*\|_{\phi^{-\frac{1}{3}}}^2 + |G_Z^*|_{1,\,\phi^{-\frac{2}{3}}}^2\right) \\
\leq\ & C\left(\|\mathcal{L} G_Z^*\|_{\phi^{-1}}^2 + \|G_Z^*\|_{\phi^{-\frac{1}{3}}}^2 + |G_Z^*|_{1,\,\phi^{-\frac{2}{3}}}^2\right) \\
\leq\ & C\left\|\delta_Z^h\right\|_{\phi^{-1}}^2 + C\left|a\left(G_Z^*, \phi^{-\frac{2}{3}} G_Z^*\right)\right| + C\left\|G_Z^*\right\|_{\phi^{-\frac{1}{3}}}^2 \\
\leq\ & C\left\|\delta_Z^h\right\|_{\phi^{-1}}^2 + C\left|\left(\delta_Z^h, \phi^{-\frac{2}{3}} G_Z^*\right)\right| + C\left\|G_Z^*\right\|_{\phi^{-\frac{1}{3}}}^2 \\
\leq\ & C\left\|\delta_Z^h\right\|_{\phi^{-1}}^2 + C\left\|G_Z^*\right\|_{\phi^{-\frac{1}{3}}}^2,
\end{aligned}$$

combined with (2.9) and (2.10), we have

$$\left\|\nabla^2 G_Z^*\right\|_{\phi^{-1}}^2 \leq C\left|\ln h\right|^{\frac{5}{3}}. \tag{2.16}$$

By (2.4), (2.15), and (2.16), we immediately obtain the result (2.14).

3 Estimates for the Discrete Green's Function

The definition (1.5) shows that G_Z^h is a finite element approximation to G_Z^*. In this section, we give the $W^{2,1}$-seminorm estimate for G_Z^h.

Lemma 3.1. *For G_Z^* and G_Z^h, the regularized Green's function and the discrete Green's function, respectively, we have the following estimate*:

$$\left|G_Z^* - G_Z^h\right|_{1,\,1} \leq Ch\left|\ln h\right|^{\frac{4}{3}}. \tag{3.1}$$

Proof. Obviously,

$$\left|G_Z^* - G_Z^h\right|_{1,\,1}^2 \leq \int_\Omega \phi\, dX \cdot \left|G_Z^* - G_Z^h\right|_{1,\,\phi^{-1}}^2. \tag{3.2}$$

Similar to the proof of (2.43) in [13], and using (2.16), we have

$$\begin{aligned}
\left|G_Z^* - G_Z^h\right|_{1,\,\phi^{-1}}^2 &\leq Ch^2\left\|\nabla^2 G_Z^*\right\|_{\phi^{-1}}^2 + \hat{C}\left\|G_Z^* - G_Z^h\right\|_{\phi^{-\frac{2}{3}}}^2 \\
&\leq Ch^2\left|\ln h\right|^{\frac{5}{3}} + \hat{C}\left\|G_Z^* - G_Z^h\right\|_{\phi^{-\frac{2}{3}}}^2.
\end{aligned} \tag{3.3}$$

In addition

$$\begin{aligned} & \left\|G_Z^* - G_Z^h\right\|^2_{\phi^{-\frac{2}{3}}} = (\phi^{-\frac{2}{3}}(G_Z^* - G_Z^h),\, G_Z^* - G_Z^h) = a(w,\, G_Z^* - G_Z^h) \\ = \;& a(w - \Pi w,\, G_Z^* - G_Z^h) \leq \varepsilon \left|G_Z^* - G_Z^h\right|^2_{1,\,\phi^{-1}} + C(\varepsilon) \left|w - \Pi w\right|^2_{1,\,\phi}. \end{aligned} \tag{3.4}$$

where $\mathcal{L}w = \phi^{-\frac{2}{3}}(G_Z^* - G_Z^h)$ and Π is an interpolation operator. Using the weighted interpolation error estimate in (3.4) (similar to pp.110 Lemma 4 in [17]) yields

$$\left\|G_Z^* - G_Z^h\right\|^2_{\phi^{-\frac{2}{3}}} \leq \varepsilon \left|G_Z^* - G_Z^h\right|^2_{1,\,\phi^{-1}} + C(\varepsilon)h^2 \left|\nabla^2 w\right|^2_{\phi}. \tag{3.5}$$

Further, from the a priori estimate (2.2), (2.5), and the Sobolev Embedding Theorem [19],

$$\begin{aligned} \left|\nabla^2 w\right|^2_{\phi} &\leq \|\phi\|_{0,\,\frac{3}{2}} \left|\nabla^2 w\right|^2_{0,\,6} \leq C\theta^{-2} \|w\|^2_{2,\,6} \leq C\theta^{-2} \left\|\phi^{-\frac{2}{3}}(G_Z^* - G_Z^h)\right\|^2_{0,\,6} \\ &\leq C\theta^{-2} \left\|\phi^{-\frac{2}{3}}(G_Z^* - G_Z^h)\right\|^2_{2} \leq C\theta^{-2} \left\|\mathcal{L}\left(\phi^{-\frac{2}{3}}(G_Z^* - G_Z^h)\right)\right\|^2_{0} \\ &\leq C\theta^{-2} \left(\left\|\phi^{-\frac{2}{3}}\delta_Z^h\right\|^2_0 + \left|G_Z^* - G_Z^h\right|^2_{1,\,\phi^{-1}} + \left\|G_Z^* - G_Z^h\right\|^2_{\phi^{-\frac{2}{3}}} \right). \end{aligned} \tag{3.6}$$

Similar to the proof of (2.9), we can obtain

$$\left\|\phi^{-\frac{2}{3}}\delta_Z^h\right\|^2_0 \leq Ch^2. \tag{3.7}$$

Combining (3.5)–(3.7) yields

$$\begin{aligned} \left\|G_Z^* - G_Z^h\right\|^2_{\phi^{-\frac{2}{3}}} \leq\; & \left(\varepsilon + C(\varepsilon)\gamma^{-2}\right) \left|G_Z^* - G_Z^h\right|^2_{1,\,\phi^{-1}} \\ & + C(\varepsilon)\gamma^{-2} \left\|G_Z^* - G_Z^h\right\|^2_{\phi^{-\frac{2}{3}}} + C(\varepsilon)\gamma^{-2}h^2. \end{aligned} \tag{3.8}$$

Choosing suitable ε and $\gamma \in [6,\, +\infty)$ such that $0 < (2\varepsilon + 1)\hat{C} < 1$ as well as $C(\varepsilon)\gamma^{-2} = \frac{1}{2}$. From (3.8),

$$\left\|G_Z^* - G_Z^h\right\|^2_{\phi^{-\frac{2}{3}}} \leq (2\varepsilon + 1) \left|G_Z^* - G_Z^h\right|^2_{1,\,\phi^{-1}} + h^2. \tag{3.9}$$

From (3.3) and (3.9),

$$\left|G_Z^* - G_Z^h\right|^2_{1,\,\phi^{-1}} \leq Ch^2 \left|\ln h\right|^{\frac{5}{3}}. \tag{3.10}$$

The result (3.1) immediately follows the results (2.4), (3.2), and (3.10).

Theorem 3.1. *For G_Z^h the discrete Green's function, we have the following estimate*:

$$\left|G_Z^h\right|^h_{2,\,1} \leq C|\ln h|^{\frac{4}{3}}. \tag{3.11}$$

Proof. By the triangle inequality, the interpolation error estimate, and the inverse property, we have

$$\begin{aligned}
\left|G_Z^h\right|_{2,1}^h &\le \left|G_Z^* - G_Z^h\right|_{2,1}^h + |G_Z^*|_{2,1} \\
&\le |G_Z^*|_{2,1} + |G_Z^* - \Pi G_Z^*|_{2,1}^h + \left|\Pi G_Z^* - G_Z^h\right|_{2,1}^h \\
&\le C\,|G_Z^*|_{2,1} + Ch^{-1}\left|\Pi G_Z^* - G_Z^h\right|_{1,1} \\
&\le C\,|G_Z^*|_{2,1} + Ch^{-1}\,|G_Z^* - \Pi G_Z^*|_{1,1} \\
&\quad + Ch^{-1}\left|G_Z^* - G_Z^h\right|_{1,1} \\
&\le C\,|G_Z^*|_{2,1} + Ch^{-1}\left|G_Z^* - G_Z^h\right|_{1,1}.
\end{aligned} \tag{3.12}$$

Combining (2.14), (3.1), and (3.12) yields the result (3.11).

4 Superconvergence of the Displacement of the Finite Element

In this section, we give an application of the estimate for the discrete Green's function in finite element superconvergence.

Let Πu and u_h be the interpolant and the finite element approximation to u, the solution of (1.1), respectively. Similar to the proof of [13], we can obtain the following lemma.

Lemma 4.1. *Let $S_0^h(\Omega)$ be the tensor-product m-degree finite element space. Suppose $v \in S_0^h(\Omega)$ and $u \in W^{m+2,\infty}(\Omega) \bigcap H_0^1(\Omega)$. Then we have the following weak estimate of the second type*:

$$|a(u - \Pi u, v)| \le Ch^{m+2}\|u\|_{m+2,\infty}|v|_{2,1}^h, \quad m \ge 2, \tag{4.1}$$

where $|v|_{2,1}^h = \sum_{e\in\mathcal{T}^h}|v|_{2,1,e}$.

Finally, we give the following superconvergent estimate.

Theorem 4.1. *Let $\{\mathcal{T}^h\}$ be a regular family of partitions of $\bar{\Omega}$ and $u \in W^{m+2,\infty}(\Omega) \bigcap H_0^1(\Omega)$. For u_h and Πu, the tensor-product m-degree finite element approximation and the corresponding interpolant to u, respectively. Then we have the following superconvergent estimates*:

$$|u_h - \Pi u|_{0,\infty,\Omega} \le Ch^{m+2}\,|\ln h|^{\frac{4}{3}}\,\|u\|_{m+2,\infty}, \quad m \ge 2. \tag{4.2}$$

Proof. For every $Z \in \Omega$, applying the definition of G_Z^h and the Galerkin orthogonality relation (1.2), we derive

$$(u_h - \Pi u)(Z) = a(u_h - \Pi u\,,\, G_Z^h) = a(u - \Pi u\,,\, G_Z^h). \tag{4.3}$$

From (3.11), (4.1), and (4.3), we immediately obtain the result (4.2).

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Estimates for Discrete Derivative Green's Function for Elliptic Equations in Dimensions Seven and Up

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This article will discuss estimates for discrete derivative Green's function for elliptic equations in dimensions seven and up. First, the definitions of some terms are given. Then the estimates for the regularized derivative Green's function are derived. Finally, using the triangular inequality, we obtain the estimates for discrete derivative Green's function. The results of the article play important roles in the research of superconvergence of finite element methods.

1 Introduction

It is well known that estimates for the Green's function play very important roles in the study of the superconvergence (especially, pointwise superconvergence) of the finite element method (see [1–8]). For one- and two-dimensional elliptic problems, one have obtained many optimal estimates for the Green's function (see [8]). Recently, for dimensions three to five, we have obtained some optimal estimates for the discrete Green's function (see [4–7]). At present, we also consider the six-dimensional discrete Green's function and its estimates, and some results have been submitted to some Journals. In this article, we will discuss estimates for the discrete derivative Green's function in dimensions seven and up.

we shall use the symbol C to denote a generic constant, which is independent from the discretization parameter h and which may not be the same in each occurrence and also use the standard notations for the Sobolev spaces and their norms.

We consider the following Poisson equation:

$$\mathcal{L}u \equiv -\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where $\Omega \subset \mathcal{R}^d (d \geq 7)$ is a bounded polytopic domain. The weak formulation of

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(1.1) reads,

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ satisfying} \\ a(u\,,\,v) = (f\,,\,v) \text{ for all } v \in H_0^1(\Omega). \end{cases}$$

where

$$a(u\,,\,v) \equiv \int_\Omega \nabla u \cdot \nabla v\, dX,$$

and

$$(f\,,\,v) \equiv \int_\Omega f v\, dX.$$

Let $\{\mathcal{T}^h\}$ be a regular family of partitions of $\bar{\Omega}$. Denote by $S^h(\Omega)$ a continuous finite elements space regarding this kind of partitions and let $S_0^h(\Omega) = S^h(\Omega) \bigcap H_0^1(\Omega)$. Discretizing the above weak formulation using $S_0^h(\Omega)$ as approximating space means,

$$\begin{cases} \text{Find } u_h \in S_0^h(\Omega) \text{ satisfying} \\ a(u_h\,,\,v) = (f\,,\,v) \text{ for all } v \in S_0^h(\Omega). \end{cases}$$

For every $Z \in \Omega$, we define the discrete derivative δ function $\partial_{Z,\ell}\delta_Z^h \in S_0^h(\Omega)$ and the L^2-projection $P_h u \in S_0^h(\Omega)$ such that

$$(v,\, \partial_{Z,\ell}\delta_Z^h) = \partial_\ell v(Z) \quad \forall\, v \in S_0^h(\Omega). \tag{1.2}$$

$$(u - P_h u,\, v) = 0 \quad \forall\, v \in S_0^h(\Omega). \tag{1.3}$$

Here, for any direction $\ell \in R^d$, $|\ell| = 1$, $\partial_{Z,\ell}\delta_Z^h$ and $\partial_\ell v(Z)$ stand for the following onesided directional derivatives, respectively.

$$\partial_{Z,\ell}\delta_Z^h = \lim_{|\Delta Z|\to 0} \frac{\delta_{Z+\Delta Z}^h - \delta_Z^h}{|\Delta Z|},\ \partial_\ell v(Z) = \lim_{|\Delta Z|\to 0} \frac{v(Z+\Delta Z) - v(Z)}{|\Delta Z|},\ \Delta Z = |\Delta Z|\ell.$$

Let $\partial_{Z,\ell}G_Z^* \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of the elliptic problem $-\Delta \partial_{Z,\ell}G_Z^* = \partial_{Z,\ell}\delta_Z^h$. We may call $\partial_{Z,\ell}G_Z^*$ the regularized derivative Green's function. Further, let the discrete derivative Green's function $\partial_{Z,\ell}G_Z^h \in S_0^h(\Omega)$ be the finite element approximation to $\partial_{Z,\ell}G_Z^*$. Thus,

$$a(\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h, v) = 0 \quad \forall\, v \in S_0^h(\Omega). \tag{1.4}$$

In this article, we will bound the terms $|\partial_{Z,\ell}G_Z^*|_{1,\,1}$ and $\left|\partial_{Z,\ell}G_Z^h\right|_{1,\,1}$.

2 Regularized Derivative Green's Function and Its Estimates

We first introduce the weight function defined by

$$\phi \equiv \phi(X) = \left(|X - \bar{X}|^2 + \theta^2\right)^{-\frac{d}{2}} \quad \forall\, X \in \bar{\Omega}, \tag{2.1}$$

where $\bar{X} \in \bar{\Omega}$ is a fixed point, $\theta = \gamma h$, and $\gamma \in [d, +\infty)$ is a suitable real number.

For every $\alpha \in \mathcal{R}$, we give the following notations:

$$|\nabla^n v|^2 = \sum_{|\beta|=n} \left|D^\beta v\right|^2, |\nabla^n v|_{\phi^\alpha} = \left(\int_\Omega \phi^\alpha |\nabla^n v|^2 \, dX\right)^{\frac{1}{2}}, \|v\|^2_{m,\,\phi^\alpha} = \sum_{n=0}^{m} |\nabla^n v|^2_{\phi^\alpha},$$

where $\beta = (\beta_1, \beta_2, \cdots, \beta_d)$, $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_d$, and $\beta_i \geq 0,\ i = 1, \cdots, d$ are integers. In particular, for the case of $m = 0$, we write

$$\|v\|_{\phi^\alpha} = \left(\int_\Omega \phi^\alpha |v|^2 \, dX\right)^{\frac{1}{2}}.$$

We assume there exists a real number q_0 $(1 < q_0 \leq \infty)$ such that

$$\|v\|_{2,\,q} \leq C(q)\|\mathcal{L}v\|_{0,\,q} \ \ \forall\, v \in W^{2,\,q}(\Omega) \cap W_0^{1,\,q}(\Omega), 1 < q < q_0, \tag{2.2}$$

which is the so-called a priori estimate (see [8]). As in the two-dimensional case (see [8]), we can obtain the following Lemma 2.1.

Lemma 2.1. *For ϕ the weight function defined by (2.1), we have the following estimates*:

$$|\nabla^n \phi^\alpha| \leq C(\alpha, n)\phi^{\alpha + \frac{n}{d}},\ \alpha \in \mathcal{R},\ n = 1, 2, \tag{2.3}$$

$$\int_\Omega \phi \, dX \leq C(k)|\ln\theta|,\ \theta \leq k < 1, \tag{2.4}$$

$$\int_\Omega \phi^\alpha \, dX \leq C(\alpha - 1)^{-1}\theta^{-d(\alpha-1)} \ \forall\, \alpha > 1. \tag{2.5}$$

$$\int_\Omega \phi^\alpha \, dX \leq C(1 - \alpha)^{-1} \ \forall\, 0 < \alpha < 1. \tag{2.6}$$

In addition, we also have the following Lemmas.

Lemma 2.2. *For $P_h w$ the L^2-projection of w, we have the following stability estimate*:

$$\|P_h w\|_{0,\,q} \leq C\|w\|_{0,\,q},\ 1 \leq q \leq +\infty. \tag{2.7}$$

Lemma 2.3. *For $\partial_{Z,\ell}\delta^h_Z$ the discrete derivative δ function defined by (1.2), we have the following estimate*:

$$|\partial_{Z,\ell}\delta^h_Z(X)| \leq Ch^{-d-1}\mathrm{e}^{-Ch^{-1}|X-Z|}, \tag{2.8}$$

where $X, Z \in \bar{\Omega}$, and C is independent of h, X, and Z.

As for $\partial_{Z,\ell}\delta^h_Z$, we have the following important estimate.

Lemma 2.4. *For $\partial_{Z,\ell}\delta^h_Z$ the discrete derivative δ function defined by (1.2) and ϕ defined by (2.1), when $\alpha > 0$, we have the following estimate*:

$$\left\|\partial_{Z,\ell}\delta^h_Z\right\|_{\phi^{-\alpha}} \leq Ch^{\frac{d(\alpha-1)-2}{2}}. \tag{2.9}$$

Proof. From (2.1) and (2.8),

$$\begin{aligned}\left\|\partial_{Z,\ell}\delta_Z^h\right\|_{\phi^{-\alpha}}^2 &\le C\int_\Omega \left(|X-Z|^2+\theta^2\right)^{\frac{d\alpha}{2}} h^{-2d-2}e^{-Ch^{-1}|X-Z|}dX\\ &\le C\int_0^\infty \left(r^2+\theta^2\right)^{\frac{d\alpha}{2}} h^{-2d-2}e^{-Ch^{-1}r}r^{d-1}dr.\end{aligned}$$

Set $h^{-1}r=t$, then

$$\left\|\partial_{Z,\ell}\delta_Z^h\right\|_{\phi^{-\alpha}}^2 \le Ch^{d(\alpha-1)-2}$$

which is the result (2.9).

Lemma 2.5. *Suppose $q_0>2$ and $0<\varepsilon<1$. For $\partial_{Z,\ell}G_Z^*$ the regularized derivative Green's function defined by (1.4) and ϕ defined by (2.1), we have the following weighted-norm estimate*:

$$\|\partial_{Z,\ell}G_Z^*\|_{\phi^{1-\varepsilon}} \le Ch^{1-d+\frac{\varepsilon d}{2}}. \tag{2.10}$$

Proof. Set $r=\frac{1+\varepsilon}{1-\varepsilon}, r'=\frac{1+\varepsilon}{2\varepsilon}$, thus $\frac{1}{r}+\frac{1}{r'}=1$. From (2.5),

$$\begin{aligned}\|\partial_{Z,\ell}G_Z^*\|_{\phi^{1-\varepsilon}}^2 &= \int_\Omega \phi^{1-\varepsilon}\left|\partial_{Z,\ell}G_Z^*\right|^2dX\\ &\le \left(\int_\Omega \phi^{1+\varepsilon}dX\right)^{\frac{1-\varepsilon}{1+\varepsilon}}\|\partial_{Z,\ell}G_Z^*\|_{0,\frac{1+\varepsilon}{\varepsilon}}^2\\ &\le C\left(\varepsilon^{-1}\theta^{-d\varepsilon}\right)^{\frac{1-\varepsilon}{1+\varepsilon}}\|\partial_{Z,\ell}G_Z^*\|_{0,\frac{1+\varepsilon}{\varepsilon}}^2 .\end{aligned}$$

Further,

$$\begin{aligned}\|\partial_{Z,\ell}G_Z^*\|_{0,\frac{1+\varepsilon}{\varepsilon}}^{\frac{1+\varepsilon}{\varepsilon}} &= \left(\partial_{Z,\ell}G_Z^*, \left|\partial_{Z,\ell}G_Z^*\right|^{\frac{1}{\varepsilon}} sgn\partial_{Z,\ell}G_Z^*\right)\\ &= a(\partial_{Z,\ell}G_Z^*,w)=\left(\partial_{Z,\ell}\delta_Z^h,w\right)=\partial_\ell P_hw(Z)\\ &\le |P_hw|_{1,\infty}\le Ch^{-\frac{d}{q}-1}\|P_hw\|_{0,q}\\ &\le Ch^{-\frac{d}{q}-1}\|w\|_{0,q},\end{aligned}$$

where $q\ge 1$, and $w\in H_0^1(\Omega)$ satisfies

$$a(v,w)=\left(v,\left|\partial_{Z,\ell}G_Z^*\right|^{\frac{1}{\varepsilon}} sgn\partial_{Z,\ell}G_Z^*\right)\ \forall\, v\in H_0^1(\Omega).$$

Taking $q=\frac{d(1+\varepsilon)}{d-2(1+\varepsilon)}>1$ and $\frac{1}{p}=\frac{1}{q}+\frac{2}{d}$, we have $p=1+\varepsilon<2$. By the a priori estimate (2.2) and the Sobolev Embedding Theorem (see [9]), we get

$$\|w\|_{0,q}\le C\|w\|_{2,p}\le C\|\partial_{Z,\ell}G_Z^*\|_{0,\frac{1+\varepsilon}{\varepsilon}}^{\frac{1}{\varepsilon}} .$$

Thus

$$\|\partial_{Z,\ell}G_Z^*\|_{0,\frac{1+\varepsilon}{\varepsilon}}^2\le Ch^{-\frac{2d}{q}-2}=Ch^{2-\frac{2d}{1+\varepsilon}},$$

which results in

$$\|\partial_{Z,\ell}G_Z^*\|_{\phi^{1-\varepsilon}}^2 \leq Ch^{2-2d+\varepsilon d}.$$

The proof of the result (2.10) is completed.

Lemma 2.6. *For $\partial_{Z,\ell}G_Z^*$ the regularized derivative Green's function defined by (1.4) and $\partial_{Z,\ell}\delta_Z^h$ the discrete derivative δ function defined by (1.2), we have the following weighted-norm estimate*:

$$\|\nabla(\partial_{Z,\ell}G_Z^*)\|_{\phi^{-\alpha}}^2 \leq C\left\|\partial_{Z,\ell}\delta_Z^h\right\|_{\phi^{-\alpha-\frac{2}{d}}}^2 + C\left\|\partial_{Z,\ell}G_Z^*\right\|_{\phi^{-\alpha+\frac{2}{d}}}^2 \quad \forall\alpha \in R. \tag{2.11}$$

Proof. Obviously,

$$\|\nabla(\partial_{Z,\ell}G_Z^*)\|_{\phi^{-\alpha}}^2 \leq a(\partial_{Z,\ell}G_Z^*\,,\,\phi^{-\alpha}\partial_{Z,\ell}G_Z^*) + C\left\|\partial_{Z,\ell}G_Z^*\right\|_{\phi^{-\alpha+\frac{2}{d}}}^2. \tag{2.12}$$

Further,

$$\begin{array}{rcl} a(\partial_{Z,\ell}G_Z^*\,,\,\phi^{-\alpha}\partial_{Z,\ell}G_Z^*) & = & (\partial_{Z,\ell}\delta_Z^h\,,\,\phi^{-\alpha}\partial_{Z,\ell}G_Z^*) \\ & \leq & \|\partial_{Z,\ell}\delta_Z^h\|_{\phi^{-\alpha-\frac{2}{d}}}\|\partial_{Z,\ell}G_Z^*\|_{\phi^{-\alpha+\frac{2}{d}}} \\ & \leq & \frac{1}{2}(\|\partial_{Z,\ell}\delta_Z^h\|_{\phi^{-\alpha-\frac{2}{d}}}^2 + \|\partial_{Z,\ell}G_Z^*\|_{\phi^{-\alpha+\frac{2}{d}}}^2). \end{array} \tag{2.13}$$

Combining (2.12) and (2.13) immediately yields the result (2.11).

Lemma 2.7. *Suppose $-\frac{2}{d} < \alpha < \frac{2}{d}$ and $q_0 > 2$. For $\partial_{Z,\ell}G_Z^*$ the regularized derivative Green's function defined by (1.4), we have the following weighted-norm estimate*:

$$\|\nabla(\partial_{Z,\ell}G_Z^*)\|_{\phi^{-\alpha}} \leq Ch^{\frac{d(\alpha-1)}{2}} \tag{2.14}$$

Proof. From (2.9),

$$\left\|\partial_{Z,\ell}\delta_Z^h\right\|_{\phi^{-\alpha-\frac{2}{d}}} \leq Ch^{\frac{d(\alpha-1)}{2}}. \tag{2.15}$$

From (2.10),

$$\|\partial_{Z,\ell}G_Z^*\|_{\phi^{-\alpha+\frac{2}{d}}} \leq Ch^{\frac{d(\alpha-1)}{2}}. \tag{2.16}$$

Combining (2.11), (2.15) and (2.16) immediately yields the result (2.14).

Theorem 2.1. *Suppose $q_0 > 2$ and $d \geq 7$. For $\partial_{Z,\ell}G_Z^*$ the regularized derivative Green's function defined by (1.4), we have the following estimate*:

$$|\partial_{Z,\ell}G_Z^*|_{1,1} \leq Ch^{\frac{2-d}{2}} \tag{2.17}$$

Proof. Obviously,

$$|\partial_{Z,\ell}G_Z^*|_{1,1} \leq \left(\int_\Omega \phi^\alpha dX\right)^{\frac{1}{2}} \|\nabla(\partial_{Z,\ell}G_Z^*)\|_{\phi^{-\alpha}}.$$

When $0 < \alpha < \frac{2}{d}$, we have by (2.6) and (2.14)

$$|\partial_{Z,\ell}G_Z^*|_{1,1} \leq C\inf_\alpha \frac{h^{\frac{d(\alpha-1)}{2}}}{1-\alpha} = Ch^{\frac{2-d}{2}},$$

which is the result (2.17).

3 Discrete Derivative Green's Function and Its Estimates

In this section, we will consider the estimates for discrete derivative Green's function. Similar to the two-dimensional setting (see [8]), the following result holds.

Lemma 3.1. *Suppose $u_h \in S_0^h(\Omega)$ is the finite element approximation to u, we have the following estimate*:

$$\|u-u_h\|^2_{1,\,\phi^{-\alpha}} \le Ch^{2s}\left\|\nabla^{s+1}u\right\|^2_{\phi^{-\alpha}} + C\gamma^{-2}\left\|u-u^h\right\|^2_{\phi^{-\alpha+\frac{2}{d}}}\,, \tag{3.1}$$

where $\gamma = \frac{\theta}{h}$. From the result (3.1), we get the following result.

Lemma 3.2. *Suppose $q_0 > 2$ and $0 < \alpha < \min\{\frac{4}{d}, 1-\frac{2}{q_0}+\frac{2}{d}\}$, then we have*

$$\left\|\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h\right\|_{1,\,\phi^{-\alpha}} \le Ch^{\frac{d(\alpha-1)}{2}}. \tag{3.2}$$

Proof. From (3.1),

$$\begin{aligned}
&\left\|\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h\right\|^2_{1,\,\phi^{-\alpha}}\\
\le\;& Ch^2\left\|\nabla^2(\partial_{Z,\ell}G_Z^*)\right\|^2_{\phi^{-\alpha}} + C\left\|\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h\right\|^2_{\phi^{-\alpha+\frac{2}{d}}}\\
\le\;& \hat{C}\left(h^2\left\|\nabla^2(\partial_{Z,\ell}G_Z^*)\right\|^2_{\phi^{-\alpha}} + \left\|\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h\right\|^2_{\phi^{-\alpha+\frac{2}{d}}}\right).
\end{aligned}$$

Similar to the Lemma 6 in [8, Chapter 3], we obtain

$$\left\|\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h\right\|^2_{\phi^{-\alpha+\frac{2}{d}}} \le \frac{2}{3\hat{C}}\left\|\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h\right\|^2_{1,\,\phi^{-\alpha}}.$$

Then we have

$$\left\|\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h\right\|^2_{1,\,\phi^{-\alpha}} \le Ch^2\left\|\nabla^2(\partial_{Z,\ell}G_Z^*)\right\|^2_{\phi^{-\alpha}}. \tag{3.3}$$

Similar to the arguments of the result (2.14), when $0 < \alpha < \frac{4}{d}$ and $q_0 > 2$, we can get

$$\left\|\nabla^2(\partial_{Z,\ell}G_Z^*)\right\|_{\phi^{-\alpha}} \le Ch^{\frac{d(\alpha-1)-2}{2}}. \tag{3.4}$$

Combining (3.3) and (3.4) immediately yields the result (3.2).

Lemma 3.3. *Suppose $q_0 > \frac{2d}{d-2}$. For $\partial_{Z,\ell}G_Z^*$ and $\partial_{Z,\ell}G_Z^h$, the regularized derivative Green's function and the discrete derivative Green's function, respectively, we have the following estimate*:

$$\left|\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h\right|_{1,\,1} \le Ch^{\frac{4-d}{2}}. \tag{3.5}$$

Proof. Obviously,

$$\left|\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h\right|^2_{1,\,1} \le \int_\Omega \phi^\alpha\,dX \cdot \left|\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h\right|^2_{1,\,\phi^{-\alpha}}. \tag{3.6}$$

When $d \geq 7$ and $0 < \alpha < \min\{\frac{4}{d}, 1 - \frac{2}{q_0} + \frac{2}{d}\}$, from (2.6), (3.2) and (3.6),

$$\left|\partial_{Z,\ell} G_Z^* - \partial_{Z,\ell} G_Z^h\right|_{1,1}^2 \leq C(1-\alpha)^{-1} h^{d(\alpha-1)}.$$

Since $q_0 > \frac{2d}{d-2}$, we have $\frac{4}{d} < 1 - \frac{2}{q_0} + \frac{2}{d}$. Thus,

$$\left|\partial_{Z,\ell} G_Z^* - \partial_{Z,\ell} G_Z^h\right|_{1,1}^2 \leq C \inf_{0<\alpha<\frac{4}{d}} (1-\alpha)^{-1} h^{d(\alpha-1)} = Ch^{4-d},$$

which shows the result (3.5) holds.

In the following, we give the estimate for the discrete derivative Green's function.

Theorem 3.1. *Suppose $q_0 > \frac{2d}{d-2}$ and $d \geq 7$. For $\partial_{Z,\ell} G_Z^h$ the discrete derivative Green's function defined by (1.4), we have the following estimate*:

$$\left|\partial_{Z,\ell} G_Z^h\right|_{1,1} \leq Ch^{\frac{2-d}{2}}. \tag{3.7}$$

Proof. By the triangular inequality,

$$\left|\partial_{Z,\ell} G_Z^h\right|_{1,1} \leq \left|\partial_{Z,\ell} G_Z^*\right|_{1,1} + \left|\partial_{Z,\ell} G_Z^* - \partial_{Z,\ell} G_Z^h\right|_{1,1}. \tag{3.8}$$

From (2.17), (3.5) and (3.8), we immediately obtain the result (3.7).

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Existence of Solutions to a Coupled System of Higher-order Nonlinear Fractional Differential Equations with Anti-periodic Boundary Conditions

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Abstract

In this paper, the authors study a coupled system of nonlinear fractional differential equations of order $\alpha, \beta \in (4, 5)$, the differential operator is taken in the Caputo sense. By using the Schauder fixed point theorem and the contraction mapping principle, the existence and uniqueness of solutions to the system with anti-periodic boundary conditions are obtained. Two examples are given to demonstrate the feasibility of the results.

Keywords: Coupled system; Fractional differential equations; Anti-periodic boundary conditions; existence; uniqueness.

1. Introduction

Recently, fractional differential equations have proved to be valuable tools in various fields of science and engineering. Indeed, we can find numerous applications in control, porous media, fluid flows, chemical physics and many other branches of science, see[1–3]. As a result, there are many papers dealing with the existence and uniqueness of solutions to nonlinear fractional differential equations, see[4–10].

Anti-periodic boundary value problems arise in the mathematical modeling of a variety of physical process, many authors have paid much attention in such problems, for examples and details of anti-periodic boundary conditions, the interested readers may refer to [11–17]. On the other hand, the coupled systems of nonlinear fractional differential equations have been a subject of intensive studies [17–21].

It should be noted that in [18–21], the study objects are coupled systems, but not the case of Caputo fractional derivatives. In [11–16], the authors only studied the existence of solutions for anti-periodic boundary value problems of fractional differential equation but not the coupled system. Motivated by [17], we consider a coupled system of nonlinear fractional differential equations in the sense of Caputo with a nonlinear term containing the derivatives of unknown functions.

In this paper, we study the existence and uniqueness of solutions to the following coupled system of

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nonlinear fractional differential equations

$$\begin{cases} {}^cD^{\alpha}x(t) + f(t, y(t), {}^cD^p y(t)) = 0, t \in [0, T], \\ {}^cD^{\beta}y(t) + g(t, x(t), {}^cD^q x(t)) = 0, t \in [0, T], \\ x^{(k)}(0) = -x^{(k)}(T), k = 0, 1, 2, 3, 4, \\ y^{(k)}(0) = -y^{(k)}(T), k = 0, 1, 2, 3, 4, \end{cases} \tag{1.1}$$

where $4 < \alpha, \beta < 5, \alpha - q \geq 1, \beta - p \geq 1$, ${}^cD^{\alpha}$ denotes the Caputo fractional derivative of order α, $f, g : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

This paper is organized as follows. In Section 2, we recall some basic definitions and preliminary results. In Section 3, we prove the existence of solutions to (1.1) by means of the Schauder fixed point theorem. Then, we obtain the uniqueness of solutions to the system by the contraction mapping principle. At the end, two examples are given to illustrate the applicability of our results.

2. Background Materials

For the convenience of the readers, we present here the necessary definitions and lemmas [2], which are used throughout this paper.

Definition 2.1. *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0, \infty) \to \mathbb{R}$ is given by*

$$I^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds,$$

provided the right hand side is pointwise defined on $(0, \infty)$.

Definition 2.2. *The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $y : (0, \infty) \to \mathbb{R}$ is given by*

$${}^cD^{\alpha}y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{y^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1, [\alpha]$ denotes the integer part of number α, provided that the right side is pointwise defined on $(0, \infty)$.

Lemma 2.3. *For any $y \in C[0, T]$, the unique solution of the boundary value problem*

$$\begin{cases} {}^cD^q x(t) = y(t), t \in [0, T], 4 < q \leq 5, \\ x^{(k)}(0) = -x^{(k)}(T), k = 0, 1, 2, 3, 4 \end{cases} \tag{2.1}$$

can be written as

$$x(t) = \int_0^T G(t, s)y(s)ds,$$

where $G(t, s)$ is the Green's function given by

$$G(t,s) = \begin{cases} \frac{2(t-s)^{q-1} - (T-s)^{q-1}}{2\Gamma(q)} + \frac{(T-2t)(T-s)^{q-2}}{4\Gamma(q-1)} + \frac{t(T-t)(T-s)^{q-3}}{4\Gamma(q-2)} \\ + \frac{(6Tt^2 - 4t^3 - T^3)(T-s)^{q-4}}{48\Gamma(q-3)} + \frac{(2Tt^3 - tT^3 - t^4)(T-s)^{q-5}}{48\Gamma(q-4)}, 0 < s < t < T, \\ -\frac{(T-s)^{q-1}}{2\Gamma(q)} + \frac{(T-2t)(T-s)^{q-2}}{4\Gamma(q-1)} + \frac{t(T-t)(T-s)^{q-3}}{4\Gamma(q-2)} \\ + \frac{(6Tt^2 - 4t^3 - T^3)(T-s)^{q-4}}{48\Gamma(q-3)} + \frac{(2Tt^3 - tT^3 - t^4)(T-s)^{q-5}}{48\Gamma(q-4)}, 0 < t < s < T. \end{cases}$$

Let

$$G_1(t,s)=\begin{cases}\frac{(t-s)^{\alpha-1}-\frac{1}{2}(T-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(T-2t)(T-s)^{\alpha-2}}{4\Gamma(\alpha-1)}+\frac{t(T-t)(T-s)^{\alpha-3}}{4\Gamma(\alpha-2)}\\ +\frac{(6Tt^2-4t^3-T^3)(T-s)^{\alpha-4}}{48\Gamma(\alpha-3)}+\frac{(2Tt^3-tT^3-t^4)(T-s)^{\alpha-5}}{48\Gamma(\alpha-4)},0<s<t<T,\\ -\frac{1}{2\Gamma(\alpha)}(T-s)^{\alpha-1}+\frac{(T-2t)(T-s)^{\alpha-2}}{4\Gamma(\alpha-1)}+\frac{t(T-t)(T-s)^{\alpha-3}}{4\Gamma(\alpha-2)}\\ +\frac{(6Tt^2-4t^3-T^3)(T-s)^{\alpha-4}}{48\Gamma(\alpha-3)}+\frac{(2Tt^3-tT^3-t^4)(T-s)^{\alpha-5}}{48\Gamma(\alpha-4)},0<t<s<T.\end{cases}$$

$$G_2(t,s)=\begin{cases}\frac{(t-s)^{\beta-1}-\frac{1}{2}(T-s)^{\beta-1}}{\Gamma(\beta)}+\frac{(T-2t)(T-s)^{\beta-2}}{4\Gamma(\beta-1)}+\frac{t(T-t)(T-s)^{\beta-3}}{4\Gamma(\beta-2)}\\ +\frac{(6Tt^2-4t^3-T^3)(T-s)^{\beta-4}}{48\Gamma(\beta-3)}+\frac{(2Tt^3-tT^3-t^4)(T-s)^{\beta-5}}{48\Gamma(\beta-4)},0<s<t<T,\\ -\frac{1}{2\Gamma(\beta)}(T-s)^{\beta-1}+\frac{(T-2t)(T-s)^{\beta-2}}{4\Gamma(\beta-1)}+\frac{t(T-t)(T-s)^{\beta-3}}{4\Gamma(\beta-2)}\\ +\frac{(6Tt^2-4t^3-T^3)(T-s)^{\beta-4}}{48\Gamma(\beta-3)}+\frac{(2Tt^3-tT^3-t^4)(T-s)^{\beta-5}}{48\Gamma(\beta-4)},0<t<s<T.\end{cases}$$

We call (G_1, G_2) Green's function for Problem (1.1).

Define the space

$$\mathcal{C}=\{x(t):x(t)\in C^4[0,T],x^{(k)}(0)=-x^{(k)}(T),k=0,1,2,3,4\},$$

and

$$X=\{x(t):x(t)\in\mathcal{C}\ and\ ({}^cD^qx)(t)\in C[0,T]\}$$

endowed with the norm

$$\|x\|_X=\max_{0\le i\le 4}\max_{t\in[0,T]}|\,x^{(i)}(t)\,|+\max_{t\in[0,T]}|\,({}^cD^qx)(t)\,|,$$

where $i\in\mathbb{N}$.

Lemma 2.4. $(X,\|\cdot\|_X)$ *is a Banach space.*

Proof. Apparently X is a subspace of $C^4[0,T]$, so we only need to prove that X is closed. Let $x_n(t)$ be a sequence converging to some $x(t)$ in $(X,\|\cdot\|_X)$, then it is clear that $x_n(t)$ is a converging sequence in the space $C^4[0,T]$ and hence $x\in\mathcal{C}$. Furthermore, the uniform convergence of $({}^cD^qx_n)(t)$ to $({}^cD^qx)(t)$ implies that $({}^cD^qx)(t)\in C[0,T]$ and therefore $x(t)\in X$. The proof is complete.

Similarly, we can define the Banach space

$$Y=\{y(t):y(t)\in\mathcal{C}\ and\ ({}^cD^p)y(t)\in C[0,T]\}$$

endowed with the norm

$$\|y\|_Y=\max_{0\le i\le 4}\max_{t\in[0,T]}|\,y^{(i)}(t)\,|+\max_{t\in[0,T]}|\,({}^cD^py)(t)\,|,$$

where $i\in\mathbb{N}$.

For $(x,y)\in(X,Y)$, let

$$\|(x,y)\|_{X\times Y}=\max\{\|x\|_X,\|y\|_Y\}.$$

Then clearly $(X\times Y,\|\cdot\|_{X\times Y})$ is a Banach space.

Consider the following coupled system of integral equations:

$$\begin{cases} x(t) = \int_0^T G_1(t,s) f(s, y(s), {}^c D^p y(t)) ds, \\ y(t) = \int_0^T G_2(t,s) g(s, x(s), {}^c D^q x(t)) ds. \end{cases} \tag{2.2}$$

The following lemma states that Problem (1.1) is equivalent to Problem (2.2) and therefore the study of a system of differential equations is turn into the study of a system of integral equations.

Lemma 2.5. *Assume that $f, g : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Then $(x,y) \in (X,Y)$ is a solution of (1.1) if and only if $(x,y) \in (X,Y)$ is a solution of system (2.2).*

Proof. The proof is immediate from the discussion above, we omit the details here.

Let $F : X \times Y \to X \times Y$ be an operator defined as $F(x,y)(t) = (F_1 y(t), F_2 x(t))$, where

$$F_1 y(t) = \int_0^T G_1(t,s) f(s, y(s), {}^c D^p y(t)) ds, \ F_2 x(t) = \int_0^T G_2(t,s) g(s, x(s), {}^c D^q x(t)) ds.$$

It is obvious that a fixed-point of the operator F is a solution of Problem (1.1).

Now we present the main results of this paper.

3. Main Results

In this section, we will discuss the existence and uniqueness of solutions to Problem (1.1).

Lemma 3.1. [17] *The Green's functions $G_1(t,s), G_2(t,s)$ satisfy the following estimates:*

$$\int_0^T | G_1(t,s) | ds \le \tfrac{T^\alpha}{\Gamma(\alpha+1)} (\tfrac{3}{2} + \tfrac{5\alpha^4 - 14\alpha^3 + 55\alpha^2 + 146\alpha}{768}) = U_1, t \in [0,T], \tag{3.1}$$

$$\int_0^T | G_2(t,s) | ds \le \tfrac{T^\beta}{\Gamma(\beta+1)} (\tfrac{3}{2} + \tfrac{5\beta^4 - 14\beta^3 + 55\beta^2 + 146\beta}{768}) = U_2, t \in [0,T], \tag{3.2}$$

$$\int_0^T \left| \tfrac{\partial G_1(t,s)}{\partial t} \right| ds \le \tfrac{T^{\alpha-1}}{\Gamma(\alpha)} (\tfrac{3}{2} + \tfrac{\alpha^3 - 3\alpha^2 + 14\alpha - 12}{48}) = U_3, t \in [0,T], \tag{3.3}$$

$$\int_0^T \left| \tfrac{\partial G_2(t,s)}{\partial t} \right| ds \le \tfrac{T^{\beta-1}}{\Gamma(\beta)} (\tfrac{3}{2} + \tfrac{\beta^3 - 3\beta^2 + 14\beta - 12}{48}) = U_4, t \in [0,T]. \tag{3.4}$$

Theorem 3.2. *Assume that one of the following conditions is satisfied:*

(H_1) *there exist positive constants A, B and constants $b_i, c_i > 0, 0 < \rho_i, \theta_i < 1$ for $i = 1,2$ such that*

$$| f(t,x,y) | \le A + b_1 | x |^{\rho_1} + b_2 | y |^{\rho_2}, | g(t,x,y) | \le B + c_1 | x |^{\theta_1} + c_2 | y |^{\theta_2};$$

(H_2) *there exist constants $l_i, k_i > 0, 0 < \gamma_i, \varphi_i < 1$ for $i = 1,2$ such that*

$$| f(t,x,y) | \le l_1 | x |^{\gamma_1} + l_2 | y |^{\gamma_2}, | g(t,x,y) | \le k_1 | x |^{\varphi_1} + k_2 | y |^{\varphi_2};$$

(H_3) *there exist constants $d_i, \sigma_i > 0, \delta_i, \varepsilon_i > 1$ for $i = 1,2$ such that*

$$| f(t,x,y) | \le d_1 | x |^{\delta_1} + d_2 | y |^{\delta_2}, | g(t,x,y) | \le \sigma_1 | x |^{\varepsilon_1} + \sigma_2 | y |^{\varepsilon_2};$$

then Problem (1.1) has a solution.

Before proving Theorem 3.2, we define a ball B in the Banach space $X \times Y$ as

$$B = \{(x(t), y(t)) | (x(t), y(t)) \in X \times Y, \|(x, y)\|_{X \times Y} \leq R, t \in [0, T]\},$$

where

$$R \geq \max\Big\{3UA\lambda_1, (3Ub_1\lambda_1)^{\frac{1}{1-\rho_1}}, (3Ub_2\lambda_1)^{\frac{1}{1-\rho_2}}, 3KB\lambda_2, (3Kc_1\lambda_2)^{\frac{1}{1-\theta_1}}, (3Kc_2\lambda_2)^{\frac{1}{1-\theta_2}}\Big\}.$$

$U = \max\{U_1, U_3, U_5, U_6, U_7\}$, where $U_5 = \frac{T^{\alpha-2}}{\Gamma(\alpha-1)}(\frac{3}{2} + \frac{\alpha^2-\alpha-2}{16})$, $U_6 = \frac{T^{\alpha-3}(\alpha+3)}{4\Gamma(\alpha-2)}$, $U_7 = \frac{3T^{\alpha-4}}{2\Gamma(\alpha-3)}$, K is defined by the expression of U by replacing the corresponding α with β in each case, $\lambda_1 = \frac{\Gamma([q]-q+2)+T^{[q]-q+1}}{\Gamma([q]-q+2)}$, $\lambda_2 = \frac{\Gamma([p]-p+2)+T^{[p]-p+1}}{\Gamma([p]-p+2)}$.

Proof.

Part 1: Let (H_1) be valid.

Step 1 : $F : B \to B$.

$$\begin{aligned}\Big|\int_0^T \frac{\partial G_1^2(t,s)}{\partial t^2} ds\Big| &\leq \int_0^T \Big|\frac{(t-s)^{\alpha-3}}{\Gamma(\alpha-2)} - \frac{(T-s)^{\alpha-3}}{2\Gamma(\alpha-2)} + \frac{(T-2t)(T-s)^{\alpha-4}}{4\Gamma(\alpha-3)} + \frac{t(T-t)(T-s)^{\alpha-5}}{4\Gamma(\alpha-4)}\Big| ds \\ &\leq \frac{T^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{T^{\alpha-2}}{2\Gamma(\alpha-1)} + \frac{T^{\alpha-2}}{4\Gamma(\alpha-2)} + \frac{T^{\alpha-2}}{16\Gamma(\alpha-3)} \\ &= \frac{T^{\alpha-2}}{\Gamma(\alpha-1)}(\frac{3}{2} + \frac{\alpha^2-\alpha-2}{16}) = U_5,\end{aligned}$$

$$\begin{aligned}\Big|\int_0^T \frac{\partial G_1^3(t,s)}{\partial t^3} ds\Big| &\leq \int_0^T \Big|\frac{(t-s)^{\alpha-4}}{\Gamma(\alpha-3)} - \frac{(T-s)^{\alpha-4}}{2\Gamma(\alpha-3)} + \frac{(T-2t)(T-s)^{\alpha-5}}{4\Gamma(\alpha-4)}\Big| ds \\ &\leq \frac{T^{\alpha-3}}{\Gamma(\alpha-2)} + \frac{T^{\alpha-3}}{2\Gamma(\alpha-2)} + \frac{T^{\alpha-3}}{4\Gamma(\alpha-3)} \\ &= \frac{T^{\alpha-3}(\alpha+3)}{4\Gamma(\alpha-2)} = U_6,\end{aligned}$$

$$\begin{aligned}\Big|\int_0^T \frac{\partial G_1^4(t,s)}{\partial t^4} ds\Big| &\leq \int_0^T \Big|\frac{(t-s)^{\alpha-5}}{\Gamma(\alpha-4)} - \frac{(T-s)^{\alpha-5}}{2\Gamma(\alpha-4)}\Big| ds \\ &\leq \frac{T^{\alpha-4}}{\Gamma(\alpha-3)} + \frac{T^{\alpha-4}}{2\Gamma(\alpha-3)} \\ &= \frac{3T^{\alpha-4}}{2\Gamma(\alpha-3)} = U_7.\end{aligned}$$

Let $U = \max\{U_1, U_3, U_5, U_6, U_7\}$, when $k = 0, 1, 2, 3, 4$, we have

$$\begin{aligned}| (F_1 y)^{(k)}(t) | &= \Big| \int_0^T \frac{\partial G_1^k(t,s)}{\partial t^k} f(s, y(s), {}^c D^p y(t)) ds \Big| \\ &\leq \int_0^T \Big| \frac{\partial G_1^k(t,s)}{\partial t^k} \Big| (A + b_1 R^{\rho_1} + b_2 R^{\rho_2}) ds \\ &\leq U(A + b_1 R^{\rho_1} + b_2 R^{\rho_2}) = M.\end{aligned}$$

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On the other hand, we can get

$$\begin{aligned}|^{c}D^{q}F_1y(t)| &= \frac{1}{\Gamma([q]+1-q)}\int_0^t (t-s)^{[q]-q}|(F_1y)^{([q]+1)}(s)|ds\\ &\leq \frac{M}{\Gamma([q]+1-q)}\int_0^t (t-s)^{[q]-q}ds\\ &\leq \frac{MT^{[q]-q+1}}{\Gamma([q]+2-q)}.\end{aligned}$$

As a result

$$\begin{aligned}\|F_1y\|_X &= \max_{0\leq i\leq 4}\max_{t\in[0,T]} \mid (F_1y)^{(i)}(t)\mid + \max_{t\in[0,T]}\mid ({}^{c}D^{q}F_1y)(t)\mid\\ &\leq M + \frac{MT^{[q]-q+1}}{\Gamma([q]+2-q)} = M\lambda_1\\ &= U(A+b_1R^{\rho_1}+b_2R^{\rho_2})\lambda_1\\ &\leq \frac{R}{3}+\frac{R}{3}+\frac{R}{3} = R.\end{aligned}$$

Similarly

$$\begin{aligned}\|F_2x\|_Y &= \max_{0\leq i\leq 4}\max_{t\in[0,T]} \mid (F_2x)^{(i)}(t)\mid + \max_{t\in[0,T]}\mid ({}^{c}D^{p}F_2x)(t)\mid\\ &\leq K(B+c_1R^{\theta_1}+c_2R^{\theta_2})\lambda_2 \leq R.\end{aligned}$$

Hence, we conclude that $\|F(x,y)\|_{X\times Y} = \max\{\|F_1y\|_X, \|F_2x\|_Y\} \leq R$, in consequence, $F: B\to B$.

Step 2: F is continuous. This follows easily from the continuity of $f, g, x(t), y(t)$ and $G_1(t,s), G_2(t,s)$.

Step 3: $F(B)$ is relatively compact. Let us set

$$M_1 = \max\{\mid f(t,y(t),{}^{c}D^{p}y(t))\mid : t\in[0,T], \|y\|_Y\leq R, \|{}^{c}D^{p}y\|\leq R\},$$

$$N_1 = \max\{\mid g(t,x(t),{}^{c}D^{q}x)\mid : t\in[0,T], \|x\|_X\leq R, \|{}^{c}D^{q}x\|\leq R\}.$$

$$\begin{aligned}\mid (F_1y)'(t)\mid &= \Big|\int_0^T \frac{\partial G_1(t,s)}{\partial t} f(s,y(s),{}^{c}D^{p}y(s))ds\Big|\\ &\leq M_1\int_0^T \Big|\frac{\partial G_1(t,s)}{\partial t}\Big|ds \leq M_1U_3.\end{aligned}$$

Hence, for $t_1, t_2\in[0,T]$, we have

$$\mid (F_1y)(t_2)-(F_1y)(t_1)\mid \leq \int_{t_1}^{t_2}\mid (F_1y)'(s)\mid ds \leq M_1U_3\mid t_2-t_1\mid.$$

Similarly, we can get

$$\mid (F_2x)(t_2)-(F_2x)(t_1)\mid \leq \int_{t_1}^{t_2}\mid (F_2x)'(s)\mid ds \leq N_1U_4\mid t_2-t_1\mid.$$

By the Arzelà-Ascoli theorem, we can obtain that $F(B)$ is an equicontinuous set, the operator $F: B\to B$ is completely continuous. Thus, Problem (1.1) has one solution by the Schauder fixed-point theorem.

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Part 2: Let (H_2) be valid. In this part, let

$$R \geq \max\Big\{(2Ul_1\lambda_1)^{\frac{1}{1-\gamma_1}}, (2Ul_2\lambda_1)^{\frac{1}{1-\gamma_2}}, (2Kk_1\lambda_2)^{\frac{1}{1-\varphi_1}}, (2Kk_2\lambda_2)^{\frac{1}{1-\varphi_2}}\Big\}.$$

We can also get the result by repeating arguments similar to part 1.

Part 3: Let (H_3) be valid. In this part, let

$$0 \leq R \leq \min\Big\{(2Ud_1\lambda_1)^{-\frac{1}{\delta_1-1}}, (2Ud_2\lambda_1)^{-\frac{1}{\delta_2-1}}, (2K\sigma_1\lambda_2)^{-\frac{1}{\varepsilon_1-1}}, (2K\sigma_2\lambda_2)^{-\frac{1}{\varepsilon_2-1}}\Big\}.$$

We can also get the result by repeating arguments similar to part 1. Here we omit it. This completes the proof.

Example 3.1. *Consider the system*

$$\begin{cases} {}^cD^{17/4}x(t) + \sin t + (y(t))^{2/3} + \big({}^cD^{5/2}y(t)\big)^{2/5} = 0, 0 < t < 1, \\ {}^cD^{9/2}y(t) + t^{1/2} + (x(t))^{1/3} + \big({}^cD^{11/4}x(t)\big)^{4/7} = 0, 0 < t < 1, \\ x^{(k)}(0) = -x^{(k)}(1), k = 0, 1, 2, 3, 4, \\ y^{(k)}(0) = -y^{(k)}(1), k = 0, 1, 2, 3, 4. \end{cases} \tag{3.5}$$

The system satisfies (H_1) and hence Theorem 3.2 implies the existence of the solution to system (3.5).

Theorem 3.3. *Let f and g satisfy the following growth conditions :*

(H_1) *there exist four positive constants L_1, L_2, H_1, H_2 such that*

$$| f(t, x_1, y_1) - f(t, x_2, y_2) | \leq L_1|x_1 - x_2| + L_2|y_1 - y_2|,$$

$$| g(t, x_1, y_1) - g(t, x_2, y_2) | \leq H_1|x_1 - x_2| + H_2|y_1 - y_2|,$$

$$t \in [0, T], x_i, y_i \in \mathbb{R}, i = 1, 2.$$

(H_2)

$$\max\{L_1, L_2\}U_1 = Q_1 < 1, \max\{H_1, H_2\}U_2 = Q_2 < 1.$$

Then Problem (1.1) has a unique solution.

Proof. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$, then

$$\begin{aligned} | (F_1y_1 - F_1y_2)(t) | &= \Big| \int_0^T G_1(t,s)f(s, y_1(s), {}^cD^py_1(s))ds - \int_0^T G_1(t,s)f(s, y_2(s), {}^cD^py_2(s))ds \Big| \\ &\leq \int_0^T | G_1(t,s) || f(s, y_1(s), {}^cD^py_1(s)) - f(s, y_2(s), {}^cD^py_2(s)) | ds \\ &\leq U_1\big(L_1|y_1(s) - y_2(s)| + L_2|{}^cD^py_1(s) - {}^cD^py_2(s)|\big) \leq \max\{L_1, L_2\}U_1\|y_1 - y_2\|_Y. \end{aligned}$$

Analogously,

$$\begin{aligned} | (F_2x_1 - F_2x_2)(t) | &\leq \int_0^T | G_2(t,s) || g(s, x_1(s), {}^cD^qx_1(s)) - g(s, x_2(s), {}^cD^qx_2(s)) | ds \\ &\leq U_2\big(H_1|x_1(s) - x_2(s)| + H_2|{}^cD^qx_1(s) - {}^cD^qx_2(s)|\big) \leq \max\{H_1, H_2\}U_2\|x_1 - x_2\|_X. \end{aligned}$$

Thus,

$$\begin{aligned}
\| F(x_1,y_1)-F(x_2,y_2) \|_{X\times Y} &= \| (F_1y_1-F_1y_2, F_2x_1-F_2x_2) \|_{X\times Y}\\
&= \max\{\| F_1y_1-F_1y_2 \|_X, \| F_2x_1-F_2x_2 \|_Y\}\\
&\le \max\{Q_1 \| y_1-y_2 \|_Y, Q_2 \| x_1-x_2 \|_X\}\\
&\le \max\{Q_1,Q_2\}\max\{\| y_1-y_2 \|_Y, \| x_1-x_2 \|_X\}\\
&= \max\{Q_1,Q_2\} \| (x_1,y_1)-(x_2,y_2) \|_{X\times Y}.
\end{aligned}$$

Hence, we conclude that Problem (1.1) has a unique solution by (H_2) and the contraction mapping principle, this ends the proof.

Example 3.2. *Consider the system*

$$\begin{cases}
{}^cD^{17/4}x(t)+L_1\sin y(t)+L_2\frac{{}^cD^{5/2}y(t)}{1+{}^cD^{5/2}y(t)}=0, 0<t<1,\\
{}^cD^{9/2}y(t)+H_1\arctan x(t)+H_2\frac{{}^cD^{11/4}x(t)}{1+{}^cD^{11/4}x(t)}=0, 0<t<1,\\
x^{(k)}(0)=-x^{(k)}(1), k=0,1,2,3,4,\\
y^{(k)}(0)=-y^{(k)}(1), k=0,1,2,3,4.
\end{cases} \tag{3.6}$$

Where $T=1$, $f(t,y(t),{}^cD^py(t))=L_1\sin y(t)+L_2\frac{{}^cD^{5/2}y(t)}{1+{}^cD^{5/2}y(t)}$, $g(t,x(t),{}^cD^qx(t))=H_1\arctan x(t)+H_2\frac{{}^cD^{11/4}x(t)}{1+{}^cD^{11/4}x(t)}$, $\alpha=\frac{17}{4}$, $\beta=\frac{9}{2}$, $p=5/2$, $q=11/4$ and $L_1,L_2,H_1,H_2>0$.

Noting that

$$|(\sin y)'|=|\cos y|\le 1, |(\arctan x)'|=\frac{1}{1+x^2}\le 1, \left|(\frac{v}{1+v})'\right|=\frac{1}{(1+v)^2}\le 1,$$

we have

$$\begin{aligned}
&| f(t,y_1(t),{}^cD^py_1(t))-f(t,y_2(t),{}^cD^py_2(t)) |\\
&\le L_1 | \sin y_1(t)-\sin y_2(t) | +L_2\left|\frac{{}^cD^{5/2}y_1(t)}{1+{}^cD^{5/2}y_1(t)}-\frac{{}^cD^{5/2}y_2(t)}{1+{}^cD^{5/2}y_2(t)}\right|\\
&\le L_1 | y_1(t)-y_2(t) | +L_2 |{}^cD^{5/2}y_1(t)-{}^cD^{5/2}y_2(t)|\\
&\le \max\{L_1,L_2\} \| y_1-y_2 \|_Y,
\end{aligned}$$

$$\begin{aligned}
&| g(t,x_1(t),{}^cD^qx_1(t))-g(t,x_2(t),{}^cD^qx_2(t)) |\\
&\le H_1 | \arctan x_1(t)-\arctan x_2(t) | +H_2\left|\frac{{}^cD^{11/4}x_1(t)}{1+{}^cD^{11/4}x_1(t)}-\frac{{}^cD^{11/4}x_2(t)}{1+{}^cD^{11/4}x_2(t)}\right|\\
&\le H_1 | x_1(t)-x_2(t) | +H_2 |{}^cD^{11/4}x_1(t)-{}^cD^{11/4}x_2(t)|\\
&\le \max\{H_1,H_2\} \| x_1-x_2 \|_X,
\end{aligned}$$

$$U_1=\frac{T^\alpha}{\Gamma(\alpha+1)}(\frac{3}{2}+\frac{5\alpha^4-14\alpha^3+55\alpha^2+146\alpha}{768})\approx 0.1229,$$

$$U_2=\frac{T^\beta}{\Gamma(\beta+1)}(\frac{3}{2}+\frac{5\beta^4-14\beta^3+55\beta^2+146\beta}{768})\approx 0.0920.$$

as long as we let $\max\{L_1,L_2\}<\frac{1}{0.1229}$, $\max\{H_1,H_2\}<\frac{1}{0.0920}$, it will have $Q_1<1, Q_2<1$, then we can conclude from Theorem 3.3 that system (3.6)has a unique solution.

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Iteration Process for Pointwise Lipschitzian Type Mappings in Hyperbolic 2-uniformly Convex Metric Spaces

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Abstract

In this paper, we discuss the existence of fixed points for a class of Lipschitzian type mappings and asymptotic pointwise Lipschitz type mappings in hyperbolic 2-uniformly convex metric spaces. In the same space setting, we deal the problem of approximation of fixed points via modified Mann iteration process. Our result generalizes and extends the corresponding results of Dehaish et al. [7], Goebel and Kirk [8], Kirk and Xu [18] and Sahu et al. [24] and many others in this direction.

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1 Introduction

Let C be a nonempty subset of a metric space X and $T : C \to C$ be a mapping. Then T is called

(1) *nonexpansive* if $d(Tx, Ty) \le d(x, y)$ for all $x, y \in C$;

(2) *asymptotically nonexpansive* [8] if for each $n \in \mathbb{N}$, there exists a constant $k_n \ge 1$ with $\lim_{n\to\infty} k_n = 1$ such that

$$d(T^n x, T^n y) \le k_n d(x, y)$$

for all $x, y \in C$;

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(3) a *pointwise contraction* [3] if there exists a function $\alpha : C \to [0, 1)$ such that

$$d(Tx, Ty) \leq \alpha(x)d(x, y)$$

for all $x, y \in C$;

(4) an *asymptotic pointwise contraction* [17] if for each $n \in \mathbb{N}$, there exists a function $\alpha_n : C \to [0, 1)$ such that

$$d(T^n x, T^n y) \leq \alpha_n(x)d(x, y)$$

for all $x, y \in C$, where $\alpha_n \to \alpha : C \to [0, 1)$ pointwise on C;

(5) *pointwise asymptotically nonexpansive* [18] if there exists a sequence $\{\alpha_n\}$ for each integer $n \in \mathbb{N}$, a function exists a function $\alpha_n : C \to [1, \infty)$

$$d(T^n x, T^n y) \leq \alpha_n(x)d(x, y)$$

for all $x, y \in C$, where $\alpha_n(x) \to 1$ pointwise on C;

(6) *asymptotically nonexpansive in the intermediate sense* [5] provided T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} d(T^n x, T^n y) - d(x, y)) \leq 0; \tag{1.1}$$

(7) *asymptotically nonexpansive type* [13, 16] if

$$\limsup_{n\to\infty} \sup_{y\in C}(d(T^n x, T^n y) - d(x, y)) \leq 0$$

for all $x \in C$.

There is a class of mappings which lies strictly between the class of contraction mappings and the class of nonexpansive mappings. The class of pointwise contractions was introduced in Belluce and Kirk [3] and later it was called generalized contraction in [12]. Banach's celebrated contraction principle was extended to this larger class of mappings as follows:

Theorem 1.1. ([3, 12]) *Let C be a nonempty weakly compact convex subset of a Banach space and $T : C \to C$ a pointwise contraction. Then T has a unique fixed point, x^*, and $\{T^n x\}$ converges strongly to x^* for each $x \in C$.*

Kirk [17] combined ideas of pointwise contraction [3] and asymptotic contraction [15] and introduced the concept of an asymptotic pointwise contraction. He announced that an asymptotic pointwise contraction defined on closed convex and bounded subset of a super-reflexive Banach space has a fixed point.

In [18], Kirk and Xu introduced the concept of asymptotically pointwise and proved that every pointwise asymptotically nonexpansive mapping defined on a closed convex Banach space has a fixed point.

The class of asymptotically nonexpansive mappings in the intermediate sense which is essentially wider than that of asymptotically nonexpansive was introduced by Bruck et al. [5]. It is known that [16] if C is a nonempty closed convex bounded subset of a uniformly convex Banach space X and T is a self mapping of C which is asymptotically nonexpansive in the intermediate sense, then T has fixed point.

On the other hand, if $c_n = \max\{\sup_{x \in C}(d(T^n x, T^n y) - d(x, y), 0\}$, then (1.1) reduces to relation

$$d(T^n x, T^n y) \leq d(x, y) + c_n \tag{1.2}$$

for all $x, y \in C$ and $n \in \mathbb{N}$. The classes of mappings more general than the class of mapping satisfying (1.2) were studied in Alber et al. [2] as the class of total asymptotically nonexpansive mappings and in Sahu [22] as the class of nearly Lipschitzian mappings.

Fix a sequence $\{a_n\}$ in $[0, \infty)$ with $a_n \to 0$. A mapping $T : C \to C$ is said to be *nearly Lipschitzian* with respect to $\{a_n\}$ ([22]) if for each $n \in \mathbb{N}$, there exists a constants $k_n > 0$ such that

$$d(T^n x, T^n y) \leq k_n(d(x, y) + a_n) \tag{1.3}$$

for all $x, y \in C$. The infimum of the constants k_n in (1.3) is called nearly Lipschitz constant and is denoted by $\eta(T^n)$. A nearly Lipschitzian mapping T with the sequence $\{a_n, \eta(T^n)\}$ is said to be

(1) *nearly contraction* if $\eta(T^n) < 1$ for all $n \in \mathbb{N}$,
(2) *nearly uniformly L-Lipschitzian* if $\eta(T^n) \leq L$ for all $n \in \mathbb{N}$,
(3) *nearly uniformly k-contraction* if $\eta(T^n) \leq k < 1$ for all $n \in \mathbb{N}$,
(4) *nearly nonexpansive* if $\eta(T^n) = 1$ for all $n \in \mathbb{N}$,
(5) *nearly asymptotically nonexpansive* if $\eta(T^n) \geq 1$ for all $n \in \mathbb{N}$ with $\lim_{n\to\infty} \eta(T^n) = 1$.

The corresponding Lipschitzian type mappings (for instance, contraction type mappings) concerning asymptotically nonexpansive mappings in the intermediate sense and total asymptotically nonexpansive mappings are not defined in Bruck et al. [5] and Alber et al. [2]. The notion of nearly Lipschitzian mappings allows to define different classes of Lipschitzian types mappings, for example, nearly contraction, nearly nonexpansive, nearly asymptotically nonexpansive, nearly uniformly L- Lipschitzian etc. Therefore, the fixed point theory of nearly Lipschitzian mappings is of fundamental importance. Some properties and existence and convergence results for nearly Lipschitzian mappings are studied in [22, 23]. The perturbation of a nonexpansive mapping as a sequence of nearly nonexpansive mappings is studied and its applications are given in [26, 25].

Recently, Sahu et al. [24] introduced some new classes of pointwise nearly Lipschitz type mappings in Banach spaces and studied some existence theorems in Banach spaces. Inspired by the work of Kirk and Xu [18] and Sahu et al. [24] studied the existence of fixed points of pointwise nearly Lipschitzian mappings in Banach spaces. In [24], it is shown that the asymptotic center of every bounded orbit of a pointwise asymptotically nonexpansive mapping is fixed point of the mapping in a uniformly convex Banach space.

In [27], Schu considered modified Mann iterations for asymptotically nonexpansive mappings on a convex subset of a Banach space. Recently, Khan et al. [11] have introduced and studied the convergence of a general iteration scheme of asymptotically quasi-nonexpansive mappings in convex metric spaces and CAT(0) spaces.

Recently, Dehaish et al. [7] studied the existence of a fixed point of a single and a family of asymptotic pointwise nonexpansive mappings defined on uniformly convex hyperbolic

spaces. They also discussed the behavior of the following modified Mann iteration process associated with asymptotic pointwise nonexpansive mapping T:

$$x_{n+1} = t_n T^n(x_n) \oplus (1 - t_n)x_n, n \in \mathbb{N}, \tag{1.4}$$

where $\{t_n\} \subset [0, 1]$ be bounded away from 0 and 1 and $x_1 \in C$ is an arbitrary point.

The purpose of this paper is to extend the notion of the pointwise Lipschitzian type mappings introduced in [24] and establish existence and convergence theorems for fixed points for the class of pointwise nearly asymptotically nonexpansive mappings in the framework of hyperbolic 2-uniformly convex metric spaces. Our results generalize, extend and unify the corresponding results of Dehaish et al. [7], Goebel and Kirk [8], Kirk and Xu [18] and Sahu et al. [24] and many others in this direction..

2 Preliminaries

2.1 Uniformly convexity in metric spaces

Let (X, d) be a metric space. Suppose that there exists a family $\mathcal{F}$ of metric space segments such that any two points $x, y \in X$ are end points of a unique metric segment $[x, y] \in \mathcal{F}$. Here $[x, y]$ is an isometric image of the real line interval $[0, d(x, y)]$. We shall denote by $tx \oplus (1 - t)y$ the unique point z of $[x, y]$ which satisfies

$$d(x, z) = (1 - t)d(x, y) \quad \text{and} \quad d(z, y) = td(x, y),$$

where $t \in [0, 1]$. Such metric spaces are usually called *convex metric spaces* [20]. Moreover, if

$$d(\alpha p \oplus (1 - \alpha)x, \alpha q \oplus (1 - \alpha)y) \leq \alpha d(p, q) + (1 - \alpha)d(x, y),$$

for all $p, q, x, y \in X$, and $\alpha \in [0, 1]$, then X is said to be a *hyperbolic metric space* (see [21]).

It is easy to see that normed linear spaces are hyperbolic spaces. As nonlinear examples, one can consider the Hadamard manifolds [6], the Hilbert open unit ball equipped with the hyperbolic metric [9], and the CAT(0) spaces [14, 16, 19] (see Example 2.8).

Definition 2.1. ([10]) A subset C of a hyperbolic metric space X is *convex* if $[x, y] \subset C$ whenever $x, y \in C$.

Definition 2.2. ([10]) Let (X, d) be a hyperbolic metric space. We say that X is *uniformly convex* if for any $a \in X$, for every $r > 0$, and for each $\varepsilon > 0$,

$$\delta(r, \varepsilon) = \inf\left\{1 - \frac{1}{r}d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) : d(x, a) \leq r, d(y, a) \leq r, d(x, y) \geq r\varepsilon\right\} > 0.$$

From now onward we assume that X is a hyperbolic metric space and if (X, d) is uniformly convex, then for every $s \geq 0, \varepsilon > 0$, there exists $\eta(s, \varepsilon) > 0$ depending on s and ε such that

$$\delta(r, \varepsilon) > \eta(s, \varepsilon) > 0 \quad \text{for any } r > s.$$

Remark 2.3. If (X,d) is uniformly convex, then we have the following:
(1) $\delta(r,0)=0$ and $\delta(r,\varepsilon)$ is an increasing function of ε for every fixed r.
(2) For $r_1 \le r_2$, the following holds:

$$1-\frac{r_2}{r_1}\left(1-\delta\left(r_2,\varepsilon\,\frac{r_1}{r_2}\right)\right)\le \delta(r_1,\varepsilon).$$

(3) If (X,d) is uniformly convex, then (X,d) is strictly convex, that is, whenever

$$d(x,a)=d(y,a)=d\left(\frac{1}{2}x\oplus\frac{1}{2}y,a\right),$$

for any $x,y,a\in X$, then we must have $x=y$.

Recall that a hyperbolic metric space X is said to have *property* (R) [10] if any non-increasing sequence of nonempty, convex, bounded, and closed sets has a nonempty intersection.

The following theorem was proved by Khamsi and Khan [10].

Theorem 2.4. ([10]) *Assume that (X,d) is complete and uniformly convex. Let C be nonempty, convex, and closed. Then for any $x\in X$, there exists a unique best approximant of x in C, that is, a unique $x_0\in C$ such that*

$$d(x,x_0)=d(x,C).$$

Note that any complete and uniformly convex metric space has the property (R) (see [10]).

We need the following results for our main results.

Lemma 2.5. ([10] Lemma 2.2) *Let (X,d) be uniformly convex. Assume that there exists $r\ge 0$ such that*

$$\limsup_{n\to\infty} d(x_n,a)\le r,\quad \limsup_{n\to\infty} d(y_n,a)\le r\quad \text{and}\quad \lim_{n\to\infty} d\left(a,\frac{1}{2}x_n\oplus\frac{1}{2}y_n\right)=r.$$

Then $\lim_{n\to\infty} d(x_n,y_n)=0$.

The following metric version of the parallelogram identity, also known as the inequality of Bruhat and Tits, has been established in [10].

Theorem 2.6. ([10]) *Let (X,d) be uniformly convex. Fix $a\in X$. For each $r>0$ and for each $\varepsilon>0$, denote*

$$\Psi(r,\varepsilon)=\inf\left\{\frac{1}{2}d^2(a,x)+\frac{1}{2}d^2(a,y)-d^2\left(a,\frac{1}{2}x\oplus\frac{1}{2}y\right)\right\},$$

where the infimum is taken over all $x,y\in X$ such that $d(a,x)\le r, d(a,y)\le r$ and $d(x,y)\ge r\varepsilon$. Then $\Psi(r,\varepsilon)>0$ for any $r>0$ and each $\varepsilon>0$. Moreover, for a fixed $r>0$, we have

(i) $\Psi(r, 0) = 0$;

(ii) $\Psi(r, \varepsilon)$ *is non-decreasing function of* ε;

(iii) *if* $\lim_{n\to\infty} \Psi(r, t_n) = 0$, *then* $\lim_{n\to\infty} t_n = 0$.

The notion of p-uniform convexity was studied extensively by Xu [28], its nonlinear version for $p = 2$ has been introduced by Khamsi and Khan [10] using the above function Ψ as follows.

Definition 2.7. ([10]) We say that (X, d) is 2-*uniformly convex* if

$$C_X = \inf\left\{\frac{\Psi(r,\varepsilon)}{r^2\varepsilon^2} : r > 0, \varepsilon > 0\right\} > 0.$$

From the definition of C_X, we obtain the following inequality:

$$d^2\left(a, \frac{1}{2}x \oplus \frac{1}{2}y\right) + C_X d^2(x, y) \le \frac{1}{2}d^2(a, x) + \frac{1}{2}d^2(a, y)$$

for any $a \in X$ and $x, y \in X$.

Example 2.8. Let (X, d) be a metric space. A geodesic from x to y in X is a mapping c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t-t'|$ for all $t, t' \in [0, l]$.

In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$, which will be denoted by $[x, y]$, and called the segment joining x to y. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ).

A comparison triangle for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3,)$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$ such triangle exists (see [4]).

A geodesic space is said to be a *CAT(0) space* if all geodesic triangles of appropriate size satisfy the following comparison axiom.

Let Δ be a geodesic triangle in X and let $\bar{\Delta} \subset \mathbb{R}^2$ be a comparison triangle for Δ. Then Δ is said to satisfy the CAT(0) inequality if

$$d(x, y) \le d(\bar{x}, \bar{y}).$$

for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$.

Complete CAT(0) spaces are often called *Hadamard spaces* (see [16]). If x, y_1, y_2 are points of a CAT(0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$, which will be denoted by $\frac{y_1 \oplus y_2}{2}$, then the CAT(0) inequality implies

$$d^2\left(x, \frac{1}{2}y_1 \oplus \frac{1}{2}y_2\right) \le \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2). \qquad (CN)$$

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This inequality is the (CN) inequality of Bruhat and Tits [4]. As for the Hilbert space, the (CN) inequality implies the CAT(0) spaces are uniformly convex with

$$\delta(r,\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}.$$

The (CN) inequality also implies that

$$\Psi(r,\varepsilon) = \frac{r^2\varepsilon^2}{4}.$$

Thus, a CAT(0) space is 2-uniformly convex with $C_X = \frac{1}{4}$.

We need the following more general inequality for convergence of Mann iterations.

Theorem 2.9. ([7]) *Let (X,d) be 2-uniformly convex. Then, for any $\alpha \in (0,1)$, there exists $C_X > 0$ such that*

$$d^2(a, \alpha x \oplus (1-\alpha)y) + C_X \min\{\alpha^2, (1-\alpha)^2\} d^2(x,y) \leq \alpha d^2(a,x) + (1-\alpha) d^2(a,y)$$

for any $a, x, y \in X$.

Recall that $\Phi : X \to \mathbb{R}^+$ is called a *type* if there exists $\{x_n\}$ in X such that

$$\Phi(x) = \limsup_{n\to\infty} d(x, x_n).$$

Theorem 2.10. ([10, Theorem 2.4]) *Assume that (X,d) is a complete and uniformly convex. Let C be a nonempty closed bounded and convex subset of X. Let Φ be a type defined on C. Then any minimizing sequence of Φ is convergent. Its limit is independent of the minimizing sequence.*

In fact, if X is 2-uniformly convex, and Φ is a type defined on a nonempty closed convex bounded subset C of X, then there exists a unique $x_0 \in C$ such that

$$\Phi^2(x_0) + 2C_X d^2(x_0, x) \leq \Phi^2(x) \tag{2.1}$$

for any $x \in C$. In this inequality, one may find an analogy with Opial property used in the study of the fixed point property in Banach and metric spaces.

2.2 Pointwise Lipschitzian type mappings and fixed points

First, we extend some wider classes of nonlinear mappings studied by Sahu et al. [24] in a metric space setting.

Definition 2.11. ([24]) Let C be a nonempty subset of a metric space (X,d). A mapping $T : C \to C$ is said to be

(1) *pointwise nearly Lipschitzian with sequence* $\{(\alpha_n(\cdot), a_n)\}$ if, there exists a sequence $\{a_n\}$ in $[0,\infty)$ with $a_n \to 0$ and for each $n \in \mathbb{N}$, there exists a function $\alpha_n(.) : C \to (0,\infty)$ such that

$$d(T^n x, T^n y) \leq \alpha_n(x)(d(x,y) + a_n)$$

for all x,$y \in C$;

(2) *pointwise nearly uniformly $\alpha(\cdot)$-Lipschitzian with sequence* $\{a_n\}$ if, there exists a sequence $\{a_n\}$ in $[0,\infty)$ with $a_n \to 0$ and there exists a function $\alpha(\cdot) : C \to (0,\infty)$ such that

$$d(T^n x, T^n y) \leq \alpha(x)(d(x,y) + a_n)$$

for all x,$y \in C$;

(3) *asymptotic pointwise nearly Lipschitzian with sequence* $\{(\alpha_n(\cdot), a_n)\}$ if, there exists a sequence $\{a_n\}$ in $[0,\infty)$ with $a_n \to 0$ and for each $n \in \mathbb{N}$, there exists a function $\alpha_n(.) : C \to (0,\infty)$ and with $\alpha_n \to \alpha : C \to (0,\infty)$ pointwise such that

$$d(T^n x, T^n y) \leq \alpha_n(x)(d(x,y) + a_n)$$

for all x,$y \in C$.

We say that, an asymptotic pointwise nearly Lipschitzian mapping is

(1) *pointwise nearly asymptotic nonexpansive* if $\alpha_n(x) \geq 1$ for all $n \in \mathbb{N}$ and $\alpha_n(x) \to 1$ pointwise,

(2) *pointwise asymptotically nonexpansive* [18] $a_n = 0$ and $\alpha_n(x) \geq 1$ for all $n \in \mathbb{N}$ and $\alpha_n(x) \to 1$ pointwise.

(3) a *asymptotic pointwise nearly contraction* if $\alpha_n \to \alpha$ pointwise and $\alpha(x) \leq k < 1$ for all $x \in C$.

A point $x \in C$ is called a *fixed point* of T if $T(x) = x$. The fixed point set of T is denoted by $Fix(T)$.

3 Existence theorem

First, we prove the existence of fixed point for a pointwise nearly asymptotically nonexpansive mapping in a 2-uniformly convex metric space.

Theorem 3.1. *Let C be nonempty closed convex and bounded subset of a complete hyperbolic 2-uniformly convex metric space (X,d). Let $T : C \to C$ be a continuous pointwise nearly asymptotically nonexpansive mapping. Then T has a fixed point in C. Moreover, the set of fixed points is closed and convex.*

Proof. Fix $x \in C$. Define the function $\Phi(y) = \limsup_{n\to\infty} d(T^n(x), y)$ on C. By (2.1), there exists a unique $\omega \in C$ such that

$$\Phi^2(\omega) + 2C_X d^2(\omega, y) \leq \Phi^2(y)$$

for all $y \in C$. In particular, we have

$$\Phi^2(\omega) + 2C_X d^2(\omega, T^n(\omega)) \leq \Phi^2(T^n(\omega)) \tag{3.1}$$

for all $n \geq 1$. Observe that

$$\begin{aligned}
\Phi(T^n(\omega)) &= \limsup_{m\to\infty} d(T^m(x), T^n(\omega)) \\
&\leq \limsup_{m\to\infty} d(T^n(T^{m-n}(x)), T^n(\omega)) \\
&\leq \limsup_{m\to\infty} \left[\alpha_n(\omega)(d(T^{m-n}(x), \omega) + a_n)\right] \\
&\leq \alpha_n(\omega)(\Phi(\omega) + a_n)
\end{aligned} \tag{3.2}$$

for all $n \geq 1$. Hence, from (3.1) and (3.2), we have

$$\begin{aligned}\Phi^2(\omega) + 2C_X d^2(\omega, T^n(\omega)) &\leq \Phi^2(T^n(\omega)) \\ &\leq \big(\alpha_n(\omega)(\Phi(\omega) + a_n)\big)^2 \\ &= \alpha_n^2(\omega)[\Phi^2(\omega) + a_n^2 + 2\Phi(\omega)a_n]\end{aligned}$$

for all $n \geq 1$. By the definition of T, $\alpha_n(\omega) \to 1$ pointwise and $a_n \to 0$ as $n \to \infty$. Thus, $\lim_{n\to\infty} d(\omega, T^n(\omega)) = 0$, i.e., $T^n(\omega) \to \omega$ as $n \to \infty$. By the continuity of T, we have

$$T(\omega) = T\Big(\lim_{n\to\infty} T^n(\omega)\Big) = \lim_{n\to\infty} T^{n+1}(\omega) = \omega.$$

Closedness of $Fix(T)$: Let $\{x_n\}$ be a sequence in $Fix(T)$ such that $\lim_{n\to\infty} x_n = x$ for some $x \in C$. Now it remains to show that $x \in Fix(T)$. Note that

$$d(T^n(x_n), T^n(x)) \leq \alpha_n(x)(d(x_n, x) + a_n),$$

which implies that

$$\lim_{n\to\infty} d(T^n(x), x_n) = 0.$$

Since

$$d(x, T^n(x)) \leq d(x, x_n) + d(x_n, T^n(x),$$

we have, $\lim_{n\to\infty} d(x, T^n(x)) = 0$. By continuity of T, we have $Tx = x$.

Convexity of $Fix(T)$: Let $x, y \in Fix(T)$. We only need to prove that $z = \frac{x\oplus y}{2} \in Fix(T)$. Without loss of generality, we assume that $x \neq y$. Note that

$$\begin{aligned}d(x, T^n(z)) &= d(T^n(x), T^n(z)) \\ &\leq \alpha_n(x)(d(x, z) + a_n) \\ &\leq \alpha_n(x)\left(d\left(x, \frac{x \oplus y}{2}\right) + a_n\right) \\ &= \alpha_n(x)\left(\frac{1}{2}d(x, y) + a_n\right)\end{aligned}$$

for all $n \geq 1$. Similarly, we have

$$d(y, T^n(z)) \leq \alpha_n(y)\left(\frac{1}{2}d(x, y) + a_n\right)$$

for all $n \geq 1$. By triangular inequality, we have

$$\begin{aligned}d(x, y) &\leq d(x, T^n(z)) + d(T^n(z), y) \\ &\leq \alpha_n(x)\left(\frac{1}{2}d(x, y) + a_n\right) + \alpha_n(y)\left(\frac{1}{2}d(x, y) + a_n\right) \\ &= (\alpha_n(x) + \alpha_n(y))\left(\frac{1}{2}d(x, y) + a_n\right),\end{aligned}$$

it follows that

$$\lim_{n\to\infty} d(x, T^n(z)) = \lim_{n\to\infty} d(T^n(z), y) = d(x, y).$$

Note

$$
\begin{aligned}
d\left(x, \frac{z \oplus T^n(z)}{2}\right) &\leq \frac{1}{2}\ d(x,z) + \frac{1}{2} d(x, T^n(z)) \\
&\leq \frac{1}{2}\ d(x,z) + \frac{1}{2}\alpha_n(x)\left(\frac{1}{2} d(x,z) + a_n\right), \\
&= \left(\frac{1+\alpha_n(x)}{2}\right) d(x,z) + \frac{\alpha_n(x) a_n}{2}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
d\left(y, \frac{z \oplus T^n(z)}{2}\right) &\leq \frac{1}{2}\ d(y,z) + \frac{1}{2} d(y, T^n(z)) \\
&\leq \frac{1}{2}\ d(y,z) + \frac{1}{2}\alpha_n(y)\left(\frac{1}{2} d(y,z) + a_n\right) \\
&= \left(\frac{1+\alpha_n(y)}{2}\right) d(y,z) + \frac{\alpha_n(y) a_n}{2}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
d(x,y) &\leq d\left(x, \frac{z \oplus T^n(z)}{2}\right) + d\left(\frac{z \oplus T^n(z)}{2}, y\right) \\
&\leq \frac{1+\alpha_n(x)}{2}\ \frac{d(x,y)}{2} + \frac{1+\alpha_n(y)}{2}\ \frac{d(x,y)}{2} + \frac{\alpha_n(x)+\alpha_n(y)}{2} a_n.
\end{aligned}
$$

Taking limit as $n \to \infty$ both the sides, Hence, we have

$$
\lim_{n\to\infty} d\left(x, \frac{z \oplus T^n(z)}{2}\right) = \lim_{n\to\infty} d\left(y, \frac{z \oplus T^n(z)}{2}\right) = \frac{d(x,y)}{2}.
$$

Using Lemma 2.5, we conclude that $\lim_{n\to\infty} d(z, T^n(z)) = 0$. Therefore, we must have $T(z) = z$, i.e., $\frac{x \oplus y}{2} \in Fix(T)$. This completes the proof. □

Remark 3.2. Theorem 3.1 is a natural generalization of Proposition 3.4 and Theorem 3.8 of Sahu et al. [24] in the framework of a hyperbolic 2-uniformly convex metric space. Theorem 3.1 extends the results of Dehaish et al. [7, Theorem 3.1], Goebel and Kirk [8, Theorem 1], and Kirk and Xu [18, Theorem 3.4] to the class of pointwise nearly Lipschitzian mappings which essentially wider than the class of mappings appearing in [7], [8] and [18].

4 Convergence of Mann iteration process

Lemma 4.1. *Let C be nonempty, closed, convex, and bounded subset of a complete hyperbolic 2-uniformly convex metric space (X,d). Let $T : C \to C$ be a pointwise nearly asymptotically nonexpansive with sequence $\{(\alpha_n(\cdot), a_n)\}$ such that T is uniformly continuous. Assume that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty}(\alpha_n(p) - 1) < \infty$ for all $p \in Fix(T)$. Let $\{t_n\} \subset [0,1]$ be bounded away from 0 and 1, i.e., there exist two real numbers a, b such*

that $0 < a \leq t_n \leq b < 1$. *The modified Mann iteration process is defined by* (1.4). *Then we have the following:*

(*a*) $\lim_{n\to\infty} d(x_n, p)$ *exists for all* $p \in Fix(T)$.

(*b*) $\lim_{n\to\infty} d(x_n, T^n(x_n)) = 0$ *and* $\lim_{n\to\infty} d(x_n, T^m(x_n)) = 0$ *for all* $m \geq 1$, *provided that* $L = \sup_{n\in\mathbb{N}} \sup_{x\in C} \alpha_n(x) < \infty$.

Proof. (*a*) Let $\delta(C) = \sup_{x,y\in C} d(x, y)$ be the diameter of C. Let $\omega \in Fix(T)$. Set $\delta_n := d(x_n, \omega)$, $\beta_n = \alpha_n(\omega)$ and $\gamma_n = a_n L$. From (1.4), we have

$$\begin{aligned}
\delta_{n+1} &= d(x_{n+1}, \omega)\\
&= d(t_n T^n(x_n) \oplus (1 - t_n)x_n, \omega)\\
&\leq t_n d(T^n(x_n), \omega) + (1 - t_n)d(x_n, \omega)\\
&\leq t_n d(T^n(x_n), T^n(\omega)) + (1 - t_n)d(x_n, \omega)\\
&\leq t_n\alpha_n(\omega)(d(x_n, \omega) + a_n) + (1 - t_n)d(x_n, \omega)\\
&\leq t_n\alpha_n(\omega)d(x_n, \omega) + a_n t_n\alpha_n(\omega) + (1 - t_n)d(x_n, \omega)\\
&\leq \alpha_n(\omega)d(x_n, \omega) + \alpha_n(\omega)a_n\\
&\leq \beta_n\delta_n + \gamma_n
\end{aligned}$$

for all $n \geq 1$. Noticing that $\sum_{n=1}^{\infty}(\beta_n - 1) < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$.for all $n \geq 1$. Therefore, from [1, Lemma 6.1.5], we conclude that $\lim_{n\to\infty} d(x_n, \omega)$ exists.

(*b*) First, we prove that $\lim_{n\to\infty} d(x_n, T^n(x_n)) = 0$. By Theorem 3.1, T has a fixed point $\omega \in C$. Lemma 4.1 implies that $\lim_{n\to\infty} d(x_n, \omega)$ exists. Set $r = \lim_{n\to\infty} d(x_n, \omega)$. Without loss of generality, we may assume $r > 0$. Note

$$\begin{aligned}
\limsup_{n\to\infty} d(T^n(x_n), \omega) &= \limsup_{n\to\infty} d(T^n(x_n), T^n(\omega))\\
&\leq \limsup_{n\to\infty}(\alpha_n(\omega)(d(x_n, \omega) + a_n)) = r.
\end{aligned}$$

On the other hand, from (1.4), we have

$$d(x_{n+1}, \omega) \leq t_n d(T^n(x_n), \omega) + (1 - t_n)d(x_n, \omega)$$

for all $n \geq 1$. Let $\mathcal{U}$ be a non-trivial ultrafilter over $\mathbb{N}$. Then $\lim_{\mathcal{U}} t_n = t \in [a, b]$. Hence

$$r = \lim_{\mathcal{U}} d(x_{n+1}, \omega) \leq t \lim_{\mathcal{U}} d(T^n(x_n), \omega) + (1 - t)r.$$

Since $t \neq 0$, we get $\lim_{\mathcal{U}} d(T^n(x_n), \omega) \geq r$. Hence

$$r \leq \liminf_{n\to\infty} d(T^n(x_n), \omega) \leq \limsup_{n\to\infty} d(T^n(x_n), \omega) \leq r.$$

So $\lim_{n\to\infty} d(T^n(x_n, \omega) = r$. Since X is 2-uniformly convex, Theorem 2.9 implies

$$\begin{aligned}
C_X \min\{t_n^2, (1 - t_n)^2\}d^2(x_n, T^n(x_n)) \leq\ & t_n d^2(x_n, \omega) + (1 - t_n)d^2(T^n(x_n), \omega)\\
& - d^2(x_{n+1}, \omega),
\end{aligned}$$

where $C_X > 0$ depends only on X. Since

$$C_X \min\{t_n^2, (1 - t_n)^2\} \geq \min\{a^2, (1 - b)^2\} > 0,$$

and

$$\lim_{n\to\infty}\left\{t_n d^2(x_n,\omega)+(1-t_n)d^2(T^n(x_n),\omega)-d^2(x_{n+1},\omega)\right\}=0,$$

we get

$$\lim_{n\to\infty} d(x_n,T^n(x_n))=0,$$

which finish the prove of our claim.

Next, we prove that $\lim_{n\to\infty} d(x_n,T^m(x_n)=0$, for all $m\geq 1$. The uniform continuity of T implies that

$$\lim_{n\to\infty} d(Tx_n,T^{n+1}(x_n))=0.$$

From (1.4), we have

$$d(x_{n+1},x_n)\leq d(x_n,T^n(x_n))\to 0 \text{ as } n\to\infty.$$

Note that

$$\begin{aligned}
d(x_n,T(x_n)) \leq\ & d(x_n,x_{n+1})+d(x_{n+1},T^{n+1}(x_{n+1}))+d(T^{n+1}(x_{n+1}),T^{n+1}(x_n))\\
& +d(T^{n+1}(x_n),T(x_n))\\
\leq\ & d(x_n,x_{n+1})+d(x_{n+1},T^{n+1}(x_{n+1}))+\alpha_{n+1}(x_n)(d(x_{n+1},x_n)\\
& +a_{n+1})+d(T^{n+1}(x_n),T(x_n))\\
\leq\ & d(x_n,x_{n+1})+d(x_{n+1},T^{n+1}(x_{n+1}))+L(d(x_{n+1},x_n)+a_{n+1})\\
& +d(T^{n+1}(x_n),T(x_n))
\end{aligned}$$

for all $n\geq 1$. Hence, we get $\lim_{n\to\infty} d(x_nT(x_n))=0$. Again, from the uniform continuity of T, we have

$$\lim_{n\to\infty} d(T(x_n),T^2(x_n))=0,$$

it follows that

$$d(x_n,T^2(x_n)\leq d(x_n,T(x_n))+d(T(x_n),T^2(x_n))\to 0 \text{ as } n\to\infty.$$

Inductively, we have

$$\lim_{n\to\infty} d(x_n,T^m(x_n)=0$$

for all $m\geq 1$. This completes the proof. □

We now establish main result of this section.

Theorem 4.2. *Let C be nonempty, closed, convex, and bounded subset of a complete hyperbolic 2-uniformly convex metric space (X,d). Let $T:C\to C$ be a pointwise nearly asymptotically nonexpansive with sequence $\{(\alpha_n(\cdot),a_n)\}$ such that T is uniformly continuous and $\sup_{n\in\mathbb{N}}\sup_{x\in C}\alpha_n(x)<\infty$. Assume that $\sum_{n=1}^{\infty}a_n<\infty$ and $\sum_{n=1}^{\infty}(\alpha_n(p)-1)<\infty$ for all $p\in Fix(T)$. Let $\{t_n\}\subset[0,1]$ be bounded away from 0 and 1, i.e., there exist two real numbers a,b such that $0<a\leq t_n\leq b<1$. The modified Mann iteration process is defined by (1.4). Consider the type $\Phi(x)=\limsup_{n\to\infty} d(x_n,x)$ on C. If ω is the minimum point of Φ, that is, $\Phi(\omega)=\inf\{\Phi(x):x\in C\}$, then $T(\omega)=\omega$.*

Proof. Suppose that ω is the minimum point of Φ. For any $m, n \geq 1$, we have

$$d^2\left(x_n, \frac{\omega \oplus T^m(\omega)}{2}\right) + C_X d^2(\omega, T^m(\omega)) \leq \frac{1}{2}d^2(x_n, \omega) + \frac{1}{2}d^2(x_n, T^m(\omega)).$$

Letting limit as $n \to \infty$, we get

$$\Phi^2\left(\frac{\omega \oplus T^m(\omega)}{2}\right) + C_X d^2(\omega, T^m(\omega)) \leq \frac{1}{2}\Phi^2(\omega) + \frac{1}{2}\Phi^2(T^m(\omega)) \tag{4.1}$$

for any $m \geq 1$. Using Lemma 4.1, we get

$$\begin{aligned}
\Phi(T^m(\omega)) &= \limsup_{n\to\infty} d(x_n, T^m(\omega)) \\
&\leq \limsup_{n\to\infty} \Big[d(x_n, T^m(x_n) + d(T^m(x_n), T^m(\omega))\Big], \\
&\leq \limsup_{n\to\infty} d(T^m(x_n), T^m(\omega)) \\
&\leq \limsup_{n\to\infty} (\alpha_m(\omega)(d(x_n, \omega) + a_m)) \\
&= \alpha_m(\omega)(\Phi(\omega) + a_m)
\end{aligned}$$

for any $m \geq 1$. Since ω is the minimum point of Φ,we have

$$\Phi(\omega) \leq \Phi\left(\frac{\omega \oplus T^m(\omega)}{2}\right)$$

for any $m \geq 1$. From (4.1), we have

$$\begin{aligned}
\Phi^2(\omega) + C_X d^2(\omega, T^m(\omega)) &\leq \Phi^2\left(\frac{\omega \oplus T^m(\omega)}{2}\right) + C_X d^2(\omega, T^m(\omega)) \\
&\leq \frac{1}{2}\Phi^2(\omega) + \frac{1}{2}\Phi^2(T^m(\omega)) \\
&\leq \frac{1}{2}\Phi^2(\omega) + \frac{1}{2}\,[\alpha_m(\omega)(\Phi(\omega) + a_m)]^2
\end{aligned}$$

for $m \geq 1$. Taking limit superior as $m \to \infty$, we get

$$\Phi^2(\omega) + C_X \limsup_{m\to\infty} d^2(\omega, T^m(\omega)) \leq \Phi^2(\omega).$$

This implies that $\lim_{m\to\infty} d(\omega, T^m(\omega)) = 0$. Therefore, $T(\omega) = \omega$, i.e., $\omega \in Fix(T)$. This completes the proof. □

Remark 4.3. Theorem 4.2 extends the result of Dehaish et al. [7, Theorem 4.1] to pointwise nearly Lipschitzian mapping which essentially wider than the mapping appearing in [7].

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Regularity of the American Option Value Function in Jump-Diffusion Model

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Abstract

This work is devoted to the regularity properties of the American options value function, when there are brusque variations in prices. We assume that there are finite number of jumps in each finite time interval and the asset price jumps in the proportions which are independent and identically distributed. These properties can be used to investigate the optimal hedging strategies, optimal exercise boundaries etc. for the options in jump-diffusion process.
Keywords: American Option, Jump-Diffusion Model, Poisson Process, Lipschitz Continuity, Weak Derivatives.

1 Introduction

The pricing of options and the corporate liabilities have been developed significantly after the classical paper by Black and Scholes (1973). Although several techniques for the calculation of the value of the European option have been proposed in closed-form, the American options are still open for further research and consideration, causing an extensive literature on numerical methods.

Recently, in Israel and Rincon (2008), the American options problem using inequality variational systems, and numerical methods based on finite elements and finite difference

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techniques is solved properly. Indeed, as it has been proposed also in Jaillet et. al. (1990), the problem to find the value of a put American option can be equivalent to getting the solution of a system of variational inequalities provided that this formulation respects some necessary hypotheses, see also Isreal and Rincon (2008). Jaillet, Lamberon and Lapeyre (1990) rely on the link between the optimal stopping and variational inequality in order to exploit the theory of American options. Pham (1997) investigated the regularity of the value function of the put American option in jump-diffusion process using the properties of the optimal exercise boundary. For more detailed discussion on the value function of the American options we refer the readers to the papers by Chiarella and Kang (2011), El-Karoui, et. al. (1998), ellot and Kopp (1990), Hussain and Shashiashvili (2010), Hussain and Rehman (2012), and books Glowinski, et. al. (1981), Karatzas and Shereve (1998), Lamberton and Lapeyre (1997), Shreve (2004) etc.

We assume the interest rate and volatility are Lipschitz functions of time, payoff is arbitrary bounded from below convex function, and use purely probabilistic approach to obtained rigorous estimates for the first and second order derivatives of the value function of the put American options in order to use these results in our next work to construct uniform approximations for the discrete time hedging strategies as well as for the investigation of the optimal exercise boundary of the put American options.

In Section 2, we set the basic notation and we formulate our model. Thus, we consider a financial market with two assets, i.e. the value of a money market account and the share of a stock whose price jumps proportionally at some times τ_j following the Poisson process, similarly as in Pham (1997). Note that in order to deal with this problem, following also the existing literature, we recall that the American options value function can be considered as the value of a function of an equivalent optimal stopping time problem. Thus, some preliminary results are presented here. Finally, in Section 3, the regularity properties of a put American option are derived solving a system of variational inequalities.

2 Notation - Preliminary Results

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space on which we define a standard Wiener process $W = (W_t)_{0 \le t \le T}$, a Poisson process $N = (N_t)_{0 \le t \le T}$ with intensity λ and a sequence $(U_j)_{j \ge 1}$ of independent, identically distributed random variables taking values in $(-1, \infty)$. Assume that the time horizon $T < \infty$ is finite and the σ-algebras generated respectively by $(W_t)_{0 \le t \le T}$, $(N_t)_{0 \le t \le T}$, and $(U_j)_{j \ge 1}$ are independent. Denote by $(\mathcal{F}_t)_{0 \le t \le T}$ the $\mathbf{P}$-completion of the natural filtration of (W_t), (N_t) and $(U_j) I_{j \le N_t}$, $j \ge 1, 0 \le t \le T$.

On a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})_{0 \le t \le T}$ consider a financial market with two assets $m_t, 0 \le t \le T$, the price of a unit of a money market account at time t, and $S_t, 0 \le t \le T$, the value at time t of the share of a stock whose price jumps in the proportions $U_1, U_2, ...,$ at some times $\tau_1, \tau_2, \ldots$, see also Pham (1997). We assume that the τ_j's correspond to the jump times of a Poisson process.

The evolution of the assets m_t and S_t obeys the following ordinary and stochastic differential equations respectively,

$$dm_t = r(t) m_t dt, \; m_0 = 1, \quad 0 \le t \le T,$$

2

$$dS_t = S_{t-}\left(b(t)dt + \sigma(t)dW_t + d\left(\sum_{j=1}^{N_t} U_j\right)\right).$$

We assume that $(b(t), \mathcal{F}_t)_{0\le t\le T}$ is a certain, progressively measurable process; the deterministic time-varying interest rate $r(t)$ and the volatility $\sigma(t)$ are continuously differentiable functions of time, and the following requirements are satisfied:

$$0 \le r(t) \le \bar{r}, \quad 0 < \underline{\sigma} \le \sigma(t) \le \bar{\sigma}, \quad |b(t)| \le \bar{r},$$

$$|r(t) - r(s)| + |\sigma(t) - \sigma(s)| \le K|t-s|, \tag{1}$$

where $s, t \in [0, T]$ and $\bar{r}$, $\underline{\sigma}$, $\bar{\sigma}$ and K are some positive constants.

From the above stochastic differential equation, the dynamics of S_t can be described by:

$$S_t = S_0\left(\prod_{j=1}^{N_t}(1+U_j)\right)\exp\left[\int_0^t \left(b(u) - \frac{\sigma^2(u)}{2}\right)du + \int_0^t \sigma(u)dW_u\right].$$

It is known, see for instance Lamberton and Lapeyre (1997), that the discounted stock price $\tilde{S}_t = e^{-\int_0^t r(u)du}S_t$ is a *martingale* if and only if

$$\int_0^t b(u)du = \int_0^t r(u)du - \lambda t E(U_1). \tag{2}$$

In this brief paper, we investigate the *regularity* properties of the American option value function with a nonnegative, non-increasing convex payoff function $g(x), x \ge 0$. We assume that $g(0) = g(0+)$. Of course, a typical example of this family of functions is the *put American option* with payoffs $g(x) = (L-x)^+$ where L is the exercise price.

In the next paragraphs of this section, we present some necessary and preliminary results for the better understanding, and evaluation of our main outputs.

First, it is necessary to recall that the American option value function $v(t,x), x \ge 0, 0 \le t \le T$, can be considered as the value function of a relevant optimal stopping problem (see, for instance Karatzas and Shereve (1998), Section 2.5). In particular

$$v(t,x) = \sup_{\tau\in\mathcal{T}_{t,T}} E\left[\exp\left(-\int_t^\tau r(v)dv\right)g(S_\tau(t,x))\right], x \ge 0, 0 \le t \le T, \tag{3}$$

where $\mathcal{T}_{t,T}$ denotes the set of all stopping times τ such that $t \le \tau \le T$, and the stochastic process $S_u(t,x), t \le u \le T$ satisfies the same stochastic differential equation as above, i.e.

$$dS_u(t,x) = S_{u-}(t,x)\left(b(u)du + \sigma(u)dW_u + d\left(\sum_{j=1+N_t}^{N_u} U_j\right)\right),\ t \le u \le T, \tag{4}$$

with the initial condition $S_t(t,x) = x, x \ge 0$.

The unique solution $(S_u(t,x), \mathcal{F}_u)_{t\le u\le T}$ of (4) is given by the exponential

$$S_u(t,x) = x\left(\prod_{j=1+N_t}^{N_u}(1+U_j)\right)\exp\left[\int_t^u \left(b(u) - \frac{\sigma^2(u)}{2}\right)du + \int_t^u \sigma(u)dW_u\right].$$

Condition (2) leads to
$S_u(t,x) = \exp\left[\ln x + \int_t^u \left(r(u) - \lambda E(U_1) - \frac{\sigma^2(u)}{2}\right) du + \int_t^u \sigma(u) dW_u + \sum_{j=N_t+1}^{N_u} \ln(1+U_j)\right].$
Now, we can introduce the new stochastic process $(X_u(t,x), \mathcal{F}_u)_{t\le u\le T}$

$$X_u(t,y) = y + \int_t^u \left(r(u) - \lambda E(U_1) - \frac{\sigma^2(v)}{2}\right) dv + \int_t^u \sigma(v) dW_v + \sum_{j=N_t+1}^{N_u} \ln(1+U_j),$$

$t \le u \le T,\ -\infty < y < \infty,\ U_j \in (-1,\infty),\ j = 1,2,\ldots.$

Remark 2.1. *Profoundly,*

$$S_u(t,x) = \exp\left[X_u(t,\ln x)\right],\ t \le u \le T\ x > 0, \tag{5}$$

and for an arbitrary stopping time τ, $t \le \tau \le T$, *we obtain*

$$g(S_\tau(t,x)) = \psi(X_\tau(t,\ln x)),$$

where $\psi(y) = g(e^y)$, $-\infty < y < \infty$ *is the new payoff function.*

Now, it is clear that the corresponding optimal stopping time problem is derived straightforwardly by just substituting (5) into (3), having now

$$u(t,y) = \sup_{\tau\in\mathcal{T}_{t,T}} E\left[\exp\left(-\int_t^\tau r(v)dv\right)\psi(X_\tau(t,y))\right], \tag{6}$$

with $0 \le t \le T$ and $-\infty < y < \infty$, then we obtain

$$v(t,x) = u(t,\ln x),\ x > 0,\ 0 \le t \le T.$$

In what follows, the next known result, from Hussain and Shashiashvili (2010), is needed

Lemma 2.2. *Let* $g(x), x \ge 0$ *be a nonnegative, non-increasing convex function. Then the new payoff function defined by* $\psi(y) = g(e^y), -\infty < y < \infty$ *is Lipschitz continuous, that is,*

$$|\psi(y_2) - \psi(y_1)| \le g(0)|y_2 - y_1|,\ y_1, y_2 \in \mathbb{R}.$$

Thus, using the *scaling* property of the Brownian motion we can express the value function $u(t,y)$ of the optimal stopping time problem (6), see Jaillet, et. al. (1990), as follows

$$u(t,y) = \sup_{\tau\in\mathcal{T}_{0,1}} E\Bigg[\exp\left(-\int_t^{t+\tau(T-t)} r(v)dv\right)\psi\Bigg(y + \int_t^{t+\tau(T-t)}\left(r(v) - \lambda E(U_1) - \frac{\sigma^2(v)}{2}\right)dv$$

$$+ \int_0^\tau \sqrt{T-t}\,\sigma(t+v(T-t))dW_v + \sum_{j=1}^{N_{t+\tau(T-t)}} \ln(1+U_j)\Bigg)\Bigg], \tag{7}$$

where $\mathcal{T}_{0,1}$ denotes the set of all stopping times τ with respect to the filtration $(\mathcal{F}_u)_{0\le u\le 1}$ taking values in $[0,1]$.

Finally, we conclude the preliminary results of this section by proving the following theorem.

Theorem 2.3. *The value function $u(t,y), 0 \le t \le T, -\infty < y < \infty$ of the optimal stopping problem (6) is Lipschitz continuous in the argument y and locally Lipschitz continuous in t i.e.*

$$|u(t,y) - u(t,z)| \le g(0)\ |y-z|,\ \ y,z \in \mathbb{R}, 0 \le t \le T, \tag{8}$$

$$|u(t,y) - u(s,y)| \le \frac{A}{\sqrt{T-t}}|t-s|, \tag{9}$$

where A is some nonnegative constant depending on parameters $\bar{r}$, $\bar{\sigma}$, $g(0)$, λ, $E(U_1)$, K and T.

Proof. Fixing any τ in $\mathcal{T}_{t,T}$ and $y,z \in \mathbb{R}$, and using Lemma 2.2, we take

$$\begin{aligned}
&\left|E\ \exp\left(-\int_t^\tau r(v)dv\right)\psi(X_\tau(t,y)) - E\ \exp\left(-\int_t^\tau r(v)dv\right)\psi(X_\tau(t,z))\right| \\
&\quad \le \ E|X_\tau(t,y) - X_\tau(t,z)| \\
&\quad \le \ g(0)\ |y-z|.
\end{aligned}$$

Benefiting ourselves by the well-known property that the difference between supremums is less or equal than the supremum of difference leads to the result (8). To show the second part of the theorem, i.e. (9), we shall use the expression (7) for the value function $u(t,y)$.

Take any $\tau \in \mathcal{T}_{0,1}$ we can write

$$\begin{aligned}
&\Bigg|Ee^{-\int_t^{t+\tau(T-t)} r(v)dv}\psi\Bigg(y + \int_t^{t+\tau(T-t)}\left(r(v) - \lambda EU_1 - \frac{\sigma^2(v)}{2}\right)dv + \sqrt{T-t}\int_0^\tau \sigma(t+v(T-t))dW_v \\
&+ \sum_{j=1}^{N_{t+\tau(T-t)}} \ln(1+U_j)\Bigg) - Ee^{-\int_s^{s+\tau(T-s)} r(v)dv}\psi\Bigg(y + \int_s^{s+\tau(T-s)}\left(r(v) - \lambda EU_1 - \frac{\sigma^2(v)}{2}\right)dv \\
&+\sqrt{T-s}\int_0^\tau \sigma(s+v(T-s))dW_v + \sum_{j=1}^{N_{s+\tau(T-s)}} \ln(1+U_j)\Bigg)\Bigg| \\
&\le E\Bigg[\left|e^{-\int_t^{t+\tau(T-t)} r(v)dv} - e^{-\int_s^{s+\tau(T-s)} r(v)dv}\right|\psi\Bigg(y+\int_t^{t+\tau(T-t)}\left(r(v) - \lambda EU_1 - \frac{\sigma^2(v)}{2}\right)dv \\
&\quad +\sqrt{T-t}\int_0^\tau \sigma(t+v(T-t))dW_v + \sum_{j=1}^{N_{t+\tau(T-t)}} \ln(1+U_j)\Bigg) + e^{-\int_s^{s+\tau(T-s)} r(v)dv}\times
\end{aligned}$$

5

$$
\begin{aligned}
&\Bigg|\psi\Bigg(y+\int_t^{t+\tau(T-t)}\left(r(v)-\lambda EU_1-\frac{\sigma^2(v)}{2}\right)dv+\sqrt{T-t}\int_0^\tau \sigma(t+v(T-t))dW_v \\
&+\sum_{j=1}^{N_{t+\tau(T-t)}}\ln(1+U_j)\Bigg) \\
&-\psi\Bigg(y+\int_s^{s+\tau(T-s)}\left(r(v)-\lambda EU_1-\frac{\sigma^2(v)}{2}\right)dv+\sqrt{T-s}\int_0^\tau \sigma(s+v(T-s))dW_v \\
&+\sum_{j=1}^{N_{s+\tau(T-s)}}\ln(1+U_j)\Bigg)\Bigg|\Bigg] \\
&\leq g(0)E\Bigg[\left|e^{-\int_t^{t+\tau(T-t)}r(v)dv}-e^{-\int_s^{s+\tau(T-s)}r(v)dv}\right|+\Bigg|\int_t^{t+\tau(T-t)}\left(r(v)-\lambda EU_1-\frac{\sigma^2(v)}{2}\right)dv \\
&-\int_s^{s+\tau(T-s)}\left(r(v)-\lambda EU_1-\frac{\sigma^2(v)}{2}\right)dv\Bigg| \\
&+\left|\int_0^\tau\left(\sqrt{T-t}\sigma(t+v(T-t))-\sqrt{T-s}\sigma(s+v(T-s))\right)dW_v\right| \\
&+\Bigg|\sum_{j=N_{s+\tau(T-s)}+1}^{N_{t+\tau(T-t)}}\ln(1+U_j)\Bigg|\Bigg]. \qquad (10)
\end{aligned}
$$

Let us denote $R(u)=\int_0^u r(v)dv, 0\leq u\leq T$, and using the mean value theorem, we can write

$$
\begin{aligned}
\left|e^{-\int_t^{t+\tau(T-t)}r(v)dv}-e^{-\int_s^{s+\tau(T-s)}r(v)dv}\right| &\leq \left|\int_t^{t+\tau(T-t)}r(v)dv-\int_s^{s+\tau(T-s)}r(v)dv\right| \\
&\leq |(R(t+\tau(T-t))-R(t))-(R(s+\tau(T-s))-R(s))| \\
&\leq 2\,\bar{r}|t-s|.
\end{aligned}
\qquad (11)
$$

Similarly, we use the same arguments and obtain

$$
\begin{aligned}
&\left|\int_t^{t+\tau(T-t)}\left(r(v)-\lambda E(U_1)-\frac{\sigma^2(v)}{2}\right)dv-\int_s^{s+\tau(T-s)}\left(r(v)-\lambda E(U_1)-\frac{\sigma^2(v)}{2}\right)dv\right| \\
&\qquad\leq 2\left(\bar{r}+\lambda E|U_1|+\frac{\bar{\sigma}^2}{2}\right)|t-s|.
\end{aligned}
\qquad (12)
$$

Moreover, we fix $\tau, 0\leq\tau\leq 1$, and $0\leq s\leq t<T$, using the requirement (1), on $\sigma(t)$ we

6

write

$$
\begin{aligned}
&E\left|\int_0^\tau \left(\sqrt{T-t}\;\sigma(t+v(T-t)) - \sqrt{T-s}\;\sigma(s+v(T-s))\right) dW_v\right|^2 \\
&\leq\; E\int_0^\tau \left(\sqrt{T-t}\;\sigma(t+v(T-t)) - \sqrt{T-s}\;\sigma(s+v(T-s))\right)^2 dv \\
&\leq\; \int_0^1 \left(\sqrt{T-t}\;\sigma(t+v(T-t)) - \sqrt{T-s}\;\sigma(s+v(T-s))\right)^2 dv \\
&\leq\; 2\int_0^1 (T-t)\left(\sigma(t+v(T-t)) - \sigma(s+v(T-s))\right)^2 dv \\
&\quad +2\int_0^1 \left(\sqrt{T-t} - \sqrt{T-s}\right)^2 \sigma^2(t+v(T-t))dv.
\end{aligned}
$$

From here we obtain

$$
\begin{aligned}
&E\left|\int_0^\tau \left(\sqrt{T-t}\;\sigma(t+v(T-t)) - \sqrt{T-s}\;\sigma(s+v(T-s))\right) dW_v\right|^2 \\
&\leq\; \frac{2K^2T^2+\bar{\sigma}^2}{T-t}(t-s)^2. \qquad (13)
\end{aligned}
$$

Since $(U_j)_{j\geq 1}$ be a sequence of independent, identically distributed, integrable random variables, therefore we can find

$$
\begin{aligned}
E\left|\sum_{j=N_{s+\tau(T-s)}+1}^{N_{t+\tau(T-t)}} \ln(1+U_j)\right| &= E\sum_{j=N_{s+\tau(T-s)}+1}^{N_{t+\tau(T-t)}} |\ln(1+U_j)| \\
&= E\sum_{j=1}^{N_{t+\tau(T-t)}-N_{s+\tau(T-s)}} |\ln(1+U_j)| \\
&= E\sum_{j=1}^{N_{(t-s)(1-\tau)}} |\ln(1+U_j)|\,.
\end{aligned}
$$

Since N_t is an increasing function of time and $\tau \leq 1$ so we can write

$$
\begin{aligned}
E\left|\sum_{j=N_{s+\tau(T-s)}+1}^{N_{t+\tau(T-t)}} \ln(1+U_j)\right| &\leq E\sum_{j=1}^{N_{(t-s)}} |\ln(1+U_j)| \\
&= E\left(N_{t-s}\right) E|\ln(1+U_1)| \\
&= \lambda\; E\left|\ln(1+U_1)\right|(t-s), \qquad (14)
\end{aligned}
$$

Substituting (11)-(14) in (10) and using the fact that the difference between supremums is less or equal than difference supremum of the difference, we complete the proof. □

In the next section, the main results of the paper are presented.

3 Variational Inequalities

In this section, the variational inequalities of the value function are developed in order to investigate the regularity results of the value function (3). Let $\tilde{S}_t = e^{-\int_0^t r(u)du}S_t$ is the

discounted stock price, then the discounted price function

$$\tilde{v}(t,x) = e^{-\int_0^t r(u)du} v(t, xe^{\int_0^t r(u)du}),\ 0 \le t \le T, x > 0 \tag{15}$$

of the option at time t is C^2 on $[0,T) \times \mathbb{R}^+$ (see, Laberton and Lapeyre (1997)) and between the jump times, satisfies

$$\begin{aligned} \tilde{v}(t,\tilde{S}_t) &= v(0,S_0) + \int_0^t \frac{\partial \tilde{v}}{\partial u}(u,\tilde{S}_u)du + \int_0^t \frac{\partial \tilde{v}}{\partial x}(u,\tilde{S}_u)\tilde{S}_u\left(-\lambda E(U_1)du + \sigma(u)dW_u\right) \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 \tilde{v}}{\partial x^2}(u,\tilde{S}_u)\sigma^2(u)\tilde{S}_u^2 du + \sum_{j=1}^{N_t}\left(\tilde{v}(\tau_j,\tilde{S}_{\tau_j}) - \tilde{v}(\tau_j,\tilde{S}_{\tau_j -})\right). \end{aligned} \tag{16}$$

The function $\tilde{v}(t,x)$ is Lipschitz of order 1 with respect to x and with $S_{\tau_{j-}} = S_{\tau_j}(1+U_j)$, $j = 1,2,\ldots$.
The process

$$M_t = \sum_{j=1}^{N_t}\left(\tilde{v}(\tau_j,\tilde{S}_{\tau_j}) - \tilde{v}(\tau_j,\tilde{S}_{\tau_j -})\right) - \lambda\int_0^t \int \left(\tilde{v}(u,\tilde{S}_u(1+z)) - \tilde{v}(u,\tilde{S}_u)\right)d\nu(z)du \tag{17}$$

is a square integrable martingale, where $\nu(z)$ is the law of the process U.

Combining (16) and (17) we obtain that

$$\begin{aligned} \tilde{v}(t,\tilde{S}_t) \quad - \quad & \int_0^t \Big[\frac{\partial \tilde{v}}{\partial u}(u,\tilde{S}_u) - \lambda EU_1\tilde{S}_u\frac{\partial \tilde{v}}{\partial x}(u,\tilde{S}_u) + \frac{1}{2}\sigma^2(u)\tilde{S}_u^2\frac{\partial^2 \tilde{v}}{\partial x^2}(u,\tilde{S}_u) \\ & -\lambda\int\left(\tilde{v}(u,\tilde{S}_u(1+z)) - \tilde{v}(u,\tilde{S}_u)\right)d\nu(z)\Big]du \end{aligned}$$

is a martingale, see Israel and Rincon (2008), and therefore

$$\begin{aligned} & \frac{\partial \tilde{v}}{\partial u}(u,\tilde{S}_u) - \lambda EU_1\tilde{S}_u\frac{\partial \tilde{v}}{\partial x}(u,\tilde{S}_u) \\ & + \frac{1}{2}\sigma^2(u)\tilde{S}_u^2\frac{\partial^2 \tilde{v}}{\partial x^2}(u,\tilde{S}_u) - \lambda\int\left(\tilde{v}(u,\tilde{S}_u(1+z)) - \tilde{v}(u,\tilde{S}_u)\right)d\nu(z) \le 0 \end{aligned} \tag{18}$$

a.e. in $[0,T) \times \mathbb{R}$.

From Pham (1997), we know that if the payoff function is convex and non-increasing then the price function of the put American contingent claim is a convex function of the stock.

Therefore, we can write

$$\frac{\partial^2 v(t,x)}{\partial x^2} \ge 0 \tag{19}$$

a.e. in $[0,T) \times \mathbb{R}$.

Theorem 3.1. *The mapping $\varsigma(t,x) = x\ v(t,x)$ is Lipschitz continuous in x and locally Lipschitz continuous in the argument of t, i.e.*

$$|\varsigma(t,x) - \varsigma(t,y)| \le 2\ g(0)|x-y|,\ 0 \le t \le T,\ 0 < x \le y < \infty, \tag{20}$$

$$|\varsigma(t,x) - \varsigma(s,x)| \le \frac{C\ x}{\sqrt{T-t}}\ |t-s|,\ 0 \le s \le t < T,\ x > 0, \tag{21}$$

where the constant C is the function of $\bar{r}$, $\bar{\sigma}$, $g(0)$, λ, $E(U_1)$, K and T.

Proof. Consider that $v(t,x) = u(t, \ln x)$, $x > 0$, $0 \le t \le T$, we can write

$$\begin{aligned} |\varsigma(t,x) - \varsigma(t,y)| &= |x\ u(t,\ln x) - y\ u(t,\ln y)| \\ &\le |x\ u(t,\ln x) - x\ u(t,\ln y)| + |x\ u(t,\ln y) - y\ u(t,\ln y)|. \end{aligned}$$

Using the bound (8) and the mean value theorem we arrive to (20).
The expression (21) derives using the same arguments as previously, and the bound (9). □

Proposition 3.2. *The second order weak partial derivative* $\frac{\partial^2 v(t,x)}{\partial x^2}$ *of the value function (3) satisfies with respect to x the local Holder estimate*

$$x^2 \left| \frac{\partial^2 v(t,x)}{\partial x^2} \right| \le \frac{D}{\sqrt{T-t}},\ x > 0,\ 0 \le t < T,$$

where D is a nonnegative constant depends on the parameters $\overline{r}$, $\overline{\sigma}$, $\underline{\sigma}$, $g(0)$, λ, $E|U_1|$, $E\left(\frac{|U_1|}{1+U_1}\right)$, K, T.

Proof. Using the expression (15), and from (18) and (19), we obtain the system of inequalities

$$\begin{cases} -r(t)v(t,x) + \frac{\partial v(t,x)}{\partial t} - \lambda x E U_1 e^{-\int_0^t r(u)du} \frac{\partial v(t,x)}{\partial x} + \frac{x^2}{2}\sigma^2(t) e^{-2\int_0^t r(u)du} \frac{\partial^2 v(t,x)}{\partial x^2} \\ -\lambda \int \left(v(t,x(1+z)) - v(t,x)\right) d\nu(z) \le 0 \qquad \text{a.e. in } [0,T) \times \mathbb{R}, \\ \frac{\partial^2 v(t,x)}{\partial x^2} \ge 0, \qquad x > 0. \end{cases} \tag{22}$$

Also since $v(t,x) = u(t,\ln x)$, $x > 0$, $0 \le t \le T$, we have

$$\begin{cases} \frac{\partial v(t,x)}{\partial t} = \frac{\partial u(t,\ln x)}{\partial t}, \quad \frac{\partial v(t,x)}{\partial x} = \frac{1}{x}\frac{\partial u(t,\ln x)}{\partial y}, \\ \frac{\partial^2 v(t,x)}{\partial x^2} = \frac{1}{x^2}\frac{\partial^2 u(t,\ln x)}{\partial y^2} - \frac{1}{x^2}\frac{\partial u(t,\ln x)}{\partial y}, \quad 0 \le t < T, x > 0. \end{cases} \tag{23}$$

Substituting the latter relations and using the results of the Theorem 2.3 in the system of inequalities (22), we have

$$\begin{aligned} &\left| \frac{\partial^2 v(t,x)}{\partial x^2} \right| \\ \le\ & \frac{2\ e^{2\int_0^t r(v)dv}}{x^2 \underline{\sigma}^2} \left[r(t)v(t,x) + \left| \frac{\partial v(t,x)}{\partial t} \right| + \lambda x E|U_1| \left| \frac{\partial v(t,x)}{\partial x} \right| + \lambda \left| \int \left(v(t,x(1+z)) - v(t,x)\right) d\nu(z) \right| \right] \\ \le\ & \frac{2\ e^{2\overline{r}T}}{x^2 \underline{\sigma}^2} \left[\overline{r} g(0) + \frac{A}{\sqrt{T-t}} + \lambda g(0) E|U_1| + \lambda g(0) E\left(\frac{|U_1|}{1+U_1}\right) \right] \\ \le\ & \frac{D}{x^2\sqrt{T-t}}. \end{aligned}$$

Thus, the required result is derived. □

Before, we proceed with the main result of this section, we need to state and prove the following result.

Lemma 3.3. *For the function $\gamma(t,y) = y\ \frac{\partial v(t,y)}{\partial y}, 0 \le t_1 \le t_2 < T, y > 0$, of the value function (3) we have the following bound*

$$
\begin{aligned}
|\gamma(t_2,y) - \gamma(t_1,y)| &\le \frac{1}{h}\bigg[\int_y^{y+h} |\gamma(t_2,y) - \gamma(t_2,z)|dz + \int_y^{y+h} |\gamma(t_1,y) - \gamma(t_1,z)|dz \\
&+ (y+h)|v(t_2,y+h) - v(t_1,y+h)| + y|v(t_2,y) - v(t_1 y)| \\
&+ \int_y^{y+h} |v(t_2,z) - v(t_1,z)|dz\bigg],
\end{aligned}
$$

where $h > 0$.

Proof. We can express the difference

$$\gamma(t_2,y) - \gamma(t_1,y) = \gamma(t_2,y) - \gamma(t_2,z) + \gamma(t_2,z) - \gamma(t_1,z) + \gamma(t_1,z) - \gamma(t_1,y),$$

for any positive real number z.

Integrating both sides with respect to z over the interval $[y, y+h]$, we obtain

$$
\begin{aligned}
\gamma(t_2,y) - \gamma(t_1,y) &= \frac{1}{h}\bigg[\int_y^{y+h} (\gamma(t_2,y) - \gamma(t_2,z))dz + \int_y^{y+h} (\gamma(t_2,z) - \gamma(t_1,z))dz \\
&+ \int_y^{y+h} (\gamma(t_1,z) - \gamma(t_1,y))dz\bigg]. \qquad (24)
\end{aligned}
$$

Simplifying the second integral, we have

$$
\begin{aligned}
\int_y^{y+h} (\gamma(t_2,z) - \gamma(t_1,z))dz &= \int_y^{y+h} z\left(\frac{\partial v(t_2,z)}{\partial z} - \frac{v(t_1,z)}{\partial z}\right)dz \\
&= (y+h)(v(t_2,y+h) - v(t_1,y+h)) - y(v(t_2,y) - v(t_1,y)) \\
&- \int_y^{y+h} (v(t_2,z) - v(t_1,z))dz.
\end{aligned}
$$

Combining the latter expression with (24), the proof is complete. ☐

In the next, a very interesting result for the value of a put American option is derived.

Theorem 3.4. *The mapping $\gamma(t,x) = x\frac{\partial v(t,x)}{\partial x}$ satisfies with respect to time argument local Hölder estimate with exponent $\frac{1}{2}$, i.e.,*

$$|\gamma(t,x) - \gamma(s,x)| \le \frac{G + x\ H}{\sqrt{T-t}}|t-s|^{\frac{1}{2}}, 0 \le s \le t < T, x > 0, \qquad (25)$$

where G and H are positive constants depend on the parameters $\bar{r}$, $\bar{\sigma}$, $\underline{\sigma}$, $g(0)$, λ, $E(U_1)$, $E\left(\frac{|U_1|}{1+U_1}\right)$, K and T.

Proof. From the continuity of $\frac{\partial v(t,x)}{\partial x}$ and the relations (23), using Proposition 3.2 we can write

$$|\gamma(t,x) - \gamma(t,y)| = \left|x\ \frac{\partial v(t,x)}{\partial x} - y\ \frac{\partial v(t,y)}{\partial y}\right| \le \frac{D}{\sqrt{T-t}}\ |x-y|,\ \ 0 \le t < T,\ 0 < x \le y < \infty. \qquad (26)$$

Application of the bounds (21) and (26) in Lemma 3.3 gives

$$\begin{aligned} |\gamma(t_2, y) - \gamma(t_1, y)| &\leq \frac{1}{h}\Big[\int_y^{y+h} \frac{D}{\sqrt{T-t_2}}(z-y)dz + \int_y^{y+h} \frac{D}{\sqrt{T-t_1}}(z-y)dz \\ &+ \frac{C(y+h)}{\sqrt{T-t_2}}|t_2-t_1| + \frac{C\ y}{\sqrt{T-t_2}}|t_2-t_1| + \int_y^{y+h} \frac{C}{\sqrt{T-t_2}}|t_2-t_1|dz\Big] \\ &= \frac{1}{h}\Big[\frac{2\ D}{\sqrt{T-t_2}}h^2 + \frac{2\ C\ y}{\sqrt{T-t_2}}\ |t_2-t_1| + \frac{C\ h}{\sqrt{T-t_2}}\ |t_2-t_1| + \frac{C\ h}{\sqrt{T-t_2}}|t_2-t_1|\Big]. \end{aligned}$$

Let us choose $h = C^*|t_2 - t_1|^{\frac{1}{2}}$ from the latter estimate we get

$$\begin{aligned} |\gamma(t_2, y) - \gamma(t_1, y)| &\leq \frac{1}{\sqrt{T-t_2}}\left[\left(2\ D\ C^* + \frac{2C\ y}{C^*}\right)|t_2-t_1|^{\frac{1}{2}} + 2\ C|t_2-t_1|\right] \\ &\leq \frac{2}{\sqrt{T-t_2}}\left(2\ D\ C^* + \frac{2\ C\ y}{C^*} + 2\ T\ C\right)|t_2-t_1|^{\frac{1}{2}}, \end{aligned}$$

the minimum of which is attained at the point $C^* = \sqrt{\frac{C\ y}{D}}$.

From here, the required result is derived. □

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On a summation boundary value problem for a second-order difference equations with resonance

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Abstract

In this paper, we obtain a sufficient condition for the existence of the solution for a second-order difference equation with summation boundary value problem at resonance, by using some properties of the Green's function, the Schaefer's fixed point theorem and intermediate value theorem. Finally, we present an example to show the importance of these result.

Keywords: boundary value problem; resonance; fixed point theorem; existence.

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1 Introduction

The study of the existence of solutions of boundary value problems for linear second-order ordinary differential and difference equations was initiated by Ilin and Moiseev [1]. Then Gupta [2] studied three-point boundary value problems for nonlinear second-order ordinary differential equations. Since then, nonlinear second-order three-point boundary value problems have also been studied by many authors, one may see [3-6] and references therein. Also, there are a lot of papers dealing with the resonant case for multi-point boundary value problems, see [7-11].

In [8], J.Liu, S.Wang and J.Zhang studied the existence of multiple solutions for boundary value problems of second-order difference equations with resonance:

$$\Delta^2 u(t-1) = g(t,u), \qquad t \in \{1,2,...,T\}, \tag{1.1}$$

$$u(0) = 0, \qquad u(T+1) = 0. \tag{1.2}$$

Using Morse theory, critical point theory, minimax methods and bifurcation theory.

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In this paper, we study the existence of solutions of a second-order difference equation with summation boundary value problem at resonance

$$\Delta^2 u(t-1) + f(t, u(t)) = 0, \qquad t \in \{1, 2, ..., T\}, \tag{1.3}$$

$$u(0) = 0, \qquad u(T+1) = \alpha \sum_{s=1}^{\eta} u(s), \tag{1.4}$$

where $\dfrac{2(T+1)}{\alpha\eta(\eta+1)} = 1$, $T \geq 3$, $\eta \in \{1, 2, ..., T-1\}$ and f is continuous function.

In this paper, we are interested in the existence of the solution for problem (1.3)-(1.4) under the condition $\frac{2(T+1)}{\alpha\eta(\eta+1)} = 1$, which is a resonant case. Using some properties of the Green's function $G(t, s)$, intermediate value theorems and Schaefer's fixed point theorem, we establish a sufficient condition for the existence of positive solutions of problem $\frac{2(T+1)}{\alpha\eta(\eta+1)} = 1$.

Let $\mathbb{N}$ be a nonnegative integer, $\mathbb{N}_{i,j} = \{k \in \mathbb{N} | \ \ i \leq k \leq j\}$ and $\mathbb{N}_p = \mathbb{N}_{0,p}$.

Throughout this paper, we suppose the following conditions hold:

(H) $f(t, u) \in C(\mathbb{N}_{T+1} \times R, R)$ and there exist two positive continuous functions $p(t), q(t) \in C(\mathbb{N}_{T+1}, R^+)$ such that

$$|f(t, tu)| \leq p(t) + q(t)|u|^m, t \in \mathbb{N}_{T+1}, \tag{1.5}$$

where $0 \leq m \leq 1$. Furthermore, $\lim\limits_{u \to \pm\infty} f(t, tu) = \infty$, for any $t \in \mathbb{N}_{1,T}$.

To accomplish this, we denote $C(\mathbb{N}_{T+1}, R)$,the Banach space of all function u with the norm defined by $\|u\| = \max\{u(t) \mid t \in \mathbb{N}_{T+1}\}$.

The proof of the main result is based upon an application of the following theorem.

Theorem 1.1. ([12]). *Let X be a Banach space and $T : X \to X$ be a continuous and compact mapping. If the set*

$$\{x \in X \ : \ x = \lambda T(x), \ for\ some\ \lambda \in (0, 1)\}$$

is bounded, then T has a fixed point.

The plan of the paper is follows. In Section 2, we recall some lemmas. In Section 3, we prove our main result. Illustrate example is presented in Section 4.

2 Preliminaries

We now state and prove several lemmas before stating our main results.

Lemma 2.1. *The problem* (1.3)-(1.4) *is equivalent to the following*

$$u(t) = \sum_{s=1}^{T} G(t,s)f(s,u(s)) + \frac{u(T+1)}{T+1}t, \tag{2.1}$$

where

$$G(t,s) = \frac{1}{(T+1)(\alpha-1)} \begin{cases} \alpha t(T+1-s) - \frac{1}{2}\alpha t(\eta-s)(\eta-s+1) & \\ \qquad - (T+1)(\alpha-1)(t-s), & s \in \mathbb{N}_{1,t-1} \cap \mathbb{N}_{1,\eta-1} \\ \alpha t(T+1-s) - \frac{1}{2}\alpha t(\eta-s)(\eta-s+1), & s \in \mathbb{N}_{t,\eta-1} \\ \alpha t(T+1-s) - (T+1)(\alpha-1)(t-s), & s \in \mathbb{N}_{\eta,t-1} \\ \alpha t(T+1-s), & s \in \mathbb{N}_{t,T} \cap \mathbb{N}_{\eta,T} \end{cases} \tag{2.2}$$

Proof. Assume that $u(t)$ is a solution of problem (1.3)-(1.4), then it satisfies the following equation:

$$u(t) = C_1 + C_2 t - \sum_{s=1}^{t-1}(t-s)f(s,u(s)),$$

where C_1, C_2 are constants. By the boundary value condition (1.3), we obtain $C_1 = 0$.

So,

$$u(t) = C_2 t - \sum_{s=1}^{t-1}(t-s)f(s,u(s)). \tag{2.3}$$

From (2.3),

$$\begin{aligned} \sum_{s=1}^{\eta} u(s) &= \frac{\eta(\eta+1)}{2}C_2 - \sum_{s=1}^{\eta-1}\sum_{l=1}^{\eta-s} ly(s) \\ &= \frac{\eta(\eta+1)}{2}C_2 - \frac{1}{2}\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)y(s). \end{aligned}$$

From the second boundary condition, we have

$$(2T+2-\alpha\eta(\eta+1))C_2 = 2\sum_{s=1}^{T}(T+1-s)f(s,u(s)) + \alpha\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)f(s,u(s)). \tag{2.4}$$

Since $\frac{2(T+1)}{\alpha\eta(\eta+1)} = 1$, then (2.4) is solvable if and only if

$$\sum_{s=1}^{T}(T+1-s)f(s,u(s)) = \frac{\alpha}{2}\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)f(s,u(s)).$$

Note that

$$u(T+1)-\sum_{s=1}^{\eta}u(s)= (T+1)C_2-\sum_{s=1}^{T}(T+1-s)f(s,u(s)) \\ -\frac{\eta(\eta+1)}{2}C_2+\frac{1}{2}\sum_{s=1}^{\eta-1}(\eta-s)(\eta+1-s)f(s,u(s)),$$

and then

$$\begin{aligned} C_2 =&\frac{2}{2T+2-\eta(\eta+1)}\Bigg[u(T+1)-\sum_{s=1}^{\eta}u(s)+\sum_{s=1}^{T}(T+1-s)f(s,u(s)) \\ &-\frac{1}{2}\sum_{s=1}^{\eta-1}(\eta-s)(\eta+1-s)f(s,u(s))\Bigg] \\ =&\frac{\alpha}{(T+1)(\alpha-1)}\Bigg[u(T+1)-\sum_{s=1}^{\eta}u(s)+\sum_{s=1}^{T}(T+1-s)f(s,u(s)) \\ &-\frac{1}{2}\sum_{s=1}^{\eta-1}(\eta-s)(\eta+1-s)f(s,u(s))\Bigg]. \end{aligned}$$

We now use that $u(T+1)=\dfrac{2(T+1)}{\eta(\eta+1)}\displaystyle\sum_{s=1}^{\eta}u(s)$ to get

$$\frac{\alpha}{(T+1)(\alpha-1)}\left[u(T+1)-\sum_{s=1}^{\eta}u(s)\right]=\frac{u(T+1)}{T+1},$$

and

$$\begin{aligned} C_2 =&\frac{\alpha}{(T+1)(\alpha-1)}\Bigg[\sum_{s=1}^{T}(T+1-s)f(s,u(s))-\frac{1}{2}\sum_{s=1}^{\eta-1}(\eta-s)(\eta+1-s)f(s,u(s))\Bigg] \\ &+\frac{u(T+1)}{T+1}. \end{aligned}$$

Hence the solution of (1.3)-(1.4) is given, implicity as

$$\begin{aligned} u(t) =&\frac{\alpha t}{(T+1)(\alpha-1)}\Bigg[\sum_{s=1}^{T}(T+1-s)f(s,u(s))-\frac{1}{2}\sum_{s=1}^{\eta-1}(\eta-s)(\eta+1-s)f(s,u(s))\Bigg] \\ &-\sum_{s=1}^{t-1}(t-s)f(s,u(s))+\frac{u(T+1)}{T+1}t. \end{aligned} \tag{2.5}$$

According to (2.5) it is easy to show that (2.1) holds. Therefore, problem (1.3)-(1.4) is equivalent to the equation (2.1) with the function $G(t,s)$ defined in (2.2). The proof is completed. □

Lemma 2.2. *For any* $(t, s) \in \mathbb{N}_{T+1} \times \mathbb{N}_{T+1}, G(t, s)$ *is continuous, and* $G(t, s) > 0$ *for any* $(t, s) \in \mathbb{N}_{1,T} \times \mathbb{N}_{1,T}$.

Proof. The continuity of $G(t, s)$ for any $(t, s) \in \mathbb{N}_{T+1} \times \mathbb{N}_{T+1}$, is obvious. Let

$$g_1(t, s) = \alpha t(T + 1 - s) - \frac{1}{2}\alpha t(\eta - s)(\eta - s + 1) - (T + 1)(\alpha - 1)(t - s),$$

where $s \in \mathbb{N}_{1,t-1} \cap \mathbb{N}_{1,\eta-1}$.

Here we only need to prove that $g_1(t, s) > 0$ for $s \in \mathbb{N}_{1,t-1} \cap \mathbb{N}_{1,\eta-1}$, the rest of the proof is similar. So, from the definition of $g_1(t, s), \eta \in \mathbb{N}_{1,T-1}$ and the resonant condition $\frac{2(T+1)}{\alpha\eta(\eta+1)} = 1$, we have

$$\begin{aligned}
g_1(t, s) &= \alpha t(T + 1 - s) - \frac{1}{2}\alpha t(\eta - s)(\eta - s + 1) - (T + 1)(\alpha - 1)(t - s) \\
&= (T + 1)(t - s) + \alpha s(T + 1 - t) - \frac{1}{2}\alpha t(\eta - s)(\eta - s + 1) \\
&> (T + 1)(t - s) - \frac{\alpha}{2}\left[t\eta(\eta + 1) - 2s(T + 1 - t)\right] \\
&> (T + 1)(t - s) - \frac{\alpha}{2} \\
&> (T + 1)(t - s) - \frac{T + 1}{\eta(\eta + 1)} \\
&> (T + 1)(t - s - 1) \\
&\geq 0,
\end{aligned}$$

for $s \in \mathbb{N}_{1,t-1} \cap \mathbb{N}_{1,\eta-1}$. Since $t > s$ and $\eta(\eta + 1) \geq 2(T + 1 - t)$ where $T \geq 3$. The proof is completed. □

Let

$$G^*(t, s) = \frac{1}{t}G(t, s). \tag{2.6}$$

Then

$$G^*(t, s) = \frac{1}{(T + 1)(\alpha - 1)} \begin{cases} \alpha(T + 1 - s) - \frac{1}{2}\alpha(\eta - s)(\eta - s + 1) & \\ \qquad\qquad - \frac{1}{t}(T + 1)(\alpha - 1)(t - s), & s \in \mathbb{N}_{1,t-1} \cap \mathbb{N}_{1,\eta-1} \\ \alpha(T + 1 - s) - \frac{1}{2}\alpha(\eta - s)(\eta - s + 1), & s \in \mathbb{N}_{t,\eta-1} \\ \alpha(T + 1 - s) - \frac{1}{t}(T + 1)(\alpha - 1)(t - s), & s \in \mathbb{N}_{\eta,t-1} \\ \alpha(T + 1 - s), & s \in \mathbb{N}_{t,T} \cap \mathbb{N}_{\eta,T}. \end{cases} \tag{2.7}$$

Thus, problem (1.3)-(1.4) is equivalent to the following equation:

$$u(t) = \sum_{s=1}^{T} tG^*(t, s)f(s, u(s)) + \frac{u(T + 1)}{T + 1}t. \tag{2.8}$$

By a simple computation, the new Green's function $G^*(t,s)$ has the following properties.

Lemma 2.3. *For any* $(t,s) \in \mathbb{N}_{T+1} \times \mathbb{N}_{T+1}, G^*(t,s)$ *is continuous, and* $G^*(t,s) > 0$ *for any* $(t,s) \in \mathbb{N}_{1,T} \times \mathbb{N}_{1,T}$. *Furthermore,*

$$\begin{aligned}\lim_{t\to 0} G^*(t,s) &:= G^*(0,s)\\ &= \frac{1}{(T+1)(\alpha-1)}\begin{cases}\alpha(T+1-s) - \frac{1}{2}\alpha(\eta-s)(\eta-s+1), & s \in \mathbb{N}_{1,\eta-1}\\ \alpha(T+1-s), & s \in \mathbb{N}_{\eta,T}.\end{cases}\end{aligned} \tag{2.9}$$

Lemma 2.4. *For any* $s \in \mathbb{N}_{1,T}, G^*(t,s)$ *is nonincreasing with respect to* $t \in \mathbb{N}_{T+1}$, *and for any* $s \in \mathbb{N}_{T+1}$, $\frac{\triangle_t G^*(t,s)}{\triangle t} < 0$, *and* $\frac{\triangle_t G^*(t,s)}{\triangle t} = 0$ *for* $t \in \mathbb{N}_s$. *That is,* $G^*(T+1,s) \leq G^*(t,s) \leq G^*(s,s)$ *where*

$$\begin{aligned}G^*(t,s) &\leq G^*(s,s)\\ &= \frac{1}{(T+1)(\alpha-1)}\begin{cases}\alpha(T+1-s) - \frac{1}{2}\alpha(\eta-s)(\eta-s+1), & s \in \mathbb{N}_{1,\eta-1}\\ \alpha(T+1-s), & s \in \mathbb{N}_{\eta,T}\end{cases}\end{aligned} \tag{2.10}$$

$$\begin{aligned}G^*(t,s) &\geq G^*(T+1,s)\\ &= \frac{1}{(T+1)(\alpha-1)}\begin{cases}(T+1)(T+1-s) - \frac{1}{2}\alpha(\eta-s)(\eta-s+1), & s \in \mathbb{N}_{1,\eta-1}\\ (T+1)(T+1-s), & s \in \mathbb{N}_{\eta,T}.\end{cases}\end{aligned} \tag{2.11}$$

Let

$$u(t) = tw(t). \tag{2.12}$$

Then $u(T+1) = (T+1)w(T+1)$, and equation (2.8) gives

$$w(t) = \sum_{s=1}^{T} G^*(t,s) f(s, sw(s)) + w(T+1). \tag{2.13}$$

Now we have

$$y(t) = w(t) - w(T+1). \tag{2.14}$$

Then $y(T+1) = w(T+1) - w(T+1) = 0$, and equation (2.13) gives

$$y(t) = \frac{1}{T+1}\sum_{s=1}^{T} G^*(t,s) f(s, s(y(s) + w(T+1))). \tag{2.15}$$

We replace $w(T+1)$ by any real number λ, then (2.15) can be rewritten as

$$y(t) = \frac{1}{T+1}\sum_{s=1}^{T} G^*(t,s) f(s, s(y(s) + \lambda)). \tag{2.16}$$

The following result is based on the Schaefer's fixed point theorem. We define an operator T on the set $\Omega = C(\mathbb{N}_{T+1})$ as follows:

$$Ty(t) = \frac{1}{T+1}\sum_{s=1}^{T} G^*(t,s) f(s, s(y(s)+\lambda)). \tag{2.17}$$

Lemma 2.5. *Assume that* $f \in C(\mathbb{N}_{T+1} \times R, R)$, $\sum_{s=1}^{T} G^*(t,s)q(s) < T+1$ *and (1.5) holds. Then the equation* (2.16) *has at least one solution for any real number* λ.

Proof. We divide the proof into four steps.

Step I. *T maps bounded sets into bounded sets in* Ω. Let us prove that for any $R > 0$, there exists a positive constant L such that for each $y \in B_R = \{y \in C(\mathbb{N}_{T+1} \times R) : \|y\| \leq R\}$, we have $\|(Ty)(t)\| \leq L$. Indeed, for any $y \in B_R$, we obtain

$$\begin{aligned}
|\,(Ty)(t)| &= \left|\frac{1}{T+1}\sum_{s=1}^{T} G^*(t,s) f(s, s(y(s)+\lambda))\right| \\
&\leq \frac{1}{T+1}\sum_{s=1}^{T} G^*(t,s)p(s) + \frac{1}{T+1}\sum_{s=1}^{T} G^*(t,s)|q(s)+\lambda|^m \\
&\leq \frac{1}{T+1}\sum_{s=1}^{T} G^*(t,s)p(s) + \frac{1}{T+1}\sum_{s=1}^{T} G^*(t,s)q(s)(\|y(s)\| + \|\lambda\|)^m \\
&\leq \frac{1}{T+1}\sum_{s=1}^{T} G^*(s,s)p(s) + \frac{(R+\|\lambda\|)^m}{T+1}\sum_{s=1}^{T} G^*(s,s)q(s) \\
&:= L.
\end{aligned} \tag{2.18}$$

Step II. *Continuity of* T. Let $\epsilon > 0$, there exists $\delta > 0$ such that for all $t \in \mathbb{N}_{T+1}$ and for all $x, y \in B_R$ with $|(t, t(x(t)+\lambda) - (t, t(y(t)+\lambda)| < \delta$, we have

$$\left|f(t, t(x(t)+\lambda) - f(t, t(y(t)+\lambda)\right| < \epsilon.$$

Then, we obtain

$$\begin{aligned}
|\,(Tx)(t) - (Ty)(t)| &\leq \left|\frac{1}{T+1}\sum_{s=1}^{T} G^*(t,s)[f(s, s(x(s)+\lambda)) - f(s, s(y(s)+\lambda))]\right| \\
&\leq \frac{\epsilon}{T+1}\left|\sum_{s=1}^{T} G^*(t,s)\right| = \epsilon.
\end{aligned}$$

This means that T is continuous in Ω.

Step III. *$T(B_R)$ is equicontinuous with B_R defined as in Step II.* Since B_R is bounded, then there exists $M > 0$ such that $|f| \leq M$.

For any $\varepsilon > 0$, there exists $\delta > 0$ such that for $t_1, t_2 \in \mathbb{N}_{T+1}$

$$|G^*(t_2, s) - G^*(t_1, s)| \leq \frac{\epsilon}{M}.$$

Then we have

$$\begin{aligned} |\,(Ty)(t_2) - (Ty)(t_1)| &\leq \left| \frac{1}{T+1} \sum_{s=1}^{T} |G^*(t_2, s) - G^*(t_1, s)||f(s, s(y(s) + \lambda))\right| \\ &\leq \frac{M}{T+1} \sum_{s=1}^{T} |G^*(t_2, s) - G^*(t_1, s)| \\ &= M \cdot \frac{\epsilon}{M} \leq \epsilon. \end{aligned}$$

This means that the set $T(B_R)$ is an equicontinuous set. As a consequence of Steps I to III together with the Arzela'-Ascoli theorem, we get that T is completely continuous in Ω.

Step IV. A priori bounds. We show that the set

$$E = \{y \in C(\mathbb{N}_{T+1}, \mathbb{R}) \ / \ y = \mu T y \ \text{ for some } \mu \in (0, 1)\} \ \text{ is bounded.}$$

By Lemma 2.1, assume that there exist $y \in \partial B_R$ with $\|y(t)\| = R$ and $\mu \in (0, 1)$ such that $y = \mu T y$. It follows that

$$\begin{aligned} |\,y(t)| &= \frac{\mu}{T+1} \left| \sum_{s=1}^{T} G^*(t, s) f(s, s(y(s) + \lambda)) \right| \\ &\leq \frac{\mu}{T+1} \sum_{s=1}^{T} G^*(s, s)\,|f(s, s(y(s) + \lambda))| \\ &< \frac{1}{T+1} \left[\sum_{s=1}^{T} G^*(s, s) p(s) + \sum_{s=1}^{T} G^*(s, s) q(s) \left(\|y(s)\| + \|\lambda\|\right)^m \right] \\ &\leq \frac{1}{T+1} \sum_{s=1}^{T} G^*(s, s) p(s) + \frac{(R + \|\lambda\|)^m}{T+1} \sum_{s=1}^{T} G^*(s, s) q(s) \\ &:= L. \end{aligned} \tag{2.19}$$

This shows that the set E is bounded. By the Schaefer's fixed point theorem, we conclude that T has a fixed point which is a solution of problem (1.1). □

3 Main Results

In this section, we prove our result by using Lemmas 2.5-2.7 and the intermediate value theorem.

Theorem 3.1. *Assume that* $(H1)$ *holds. If* $\sum_{s=1}^{T} G^*(s,s)q(s) < 1$*, then the problem* (1.3)-(1.4) *has at least one solution, where*

$$G^*(s,s) \;=\; \frac{1}{(T+1)(\alpha-1)}\begin{cases}\alpha(T+1-s)-\frac{1}{2}\alpha(\eta-s)(\eta-s+1), & s\in\mathbb{N}_{1,\eta-1}\\ \alpha(T+1-s), & s\in\mathbb{N}_{\eta,T}\end{cases}$$

Proof. Since (2.19) is continuously dependent on the parameter λ. So, we should only investigate λ such that $y(T+1)=0$ in order that $u(T+1)=\lambda$.

Equation (2.16) is rewrite as

$$y_\lambda(t) = \frac{1}{T+1}\sum_{s=1}^{T} G^*(t,s)f(s,s(y_\lambda(s)+\lambda)),\quad t\in\mathbb{N}_{T+1}. \tag{3.1}$$

where λ is any given real number.

Equation(3.1) show that there exists λ such that

$$L(\lambda) := y_\lambda(T+1) = \frac{1}{T+1}\sum_{s=1}^{T} G^*(T+1,s)f(s,s(y_\lambda(s)+\lambda)) \tag{3.2}$$

and we can observe that, $y_\lambda(T+1)$ is continuously dependent on the parameter λ.

To prove that there exists λ^* such that $y_{\lambda^*}(T+1)=0$, we must to show that $\lim\limits_{\lambda\to\infty} L(\lambda)=\infty$ and $\lim\limits_{\lambda\to-\infty} L(\lambda)=-\infty$.

Firstly, we prove that $\lim\limits_{\lambda\to\infty} L(\lambda)=\infty$ by supposing that $\lim\limits_{\lambda\to\infty} L(\lambda)<\infty$ as acontradiction. Therefore there exists a sequence $\{\lambda_n\}$ with $\lim\limits_{n\to\infty} L(\lambda)=\infty$ such that $\lim\limits_{\lambda_n\to\infty} L(\lambda_n)<\infty$. This implies that the sequence $\{L(\lambda_n)\}$ is bounded. Since the function $f(t,ty)$ is continuous with respect to $t\in\mathbb{N}_{T+1}$ and $y\in R$, we have

$$f(t,t(y_{\lambda_n}(t)+\lambda_n))\geq 0\ ,\quad t\in\mathbb{N}_{T+1} \tag{3.3}$$

where λ_n is large enough, Assuminh that (3.3) is true and using (3.1), we have

$$y_\lambda \geq 0\ ,\quad t\in\mathbb{N}_{T+1}. \tag{3.4}$$

Therefore,

$$\lim_{\lambda_n\to\infty} f(t,t(y_{\lambda_n}(t)+\lambda_n))=\infty\ ,\quad t\in\mathbb{N}_{T+1}. \tag{3.5}$$

From (H), we get

$$\lim_{\lambda\to\infty} f(t,tu)=\infty\ ,\quad t\in\mathbb{N}_{T+1}. \tag{3.6}$$

From (3.2),(3.5) and (3.6), we have

$$\lim_{\lambda_n\to\infty} y_{\lambda_n}(T+1) \;=\; \lim_{\lambda_n\to\infty}\sum_{s=1}^{T} G^*(T+1,s)f(s,s(y_{\lambda_n}(s)+\lambda_n)) \tag{3.7}$$

$$\begin{aligned}
&\geq \lim_{\lambda_n\to\infty} \sum_{s=\frac{1}{4}(T-1)}^{\frac{3}{4}(T-1)} G^*(T+1,s)f(s,s(y_{\lambda_n}(s)+\lambda_n)) \\
&= \infty, \qquad (3.8)
\end{aligned}$$

we find that this result contradicts our assumption.

We define

$$S_n = \{t \in \mathbb{N}_{T+1} \mid f(t,t(y_{\lambda_n}(t)+\lambda_n)) < 0\}.$$

where λ_n is large. Note that S_n is not empty.

Secondly, we divide the set S_n into set $\widetilde{S}_n$ and set $\widehat{S}_n$ as follows:

$$\widetilde{S}_n = \{t \in S_n \mid y_{\lambda_n} + \lambda_n > 0\} \quad \text{and} \quad \widehat{S}_n = \{t \in S_n \mid y_{\lambda_n} + \lambda_n \leq 0\}$$

where $\widetilde{S}_n \cap \widehat{S}_n = \emptyset$, $\widetilde{S}_n \cup \widehat{S}_n = S_n$. So, we have from (H) that $\widehat{S}_n$ is not empty.

In addition, we find from (H) that the function $f(t,tu)$ is bounded below by a constant for $t \in \mathbb{N}_{T+1}$ and $\lambda \in [0,\infty)$. Thus, there exists a constant $M(<0)$ which is independent of t and λ_n, such that

$$f(t,t(y_{\lambda_n}(t)+\lambda_n)) \geq M\ ,\quad t \in \widetilde{S}_n, \qquad (3.9)$$

Let $h(\lambda_n) = \min_{t\in S_n} y_{\lambda_n}(t)$ and using the definitions of $\widetilde{S}_n$ and set $\widehat{S}_n$, we have

$$h(\lambda_n) = \min_{t\in\widehat{S}_n} y_{\lambda_n}(t) = -\|y_{\lambda_n}(t)\|_{\widehat{S}_n}.$$

It follows that $h(\lambda_n) \to -\infty$ as $\lambda_n \to \infty$ since if $h(\lambda_n)$ is bounded below by a constant as $\lambda_n \to \infty$, then (3.7) holds. Therefore, we can choose largeλ_{n_1} such that

$$h(\lambda_n) < \frac{1}{T+1}\max\left\{-1, \frac{M\sum_{s=1}^{T} G^*(s,s) - \sum_{s=1}^{T} G^*(s,s)p(s)}{1-\sum_{s=1}^{T} G^*(s,s)q(s)}\right\} \qquad (3.10)$$

for $n > n_1$. Employing (H), (3.1), (3.8), (3.9), the definitions of $\widetilde{S}_n$, and set $\widehat{S}_n$, for any $\lambda_n > \lambda_{n_1}$, we have

$$\begin{aligned}
y_{\lambda_n}(t) &\geq \frac{1}{T+1}\sum_{s\in S_n} G^*(s,s)f(s,s(y_{\lambda_n}(s)+\lambda_n)) \\
&\geq \frac{1}{T+1}\sum_{s\in\widetilde{S}_n} G^*(s,s)f(s,s(y_{\lambda_n}(s)+\lambda_n)) \\
&\quad + \frac{1}{T+1}\sum_{s\in\widehat{S}_n} G^*(s,s)\left(-p(s) - q(s)|y_{\lambda_n}(s)+\lambda_n|^m\right)
\end{aligned}$$

$$\geq \frac{1}{T+1}\left[M\sum_{s\in\widetilde{S}_n} G^*(s,s) - \sum_{s\in\widehat{S}_n} G^*(s,s)p(s) - \sum_{s\in\widehat{S}_n} G^*(s,s)q(s)\|y_{\lambda_n}(s)+\lambda_n\|^m\right].$$

It follows that

$$\begin{aligned} y_{\lambda_n}(t) &\geq \frac{1}{T+1}\left[M\sum_{s=1}^{T} G^*(s,s) - \sum_{s=1}^{T} G^*(s,s)p(s) - \sum_{s=1}^{T} G^*(s,s)q(s)\|y_{\lambda_n}(s)+\lambda_n\|^m_{S_n}\right], \\ &\geq \frac{1}{T+1}\left[M\sum_{s=1}^{T} G^*(s,s) - \sum_{s=1}^{T} G^*(s,s)p(s) - \sum_{s=1}^{T} G^*(s,s)q(s)h(\lambda_n)\right], \quad t\in S_n, \end{aligned}$$

which implies that

$$h(\lambda_n) \geq \frac{1}{T+1}\left[\frac{M\sum_{s=1}^{T} G^*(s,s) - \sum_{s=1}^{T} G^*(s,s)p(s)}{1-\sum_{s=1}^{T} G^*(s,s)q(s)}\right].$$

This result contradicts (3.9). Thus, the proof that $\lim_{\lambda\to\infty} L(\lambda)=\infty$ is done. using a similar method, we can prove that $\lim_{\lambda\to-\infty} L(\lambda)=-\infty$.

Notice that $L(\lambda)$ is continuous with respect to $\lambda\in(-\infty,\infty)$. From the intermediate value theorem, there exists $\lambda^*\in(-\infty,\infty)$ such that $L(\lambda^*)=0$, that is, $y(T+1)=y_{\lambda^*}(T+1)=0$, which satisfies the second boundary value condition of (1.2). The proof is completed. □

4 Example

In this section, we give an example to illustrate our result.

Example Consider the BVP

$$\Delta^2 u(t-1) + t^2 + \frac{1}{2}u(t) = 0, \qquad t\in N_{1,4}, \tag{4.1}$$

$$u(0)=0, \qquad u(5)=\frac{5}{6}\sum_{s=1}^{2} u(s). \tag{4.2}$$

Set $\alpha=\frac{5}{6},\ \eta=2,\ T=4,\ f(t,u)=t^2+\frac{1}{2}u(t)$. So we have

$$\frac{\alpha\eta(\eta+1)}{2(T+1)}=1 \text{ and } f(t,tu)=t^2+\tfrac{t}{2}u(t).$$

Now we take $q(t)=\frac{t}{5}$. It is easy to check that

$$\lim_{u\to\pm\infty} f(t,tu)=\pm\infty \text{ and } \sum_{s=1}^{4} G^*(s,s)q(s) \leq \frac{1}{25}\sum_{s=1}^{4}(5-s)s = \frac{4}{5} < 1.$$

Thus the conditions of Theorem 3.1 are satisfied. Therefore problem (4.1)-(4.2) has at least a nontrivial solution. □

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Fuzzy quadratic mean operators and their use in group decision making

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Abstract

Quadratic mean in statistics is a statistical measure defined as the square root of the mean of the squares of a sample. In this paper, we investigate the situations in which the input data are expressed in fuzzy values and develop some fuzzy quadratic mean operators, such as fuzzy weighted quadratic mean operator, fuzzy ordered weighted quadratic mean operator, and fuzzy hybrid quadratic mean operator. Especially, all these operators can reduce to aggregate interval or real numbers. Then based on the developed operators, we present an approach to group decision making and illustrate it with a practical example.

1 Introduction

Information aggregation is an essential process of gathering relevant information from multiple sources by using a proper aggregation technique. Many techniques, such as the weighted average operator [5], the weighted geometric mean operator [1], harmonic mean operator [2], weighted harmonic mean (WHM) operator [2], ordered weighted average (OWA) operator [17], ordered weighted geometric operator [3, 13], weighted OWA operator [8], induced OWA operator [21], induced ordered weighted geometric operator [15], uncertain OWA operator [14], hybrid aggregation operator [10] and so on, have been developed to aggregate data information. However, yet most of existing aggregation operators do not take into account the information about the relationship between the values being fused. Yager [18] introduced a tool to provide more versatility in the

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information aggregation process, i.e., developed a power-average (PA) operator and a power OWA (POWA) operator. In some situations, however, these two operators are unsuitable to deal with the arguments taking the forms of multiplicative variables because of lack of knowledge, or data, and decision makers' limited expertise related to the problem domain. Based on this tool, Xu and Yager [16] developed additional new geometric aggregation operators, including the power-geometric (PG) operator, weighted PG operator and power-ordered weighted geometric (POWG) operator, whose weighting vectors depend upon the input arguments and allow values being aggregated to support and reinforce each other.

Quadratic mean in statistics is a statistical measure defined as the square root of the mean of the squares of a sample, which is a conservative average to be used to provide for aggregation lying between the max and min operators. Consider that, in the existing literature, the quadratic mean is generally considered as a fusion technique of numerical data, in the real-life situations, the input data sometimes cannot be obtained exactly, but fuzzy data can be given. Therefore, how to aggregate fuzzy data by using the quadratic mean? is an interesting research topic and is worth paying attention to. In this paper, we develop some fuzzy quadratic mean (FQM) operators. To do so, the remainder of this paper is arranged in the following sections. Section 2 reviews some basic aggregation operators. Section 3 develops some FQM operators, such as fuzzy weighted quadratic mean (FWQM) operator, fuzzy ordered weighted quadratic mean (FOWQM) operator, fuzzy hybrid quadratic mean (FHQM) operator, and so on. Section 4 presents an approach to multiple attribute group decision making based on the developed operators. Section 5 illustrates the presented approach with a practical example. Section 6 ends the paper with some concluding remarks.

2 Basic aggregation operators

We review some basic aggregation techniques and some of their fundamental characteristics.

Definition 2.1 [5] Let WAA : $R^n \to R$, if

$$\text{WAA}(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j a_j, \tag{1}$$

where R is the set of real numbers, a_j $(j = 1, 2, \ldots, n)$ is a collection of positive real numbers, and $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of a_j $(j = 1, 2, \ldots, n)$, with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$, then WAA is called the weighted arithmetic averaging (WAA) operator. Especially, if $w_i = 1$, $w_j = 0$, $j \neq i$, then $\text{WAA}(a_1, a_2 \ldots, a_n) = a_i$; if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the WAA

operator is reduced to the arithmetic averaging (AA) operator, i.e.,

$$\mathrm{AA}(a_1, a_2, \ldots, a_n) = \frac{1}{n}\sum_{j=1}^{n} a_j. \tag{2}$$

Definition 2.2 [2] Let WQM : $(R^+)^n \to R^+$, if

$$\mathrm{WQM}(a_1, a_2, \ldots, a_n) = \left(\sum_{j=1}^{n} w_j a_j^2\right)^{\frac{1}{2}}, \tag{3}$$

where R^+ is the set of all positive real numbers, a_j $(j = 1, 2, \ldots, n)$ is a collection of positive real numbers, and $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of a_j $(j = 1, 2, \ldots, n)$, with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$, then WQM is called the weighted quadratic mean (WQM) operator. Especially, if $w_i = 1$, $w_j = 0$, $j \neq i$, then $\mathrm{WQM}(a_1, a_2 \ldots, a_n) = a_i$; if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the WQM operator is reduced to the quadratic mean (QM) operator, i.e.,

$$\mathrm{QM}(a_1, a_2, \ldots, a_n) = \left(\frac{\sum_{j=1}^{n} a_j^2}{n}\right)^{\frac{1}{2}}. \tag{4}$$

The WAA and WQM operators first weight all the given data, and then aggregate all these weighted data into a collective one. Yager [17] introduced and studied the OWA operator that weights the ordered positions of the data instead of weighting the data themselves.

Definition 2.3 [17] An OWA operator of dimension n is a mapping OWA : $R^n \to R$ that has an associated vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that $\omega_j \geq 0$ and $\sum_{j=1}^{n} \omega_j = 1$. Furthermore,

$$\mathrm{OWA}(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j, \tag{5}$$

where b_j is the jth largest of a_i $(i = 1, 2, \ldots, n)$. Especially, if $w_i = 1$, $w_j = 0$, $j \neq i$, then $b_n \leq \mathrm{OWA}(a_1, a_2, \ldots, a_n) = b_i \leq b_1$; if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then

$$\mathrm{OWA}(a_1, a_2, \ldots, a_n) = \frac{1}{n}\sum_{j=1}^{n} b_j = \frac{1}{n}\sum_{j=1}^{n} a_j = \mathrm{AA}(a_1, a_2, \ldots, a_n). \tag{6}$$

3 Fuzzy quadratic mean operators

The above aggregation techniques can only deal with the situation that the arguments are represented by the exact numerical values, but are invalid if the aggregation information is given in other forms, such as triangular fuzzy number [9], which is a widely used tool to deal with uncertainty and fuzzyness, described as follows:

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Definition 3.1 [9] A triangular fuzzy number $\hat{a}$ can be defined by a triplet $[a^L, a^M, a^U]$. The membership function $\mu_{\hat{a}}(x)$ is defined as:

$$\mu_{\hat{a}}(x) = \begin{cases} 0, & x < a^L; \\ \frac{x-a^L}{a^M-a^L}, & a^L \leq x \leq a^M; \\ \frac{x-a^U}{a^M-a^U}, & a^M \leq x \leq a^U; \\ 0, & x > a^U, \end{cases}$$

where $a^U \geq a^M \geq a^L \geq 0$, a^L and a^U stand for the lower and upper values of $\hat{a}$, respectively, and a^M stands for the modal value [9]. Especially, if and two of a^L, a^M and a^U are equal,then $\hat{a}$ is reduced to an interval number; if all a^L, a^M and a^U are equal, then $\hat{a}$ is reduced to a real number. For convenience, we let Ω be the set of all triangular fuzzy numbers.

Let $\hat{a} = [a^L, a^M, a^U]$ and $\hat{b} = [b^L, b^M, b^U]$ be two triangular fuzzy numbers, then some operational laws defined as follows [9]:
(1) $\hat{a} + \hat{b} = [a^L, a^M, a^U] + [b^L, b^M, b^U] = [a^L + b^L, a^M + b^M, a^U + b^U]$;
(2) $\lambda\hat{a} = \lambda[a^L, a^M, a^U] = [\lambda a^L, \lambda a^M, \lambda a^U]$;
(3) $\hat{a} \times \hat{b} = [a^L, a^M, a^U] \times [b^L, b^M, b^U] = [a^L b^L, a^M b^M, a^U b^U]$
(4) $\frac{1}{\hat{a}} = \frac{1}{[a^L, a^M, a^U]} = [\frac{1}{a^U}, \frac{1}{a^M}, \frac{1}{a^L}]$.

In order to compare two triangular fuzzy numbers, Xu [12] provided the following definition:

Definition 3.2 [12] Let $\hat{a} = [a^L, a^M, a^U]$ and $\hat{b} = [b^L, b^M, b^U]$ be two triangular fuzzy numbers, then the degree of possibility of $\hat{a} \geq \hat{b}$ is defined as follows:

$$p(\hat{a} \geq \hat{b}) = \delta \max\left\{1 - \max\left(\frac{b^M - a^L}{a^M - a^L + b^M - b^L}, 0\right), 0\right\}$$
$$+(1-\delta)\max\left\{1 - \max\left(\frac{b^U - a^M}{a^U - a^M + b^U - b^M}, 0\right), 0\right\}, \delta \in [0,1] \quad (7)$$

which satisfies the following properties:

$$0 \leq p(\hat{a} \geq \hat{b}) \leq 1, \quad p(\hat{a} \geq \hat{a}) = 0.5, \quad p(\hat{a} \geq \hat{b}) + p(\hat{b} \geq \hat{a}) = 1. \quad (8)$$

Here, δ reflects the decision maker's risk-bearing attitude. If $\delta > 0.5$, then the decision maker is risk lover; If $\delta = 0.5$, then the decision maker is neutral to risk; If $\delta < 0.5$, then the decision maker is risk avertor.

In the following, we shall give a simple procedure for ranking of the triangular fuzzy numbers $\hat{a}_i$ $(i = 1, 2, \ldots, n)$. First, by using Eq. (7), we compare each $\hat{a}_i$ with all the $\hat{a}_j$ $(j = 1, 2, \ldots, n)$, for simplicity, let $p_{ij} = p(\hat{a}_i \geq \hat{a}_j)$, then we develop a possibility matrix [14] as

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ & & \vdots & \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}, \quad (9)$$

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where $p_{ij} \geq 0$, $p_{ij} + p_{ji} = 1$, $p_{ii} = \frac{1}{2}$, $i, j = 1, 2, \ldots, n$.

Summing all elements in each line of matrix P, we have $p_i = \sum_{j=1}^{n} p_{ij}$, $i = 1, 2, \ldots, n$. Then, in accordance with the values of p_i $(i = 1, 2, \ldots, n)$, we rank the $\hat{a}_i$ $(i = 1, 2, \ldots, n)$ in descending order.

Now, based on operational laws, we extend the WQM operator (3) to fuzzy environment:

Definition 3.3 Let $\hat{a}_j = [a_j^L, a_j^M, a_j^U]$ $(j = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers, and let $\mathrm{FWQM} : \Omega^n \to \Omega$, if

$$\mathrm{FWQM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \left(\sum_{j=1}^{n} w_j \hat{a}_j^2 \right)^{\frac{1}{2}}, \tag{10}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $\hat{a}_j$ $(j = 1, 2, \ldots, n)$, with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$, then FWQM is called a fuzzy weighted quadratic mean (FWQM) operator.

Especially, if $w_i = 1$, $w_j = 0$, $j \neq i$, then $\mathrm{FWQM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \hat{a}_i$; if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the FWQM operator is reduced to the fuzzy quadratic mean (FQM) operator:

$$\mathrm{FQM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \left(\frac{\sum_{j=1}^{n} \hat{a}_j^2}{n} \right)^{\frac{1}{2}}. \tag{11}$$

By the operational laws and Eq. (10), we have

$$\begin{aligned} \mathrm{FWQM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) &= \left(\sum_{j=1}^{n} w_j \hat{a}_j^2 \right)^{\frac{1}{2}} = \left(\sum_{j=1}^{n} w_j [a_j^L, a_j^M, a_j^U]^2 \right)^{\frac{1}{2}} \\ &= \left[\left(\sum_{j=1}^{n} w_j (a_j^L)^2 \right)^{\frac{1}{2}}, \left(\sum_{j=1}^{n} w_j (a_j^M)^2 \right)^{\frac{1}{2}}, \left(\sum_{j=1}^{n} w_j (a_j^U)^2 \right)^{\frac{1}{2}} \right] \end{aligned} \tag{12}$$

and then by Eq. (12), we have

$$\begin{aligned} &\mathrm{FQM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) \\ &= \left[\left(\frac{\sum_{j=1}^{n} (a_j^L)^2}{n} \right)^{\frac{1}{2}}, \left(\frac{\sum_{j=1}^{n} (a_j^M)^2}{n} \right)^{\frac{1}{2}}, \left(\frac{\sum_{j=1}^{n} (a_j^U)^2}{n} \right)^{\frac{1}{2}} \right]. \end{aligned} \tag{13}$$

Especially, if the triangular fuzzy numbers $\hat{a}_j = [a_j^L, a_j^M, a_j^U]$ $(j = 1, 2, \ldots, n)$ are reduced to the interval numbers $\tilde{a}_j = [a_j^L, a_j^U]$ $(j = 1, 2, \ldots, n)$, then the

FWQM operator is reduced to the uncertain weighted quadratic mean(UWQM) operator:

$$\begin{aligned}\mathrm{UWQM}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) &= \left(\sum_{j=1}^{n} w_j \tilde{a}_j^2\right)^{\frac{1}{2}} \\ &= \left[\left(\sum_{j=1}^{n} w_j (a_j^L)^2\right)^{\frac{1}{2}}, \left(\sum_{j=1}^{n} w_j (a_j^U)^2\right)^{\frac{1}{2}}\right]. \quad (14)\end{aligned}$$

If $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the UWQM operator is reduced to the uncertain quadratic mean(UQM) operator:

$$\begin{aligned}\mathrm{UQM}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) &= \left(\frac{\sum_{j=1}^{n} (\tilde{a}_j^2)}{n}\right)^{\frac{1}{2}} \\ &= \left[\left(\frac{\sum_{j=1}^{n} (a_j^L)^2}{n}\right)^{\frac{1}{2}}, \left(\frac{\sum_{j=1}^{n} (a_j^U)^2}{n}\right)^{\frac{1}{2}}\right]. \quad (15)\end{aligned}$$

If $a_j^L = a_j^U = a_j$, for all $j = 1, 2, \ldots, n$, then Eqs. (14) and (15) are, respectively, reduced to the WQM operator (3) and the QM operator (4).

Example 3.4 Given a collection of triangular fuzzy numbers: $\hat{a}_1 = [2, 3, 4], \hat{a}_2 = [1, 2, 4], \hat{a}_3 = [2, 4, 6], \hat{a}_4 = [1, 3, 5]$, let $w = (0.3, 0.1, 0.2, 0.4)^T$ be the weight vector of $\hat{a}_i$ $(i = 1, 2, 3, 4)$, then by Eq. (12), we have

$$\begin{aligned}\mathrm{FWQM}(\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4) &= \left[\left(\sum_{j=1}^{n} w_j (a_j^L)^2\right)^{\frac{1}{2}}, \left(\sum_{j=1}^{n} w_j (a_j^M)^2\right)^{\frac{1}{2}}, \left(\sum_{j=1}^{n} w_j (a_j^U)^2\right)^{\frac{1}{2}}\right] \\ &= [1.5811, 3.1464, 4.8580].\end{aligned}$$

Based on the OWA and FQM operators and Definition 3.2, we define a fuzzy ordered weighted quadratic mean (FOWQM) operator as below:

Definition 3.5 Let $\hat{a}_j = [a_j^L, a_j^M, a_j^U]$ $(j = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers. A FOWQM operator of dimension n is a mapping FOWQM : $\Omega^n \to \Omega$, that has an associated vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that $\omega_j \geq 0$ and $\sum_{j=1}^{n} \omega_j = 1$. Furthermore,

$$\begin{aligned}\mathrm{FOWQM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) &= \left(\sum_{j=1}^{n} \omega_j \hat{a}_{\sigma(j)}^2\right)^{\frac{1}{2}} \\ &= \left[\left(\sum_{j=1}^{n} \omega_j (a_{\sigma(j)}^L)^2\right)^{\frac{1}{2}}, \left(\sum_{j=1}^{n} w_j (a_{\sigma(j)}^M)^2\right)^{\frac{1}{2}}, \left(\sum_{j=1}^{n} w_j (a_{\sigma(j)}^U)^2\right)^{\frac{1}{2}}\right], \quad (16)\end{aligned}$$

where $a_{\sigma(j)} = \left[a^L_{\sigma(j)}, a^M_{\sigma(j)}, a^U_{\sigma(j)}\right]$ $(j = 1, 2, \ldots, n)$, and $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $\hat{a}_{\sigma(j-1)} \geq \hat{a}_{\sigma(j)}$ for all j.

However, if there is a tie between $\hat{a}_i$ and $\hat{a}_j$ by their average $(\hat{a}_i + \hat{a}_j)/2$ in process of aggregation. If k items are tied, then we replace these by k replicas of their average. The weighting vector $w = (w_1, w_2, \ldots, w_n)^T$ can be determined by using some weight determining methods like the normal distribution based method, see Refs [11, 20] for more details.

Similarly to the OWA operator, the FOWQM operator has the following properties:

Theorem 3.6 Let $\hat{a}_j = [a^L_j, a^M_j, a^U_j]$ $(j = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers, the following are valid:

(1) **Idempotency**: If all $\hat{a}_j$ $(j = 1, 2, \ldots, n)$ are equal, i.e., $\hat{a}_j = \hat{a}$, for all i, then

$$\mathrm{FOWQM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \hat{a}.$$

(2) **Boundedness**: Let $\hat{a}^- = [\min_j(a^L_j), \min_j(a^M_j), \min_j(a^U_j)]$ and $\hat{a}^+ = [\max_j(a^L_j), \max_j(a^M_j), \max_j(a^U_j)]$, then

$$\hat{a}^- \leq \mathrm{FOWQM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) \leq \hat{a}^+.$$

(3) **Monotonicity**: Let $\hat{a}^*_j = [a^{L*}_j, a^{M*}_j, a^{U*}_j]$ $(j = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers, then if $a^L_j \leq a^{L*}_j$, $a^M_j \leq a^{M*}_j$ and $a^U_j \leq a^{U*}_j$ for all j, then

$$\mathrm{FOWQM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) \leq \mathrm{FOWQM}(\hat{a}^*_1, \hat{a}^*_2, \ldots, \hat{a}^*_n).$$

(4) **Commutativity**: Let $\hat{a}'_j = [a^{L'}_j, a^{M'}_j, a^{U'}_j]$ $(j = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers, then

$$\mathrm{FOWQM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \mathrm{FOWQM}(\hat{a}'_1, \hat{a}'_2, \ldots, \hat{a}'_n),$$

where $(\hat{a}'_1, \hat{a}'_2, \ldots, \hat{a}'_n)$ is any permutation of $(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n)$.

Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the FOWQM operator is reduced to the FQM operator; if the triangular fuzzy numbers $\hat{a}_j = [a^L_j, a^M_j, a^U_j]$ $(j = 1, 2, \ldots, n)$ are reduced to the interval numbers $\tilde{a}_j = [a^L_j, a^U_j]$ $(j = 1, 2, \ldots, n)$, then the FOWQM operator is reduced to the uncertain ordered weighted quadratic mean (UOWQM) operator:

$$\begin{aligned}\mathrm{UOWQM}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) &= \left(\sum_{j=1}^{n} \omega_j \tilde{a}^2_{\sigma(j)}\right)^{\frac{1}{2}} \\ &= \left[\left(\sum_{j=1}^{n} \omega_j (a^L_{\sigma(j)})^2\right)^{\frac{1}{2}}, \left(\sum_{j=1}^{n} \omega_j (a^U_{\sigma(j)})^2\right)^{\frac{1}{2}}\right], \quad (17)\end{aligned}$$

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where $\tilde{a}_\sigma(j) = [a^L_{\sigma(j)}, a^U_{\sigma(j)}]$, $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $\tilde{a}_{\sigma(j-1)} \geq \tilde{a}_{\sigma(j)}$ for all j. If there is a tie between $\tilde{a}_i$ and $\tilde{a}_j$, then we replace each of $\tilde{a}_i$ and $\tilde{a}_j$ by their average $(\tilde{a}_i + \tilde{a}_j)/2$ in process of aggregation. If k items are tied, then we replace these by k replicas of their average.

If $a^L_i = a^U_i = a_i$, for all $i = 1, 2, \ldots, n$, then the UOWQM operator is reduced to the ordered weighted quadratic mean (OWQM)operator:

$$\text{OWQM}(a_1, a_2, \ldots, a_n) = \left(\sum_{j=1}^{n} \omega_j b_j^2\right)^{\frac{1}{2}}, \tag{18}$$

where b_j is the jth largest of $a_j (j = 1, 2, \ldots, n)$. The OWQM operator (18) has some special cases:

(1) If $\omega = (1, 0, \ldots, 0)^T$, then

$$\text{OWQM}(a_1, a_2, \ldots, a_n) = \max\{a_i\} = b_1. \tag{19}$$

(2) If $\omega = (0, 0, \ldots, 1)^T$, then

$$\text{OWQM}(a_1, a_2, \ldots, a_n) = \min\{a_i\} = b_n. \tag{20}$$

(3) If $\omega_j = 1,\ w_i = 0,\ i \neq j$, then

$$b_n \leq \text{OWQM}(a_1, a_2, \ldots, a_n) = b_j \leq b_1. \tag{21}$$

(4) If $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then

$$\begin{aligned}\text{OWQM}(a_1, a_2, \ldots, a_n) &= \left(\frac{\sum_{j=1}^n b_j^2}{n}\right)^{\frac{1}{2}} = \left(\frac{\sum_{j=1}^n a_j^2}{n}\right)^{\frac{1}{2}} \\ &= \text{QM}(a_1, a_2, \ldots, a_n). \end{aligned} \tag{22}$$

Clearly, the fundamental characteristic of the FWQM operator is that it considers the importance of each given triangular fuzzy number, whereas the fundamental characteristic of the FOWQM operator is the reordering step, and it weights all the ordered positions of the triangular fuzzy numbers instead of weighing the given triangular fuzzy numbers themselves. By combining the advantages of the FWQM and FOWQM operators, in the following, we develop a fuzzy hybrid quadratic mean (FHQM) operator that weights both the given triangular fuzzy numbers and their ordered positions.

Definition 3.7 Let $\hat{a}_j = [a^L_j, a^M_j, a^U_j]$ $(j = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers. A FHQM operator of dimension n is a mapping FHQM : $\Omega^n \to \Omega$, which has an associated vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ with $\omega_j \geq 0$ and $\sum_{j=1}^n \omega_j = 1$, such that

$$\text{FHQM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \left(\sum_{j=1}^{n} \omega_j \dot{\hat{a}}^2_{\sigma(j)}\right)^{\frac{1}{2}}$$

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$$= \left[\left(\sum_{j=1}^{n} \omega_j (\dot{a}^L_{\sigma(j)})^2 \right)^{\frac{1}{2}}, \left(\sum_{j=1}^{n} \omega_j (\dot{a}^M_{\sigma(j)})^2 \right)^{\frac{1}{2}}, \left(\sum_{j=1}^{n} \omega_j (\dot{a}^U_{\sigma(j)})^2 \right)^{\frac{1}{2}} \right], \quad (23)$$

where $\dot{\hat{a}}_{\sigma(j)} = \left[\dot{a}^L_{\sigma(j)}, \dot{a}^M_{\sigma(j)}, \dot{a}^U_{\sigma(j)} \right]$ is the jth largest of the weighted triangular fuzzy numbers $\dot{\hat{a}}_j$ $(\dot{\hat{a}}_j = nw_j\hat{a}_j,\ j = 1, 2, \ldots, n)$, $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $\hat{a}_j$ $(j = 1, 2, \ldots, n)$ with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$, and n is the balancing coefficient.

Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then $\dot{\hat{a}}_j = \hat{a}_j$, $j = 1, 2, \ldots, n$, in this case, the FHQM operator is reduced to the FOWQM operator; if $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then

$$\mathrm{FHQM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \left(\sum_{j=1}^{n} w_j \dot{\hat{a}}^2_{\sigma(j)} \right)^{\frac{1}{2}}$$

$$= \left[\left(\sum_{j=1}^{n} nw_j^2 (a^L_{\sigma(j)})^2 \right)^{\frac{1}{2}}, \left(\sum_{j=1}^{n} nw_j^2 (a^M_{\sigma(j)})^2 \right)^{\frac{1}{2}}, \left(\sum_{j=1}^{n} nw_j^2 (a^U_{\sigma(j)})^2 \right)^{\frac{1}{2}} \right] \quad (24)$$

which we call the generalized fuzzy weighted quadratic mean (GFWQM) operator.

Moreover, if the triangular fuzzy numbers $\hat{a}_j = [a^L_j, a^M_j, a^U_j]$ $(j = 1, 2, \ldots, n)$ are reduced to the interval numbers $\tilde{a}_j = [a^L_j, a^U_j]$ $(j = 1, 2, \ldots, n)$, then the FHQM operator is reduced to the uncertain hybrid quadratic mean (UHQM) operator:

$$\mathrm{UHQM}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left(\sum_{j=1}^{n} \omega_j \dot{\tilde{a}}^2_{\sigma(j)} \right)^{\frac{1}{2}}$$

$$= \left[\left(\sum_{j=1}^{n} nw_j^2 (a^L_{\sigma(j)})^2 \right)^{\frac{1}{2}}, \left(\sum_{j=1}^{n} nw_j^2 (a^U_{\sigma(j)})^2 \right)^{\frac{1}{2}} \right], \quad (25)$$

where $\dot{\tilde{a}}_{\sigma(j)}$ is the jth largest of the weighted interval numbers $\dot{\tilde{a}}_j$ $(\dot{\tilde{a}}_j = nw_j\tilde{a}_j, j = 1, 2, \ldots, n)$, $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $\tilde{a}_j$ $(j = 1, 2, \ldots, n)$ with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$, and n is the balancing coefficient. Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then $\dot{\tilde{a}}_j = \tilde{a}_j$, $j = 1, 2, \ldots, n$, in this case, the UHQM operator is reduced to the UOWQM operator.

If $a^L_i = a^U_i = a_i$, for all $i = 1, 2, \ldots, n$, then the UHQM operator is reduced to the hybrid quadratic mean (HQM) operator:

$$\mathrm{HQM}(a_1, a_2, \ldots, a_n) = \left(\sum_{j=1}^{n} \omega_j \dot{a}^2_{\sigma(j)} \right)^{\frac{1}{2}}, \quad (26)$$

where $\dot{a}_{\sigma(j)}$ is the jth largest of the weighted interval numbers $\dot{a}_j$ ($\dot{a}_j = nw_j\tilde{a}_j, j = 1, 2, \ldots, n$), $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of a_j ($j = 1, 2, \ldots, n$) with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$, and n is the balancing coefficient. Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then $\dot{a}_j = a_j$, $j = 1, 2, \ldots, n$, in this case, the HQM operator is reduced to the OWQM operator.

Example 3.8 Given a collection of triangular fuzzy numbers: $\hat{a}_1 = [2, 4, 5]$, $\hat{a}_2 = [1, 3, 4]$, $\hat{a}_3 = [2, 3, 5]$, $\hat{a}_4 = [3, 4, 5]$, and $\hat{a}_5 = [2, 5, 8]$, and $w = (0.20, 0.25, 0.15, 0.25, 0.15)^T$ be the weight vector of $\hat{a}_j$ ($j = 1, 2, 3, 4, 5$). Then we get the weighted triangular fuzzy numbers:

$$\dot{\hat{a}}_1 = [2, 4, 5],\ \dot{\hat{a}}_2 = [1.25, 3.75, 5],\ \dot{\hat{a}}_3 = [1.5, 2.25, 3.75],$$
$$\dot{\hat{a}}_4 = [3.75, 5, 6.25],\ \dot{\hat{a}}_5 = [1.5, 3.75, 6].$$

By using Eq. (9) (without loss of generality, set $\delta = 0.5$), we construct the following matrix:

$$P = \begin{pmatrix} 0.5000 & 0.5833 & 0.9545 & 0.0385 & 0.4864 \\ 0.4167 & 0.5000 & 0.8462 & 0 & 0.4154 \\ 0.0455 & 0.1538 & 0.5000 & 0 & 0.1250 \\ 0.9615 & 1 & 1 & 0.5000 & 0.8571 \\ 0.5136 & 0.5846 & 0.8750 & 0.1429 & 0.5000 \end{pmatrix}.$$

Summing all elements in each line of matrix P, we have

$$p_1 = 2.5628, \quad p_2 = 2.1782, \quad p_3 = 0.8243, \quad p_4 = 4.3187, \quad p_5 = 2.6160$$

and then we rank the triangular fuzzy number $\hat{a}_i$ ($i = 1, 2, 3, 4, 5$) in descending order in accordance with the values of p_i ($i = 1, 2, 3, 4, 5$):

$$\dot{\hat{a}}_{\sigma(1)} = \dot{\hat{a}}_4,\ \dot{\hat{a}}_{\sigma(2)} = \dot{\hat{a}}_5,\ \dot{\hat{a}}_{\sigma(3)} = \dot{\hat{a}}_1,\ \dot{\hat{a}}_{\sigma(4)} = \dot{\hat{a}}_2,\ \dot{\hat{a}}_{\sigma(5)} = \dot{\hat{a}}_3.$$

Suppose that $\omega = (0.1117, 0.2365, 0.3036, 0.3265, 0.1117)^T$ is the weighting vector of the FHQM operator (derived by the normal distribution based method [11]), then by Eq. (23), we get

$$\begin{aligned} \mathrm{FHQM}(\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4, \hat{a}_5) &= \left(\sum_{j=1}^{n} \omega_j \dot{\hat{a}}_{\sigma(j)}^2 \right)^{\frac{1}{2}} \\ &= \left[\left(\sum_{j=1}^{n} \omega_j (\dot{a}_{\sigma(j)}^L)^2 \right)^{\frac{1}{2}}, \left(\sum_{j=1}^{n} \omega_j (\dot{a}_{\sigma(j)}^M)^2 \right)^{\frac{1}{2}}, \left(\sum_{j=1}^{n} \omega_j (\dot{a}_{\sigma(j)}^U)^2 \right)^{\frac{1}{2}} \right] \\ &= [2.0196, 4.0166, 5.4955]. \end{aligned}$$

4 Approaches to multiple attribute group decision making with triangular fuzzy information

For a group decision making with triangular fuzzy information, let X=$\{x_1, x_2, \ldots, x_n\}$ be a discrete set of n alternatives, and $G = \{G_1, G_2, \ldots, G_m\}$ be the set of

m attributes, whose weight vector is $w = (w_1, w_2, \ldots, w_m)^T$ with $w_i \geq 0$ and $\sum_{i=1}^m w_i = 1$, and let $D = \{d_1, d_2, \ldots, d_s\}$ be the set of decision makers, whose weight vector is $v = (v_1, v_2, \ldots, v_s)^T$, where $v_k \geq 0$ and $\sum_{k=1}^s v_k = 1$. Suppose that $A^{(k)} = (\hat{a}_{ij}^{(k)})_{m\times n}$ is the decision matrix, where $\hat{a}_{ij}^{(k)} = \left[a_{ij}^{L(k)}, a_{ij}^{M(k)}, a_{ij}^{U(k)}\right]$ is an attribute value, which takes the form of triangular fuzzy number, of the alternative $x_j \in X$ with respect to the attribute $G_i \in G$.

Then, we utilize the FWQM and FHQM operators to propose an approach to multiple attribute group decision making with triangular fuzzy information, which involves the following steps:

Step 1. Normalize each attribute value $\hat{a}_{ij}^{(k)}$ in the matrix $A^{(k)}$ into a corresponding element in the matrix $R^{(k)} = (\hat{r}_{ij}^{(k)})_{m\times n}$ ($\hat{r}_{ij}^{(k)} = \left[{r_{ij}}^{L(k)}, {r_{ij}}^{M(k)}, {r_{ij}}^{U(k)}\right]$) using the following formulas:

$$\hat{r}_{ij}^{(k)} = \frac{\hat{a}_{ij}^{(k)}}{\sum_{j=1}^n \hat{a}_{ij}^{(k)}} = \left[\frac{{a_{ij}}^{L(k)}}{\sum_{j=1}^n {a_{ij}}^{U(k)}}, \frac{{a_{ij}}^{M(k)}}{\sum_{j=1}^n {a_{ij}}^{M(k)}}, \frac{{a_{ij}}^{U(k)}}{\sum_{j=1}^n {a_{ij}}^{L(k)}}\right],$$
$$\text{for benefit attribute } G_i, \tag{27}$$

$$\hat{r}_{ij}^{(k)} = \frac{1/\hat{a}_{ij}^{(k)}}{\sum_{j=1}^n (1/\hat{a}_{ij}^{(k)})} = \left[\frac{1/{a_{ij}}^{U(k)}}{\sum_{j=1}^n (1/{a_{ij}}^{L(k)})}, \frac{1/{a_{ij}}^{M(k)}}{\sum_{j=1}^n (1/{a_{ij}}^{M(k)})}, \frac{1/{a_{ij}}^{L(k)}}{\sum_{j=1}^n (1/{a_{ij}}^{U(k)})}\right],$$
$$\text{for cost attribute } G_i, \tag{28}$$

where $i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n,\ k = 1, 2, \ldots, s$.

Step 2. Utilize the FWQM operator:

$$\hat{r}_j^{(k)} = \mathrm{FWQM}(\hat{r}_{1j}^{(k)}, \hat{r}_{2j}^{(k)}, \ldots, \hat{r}_{mj}^{(k)}) = \left(\sum_{i=1}^m w_i (\hat{r}_{ij}^{(k)})^2\right)^{\frac{1}{2}}$$
$$= \left[\left(\sum_{i=1}^m w_i (\hat{r}_{ij}^{L(k)})^2\right)^{\frac{1}{2}}, \left(\sum_{i=1}^m w_i (\hat{r}_{ij}^{M(k)})^2\right)^{\frac{1}{2}}, \left(\sum_{i=1}^m w_i (\hat{r}_{ij}^{U(k)})^2\right)^{\frac{1}{2}}\right] \tag{29}$$

to aggregate all the elements in the jth column of $R^{(k)}$ and get the overall attribute value $\hat{r}_j^{(k)}$ of the alternative x_j corresponding to the decision maker d_k.

Step 3. Utilize the FHQM operator:

$$\hat{r}_j = \mathrm{FHQM}(\hat{r}_j^{(1)}, \hat{r}_j^{(2)}, \ldots, \hat{r}_j^{(s)}) = \left(\sum_{k=1}^s \omega_k (\dot{\hat{r}}_j^{(\sigma(k))})^2\right)^{\frac{1}{2}}$$
$$= \left[\left(\sum_{k=1}^s \omega_k (\dot{r}_j^{L(\sigma(k))})^2\right)^{\frac{1}{2}}, \left(\sum_{k=1}^s \omega_k (\dot{r}_j^{M(\sigma(k))})^2\right)^{\frac{1}{2}}, \left(\sum_{k=1}^s \omega_k (\dot{r}_j^{U(\sigma(k))})^2\right)^{\frac{1}{2}}\right] \tag{30}$$

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to aggregate the overall attribute values $\hat{r}_j^{(k)}$ $(k = 1, 2, \ldots, s)$ corresponding to the decision maker d_k $(k = 1, 2, \ldots, s)$ and get the collective overall attribute value $\hat{r}_j$, where $\dot{\hat{r}}_j^{(\sigma(k))} = [\dot{r}_j^{L(\sigma(k))}, \dot{r}_j^{M(\sigma(k))}, \dot{r}_j^{U(\sigma(k))}]$ is the kth largest of the weighted data $\dot{\hat{r}}_j^{(k)}$ $(\dot{\hat{r}}_j^{(k)} = sv_k\hat{r}_j^{(k)}$, $k = 1, 2, \ldots, s)$, $\omega = (\omega_1, \omega_2, \ldots, \omega_s)^T$ is the weighting vector of the FHQM operator, with $\omega_k \geq 0$ and $\sum_{k=1}^{s} \omega_k = 1$.

Step 4. Compare each $\hat{r}_j$ with all $\hat{r}_i$ $(i = 1, 2, \ldots, n)$ by using Eq. (9), and let $p_{ij} = p(\hat{r}_i \geq \hat{r}_j)$, and then construct a possibility matrix $P = (p_{ij})_{n\times n}$, where $p_{ij} \geq 0$, $p_{ij} + p_{ji} = 1$, $p_{ii} = 0.5$, $i, j = 1, 2, \ldots, n$. Summing all elements in each line of matrix P, we have $p_i = \sum_{j=1}^{n} p_{ij}$, $i = 1, 2, \ldots, n$, and then reorder $\hat{r}_j$ $(j = 1, 2, \ldots, n)$ in descending order in accordance with the values of p_j $(j = 1, 2, \ldots, n)$.

Step 5. Rank all the alternatives x_j $(j = 1, 2, \ldots, n)$ by the ranking of $\hat{r}_j$ $(j = 1, 2, \ldots, n)$, and then select the most desirable one.

Step 6. End.

5 Illustrative example

In this section, we use a multiple attribute group decision making problem of determining what kind of air-conditioning systems should be installed in a library(adopted from [6, 7, 12, 22]) to illustrate the proposed approach.

A city is planning to build a municipal library. One of the problems facing the city development commissioner is to determine what kind of air-conditioning systems should be installed in the library. The contractor offers five feasible alternatives, which might be adapted to the physical structure of the library. The alternatives x_j $(j = 1, 2, 3, 4, 5)$ are to be evaluated using triangular fuzzy numbers by the three decision makers d_k $(k = 1, 2, 3)$ (whose weight vector is $v = (0.4, 0.3, 0.3)^T$) under three major impacts: economic, functional, and operational. Two monetary attributes and six nonmonetary attributes (that is, G_1: owning cost ($/ft^2), G_2: operating cost ($/ft^2), G_3: performance (*), G_4: noise level (Db), G_5: maintainability (*), G_6: reliability (%), G_7: flex-

Table 1: Triangular fuzzy number decision matrix $A^{(1)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$[3.5, 4.0, 4.7]$	$[1.7, 2.0, 2.3]$	$[3.5, 3.8, 4.2]$	$[3.5, 3.8, 4.5]$	$[3.3, 3.8, 4.0]$
G_2	$[5.5, 6.0, 6.5]$	$[4.8, 5.1, 5.5]$	$[4.5, 5.2, 5.5]$	$[4.5, 4.7, 5.0]$	$[5.5, 5.7, 6.0]$
G_3	$[0.7, 0.8, 0.9]$	$[0.5, 0.56, 0.6]$	$[0.5, 0.6, 0.7]$	$[0.7, 0.85, 0.9]$	$[0.6, 0.7, 0.8]$
G_4	$[35, 40, 45]$	$[70, 73, 75]$	$[65, 68, 70]$	$[40, 42, 45]$	$[50, 55, 60]$
G_5	$[0.4, 0.45, 0.5]$	$[0.4, 0.44, 0.6]$	$[0.7, 0.76, 0.8]$	$[0.9, 0.97, 1.0]$	$[0.5, 0.54, 0.6]$
G_6	$[95, 98, 100]$	$[70, 73, 75]$	$[80, 83, 90]$	$[90, 93, 95]$	$[85, 90, 95]$
G_7	$[0.3, 0.35, 0.5]$	$[0.7, 0.75, 0.8]$	$[0.8, 0.9, 1.0]$	$[0.6, 0.75, 0.8]$	$[0.4, 0.5, 0.6]$
G_8	$[0.7, 0.74, 0.8]$	$[0.5, 0.53, 0.6]$	$[0.6, .68, 0.7]$	$[0.7, 0.8, 0.9]$	$[0.8, .85, 0.9]$

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ibility (*), G_8: safety (*), where * unit is from $0-1$ scale, three attributes G_1, G_2, and G_4 are cost attributes, and the other five attributes are benefit attributes, suppose that the weight vector of the attributes G_i $(i=1,2,\ldots,8)$ is $w=(0.05,0.08,0.14,0.12,0.18,0.21,0.05,0.17)^T$) emerged from three impacts is Tables 1-3.

Table 2: Triangular fuzzy number decision matrix $A^{(2)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$[4.0, 4.3, 4.5]$	$[2.1, 2.2, 2.4]$	$[5.0, 5.1, 5.2]$	$[4.3, 4.4, 4.5]$	$[3.0, 3.3, 3.5]$
G_2	$[6.0, 6.3, 6.5]$	$[5.0, 5.1, 5.2]$	$[4.5, 4.7, 5.0]$	$[5.0, 5.1, 5.3]$	$[7.0, 7.5, 8.0]$
G_3	$[0.7, 0.8, 0.9]$	$[0.4, 0.5, 0.6]$	$[0.5, .55, 0.6]$	$[0.7, 0.75, 0.8]$	$[0.7, 0.8, 0.9]$
G_4	$[37, 38, 39]$	$[70, 73, 75]$	$[65, 66, 67]$	$[40, 42, 45]$	$[50, 52, 55]$
G_5	$[0.4, 0.5, 0.6]$	$[0.5, 0.55, 0.6]$	$[0.8, 0.85, 0.9]$	$[0.8, 0.95, 1.0]$	$[0.4, 0.44, 0.5]$
G_6	$[92, 93, 95]$	$[70, 75, 80]$	$[83, 84, 85]$	$[90, 91, 92]$	$[90, 93, 95]$
G_7	$[0.4, 0.45, 0.5]$	$[0.8, 0.85, 0.9]$	$[0.7, 0.73, 0.8]$	$[0.7, 0.85, 0.9]$	$[0.4, 0.45, 0.5]$
G_8	$[0.6, 0.7, 0.8]$	$[0.6, 0.65, 0.7]$	$[0.5, 0.6, 0.7]$	$[0.7, 0.76, 0.8]$	$[0.7, 0.8, 0.9]$

Table 3: Triangular fuzzy number decision matrix $A^{(3)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$[4.3, 4.4, 4.6]$	$[2.2, 2.4, 2.5]$	$[4.5, 4.8, 5.0]$	$[4.7, 4.9, 5.0]$	$[3.1, 3.2, 3.4]$
G_2	$[6.4, 6.7, 7.0]$	$[5.0, 5.2, 5.5]$	$[4.7, 4.8, 4.9]$	$[5.5, 5.7, 6.0]$	$[6.0, 6.5, 7.0]$
G_3	$[0.8, 0.85, 0.9]$	$[0.5, 0.6, 0.7]$	$[0.6, 0.7, 0.8]$	$[0.7, 0.8, 0.9]$	$[0.7, 0.75, 0.8]$
G_4	$[36, 38, 40]$	$[72, 73, 75]$	$[67, 68, 70]$	$[45, 48, 50]$	$[55, 57, 60]$
G_5	$[0.4, 0.46, 0.5]$	$[0.4, 0.45, 0.6]$	$[0.8, 0.95, 1.0]$	$[0.8, 0.85, 0.9]$	$[0.5, 0.55, 0.6]$
G_6	$[93, 94, 95]$	$[77, 78, 80]$	$[85, 87, 90]$	$[90, 94, 95]$	$[90, 96, 100]$
G_7	$[0.4, 0.5, 0.6]$	$[0.8, 0.9, 1.0]$	$[0.8, 0.86, 0.9]$	$[0.6, 0.7, 0.8]$	$[0.5, 0.57, 0.6]$
G_8	$[0.7, 0.78, 0.8]$	$[0.5, 0.55, 0.6]$	$[0.6, 0.68, 0.7]$	$[0.8, 0.85, 0.9]$	$[0.8, 0.85, 0.9]$

To select the best air-conditioning system, we first utilize the approach based on the FWQM and FHQM operators, the main steps are as follows:

Step 1. By using Eqs. (27) and (28), we normalize each attribute value $\hat{a}_{ij}^{(k)}$ in the matrices $A^{(k)}$ $(k=1,2,3)$ into the corresponding element in the matrices $R^{(k)}=(\hat{r}_{ij})_{8\times 5}$ $(k=1,2,3)$ (Tables 4-6):

Step 2. Utilize the FWQM operator (29) to aggregate all elements in the jth column $R^{(K)}$ and get the overall attribute value $\hat{r}_j^{(k)}$:

$$\hat{r}_1^{(1)}=[0.1736,0.2029,0.2436]\,,\hat{r}_2^{(1)}=[0.1473,0.1751,0.2167]\,,$$
$$\hat{r}_3^{(1)}=[0.1689,0.1985,0.2354]\,,\hat{r}_4^{(1)}=[0.2043,0.2422,0.2759]\,,$$
$$\hat{r}_5^{(1)}=[0.1687,0.1991,0.2370]\,,$$
$$\hat{r}_1^{(2)}=[0.1770,0.2044,0.2417]\,,\hat{r}_2^{(2)}=[0.1622,0.1878,0.2191]\,,$$

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Table 4: Normalized triangular fuzzy number decision matrix $R^{(1)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$[0.12, 0.16, 0.21]$	$[0.25, 0.32, 0.43]$	$[0.14, 0.17, 0.21]$	$[0.13, 0.17, 0.21]$	$[0.14, 0.17, 0.22]$
G_2	$[0.15, 0.18, 0.21]$	$[0.18, 0.21, 0.24]$	$[0.18, 0.20, 0.25]$	$[0.20, 0.23, 0.25]$	$[0.16, 0.19, 0.21]$
G_3	$[0.18, 0.23, 0.30]$	$[0.13, 0.16, 0.20]$	$[0.13, 0.17, 0.23]$	$[0.18, 0.24, 0.30]$	$[0.15, 0.20, 0.27]$
G_4	$[0.22, 0.26, 0.32]$	$[0.13, 0.14, 0.16]$	$[0.14, 0.15, 0.17]$	$[0.22, 0.25, 0.28]$	$[0.16, 0.19, 0.23]$
G_5	$[0.11, 0.14, 0.17]$	$[0.11, 0.14, 0.21]$	$[0.20, 0.24, 0.28]$	$[0.26, 0.31, 0.34]$	$[0.14, 0.17, 0.21]$
G_6	$[0.21, 0.22, 0.24]$	$[0.15, 0.17, 0.18]$	$[0.18, 0.19, 0.21]$	$[0.20, 0.21, 0.23]$	$[0.19, 0.21, 0.23]$
G_7	$[0.08, 0.11, 0.18]$	$[0.19, 0.23, 0.29]$	$[0.22, 0.28, 0.36]$	$[0.16, 0.23, 0.29]$	$[0.11, 0.15, 0.21]$
G_8	$[0.18, 0.21, 0.24]$	$[0.13, 0.15, 0.18]$	$[0.15, 0.19, 0.21]$	$[0.18, 0.22, 0.27]$	$[0.21, 0.24, 0.27]$

Table 5: Normalized triangular fuzzy number decision matrix $R^{(2)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$[0.15, 0.16, 0.19]$	$[0.28, 0.32, 0.36]$	$[0.13, 0.14, 0.15]$	$[0.15, 0.16, 0.17]$	$[0.19, 0.21, 0.25]$
G_2	$[0.17, 0.18, 0.19]$	$[0.21, 0.22, 0.23]$	$[0.21, 0.24, 0.26]$	$[0.20, 0.22, 0.23]$	$[0.13, 0.15, 0.17]$
G_3	$[0.18, 0.24, 0.30]$	$[0.11, 0.15, 0.20]$	$[0.13, 0.16, 0.20]$	$[0.18, 0.22, 0.27]$	$[0.18, 0.24, 0.30]$
G_4	$[0.25, 0.27, 0.29]$	$[0.13, 0.14, 0.15]$	$[0.15, 0.15, 0.16]$	$[0.22, 0.24, 0.27]$	$[0.18, 0.20, 0.21]$
G_5	$[0.11, 0.15, 0.21]$	$[0.14, 0.17, 0.21]$	$[0.22, 0.26, 0.31]$	$[0.22, 0.29, 0.34]$	$[0.11, 0.13, 0.17]$
G_6	$[0.21, 0.21, 0.22]$	$[0.16, 0.17, 0.19]$	$[0.19, 0.19, 0.20]$	$[0.20, 0.21, 0.22]$	$[0.20, 0.21, 0.22]$
G_7	$[0.11, 0.14, 0.17]$	$[0.22, 0.26, 0.30]$	$[0.19, 0.22, 0.27]$	$[0.19, 0.26, 0.30]$	$[0.19, 0.14, 0.17]$
G_8	$[0.15, 0.20, 0.26]$	$[0.15, 0.19, 0.23]$	$[0.13, 0.17, 0.23]$	$[0.18, 0.22, 0.26]$	$[0.18, 0.23, 0.29]$

Table 6: Normalized triangular fuzzy number decision matrix $R^{(3)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$[0.15, 0.17, 0.18]$	$[0.28, 0.30, 0.35]$	$[0.14, 0.15, 0.17]$	$[0.14, 0.15, 0.16]$	$[0.20, 0.23, 0.25]$
G_2	$[0.16, 0.17, 0.19]$	$[0.20, 0.22, 0.24]$	$[0.22, 0.24, 0.25]$	$[0.18, 0.20, 0.22]$	$[0.16, 0.17, 0.20]$
G_3	$[0.20, 0.23, 0.27]$	$[0.12, 0.16, 0.21]$	$[0.15, 0.19, 0.24]$	$[0.17, 0.22, 0.27]$	$[0.17, 0.20, 0.24]$
G_4	$[0.26, 0.28, 0.31]$	$[0.14, 0.15, 0.16]$	$[0.15, 0.16, 0.17]$	$[0.21, 0.22, 0.25]$	$[0.17, 0.19, 0.20]$
G_5	$[0.11, 0.14, 0.17]$	$[0.11, 0.14, 0.21]$	$[0.20, 0.24, 0.28]$	$[0.26, 0.31, 0.34]$	$[0.14, 0.17, 0.21]$
G_6	$[0.21, 0.22, 0.24]$	$[0.15, 0.17, 0.18]$	$[0.18, 0.19, 0.21]$	$[0.20, 0.21, 0.23]$	$[0.19, 0.21, 0.23]$
G_7	$[0.08, 0.11, 0.18]$	$[0.19, 0.23, 0.29]$	$[0.22, 0.28, 0.36]$	$[0.16, 0.23, 0.29]$	$[0.11, 0.15, 0.21]$
G_8	$[0.18, 0.21, 0.24]$	$[0.13, 0.15, 0.18]$	$[0.15, 0.19, 0.21]$	$[0.18, 0.22, 0.27]$	$[0.21, 0.24, 0.27]$

$$\begin{aligned}
\hat{r}_3^{(2)} &= [0.1744, 0.1974, 0.2314]\,, \hat{r}_4^{(2)} = [0.1977, 0.2342, 0.2676]\,,\\
\hat{r}_5^{(2)} &= [0.1717, 0.1979, 0.2333]\,,\\
\hat{r}_1^{(3)} &= [0.0714, 0.0795, 0.0892]\,, \hat{r}_2^{(3)} = [0.0573, 0.0638, 0.0772]\,,\\
\hat{r}_3^{(3)} &= [0.0699, 0.0831, 0.0959]\,, \hat{r}_4^{(3)} = [0.0782, 0.0879, 0.1004]\,,\\
\hat{r}_5^{(3)} &= [0.0704, 0.0781, 0.0890]\,.
\end{aligned}$$

Step 3. Utilize the FHQM operator (30) (suppose that its weight vector is $w = (0.243, 0514, 0.243)^T$ determined by using the normal distribution based method [11], let $\sigma = 0.5$) to aggregate the overall attribute value $\hat{r}_j^{(k)}$ $(k = 1, 2, 3)$ corresponding to the decision maker d_k $(k = 1, 2, 3)$, and get the collective overall

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attribute value $\hat{r}_j$:

$$\hat{r}_1 = [0.1568, 0.1818, 0.2160], \hat{r}_2 = [0.1385, 0.1619, 0.1939],$$
$$\hat{r}_3 = [0.1536, 0.1771, 0.2086], \hat{r}_4 = [0.1791, 0.2119, 0.2417],$$
$$\hat{r}_5 = [0.1523, 0.1771, 0.2095].$$

Step 4. Compare each $\hat{r}_j$ with all $\hat{r}_i$ $(i = 1, 2, 3, 4, 5)$ by using Eq. (9) (without loss of generality, set $\delta = 0.5$), and let $p_{ij} = p(\hat{r}_i \geq \hat{r}_j)$, and then construct a possibility matrix:

$$P = \begin{pmatrix} 0.5 & 0.8558 & 0.5869 & 0.0553 & 0.5882 \\ 0.1442 & 0.5 & 0.2209 & 0 & 0.2301 \\ 0.4131 & 0.7791 & 0.5 & 0 & 0.5031 \\ 0.9447 & 1 & 1 & 0.5 & 1 \\ 0.4118 & 0.7699 & 0.4969 & 0 & 0.5 \end{pmatrix}.$$

Summing all elements in each line of matrix P, we have

$$p_1 = 2.5861,\ p_2 = 1.0952,\ p_3 = 2.1953,\ p_4 = 4.4447,\ p_5 = 2.1786$$

and then we reorder $\hat{r}_j$ $(j = 1, 2, 3, 4, 5)$ in descending order in accordance with the values of p_j $(j = 1, 2, 3, 4, 5)$:

$$\hat{r}_4 > \hat{r}_1 > \hat{r}_3 > \hat{r}_5 > \hat{r}_2.$$

Step 5. Rank all the alternatives x_j $(j = 1, 2, 3, 4, 5)$ by the ranking of $\hat{r}_j$ $(j = 1, 2, 3, 4, 5)$:

$$x_4 \succ x_1 \succ x_3 \succ x_5 \succ x_2$$

and thus the most desirable alternative is x_4.

Table 7: Comparison of the proposed approach with other approaches

	Xu's approach [12]	Park et al.'s approach [7]	Proposed approach
Solution method			
Aggregation stage	FWHM operator	FWCHM operator	FWQM operator
Exploitation stage	FHHM operator	FHCHM operator	FHQM operator
Ranking of alternatives	$x_4 \succ x_5 \succ x_3 \succ x_1 \succ x_2$	$x_4 \succ x_1 \succ x_3 \succ x_5 \succ x_2$	$x_4 \succ x_1 \succ x_3 \succ x_5 \succ x_2$

From the above analysis, the results obtained by the proposed approach are slightly different to the ones obtained Xu's [12] approach but the same with Park et al. [7] approach (see Table 7). It perfectly depends on how we look at things, and not on how they are themselves. Therefore, depending on aggregation operators used, the results may lead to different decisions. However, the best alternative is x_4.

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6 Conclusions

In this paper, we have extended the traditional quadratic mean to fuzzy environments and introduced the FWQM operator. Based on the FWQM operator and Yager's OWA operator [17], we have developed the FOWQM operator and the FHQM operator. It has been shown that both the FOWQM and FWQM operators are the special cases of the FHQM operator. It has also been pointed out that if all the input fuzzy data are reduced to the interval or numerical data, then the FHQM operator is reduced to the UHQM operator and the HQM operator, respectively. In these situations, the WQM operator and the OWQM operator are the two special cases of the HQM operator; the UWQM operator and the UOWQM operator are the two special cases of the UHQM operator. Then, based on the FWQM and FHQM operators, we present an approach to multiple attribute group decision making with triangular fuzzy information and illustrate it with a practical example.

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Sensitivity Analysis for General Nonlinear Nonconvex Set-Valued Variational Inequalities in Banach Spaces

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Abstract. In this paper, we show that the parametric general nonlinear nonconvex set-valued variational inequality is equivalent to the parametric general Wiener-Hopf equations. We used the equivalence formulation to study the sensitivity analysis for general nonlinear nonconvex set-valued variational inequalities without assuming the differentiability of the given data.

Keywords: Sensitivity analysis, general nonlinear nonconvex variational set-valued inequalities, fixed point, general Wiener-Hopf equations, relaxed φ-accretive mapping, locally Lipschitz continuous mappings, uniformly r-prox regular sets, uniformly smooth Banach spaces.

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1 Introduction

Variational inequality theory has become a very effective and powerful tools for studying a wide range of problems arising in pure and applied sciences which include the work on differential equations, mechanics, control problems in elasticity, general equilibrium problems in economics and transportation, obstacle, moving, and free boundary problems (see [1,3,5,8-10]).

Sensitivity analysis for the solutions of variational inequalities with single-valued mappings has been studied by many authors by quite different techniques. By using the projection methods, Anastassiou *et al.* [2], Agarwal *et al.* [4], Dafermos [6], Faraj and Salahuddin [7], Kim *et al.* [11], Kyparisis [12], Khan and Salahuddin [13], Liu [14], Lee and Salahuddin [15], Noor and Noor [16], Qiu and Magnanti [18], Salahuddin [19,20], Yen and Lee [23], and Verma [24] studied the sensitivity analysis for the solutions of some variational inequalities with single-valued mappings in finite dimensional spaces, Hilbert spaces and Banach spaces.

Noor and Noor [16] introduced and considered a new class of variational inequalities on the uniformly prox regular sets which are called the general nonlinear nonconvex variational inequalities. We note that the uniformly prox regular sets are nonconvex and include the convex sets as a special cases (see [5,17]).

In this paper, we developed the general framework of sensitivity analysis for the general nonlinear nonconvex set-valued variational inequalities. For this, we established the equivalence between the parametric general nonlinear nonconvex set-valued variational inequalities and parametric general Wiener-Hopf equations by using the

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projection techniques (see [11,21]). This fixed point formulation is obtained by a suitable and approximate rearrangement of the parametric general Wiener-Hopf equations. We would like to point out that the Wiener-Hopf equations technique is quite general unified flexible and provides us with new approach to study the sensitivity analysis of general nonlinear nonconvex set-valued variational inequalities and related optimization problems. We used this equivalence to develop the sensitivity analysis for general nonlinear nonconvex set-valued variational inequalities without assuming the differentiability of the given data.

2 Preliminaries

Let X be a real Banach space with dual space X^*, $\langle\cdot,\cdot\rangle$ be the dual pairing between X and X^*, and $CB(X)$ denotes the family of all nonempty closed bounded subsets of X. The generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(u) = \{f^* \in X^* : \langle u, f^*\rangle = \|u\|^q, \|f^*\| = \|u\|^{q-1}\},\ \forall u \in X,$$

where $q > 1$ is a constant. In particular J_2 is a usual normalized duality mapping. It is known that in general $J_q(u) = \|u\|^{q-2}J_2(u)$ for all $u \neq 0$ and J_q is single-valued if X^* is strictly convex. In the sequel, we always assume that X is a real Banach space such that J_q is a single-valued. If X is a Hilbert space then J_q becomes the identity mapping on . The modulus of smoothness of X is the function $\rho_X : [0,\infty) \to [0,\infty)$ is defined by

$$\rho_X(t) = \sup\left\{\frac{1}{2}(\|u+v\| + \|u-v\|) - 1 : \|u\| \le 1, \|v\| \le t\right\}.$$

A Banach space X is called uniformly smooth if

$$\lim_{t\to 0}\frac{\rho_X(t)}{t} = 0.$$

X is called q-uniformly smooth if there exists a constant $c > 0$ such that

$$\rho_X(t) < ct^q, q > 1.$$

It is well known that the Hilbert spaces, $L_p($ or $l_p)$ spaces, $1 < p < \infty$ and the Sobolev spaces $W^{m,p}, 1 < p < \infty$ are all q-uniformly smooth. Note that J_q is single-valued if X is uniformly smooth. Concerned with the characteristic inequalities in q-uniformly smooth Banach spaces. Xu [22] proved the following results.

Lemma 2.1. [22] *The real Banach space X is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $u, v \in X$,*

$$\|u+v\|^q \le \|u\|^q + q\langle v, J_q(u)\rangle + c_q\|v\|^q.$$

Let $\mathcal{K}$ be a nonempty closed subsets of X and we denote $d_{\mathcal{K}}(\cdot)$ or $d(\cdot,\mathcal{K})$ the usual distance function to the subset $\mathcal{K}$, that is,

$$d_{\mathcal{K}}(u) = \inf_{v\in\mathcal{K}}\|u-v\|.$$

The set of all projections of u onto $\mathcal{K}$ is given by

$$P_{\mathcal{K}}(u) = \big\{v \in \mathcal{K} : d_{\mathcal{K}}(u) = \|u-v\|\big\}.$$

Definition 2.2. The proximal normal cone of $\mathcal{K}$ at a point $u \in X$ is given by

$$N_{\mathcal{K}}^{P}(u) = \{\zeta \in X : u \in P_{\mathcal{K}}(u + \alpha\zeta) \text{ for some } \alpha > 0\}.$$

Lemma 2.3. [5] *Let $\mathcal{K}$ be a nonempty closed subset of X. Then $\zeta \in N_{\mathcal{K}}^{P}(u)$ if and only if there exists a constant $\alpha = \alpha(\zeta, u) > 0$ such that*

$$\langle \zeta, v - u \rangle \leq \alpha \|v - u\|^2, \quad \forall v \in \mathcal{K}.$$

Lemma 2.4. [5] *Let $\mathcal{K}$ be a nonempty closed and convex subset in X. Then $\zeta \in N_{\mathcal{K}}^{P}(u)$ if and only if*

$$\langle \zeta, v - u \rangle \leq 0, \quad \forall v \in \mathcal{K}.$$

Definition 2.5. Let $f : X \to R$ be a locally Lipschitz continuous mapping with constant τ near a given point $u \in X$, i.e., for some $\epsilon > 0$,

$$| f(v) - f(w) | \leq \tau \|v - w\|, \forall\ v, w \in B(u; \epsilon),$$

where $B(u; \epsilon)$ denotes the open ball of radius $r > 0$ and centered at u. The generalized directional derivative of f at u in the direction z, denoted by $f^o(u; z)$ is defined as follows:

$$f^o(u; z) = \limsup_{v \to u_{t \downarrow 0}} \frac{f(v + tz) - f(v)}{t},$$

where v is a vector in X and t is a positive scalar.

Definition 2.6. The tangent cone $T_{\mathcal{K}}(u)$ to $\mathcal{K}$ at a point $u \in \mathcal{K}$ is defined as follows:

$$T_{\mathcal{K}}(u) = \{v \in X : d_{\mathcal{K}}^{o}(u; v) = 0\}.$$

The normal cone of $\mathcal{K}$ at u by polarity with $T_{\mathcal{K}}(u)$ is defined as follows:

$$N_{\mathcal{K}}(u) = \{\zeta : \langle \zeta, v \rangle \leq 0, \quad \forall v \in T_{\mathcal{K}}(u)\}.$$

The Clarke normal cone $N_{\mathcal{K}}^{C}(u)$ is given by

$$N_{\mathcal{K}}^{C}(u) = \overline{co}\{N_{\mathcal{K}}^{P}(u)\},$$

where $\overline{co}(S)$ is the closure of the convex hull of S.

It is clear that $N_{\mathcal{K}}^{P}(u) \subseteq N_{\mathcal{K}}^{C}(u)$. The converse is not true in general. Note that $N_{\mathcal{K}}^{C}(u)$ is always closed and convex, where as $N_{\mathcal{K}}^{P}(u)$ is always convex but may not be closed (see [5,17]).

Definition 2.7. [17] For any $r \in (0, +\infty]$, a subset $\mathcal{K}_r$ of X is said to be normalized uniformly r-prox regular (or uniformly r-prox regular) if every nonzero proximal normal to $\mathcal{K}_r$ can be realized by an r-ball, that is, for all $u \in \mathcal{K}_r$ and all $0 \neq \zeta \in N_{\mathcal{K}_r}^{P}(u)$ with $\|\zeta\| = 1$,

$$\langle \zeta, v - u \rangle \leq \frac{1}{2r} \|v - u\|^2, \ \forall\ v \in \mathcal{K}_r.$$

Proposition 2.8. [17] *Let $r > 0$ and $\mathcal{K}_r$ be a nonempty closed and uniformly r-prox regular subset of X. Set*

$$\mathcal{U}(r) = \{u \in X : 0 \leq d_{\mathcal{K}_r}(u) < r\}.$$

Then we have the following statements:

(i) *For all $u \in \mathcal{U}(r)$, we have $P_{\mathcal{K}_r}(u) \neq \emptyset$;*

(ii) *For all $r' \in (0, r)$, $P_{\mathcal{K}_r}$ is Lipschitz continuous with constant $\delta = \frac{r}{r-r'}$ on $\mathcal{U}(r')$;*

(iii) *The proximal normal cone is closed as a set-valued mapping.*

Assume that $T : X \to 2^{X^*}$ is a set-valued mapping and $h : X \to X$ is a nonlinear single-valued mapping. For any constant $\rho > 0$, we consider the problem of finding $u \in X, x \in T(u)$ such that $h(u) \in \mathcal{K}_r$ and

$$\langle \rho x + h(u) - u, v - h(u) \rangle + \frac{1}{2r}\|v - h(u)\|^2 \geq 0, \quad \forall v \in \mathcal{K}_r. \tag{2.1}$$

The equation (2.1) is called a general nonlinear nonconvex set-valued variational inequality.

Now we consider the problem of solving general Wiener-Hopf equations. To be more precise, let $Q_{\mathcal{K}_r} = I - h^{-1}P_{\mathcal{K}_r}$ where $P_{\mathcal{K}_r}$ is the projection operator, h^{-1} is the inverse of nonlinear operator h and I is an identity operator. For given nonlinear operators, $z, u \in X, x \in T(u)$ such that

$$TP_{\mathcal{K}_r}z + \rho^{-1}Q_{\mathcal{K}_r}z = 0 \tag{2.2}$$

is a called general Wiener-Hopf equation.

Lemma 2.9. [17] *$u \in X, x \in T(u), h(u) \in \mathcal{K}_r$ is a solution of (2.1) if and only if $u \in X, x \in T(u), h(u) \in \mathcal{K}_r$ satisfies the relation*

$$h(u) = P_{\mathcal{K}_r}[u - \rho x], \tag{2.3}$$

where $P_{\mathcal{K}_r}$ is the projection of X onto the uniformly r-prox regular set $\mathcal{K}_r$.

Lemma 2.9 implies that the general nonlinear nonconvex set-valued variational inequality (2.1) is equivalent to the fixed point problem (2.3).

Now, we consider the parametric version of equations (2.1) and (2.2). To formulate the problem, let Γ be an open subset of X in which parameter λ takes values. Let $x_\lambda(u) \in T_\lambda(u)$ be a given operator defined on $X \times \Gamma$ and takes values in $X \times X$.From now, we denote $x_\lambda(u) \in T_\lambda(u)$ unless otherwise specified. The parametric general nonlinear nonconvex set-valued variational inequality problem is to find $(u, \lambda) \in X \times \Gamma, x_\lambda(u) \in T_\lambda(u)$ such that

$$\langle \rho x_\lambda(u) + h_\lambda(u) - u, v - h_\lambda(u) \rangle \geq 0, \quad \forall v \in \mathcal{K}_r. \tag{2.4}$$

We also assume that for some $\overline{\lambda} \in \mathcal{B}$, problem (2.4) has a unique solution $\overline{u}$. Related to parametric general nonlinear nonconvex set-valued variational inequality problem (2.4), we consider the parametric general Wiener-Hopf equation. We consider the problem of finding $(z, \lambda) \in X \times \Gamma, x_\lambda(u) \in T_\lambda(u)$ such that

$$T_\lambda P_{\mathcal{K}_r}z + \rho^{-1}Q_{\mathcal{K}_r}z = 0, \tag{2.5}$$

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where $\rho > 0$ is a constant and $Q_{\mathcal{K}_r} z$ is defined on the set (z, λ) with $\lambda \in \Gamma$ and takes values in $\mathcal{B}$. Equation (2.5) is called a parametric general Wiener-Hopf equation.

Lemma 2.10. *Let X be a real Banach space. Than the following two statements are equivalent:*

(i) *An element $u \in X, x_\lambda(u) \in T_\lambda(u)$ is a solution of (2.4),*

(ii) *The mapping*
$$F_\lambda(u) = u - h_\lambda(u) + P_{\mathcal{K}_r}[u - \rho x_\lambda(u)]$$
has a fixed point.

One can established the equivalence relation between inequality (2.4) and equation (2.5) by using the projection techniques.

Lemma 2.11. *Parametric general nonlinear nonconvex set-valued variational inequality (2.4) has a solution $(u, \lambda) \in X \times \Gamma, x_\lambda(u) \in T_\lambda(u)$ if and only if parametric general Wiener-Hopf equation (2.5) has a solution $(z, \lambda) \in X \times \Gamma, x_\lambda(u) \in T_\lambda(u)$, where*
$$h_\lambda(u) = P_{\mathcal{K}_r} z \tag{2.6}$$
and
$$z = u - \rho x_\lambda(u). \tag{2.7}$$

From Lemma 2.11, we know that Parametric general nonlinear nonconvex set-valued variational inequality (2.4) and parametric general Wiener-Hopf equation (2.5) are equivalent.

We used these equivalence to study the sensitivity analysis of general nonlinear nonconvex set-valued variational inequalities. We assume that for some $\overline{\lambda} \in \Gamma$, problem (2.5) has a solution $\overline{z}$ and $\mathcal{B}$ is a closure of a ball in X centered at $\overline{z}$. We want to investigate those condition under which for each λ in a neighbourhood of $\overline{\lambda}$, then (2.5) has a unique solution $z(\lambda)$ near $\overline{z}$ and the function $z(\lambda)$ is (Lipschitz) continuous and differentiable.

Definition 2.12. Let $T : X \times \Gamma \to 2^{X^*}$ be a set-valued mapping. Then the operator $T_\lambda(\cdot)$ is said to be locally relaxed φ-accretive if there exists a constant $\varphi > 0$ such that
$$\langle x_\lambda(u) - x_\lambda(v), j_q(u - v)\rangle \geq -\varphi\|u - v\|^q,\ \forall u, v \in X, \lambda \in \Gamma,$$
and locally D-Lipschitz continuous if there exists a constant $\beta > 0$ such that
$$\|x_\lambda(u) - x_\lambda(v)\| \leq D(T_\lambda(u), T_\lambda(v)) \leq \beta\|u - v\|,$$
where $D : 2^{X^*} \times 2^{X^*} \to (-\infty, \infty) \bigcup \{+\infty\}$ is the Hausdorff metric *i.e.,*
$$D(A, B) = \left\{ \sup_{u\in A} \inf_{v\in B} \|u - v\|,\ \sup_{u\in B} \inf_{v\in A} \|u - v\| \right\},\ \forall A, B \in 2^{X^*}.$$

Definition 2.13. A single-valued mapping $h : X \times \Omega \to X$ is said to be locally Lipschitz continuous if there exists a constant $\gamma > 0$ such that
$$\|h_\lambda(u) - h_\lambda(v)\| \leq \gamma\|u - v\|, \forall u, v \in X,$$
and locally strongly accretive if there exists a constant $\xi > 0$ such that
$$\langle h_\lambda(u) - h_\lambda(v), j_q(u - v)\rangle \geq \xi\|u - v\|^q, \forall u, v \in X, \lambda \in \Gamma.$$

3 Main Results

In this section, we derive the main results of this paper. We consider the case when the solutions of the parametric general Wiener-Hopf equation (2.5) lies in the interior of $\mathcal{B}$.

We consider the map: for all $(z,\lambda) \in X \times \Gamma, x_\lambda(u) \in T_\lambda(u)$,

$$F_\lambda(z) = P_{\mathcal{K}_r} z - \rho x_\lambda(u) = u - \rho x_\lambda(u), \tag{3.1}$$

where

$$h_\lambda(u) = P_{\mathcal{K}_r} z. \tag{3.2}$$

We have to show that the map $F_\lambda(z)$ has a fixed point, which is a solution of parametric general Wiener-Hopf equation (2.5). First of all we prove the map $F_\lambda(z)$ defined by (3.1) is contractive with respect to z uniformly in $\lambda \in \Gamma$.

Lemma 3.1. *Let $P_{\mathcal{K}_r}$ be a locally Lipschitz continuous operator with constant $\delta = \frac{r}{r-r'}$. Let $h : X \times \Gamma \to X$ be a locally Lipschitz continuous with constant $\gamma > 0$ and locally strongly accretive mapping with respect to the constant $\xi > 0$. Let $T : \Gamma \times X \to 2^{X^*}$ be a locally D-Lipschitz continuous with respect to the constant $\beta > 0$ and locally relaxed φ-accretive mapping with respect to the constant $\varphi > 0$. Then for all $z_1, z_2 \in X$ and $\lambda \in \Gamma$, we have*

$$\|F_\lambda(z_1) - F_\lambda(z_2)\| \le \theta \|z_1 - z_2\|, \tag{3.3}$$

where

$$\theta = \frac{\delta \sqrt[q]{1 + q\rho\varphi + c_q \rho^q \beta^q}}{1-\kappa}, \ \kappa = \sqrt[q]{1 - q\xi + c_q \gamma^q} \tag{3.4}$$

for

$$\sqrt[q]{\rho^q c_q \beta^q + \rho q \varphi + 1} < \frac{1-\kappa}{\delta}. \tag{3.5}$$

Proof. For all $z_1, z_2 \in \mathcal{B}, \lambda \in \Gamma$, from (3.1) we have

$$\|F_\lambda(z_1) - F_\lambda(z_2)\| = \|u - v - \rho(x_\lambda(u) - x_\lambda(v))\|. \tag{3.6}$$

Since $T_\lambda(\cdot)$ is a locally D-Lipschitz continuous mapping, we have

$$\|x_\lambda(u) - x_\lambda(v)\| \le D(T_\lambda(u), T_\lambda(v)) \le \beta \|u - v\|. \tag{3.7}$$

Using the locally relaxed φ-accretivity and locally D-Lipschitz continuity of $T_\lambda(\cdot)$, we have

$$\begin{aligned} \|u - v - \rho(x_\lambda(u) - x_\lambda(v))\|^q &\le \|u-v\|^q - q\rho\langle x_\lambda(u) - x_\lambda(v), j_q(u-v)\rangle \\ &\quad + c_q \rho^q \|x_\lambda(u) - x_\lambda(v)\|^q \\ &\le \|u-v\|^q - q\rho(-\varphi\|u-v\|^q) + c_q\rho^q\beta^q\|u-v\|^q \\ &\le (1 + q\rho\varphi + c_q\rho^q\beta^q)\|u-v\|^q. \end{aligned} \tag{3.8}$$

From (3.6) and (3.8), we have

$$\|F_\lambda(z_1) - F_\lambda(z_2)\| \le \sqrt[q]{1 + q\rho\varphi + c_q\rho^q\beta^q}\|u - v\|. \tag{3.9}$$

Also from (3.2) and locally Lipschitz continuity of projection operator $P_{\mathcal{K}_r}$ with constant δ, we have

$$\begin{aligned} \|u-v\| &\leq \|u-v-(h_\lambda(u)-h_\lambda(v))\| + \|P_{\mathcal{K}_r}(z_1)-P_{\mathcal{K}_r}(z_2)\| \\ &\leq \|u-v-(h_\lambda(u)-h_\lambda(v))\| + \delta\|z_1-z_2\|. \end{aligned} \tag{3.10}$$

Since h_λ is a locally Lipschitz continuous with constant $\gamma > 0$ and locally strongly accretive mapping with constant $\xi > 0$, we have

$$\begin{aligned} \|u-v-(h_\lambda(u)-h_\lambda(v))\|^q &\leq \|u-v\|^q - q\langle h_\lambda(u)-h_\lambda(v), j_q(u-v)\rangle \\ &\quad + c_q\|h_\lambda(u)-h_\lambda(v)\|^q \\ &\leq \|u-v\|^q - q\xi\|u-v\|^q + c_q\gamma^q\|u-v\|^q \\ &\leq (1-q\xi+c_q\gamma^q)\|u-v\|^q. \end{aligned}$$

It implies that

$$\|u-v-(h_\lambda(u)-h_\lambda(v))\| \leq \sqrt[q]{1-q\xi+c_q\gamma^q}\|u-v\|. \tag{3.11}$$

From (3.10) and (3.11), we have

$$\|u-v\| \leq \kappa\|u-v\| + \delta\|z_1-z_2\|,$$

where $\kappa = \sqrt[q]{1-q\xi+c_q\gamma^q}$. From which we have

$$\|u-v\| \leq \frac{\delta}{1-\kappa}\|z_1-z_2\|. \tag{3.12}$$

Combining (3.9),(3.12) and (3.3), we have

$$\begin{aligned} \|F_\lambda(z_1)-F_\lambda(z_2)\| &\leq (1-\alpha)\|z_1-z_2\| + \alpha\delta\frac{\sqrt[q]{1+q\rho\varphi+c_q\rho^q\beta^q}}{1-\kappa}\|z_1-z_2\| \\ &= (1-\alpha)\|z_1-z_2\| + \alpha\theta\|z_1-z_2\|. \end{aligned} \tag{3.13}$$

It follows from (3.4) that $\theta < 1$. Hence the mapping $F_\lambda(z)$ defined by (3.1) is contractive and has a fixed point $z(\lambda)$ which is the solution of parametric general Wiener-Hopf equation (2.5).

Remark 3.2. From Lemma 3.1, we see that the map $F_\lambda(z)$ defined by (2.4) has a unique fixed point $z(\lambda)$, that is, $z(\lambda) = F_\lambda(z)$. Also by assumptions, the function $\overline{z}$ for $\lambda = \overline{\lambda}$ is a solutions of parametric general Wiener-Hopf equation (2.5). Again by Lemma 3.1, we know that $\overline{z}$ for $\lambda = \overline{\lambda}$ is a fixed point of $F_\lambda(z)$ and it is also a fixed point of $F_{\overline{\lambda}}(z)$. Consequently, we conclude that

$$z(\overline{\lambda}) = \overline{z} = F_{\overline{\lambda}}(z(\overline{\lambda})).$$

Using Lemma 3.1, we can prove the continuity of the solution $z(\lambda)$ of parametric general Wiener-Hopf equation (2.5). However for the sake of completeness and to convey the idea of the technique involved, we give the proof.

Lemma 3.3. *Assume that the operator $T_\lambda(\cdot)$ is locally D-Lipschitz continuous with respect to the parameter λ and $h_\lambda(\cdot)$ is a locally Lipschitz continuous mapping. If $T_\lambda(\cdot)$ is a locally Lipschitz continuous mapping and the mapps $\lambda \to P_{\mathcal{K}_r\lambda}z, \lambda \to h_\lambda(u), \lambda \to T_\lambda(u)$ are continuous (or Lipschitz continuous), then the function $z(\lambda)$ satisfying (3.3) is (Lipschitz) continuous at $\lambda = \overline{\lambda}$.*

Proof. For all $\lambda \in \Gamma$ invoking Lemma 3.1 and the triangle inequality, we have

$$\begin{aligned} \|z(\lambda) - z(\overline{\lambda})\| &\leq \|F_\lambda(z(\lambda)) - F_\lambda(z(\overline{\lambda}))\| + \|F_\lambda(z(\overline{\lambda})) - F_{\overline{\lambda}}(z(\overline{\lambda}))\| \\ &\leq \theta\|z(\lambda) - z(\overline{\lambda})\| + \|F_\lambda(z(\overline{\lambda})) - F_{\overline{\lambda}}(z(\overline{\lambda}))\|. \end{aligned} \tag{3.14}$$

From (3.1) and the fact that the operator $T_\lambda(\cdot)$ is locally D-Lipschitz continuous with respect to the parameter λ, we have

$$\begin{aligned} \|F_\lambda(z(\overline{\lambda})) - F_{\overline{\lambda}}(z(\overline{\lambda}))\| &= \|u(\overline{\lambda}) - u(\overline{\lambda}) - \rho(T_\lambda(u(\overline{\lambda})) - T_{\overline{\lambda}}(u(\overline{\lambda})))\| \\ &\leq \rho\beta\|\lambda - \overline{\lambda}\|. \end{aligned} \tag{3.15}$$

Combining (3.14) and (3.15), we obtain

$$\|z(\lambda) - z(\overline{\lambda})\| \leq \frac{\rho\beta}{1-\theta}\|\lambda - \overline{\lambda}\|, \quad \forall \lambda, \overline{\lambda} \in \Gamma.$$

This completes the proof.

Now, we are in a position to state and prove the main result of this paper.

Theorem 3.4. *Let $\overline{u}$ be a solution of parametric general nonlinear nonconvex set-valued variational inequality (2.4) and $\overline{z}$ be a solution of parametric general Wiener-Hopf equation (2.5) for $\lambda = \overline{\lambda}$. Let $h_\lambda(u)$ be a locally strongly accretive and locally Lipschitz continuous mapping. Let $T_\lambda(u)$ be a locally D-Lipschitz continuous and locally relaxed φ-accretive mapping with respect to $\varphi > 0$ for all $u \in \mathcal{B}$. If the mapps $\lambda \to P_{\mathcal{K}_r}, \lambda \to h_\lambda(u), \lambda \to T_\lambda(u)$ are Lipschitz (continuous) at $\lambda = \overline{\lambda}$, then there exists a neighbourhood $\mathcal{M}$ of Γ of $\overline{\lambda}$ such that for $\lambda \in \mathcal{M}$, parametric general Wiener-Hopf equation (2.5) has a unique solution $z(\lambda)$ in the interior of $\mathcal{B}, z(\overline{\lambda}) = \overline{z}$ and $z(\lambda)$ is (Lipschitz) continuous at $\lambda = \overline{\lambda}$.*

Poof. The proof follows from Lemma 3.1, 3.3 and Remark 3.2.

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Common Fixed Point Theorems for Non-compatible Self-mappings in b-Fuzzy Metric Spaces

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Abstract. By using R-weak commutativity of type (A_g) and non-compatible conditions of self-mapping pairs in a b-fuzzy metric space, without the conditions for the completeness of space and the continuity of mappings, we establish some new common fixed point theorems for two self-mappings. An example is provided to support our new result.

Keywords: b-fuzzy metric space, common fixed point theorem, R-weakly commuting mappings of type (A_g), non-compatible mapping pairs.

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1 Introduction and Preliminaries

The concept of fuzzy sets was introduced initially by Zadeh [17] in 1965. Since then, to use this concept in topology and analysis, many authors have expansively developed the theory of fuzzy sets and applications. George and Veeramani [5], Kramosil and Michalek [7] have introduced the concept of fuzzy topological spaces induced by fuzzy metric which have very important applications in quantum particle physics, particularly in connections with both string and E-infinity theory which were given and studied by El Naschie [1-4]. Many authors [6,9,10,13-15] have proved fixed point theorems in fuzzy (probabilistic) metric spaces.

Definition 1.1. A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if it satisfies the following conditions:

(1) $*$ is associative and commutative,

(2) $*$ is continuous,

(3) $a * 1 = a$, for all $a \in [0,1]$,

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(4) $a * b \leq c * d$, whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$.

Two typical examples of a continuous t-norm are $a * b = ab$ and $a * b = \min(a, b)$.

Definition 1.2. [11] A 3-tuple $(X, M, *)$ is called a *fuzzy metric space* if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,

(1) $M(x, y, t) > 0$,

(2) $M(x, y, t) = 1$ if and only if $x = y$,

(3) $M(x, y, t) = M(y, x, t)$,

(4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(5) $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous.

Definition 1.3. [11] A 3-tuple $(X, M, *)$ is called a b-fuzzy metric space for $b \geq 1$ if X is an arbitrary nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,

(1) $M(x, y, t) > 0$,

(2) $M(x, y, t) = 1$ if and only if $x = y$,

(3) $M(x, y, t) = M(y, x, t)$,

(4) $M(x, y, \frac{t}{b}) * M(y, z, \frac{s}{b}) \leq M(x, z, t + s)$,

(5) $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous.

It should be noted that, the class of b-fuzzy metric spaces is effectively larger than that of fuzzy metric spaces, since a b-fuzzy metric is a fuzzy metric when $b = 1$.

We present an example shows that a b-fuzzy metric on X need not be a fuzzy metric on X.

Example 1.4. Let $M(x, y, t) = e^{\frac{-|x-y|^p}{t}}$, where $p > 1$ is a real number. We show that M is a b-fuzzy metric with $b = 2^{p-1}$. In fact, obviously conditions (1),(2),(3) and (5) of definition 1.3 are satisfied. Let $f(x) = x^p$ $(x > 0)$. Then we know that it is a convex function, for $1 < p < \infty$. So, we have

$$\left(\frac{a+c}{2}\right)^p \leq \frac{1}{2}\left(a^p + c^p\right),$$

it implies that $(a + c)^p \leq 2^{p-1}(a^p + c^p)$. Therefore, we have

$$\begin{aligned}
\frac{|x-y|^p}{t+s} &\leq 2^{p-1}\frac{|x-z|^p}{t+s} + 2^{p-1}\frac{|z-y|^p}{t+s} \\
&\leq 2^{p-1}\frac{|x-z|^p}{t} + 2^{p-1}\frac{|z-y|^p}{s} \\
&= \frac{|x-z|^p}{t/2^{p-1}} + \frac{|z-y|^p}{s/2^{p-1}}.
\end{aligned}$$

Thus, for each $x, y, z \in X$ we obtain

$$\begin{aligned} M(x,y,t+s) &= e^{\frac{-|x-y|^p}{t+s}} \\ &\geq M(x,z,\frac{t}{2^{p-1}}) * M(z,y,\frac{s}{2^{p-1}}), \end{aligned}$$

where $a * c = ac$ for all $a, c \in [0,1]$. So condition (4) of definition 1.3 is hold and M is a b-fuzzy metric.

It should be noted that in preceding example, for $p = 2$ it is easy to see that $(X, M, *)$ is not a fuzzy metric space.

Example 1.5. Let $M(x,y,t) = e^{\frac{-d(x,y)}{t}}$ or $M(x,y,t) = \frac{t}{t+d(x,y)}$, where d is a b-metric on X and $a * c = ac$ for all $a, c \in [0,1]$. Then it is easy to show that M is a b-fuzzy metric. In fact, obviously conditions (1),(2),(3) and (5) of definition 1.3 are satisfied. Since d is a b-metric, for each $x, y, z \in X$ we have

$$d(x,y) \leq b\big[d(x,z) + d(z,y)\big].$$

Therefore, we obtain

$$\begin{aligned} M(x,y,t+s) &= e^{\frac{-d(x,y)}{t+s}} \\ &\geq e^{-b\frac{d(x,z)+d(z,y)}{t+s}} \\ &= \left(e^{-b\frac{d(x,z)}{t+s}}\right)\left(e^{-b\frac{d(z,y)}{t+s}}\right) \\ &\geq \left(e^{\frac{-d(x,z)}{t/b}}\right)\left(e^{\frac{-d(z,y)}{s/b}}\right) \\ &= M(x,z,\frac{t}{b}) * M(z,y,\frac{s}{b}). \end{aligned}$$

So condition (4) of definition 1.3 is hold and M is a b-fuzzy metric. Similarly, we can show that $M(x,y,t) = \frac{t}{t+d(x,y)}$ is also a b-fuzzy metric.

Next, we need the following definitions and propositions in b-metric spaces for our main theorems.

Definition 1.6. Let $f : R \to R$ be a function. Then f is called b-nondecreasing, if $x > by$ implies that $f(x) \geq f(y)$ for each $x, y \in R$.

Lemma 1.7. [11] *Let $(X, M, *)$ be a b-fuzzy metric space. Then $M(x,y,t)$ is b-nondecreasing with respect to t, for all x, y in X. Also,*

$$M(x,y,b^n t) \geq M(x,y,t), \forall n \in N.$$

Let $(X, M, *)$ be a b-fuzzy metric space. For $t > 0$, the open ball $B(x,r,t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B(x,r,t) = \big\{y \in X : M(x,y,t) > 1 - r\big\}.$$

We recall the notions of convergence and completeness in a $b-$fuzzy metric space.

Let $(X, M, *)$ be a b-fuzzy metric space. Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exists $t > 0$ and $0 < r < 1$ such that $B(x,r,t) \subset A$. Then

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τ is a topology on X (induced by the b-fuzzy metric M). A sequence $\{x_n\}$ in X converges to x if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$, for each $t > 0$. It is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \varepsilon$, for each $n, m \geq n_0$. The b-fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence is convergent. A subset A of X is said to be F-bounded if there exists $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1 - r$, for all $x, y \in A$.

Lemma 1.8. [11] *Let $(X, M, *)$ be a b-fuzzy metric space. Then the following assertions hold:*
(i) *If sequence $\{x_n\} \subset X$ converges to x, then x is unique,*
(ii) *The convergent sequence $\{x_n\} \subset X$ is Cauchy.*

We have the following propositions in a b-fuzzy metric space.

Proposition 1.9. [11] *Let $(X, M, *)$ be a b-fuzzy metric space and suppose that $\{x_n\}$ and $\{y_n\}$ are convergent to x, y respectively. Then we have*

$$M(x, y, \frac{t}{b^2}) \leq \limsup_{n\to\infty} M(x_n, y_n, t) \leq M(x, y, b^2 t)$$

and

$$M(x, y, \frac{t}{b^2}) \leq \liminf_{n\to\infty} M(x_n, y_n, t) \leq M(x, y, b^2 t).$$

Proposition 1.10. [12] *Let $(X, M, *)$ be a b-fuzzy metric space and suppose that $\{x_n\}$ is convergent to x. Then, for all $y \in X$ we have*

$$M(x, y, \frac{t}{b}) \leq \limsup_{n\to\infty} M(x_n, y, t) \leq M(x, y, bt)$$

and

$$M(x, y, \frac{t}{b}) \leq \liminf_{n\to\infty} M(x_n, y, t) \leq M(x, y, bt).$$

Lemma 1.11. *A b-fuzzy metric is not continuous in general.*

Throughout, in this paper we assume that $\lim_{t\to\infty} M(x, y, t) = 1$.

Lemma 1.12. *Let $(X, M, *)$ be a b-fuzzy metric space and suppose that $M(x, y, kt) \geq M(x, y, t)$, for all $x, y \in X$, $0 < k < 1$ and $t > 0$. Then $x = y$.*

Proof. Since, $M(x, y, kt) \geq M(x, y, t)$, it follows that

$$M(x, y, t) \geq M(x, y, \frac{t}{k}) \geq \cdots \geq M(x, y, \frac{t}{k^n}).$$

Hence, we can get $M(x, y, t) \geq \lim_{n\to\infty} M(x, y, \frac{t}{k^n}) = 1$, therefore, $x = y$.

In 2010, Vats et al. [16] introduced the concept of weakly compatible. Also, in 2010, Manro et al. [8] introduced the concepts of weakly commuting, R-weakly commuting mappings, and R-weakly commuting mappings of type (P), (A_f), and (A_g) in a G-metric space.

We will introduce these concepts in a b-fuzzy metric space.

Definition 1.13. The self-mappings f and g of a b-fuzzy metric space $(X, M, *)$ are said to be compatible if

$$\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$$

and

$$\lim_{n\to\infty} M(gfx_n, fgx_n, t) = 1,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$, for some $z \in X$.

Definition 1.14. A pair of self-mappings (f, g) of a b-fuzzy metric space $(X, M, *)$ is said to be

(1) weakly commuting if $M(fgx, gfx, t) \geq M(fx, gx, t)$, for all $x \in X$.

(2) R-weakly commuting if there exists some positive real number R such that $M(fgx, gfx, t) \geq M(fx, gx, \frac{t}{R})$, $\quad \forall x \in X$.

Remark 1.15. If $R \leq 1$, then R-weakly commuting mappings are weakly commuting.

Definition 1.16. A pair of self-mappings (f, g) of a b-fuzzy metric space $(X, M, *)$ are said to be

(1) R-weakly commuting mappings of type (A_f) if there exists some positive real number R such that $M(fgx, ggx, t) \geq M(fx, gx, \frac{t}{R})$, for all $x \in X$.

(2) R-weakly commuting mappings of type (A_g) if there exists some positive real number R such that $M(gfx, ffx, t) \geq M(gx, fx, \frac{t}{R})$, for all $x \in X$.

(3) R-weakly commuting mappings of type (P) if there exists some positive real number R such that $M(ffx, ggx, t) \geq M(fx, gx, \frac{t}{R})$, for all $x \in X$.

Remark 1.17. The self-mapping f of a b-fuzzy metric space $(X, M, *)$ is said to be b-continuous at $x \in X$ if and only if it is b-sequentially continuous at x, that is, whenever $\{x_n\}$ is b-convergent to x, $\{f(x_n)\}$ is b-convergent to $f(x)$.

Example 1.18. Let $M(x, y, t) = e^{\frac{-|x-y|^2}{t}}$, $fx = 1$ and

$$gx = \begin{cases} 1, & x \in Q, \\ -1, & otherwise, \end{cases}$$

for each $x, y \in R$, where $a * c = ac$. Then it is easy to see that a pair of self-mappings (f, g) of a b-fuzzy metric space is weakly commuting, R-weakly commuting, and R-weakly commuting of type (P), (A_f), and (A_g).

2 The Main Results

Now we are in a position to introduce the main results of this paper.

Theorem 2.1. *Let $(X, M, *)$ be a b-fuzzy metric space and (f, g) be a pair of non-compatible self-mappings with $\overline{fX} \subseteq gX$ ($\overline{fX}$ denotes the closure of fX). Assume*

that the following condition is satisfied:

$$M(fx, fy, kt) \geq \min\{M(gx, gy, b^2t), M(fx, gx, b^2t), M(fy, gy, b^2t)\}, \tag{2.1}$$

for all $x, y \in X$ and $0 < k < 1$. If (f, g) is a pair of R-weakly commuting mappings of type (A_g), then f and g have a unique common fixed point (say z) and both f and g are not b-continuous at z.

Proof. Since f and g are non-compatible mappings, there exists a sequence $\{x_n\} \subset X$, such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z, \quad z \in X,$$

but either $\lim_{n\to\infty} M(fgx_n, gfx_n, t)$ or $\lim_{n\to\infty} M(gfx_n, fgx_n, t)$ does not exist or exists and is different from 1. Since $z \in \overline{fX} \subset gX$, there must exist a $u \in X$ satisfying $z = gu$. We can assert that $fu = gu$. If not, from condition (2.1) and Propsition 1.10, we obtain

$$\begin{aligned}
& M(fu, gu, bkt) \\
\geq\ & \limsup_{n\to\infty} M(fu, fx_n, kt) \\
\geq\ & \limsup_{n\to\infty} \min\left\{M(gu, gx_n, b^2t), M(fu, gx_n, b^2t), M(fx_n, gu, b^2t)\right\} \\
\geq\ & \min\left\{M(gu, gu, bt), M(fu, gu, bt), M(fu, gu, bt)\right\} \\
=\ & M(fu, gu, bt),
\end{aligned}$$

that is, $M(fu, gu, kt) \geq M(fu, gu, t)$. Hence, by Lemma 1.12, we get $fu = gu$. Since (f, g) is a pair of R-weakly commuting mappings of type (A_g), we have

$$M(gfu, ffu, t) \geq M(gu, fu, \frac{t}{R}) = 1.$$

It means that $ffu = gfu$. Next, we prove $ffu = fu$. From condition (2.1), $fu = gu$ and $ffu = gfu$, we have

$$\begin{aligned}
M(fu, ffu, kt) &\geq \min\{M(gu, gfu, b^2t), M(fu, gfu, b^2t), M(gu, ffu, b^2t)\} \\
&= M(fu, ffu, b^2t) \\
&\geq M(fu, ffu, t).
\end{aligned}$$

From Lemma 1.12, we have $fu = ffu$, which implies that $fu = ffu = gfu$, and so $z = fu$ is a common fixed point of f and g.

Next we prove that the common fixed point z is unique. Actually, suppose that w is also a common fixed point of f and g. Then using the condition (2.1), we have

$$\begin{aligned}
M(z, w, kt) &= M(fz, fw, kt) \\
&\geq \min\{M(gz, gw, b^2t), M(fz, gw, b^2t), M(fw, gz, b^2t)\} \\
&= M(z, w, b^2t) \\
&\geq M(z, w, t),
\end{aligned}$$

which implies that $z = w$, so that uniqueness is proved.

Now, we prove that f and g are not b-continuous at z. In fact, if f is b-continuous at z, then for the b-convergent sequence $\{x_n\}$ to x, we have

$$\lim_{n\to\infty} ffx_n = fz = z \text{ and } \lim_{n\to\infty} fgx_n = fz = z.$$

Since f and g are R-weakly commuting mappings of type (A_g), we get

$$M(gfx_n, ffx_n, t) \geq M\big(gx_n, fx_n, \frac{t}{R}\big).$$

Hence, by Proposition 1.9, we have

$$\begin{aligned} M(\lim_{n\to\infty} gfx_n, z, b^2t) &\geq \limsup_{n\to\infty} M(gfx_n, ffx_n, t) \\ &\geq \limsup_{n\to\infty} M\big(gx_n, fx_n, \frac{t}{R}\big) \\ &\geq M\big(z, z, \frac{t}{Rb^2}\big) = 1, \end{aligned}$$

it follows that $\lim_{n\to\infty} gfx_n = z$. Hence, we can get

$$\lim_{n\to\infty} M(fgx_n, gfx_n, t) \geq M(z, z, \frac{t}{b^2}) = 1.$$

Therefore, we have

$$\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1.$$

This contradicts with f and g being non-compatible. So f is not b-continuous at z. If g is b-continuous at z, then for the b-convergent sequence $\{x_n\}$ to x, we have

$$\lim_{n\to\infty} gfx_n = gz = z \text{ and } \lim_{n\to\infty} ggx_n = gz = z.$$

Since f and g are R-weakly commuting mappings of type (A_g), we get

$$M(gfx_n, ffx_n, t) \geq M\big(gx_n, fx_n, \frac{t}{R}\big).$$

Hence, we have

$$\begin{aligned} M(z, \lim_{n\to\infty} ffx_n, b^2t) &\geq \limsup_{n\to\infty} M(gfx_n, ffx_n, t) \\ &\geq \limsup_{n\to\infty} M\big(gx_n, fx_n, \frac{t}{R}\big) \\ &\geq M\big(z, z, \frac{t}{Rb^2}\big) = 1, \end{aligned}$$

it implies that

$$\lim_{n\to\infty} ffx_n = z = fz.$$

This contradicts with f being not b-continuous at z, which implies that g is not b-continuous at z. This completes the proof.

For the case $b = 1$ in Theorem 2.1, we have the following corollary.

Corollary 2.2. *Let $(X, M, *)$ be a fuzzy metric space and (f, g) be a pair of non-compatible selfmappings with $\overline{fX} \subseteq gX$. Assume that the following condition is satisfied:*

$$M(fx, fy, kt) \geq \min\big\{M(gx, gy, t), M(fx, gx, t), M(fy, gy, t)\big\}, \tag{2.2}$$

for all $x, y \in X$ and $0 < k < 1$. If (f, g) is a pair of R-weakly commuting mappings of type (A_g), then f and g have a unique common fixed point (say z) and both f and g are not b-continuous at z.

3 Example

Next, we give an example to support for the main Theorem 2.1.

Example 3.1. Let $X = [2, 20]$ and $a * c = ac$, for all $a, c \in [0, 1]$ and let M be a fuzzy set on $X \times X \times (0, +\infty)$ defined as follows:

$$M(x, y, t) = e^{\frac{-|x-y|}{t}},$$

for all $t \in R^+$. Then $(X, M, *)$ is a fuzzy metric space. We define mappings f and g on X by

$$fx = \begin{cases} 2, & x = 2 \ or \ x \in (5, 20], \\ 6, & x \in (2, 5] \end{cases}$$

and

$$gx = \begin{cases} 2, & x = 2, \\ 18, & x \in (2, 5], \\ \frac{x+1}{3}, & x \in (5, 20]. \end{cases}$$

Clearly, from the above definitions, we know that $\overline{f(X)} \subseteq g(X)$, and (f, g) is a pair of non-compatible self-mappings. To see that f and g are non-compatible, consider a sequence $\{x_n\} = \{5 + \frac{1}{n}\}$. Then we have $fx_n \to 2, gx_n \to 2, fgx_n \to 6$ and $gfx_n \to 2$. Thus

$$\lim_{n\to\infty} M(gfx_n, fgx_n, t) = e^{-\frac{4}{t}} \neq 1.$$

On the other hand, there exists $R = 1$ such that

$$M(gfx, ffx, t) = \begin{cases} e^{-\frac{(2-2)}{t}}, & x = 2, \\ e^{-\frac{(\frac{7}{3}-2)}{t}}, & x \in (2, 5], \\ e^{-\frac{(2-2)}{t}}, & x \in (5, 20] \end{cases}$$

and

$$M(fx, gx, t) = \begin{cases} e^{-\frac{(2-2)}{t}}, & x = 2, \\ e^{-\frac{(18-6)}{t}}, & x \in (2, 5], \\ e^{-\frac{(\frac{x+1}{3}-2)}{t}}, & x \in (5, 20], \end{cases}$$

for all $x \in X$. Hence, it is easy to see that in every cases, we have

$$M(gfx, ffx, t) \geq M(gx, fx, t).$$

That is, (f, g) is a pair of R-weakly commuting mappings of type (A_g).

Now we prove that the mappings f and g satisfy the condition (2.1) of Theorem 2.1 with $k = \frac{1}{2}$. To do this, we consider the following cases:

Case (1) If $x, y \in \{2\} \cup (5, 20]$, then we have

$$\begin{aligned} M(fx, fy, kt) &= M(2, 2, kt) = 1 \\ &\geq \min\left\{M(gx, gy, t), M(fx, gx, t), M(fy, gy, t)\right\}, \end{aligned}$$

and hence (2.1) is obviously satisfied.

Case (2) If $x, y \in (2, 5]$, then we have

$$\begin{aligned} M(fx, fy, kt) &= M(6, 6, kt) = 1 \\ &\geq \min\left\{M(gx, gy, t), M(fx, gx, t), M(fy, gy, t)\right\}, \end{aligned}$$

and hence (2.1) is obviously satisfied.

Case (3) If $x \in \{2\} \cup (5, 20]$ and $y \in (2, 5]$, then we have

$$M(fx, fy, kt) = M(2, 6, kt) = e^{-\frac{4}{kt}}$$

and

$$M(gx, gy, t) = \begin{cases} e^{-\frac{|2-18|}{t}}, & x = 2, \\ e^{-\frac{|\frac{x+1}{3}-18|}{t}}, & x \in (5, 20]. \end{cases}.$$

Thus we obtain

$$M(fx, fy, t) \geq \min\left\{M(gx, gy, t), M(fx, gx, t), M(fy, gy, t)\right\},$$

for all x, y in X. Thus all the conditions of Theorem 2.1 are satisfied and 2 is a uniquecommon fixed point of f and g.

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On hesitant fuzzy filters in BE-algebras

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Abstract. The notions of hesitant fuzzy subalgebras and hesitant fuzzy filters are introduced and related properties are investigated. Relations between a hesitant fuzzy subalgebras and a hesitant fuzzy filters are discussed. The problem of classifying hesitant fuzzy filters by their γ-inclusive filter will be solved. Given a special set, we provide conditions for this set to be a hesitant fuzzy filter.

1. **Introduction**

In 2007, Kim and Kim [4] introduced the notion of a BE-algebra, and investigated several properties. In [1], Ahn and So introduced the notion of ideals in BE-algebras. They gave several descriptions of ideals in BE-algebras. Song et al. [7] considered the fuzzification of ideals in BE-algebras. They introduced the notion of fuzzy ideals in BE-algebras, and investigated related properties. They gave characterizations of a fuzzy ideal in BE-algebras.

The notions of Atanassov's intuitionistic fuzzy sets, type 2 fuzzy sets and fuzzy multisets etc. are a generalization of fuzzy sets. As another generalization of fuzzy sets, Torra [8] introduced the notion of hesitant fuzzy sets which are a very useful to express peoples hesitancy in daily life. The hesitant fuzzy set is a very useful tool to deal with uncertainty, which can be accurately and perfectly described in terms of the opinions of decision makers. Also, hesitant fuzzy set theory is used in decision making problem etc. (see [6, 10, 11, 12, 13, 14]), and is applied to residuated lattices and MTL-algebras (see [3, 5]).

In this paper, we introduce the notions of hesitant fuzzy subalgebras and hesitant fuzzy filters of BE-algebras, and investigate their relations and properties. We consider characterizations of hesitant fuzzy fuzzy subalgebras and hesitant fuzzy filters of BE-algebras. Given a special set, we provide conditions for this set to be a hesitant fuzzy filter. Given a special set, we provide conditions for this set to be a hesitant fuzzy filter.

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2. Preliminaries

Let $K(\tau)$ be the class of all algebras of type $\tau = (2, 0)$. By a *BE-algebra* we mean a system $(X; *, 1) \in K(\tau)$ in which the following axioms hold (see [4]):

$$(\forall x \in X)\,(x * x = 1), \tag{2.1}$$

$$(\forall x \in X)\,(x * 1 = 1), \tag{2.2}$$

$$(\forall x \in X)\,(1 * x = x), \tag{2.3}$$

$$(\forall x, y, z \in X)\,(x * (y * z) = y * (x * z)). \quad \text{(exchange)} \tag{2.4}$$

A relation "$\leq$" on a BE-algebra X is defined by

$$(\forall x, y \in X)\,(x \leq y \iff x * y = 1). \tag{2.5}$$

A BE-algebra $(X; *, 1)$ is said to be *transitive* (see [1]) if it satisfies:

$$(\forall x, y, z \in X)\,(y * z \leq (x * y) * (x * z)). \tag{2.6}$$

A BE-algebra $(X; *, 1)$ is said to be *self distributive* (see [4]) if it satisfies:

$$(\forall x, y, z \in X)\,(x * (y * z) = (x * y) * (x * z)). \tag{2.7}$$

Every self distributive BE-algebra $(X; *, 1)$ satisfies the following properties:

$$(\forall x, y, z \in X)\,(x \leq y \;\Rightarrow\; z * x \leq z * y \text{ and } y * z \leq x * z)\,, \tag{2.8}$$

$$(\forall x, y \in X)\,(x * (x * y) = x * y)\,, \tag{2.9}$$

$$(\forall x, y, z \in X)\,(x * y \leq (z * x) * (z * y))\,. \tag{2.10}$$

Note that every self distributive BE-algebra is transitive, but the converse is not true in general (see [1]).

Definition 2.1. ([4]) Let $(X; *, 1)$ be a BE-algebra and let F be a non-empty subset of X. Then F is a *filter* of X if

(F1) $1 \in F$;
(F2) $(\forall x, y \in X)(x * y, x \in F \Rightarrow y \in F)$.

3. Hesitant fuzzy filters

Definition 3.1. ([8]) Let E be a reference set. A *hesitant fuzzy set* on E is defined in terms of a function that when applied to E returns a subset of $[0, 1]$, which can be viewed as the following mathematical representation:

$$H_E := \{(e, h_E(e)) | e \in E\}$$

where $h_E : E \to \mathscr{P}([0, 1])$.

Definition 3.2. Given a non-empty subset A of X, a hesitant fuzzy set

$$H_X := \{(x, h_X(x)) | x \in X\}$$

on satisfying the following condition:

$$h_X(x) = \emptyset \text{ for all } x \notin A \tag{3.1}$$

is called a *hesitant fuzzy set related to* A (briefly, A-*hesitant fuzzy set*) on X, and is represented by $H_A := \{(x, h_A(x)) \mid x \in X\}$, where h_A is a mapping from X to $\mathscr{P}([0,1])$ with $h_A(x) = \emptyset$ for all $x \notin A$.

Definition 3.3. Given a non-empty subset (subalgebra as much as possible) A of X, let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy set on X. Then $H_A := \{(x, h_A(x)) \mid x \in X\}$ is called a *hesitant fuzzy subalgebra of* X *related to* A (briefly, A-*hesitant fuzzy subalgebra* of X) if it satisfies the following condition:

$$(\forall x, y \in A)\,(h_A(x * y) \supseteq h_A(x) \cap h_A(y))\,. \tag{3.2}$$

An A-hesitant fuzzy subalgebra of X with $A = X$ is called a *hesitant fuzzy subalgebra* of X.

Example 3.4. Let $X = \{0, 1, a, b, c\}$ be a BE-algebra with the following Cayley table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

For a subalgebra $A = \{1, a, b\}$ of X, let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy set on X defined by

$$H_A = \left\{(1, [0, 1]), (a, (0, \tfrac{1}{2}]), (b, (\tfrac{1}{4}, \tfrac{3}{4}]), (c, \emptyset)\right\}$$

Then H_A is an A-hesitant fuzzy subalgebra of X.

Definition 3.5. Given a non-empty subset (subalgebra as much as possible) A of X, let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy set on X. Then $H_A := \{(x, h_A(x)) \mid x \in X\}$ is called a *hesitant fuzzy filter of* X *related to* A (briefly, A-*hesitant fuzzy filter* of X) if it satisfies the following condition:

$$(\forall x \in A)\,(h_A(x) \subseteq h_A(1))\,, \tag{3.3}$$

$$(\forall x, y \in A)\,(h_A(x * y) \cap h_A(x) \subseteq h_A(y))\,. \tag{3.4}$$

An A-hesitant fuzzy filter of X with $A = X$ is called a *hesitant fuzzy filter* of X.

Example 3.6. (1) Consider a BE-algebra $X = \{1, a, b, c\}$ as in Example 3.4. Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H_X = \{(1, [0,1]), (a, (0, \tfrac{1}{8})), (b, (\tfrac{1}{4}, \tfrac{3}{4}])), (c, (0, \tfrac{1}{4}))\}$$

Then H_X is a hesitant fuzzy subalgebra of X, but not a hesitant fuzzy filter of X since $h_A(b * a) \cap h_A(b) = h_A(1) \cap h_A(b) = [0,1] \cap (\frac{1}{4}, \frac{3}{4}] \nsubseteq h_A(a) = (0, \frac{1}{8})$.
(2) Let $X = \{0, 1, a, b, c\}$ be a BE-algebra with the following Cayley table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	a	a	1

Let $H_X := \{(x, h_X(x))|x \in X\}$ be a hesitant fuzzy set defined by

$$H_X = \{(1, [0,1]), (a, (0, \tfrac{1}{4})), (b, (0, \tfrac{1}{4})), (c, (0, \tfrac{1}{2}))\}$$

It is routine to verify that $H_X := \{(x, h_X(x))|x \in X\}$ is a hesitant fuzzy filter of X.

Proposition 3.7. *Let $H_A := \{(x, h_A(x))|x \in X\}$ be an A-hesitant fuzzy filter of X where A is a subalgebra of X. Then the following assertions are valid.*

(i) $(\forall x, y \in A)(x \leq y \Rightarrow h_A(x) \subseteq h_A(y))$,
(ii) $(\forall x, y, z \in A)(h_A(x * (y * z)) \cap h_A(y) \subseteq h_A(x * z))$,
(iii) $(\forall a, x \in A)(h_A(a) \subseteq h_A((a * x) * x)$.

Proof. Let $x, y \in A$ be such that $x \leq y$. Then $x * y = 1$. It follows from (3.3) and (3.4) that $h_A(x) = h_A(1) \cap h_A(x) = h_A(x * y) \cap h_A(x) \subseteq h_A(y)$.
(ii) Using (3.4) and (2.4), we have $h_A(x * z) \supseteq h_A(y * (x * z)) \cap h_A(y) = h_A(x * (y * z)) \cap h_A(y)$ for all $x, y, z \in A$.
(iii) Take $y := (a * x) * x$ and $x := a$ in (3.4). Then we have

$$\begin{aligned} h_A((a * x) * x)) &\supseteq h_A(a * ((a * x) * x)) \cap h_A(a) \\ &= h_A((a * x) * (a * x)) \cap h_A(a) \\ &= h_A(1) \cap h_A(a) = h_A(a) \end{aligned}$$

by using (2.4), (2.1) and (3.3). □

Corollary 3.8. *Every hesitant fuzzy filter $H_X := \{(x, h_X(x))|x \in X\}$ of X satisfies the following properties:*

(i) $(\forall x, y \in X)(x \leq y \Rightarrow h_X(x) \subseteq h_X(y))$,
(ii) $(\forall x, y, z \in X)(h_X(x * (y * z)) \cap h_X(y) \subseteq h_X(x * z))$,

(iii) $(\forall a, x \in X)(h_A(a) \subseteq h_A((a * x) * x)$.

We provide conditions for a hesitant fuzzy set to be a hesitant filter.

Theorem 3.9. *Let A be a subalgebra of a BE-algebra X. Every A-hesitant fuzzy set satisfies (3.3) and Proposition 3.7(ii). Then it is an A-hesitant fuzzy filter of X.*

Proof. Taking $x := 1$ in Proposition 3.7(ii) and using (2.3), we obtain $h_A(z) = h_A(1 * z) \supseteq h_A(1 * (y * z)) \cap h_A(y) = h_A(y * z) \cap h_A(y)$ for all $y, z \in A$. Hence $H_A = \{(x, h_A(x))|x \in X\}$ is an A-hesitant fuzzy filter of X. $\square$

Corollary 3.10. *Let $H_A := \{(x, h_A(x))|x \in X\}$ be an A-hesitant fuzzy set for a subalgebra A of X. Then h_A is an A-hesitant fuzzy filter of X if and only if it satisfies (3.3) and Proposition 3.7(ii).*

Theorem 3.11. *An hesitant fuzzy set H_A of X, where A is a subalgebra of X, is an A-hesitant fuzzy filter of X if and only if it satisfies the following conditions:*

(i) $(\forall x, y \in A)(h_A(y * x) \supseteq h_A(x))$,

(ii) $(\forall x, a, b \in A)(h_A((a * (b * x)) * x) \supseteq h_A(a) \cap h_A(b))$.

Proof. Assume that $H_A := \{(x, h_A(x))|x \in X\}$ is an A-hesitant fuzzy filter of X. Using (3.3), (3.4), (2.4),(2.1) and (2.2), we get $h_A(y*x) \supseteq h_A(x*(y*x)) \cap h_A(x) = h_A(1) \cap h_A(x) = h_A(x)$ for all $x, y \in A$. It follows from Proposition 3.7 that $h_A((a*(b*x))*x) \supseteq h_A((a*(b*x))*(b*x)) \cap h_A(b) \supseteq h_A(a) \cap h_a(b)$ for all $x, a, b \in X$.

Conversely, let $H_A(X) = \{(x, h_A(x))|x \in A\}$ be an A-hesitant fuzzy set of X satisfying conditions (i) and (ii). If we take $y := x$ in (i), then $h_A(1) = h_A(x * x) \supseteq h_A(x)$ for all $x \in A$. Using (ii), we obtain $h_A(y) = h_A(1 * y) = h_A(((x * y) * (x * y)) * y) \supseteq h_A(x * y) \cap h_A(x)$ for all $x, y \in A$. Hence H_A is an A-hesitant fuzzy filter of X. $\square$

Proposition 3.12. *Let $H_X := \{(x, h_X(x)|x \in X\}$ be a hesitant fuzzy set on X. Then H_X is a hesitant fuzzy filter of X if and only if*

$$(\forall x, y, z \in X)(z \leq x * y \Rightarrow h_X(y) \supseteq h_X(x) \cap h_X(z)). \tag{3.5}$$

Proof. Assume that H_X is a hesitant fuzzy filter of X. Let $x, y, z \in X$ be such that $z \leq x * y$. By Proposition 3.7 and Definition 3.5, we have $h_X(y) \supseteq h_X(x * y) \cap h_X(x) \supseteq h_X(z) \cap h_X(x)$.

Conversely, suppose that H_X satisfies (3.5). By (2.2), we have $x \leq x * 1 = 1$. Using (3.5), we have $h_X(1) \supseteq h_X(x)$ for all $x \in X$. It follows from (2.1) and (2.4) that $x \leq (x * y) * y$ for all $x, y \in X$. Using (3.5), we have $h_X(y) \supseteq h_X(x * y) \cap h_X(x)$. Therefore H_X is a hesitant fuzzy filter of X. $\square$

As a generalization of Proposition 3.12, we have the following results.

Theorem 3.13. *Let $H_X := \{(x, h_X(x)|x \in X\}$ be a hesitant fuzzy filter of X. Then*

$$\prod_{i=1}^{n} w_i * x = 1 \Rightarrow h_X(x) \supseteq \cap_{i=1}^{n} h_X(w_i) \tag{3.6}$$

for all $x, w_1, \cdots, w_n \in X$, where $\prod_{i=1}^{n} w_i * x = w_n * (w_{n-1} * (\cdots w_1 * x) \cdots))$.

Proof. The proof is by induction on n. Let H_X be a hesitant fuzzy filter of X. By Proposition 3.7(i) and (3.6), we know that the condition (3.6) is true for $n = 1, 2$. Assume that H_X satisfies the condition (3.6) for $n = k$, i.e., $\prod_{i=1}^{k} w_i * x = 1 \Rightarrow \cap_{i=1}^{k} h_X(w_i)$ for all $x, w_1, \cdots, w_k \in X$. Suppose that $\prod_{i=1}^{k+1} w_i * x = 1$ for all $x, w_1, \cdots, w_k, w_{k+1} \in X$. Then

$$h_X(w_1 * x) \supseteq \cap_{i=2}^{k+1} h_X(w_i).$$

Since H_X is a hesitant fuzzy filter of X, it follows form (3.4) that

$$\begin{aligned} h_X(x) &\supseteq h_X(w_1 * x) \cap h_X(w_1) \\ &\supseteq (\cap_{i=2}^{k+1} h_X(w_i)) \cap h_X(w_1) \\ &= \cap_{i=1}^{k+1} h_X(w_i). \end{aligned}$$

This completes the proof. □

Theorem 3.14. *Let $H_X = \{(x, h_X(x))|x \in X\}$ be a hesitant fuzzy set of a BE-algebra satisfying (3.6). Then H_X is a hesitant fuzzy filter of X.*

Proof. Let $x, y, z \in X$ be such that $z \leq x * y$. Then $z * (x * y) = 1$ and so $h_X(y) \supseteq h_X(x) \cap h_X(z)$ by (3.6). Using Proposition 3.12, H_X is a hesitant fuzzy filter of X. □

Theorem 3.15. *A hesitant fuzzy set $H_X := \{(x, h_X(x)|x \in X\}$ of a BE-algebra X is a hesitant fuzzy filter of X if and only if the set $H_X(\gamma) := \{x \in X | h_X(x) \supseteq \gamma\}$ is a filter of X for all $\gamma \in \mathscr{P}([0, 1])$ whenever it is nonempty.*

Proof. Suppose that H_X is a hesitant fuzzy filter of X. Let $x, y \in X$ and $\gamma \in \mathscr{P}([0, 1])$ be such that $x * y \in H_X(\gamma)$ and $x \in H_X(\gamma)$. Then $h_X(x * y) \supseteq \gamma$ and $h_X(x) \supseteq \gamma$. It follows from (3.3) and (3.4) that $h_X(1) \supseteq h_X(y) \supseteq h_X(x * y) \cap h_X(x) \supseteq \gamma$. Hence $1 \in H_X(\gamma)$ and $y \in H_X(\gamma)$, and therefore $H_X(\gamma)$ is a filter of X.

Conversely, assume that $H_X(\gamma)$ is a filter of X for all $\gamma \in \mathscr{P}([0, 1])$ with $H_X(\gamma) \neq \emptyset$. For any $x \in X$, let $h_X(x) = \gamma$. Then $x \in H_X(\gamma)$. Since $H_X(\gamma)$ is a filter of X, we have $1 \in h_X(\gamma)$ and so $h_X(x) = \gamma \subseteq h_X(1)$. For any $x, y \in X$, let $h_X(x * y) = \gamma_{x*y}$ and $h_X(x) = \gamma_x$. Take $x * y \in H_X(\gamma)$ and $x \in H_X(\gamma)$ which imply that $y \in H_X(\gamma)$. Hence $h_X(y) \supseteq \gamma = \gamma_{x*y} \cap \gamma_x = h_X(x * y) \cap h_X(x)$. Thus H_X is a hesitant fuzzy filter of X. □

The filter $H_X(\gamma)$ in Theorem 3.15 is called the *hesitant γ-inclusive set* of $H_X := \{(x, h_X(x))|x \in X\}$.

We make a new hesitant fuzzy filter from old one.

Theorem 3.16. *Let $H_X := \{(x, h_X(x))|x \in X\}$ be a hesitant fuzzy set on a BE-algebra X. Define a hesitant fuzzy set H_X^* on X by*

$$h_X^* : X \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} h_X(x) & \text{if } x \in H_X(\gamma) \\ \delta & \text{otherwise} \end{cases}$$

where γ is any subset of $[0,1]$ and δ is a subset of $[0,1]$ satisfying $\delta \subsetneq \cap_{x \notin H_X(\gamma)} h_X(x)$. If H_X is a hesitant fuzzy filter of X, then so is H_X^*.

Proof. Assume that H_X is a hesitant fuzzy filter of X. Then $H_X(\gamma)$ is a filter of X for all $\gamma \in \mathscr{P}([0,1])$ by Theorem 3.15. Hence $1 \in H_X(\gamma)$ and so $h_X^*(1) = h_X(1) \supseteq h_X(x) \supseteq h_X^*(x)$ for all $x \in X$. Let $x, y \in X$. If $x * y \in H_X(\gamma)$ and $x \in H_X(\gamma)$, then $y \in H_X(\gamma)$. Hence $h_X^*(y) = h_X(y) \supseteq h_X(x*y) \cap h_X(x) = h_X^*(x*y) \cap h_X^*(x)$. If $x * y \notin H_X(\gamma)$ or $x \notin H_X(\gamma)$, then $h_X^*(x*y) = \delta$ or $h_X^*(x) = \delta$. Thus $h_X^*(y) \supseteq \delta = h_X^*(x*y) \cap h_X^*(x)$. Therefore H_X^* is a hesitant fuzzy filter of X. □

For two elements a and B of X, consider a hesitant fuzzy set $H_X^{a,b} = \{(x, h_X(x))|x \in X\}$ where

$$h_X^{a,b} : X \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} \gamma_1 & \text{if } a*(b*x) = 1 \\ \gamma_2 & \text{otherwise} \end{cases}$$

where γ_1 and γ_2 are subsets of $[0,1]$ with $\gamma_2 \subsetneq \gamma_1$. In the following example, we know that there exist $a, b \in X$ such that $H_X^{a,b}$ is not a hesitant fuzzy filter of X.

Example 3.17. Let $X = \{0, 1, a, b, c\}$ be a BE-algebra with the following Cayley table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	c
b	1	1	1	c
c	1	a	b	1

Let $H_X := \{(x, h_X(x))|x \in X\}$ be a hesitant fuzzy set defined by

$$H_X = \left\{(1, [0,1]), (a, (0, \tfrac{1}{4})), (b, (\tfrac{1}{4}, \tfrac{3}{4})), (c, (\tfrac{6}{8}, \tfrac{7}{8}))\right\}$$

Then $H_X^{1,a}$ is not a hesitant fuzzy filter of X since $h_X^{1,a}(a*b) \cap h_X^{1,a}(a) = [0,1] \nsubseteq h_X^{1,a}(b) = (0, \frac{1}{4})$.

Now we provide a condition for the hesitant fuzzy set $H_X^{a,b}$ to be a hesitant fuzzy filter of X for all $a, b \in X$.

Theorem 3.18. *If X is a self distributive BE-algebra, then the hesitant fuzzy set $H_X^{a,b}$ is a hesitant fuzzy filter of X for all $a, b \in X$.*

Proof. Let $a, b \in X$. Obviously, $h_X^{a,b}(1) \supseteq h_X^{a,b}(x)$ for all $x \in X$. Let $x, y \in X$ be such that $a * (b * (x * y)) \neq 1$ or $a * (b * x) \neq 1$. Then $h_X^{a,b}(x * y) = \gamma_2$ or $h_X^{a,b}(x) = \gamma_2$. Hence $h_X^{a,b}(x * y) \cap h_X^{a,b}(x) = \gamma_2 \subseteq h_X^{a,b}(y)$. Assume that $a * (b * (x * y)) = 1$ and $a * (b * x) = 1$. Then

$$\begin{aligned} 1 &= a * (b * (x * y)) \\ &= a * ((b * x) * (b * y)) \\ &= (a * (b * x)) * (a * (b * y)) \\ &= 1 * (a * (b * y)) \\ &= a * (b * y) \end{aligned}$$

and so $h_X^{a,b}(x * y) \cap h_X^{a,b}(x) = \gamma_1 = h_X^{a,b}(y)$. Therefore $H_X^{a,b}$ is a hesitant fuzzy filter of X for all $a, b \in X$. □

Theorem 3.19. *Every filter of a BE-algebra can be represented as γ-inclusive set of a hesitant fuzzy filter.*

Proof. Let F be a filter of a BE-algebra X. For a subset γ of $[0, 1]$, define a hesitant set H_X by

$$h_X : X \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} \gamma & \text{if } x \in F \\ \emptyset & \text{if } x \notin F \end{cases}$$

Obviously, $F = H_X(\gamma)$. We now prove that H_X is a hesitant fuzzy filter of X. Since $1 \in H_X(\gamma)$, we have $H_X(1) = \gamma \supseteq h_X(x)$ for all $x \in X$. Let $x, y \in X$. If $x * y, x \in F$, then $y \in F$ since F is a filter of X. Hence $h_X(x * y) = h_X(x) = h_X(y) = \gamma$ and so $h_X(x * y) \cap h_X(x) \subseteq h_X(y)$. If $x * y \in F$ and $x \notin F$, then $h_X(x * y) = \gamma$ and $h_X(x) = \emptyset$ which imply that $h_X(x * y) \cap h_X(x) = \gamma \cap \emptyset = \emptyset \subseteq h_X(y)$. Similarly, if $x * y \notin F$ and $x \in F$, then $h_X(x * y) \cap h_X(x) \subseteq h_X(y)$. Obviously, if $x * y \notin F$ and $x \notin F$, then $h_X(x * y) \cap h_X(x) \subseteq h_X(y)$. Therefore H_X is a hesitant fuzzy filter of X. □

Let $H_X = \{(x, h(x)) | x \in X\}$ be a hesitant fuzzy set on X. For any $a, b \in X$ and $k \in \mathbb{N}$, consider the set

$$h_X[a^k; b] := \{x \in X | h_X(a^k * (b * x)) = h_X(1)\}$$

where $h_X(a * (a * (\cdots * (a * (a * x)) \cdots)))$ in which a appears k-times. Note that $1, a, b \in H_X[a^k; b]$ for all $a, b \in X$ and $k \in \mathbb{N}$.

Proposition 3.20. *Let $H_X := \{(x, h_x(x)) | x \in X\}$ be a hesitant fuzzy set on X satisfying (3.3) and $h_X(x * y) = h_X(x) \cup h_X(y)$ for all $x, y \in X$. For any $a, b \in X$ and $k \in \mathbb{N}$, if $x \in h_X[a^k; b]$, then $y * x \in h_X[a^k; b]$ for all $y \in X$.*

Proof. Assume that $x \in h_X[a^k; b]$. Then $h_X(a^k * (b * x)) = h_X(1)$ and so

$$\begin{aligned} h_X(a^k * (b * (y * x))) &= h_X(a^k * (y * b * x)) \\ &= h_X(y * (a^k * (b * x))) \\ &= h_X(y) \cup h_X(a^k * (b * x)) \\ &= h_X(y) \cup h_X(1) = h_X(1) \end{aligned}$$

for all $y \in X$ by (2.4). Hence $y * x \in h_X[a^k; b]$ for all $y \in X$. □

Proposition 3.21. *Let $H_X := \{(x, h_X))|x \in X\}$ be a hesitant fuzzy set on a BE-algebra X. If an element $a \in X$ satisfies $a * x = 1$ for all $x \in X$, then $h_X[a^k; b] = X = [b^k; a]$ for all $b \in X$ and $k \in \mathbb{N}$.*

Proof. For any $x \in X$, we have

$$\begin{aligned} h_X(a^k * (b * x)) =& h_X(a^{k-1} * (a * (b * x))) \\ =& h_X(a^{k-1} * (b * (a * x))) \\ =& h_X(a^{k-1} * (b * 1)) \\ =& h_X(1), \end{aligned}$$

and so $x \in h_X[a^k; b]$. Similarly, $x \in h_X[b^k; a]$. □

Proposition 3.22. *Let X be a self distributive BE-algebra and let $H_X := \{(x, h_X(x))|x \in X\}$ be a order reversing hesitant fuzzy set of X with the property (3.3). If $b \leq c$ in X, then $h_X[a^k; c] \subseteq h_X[a^k; b]$ for all $a \in X$ and $k \in \mathbb{N}$.*

Proof. Let $a, b, c \in X$ be such that $b \leq c$. For any $k \in \mathbb{N}$, if $x \in h_X[a^k; c]$, then

$$\begin{aligned} h_X(1) &= h_X(a^k * (c * x)) \\ &= h_X(c * (a^k * x)) \\ &\subseteq h_X(b * (a^k * x)) \\ &= h_X(a^k * (b * x)) \end{aligned}$$

by (2.4) and (2.8). Hence $h_X(a^k * (b * x)) = h_X(1)$. Thus $x \in h_X[a^k; b]$, which completes the proof. □

The following example shows that there exists a hesitant fuzzy set H_X of X, $a, b \in X$ and $k \in \mathbb{N}$ such that $h_X[a^k; b]$ is not a filter of X.

Example 3.23. *Let $X = \{0, 1, a, b, c\}$ be a BE-algebra with the following Cayley table:*

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	a	a	1

Let $H_X := \{(x, h_X(x)) | x \in X\}$ be a hesitant fuzzy set defined by

$$H_X = \{(1, [0, 1]), (a, (\tfrac{1}{4}, \tfrac{3}{4})), (b, (\tfrac{3}{4}, \tfrac{1}{2})), (c, (\tfrac{6}{8}, \tfrac{7}{8}))\}$$

Then $h_X[c; b] = \{x \in X | h_X(c*(b*x)) = h_X(1)\} = \{1, a, b\}$ is not a filter, since $a*c = a \in h_X[c; b]$ and $c \notin h_X[c; b]$.

Theorem 3.24. *$H_X := \{(x, h_X(x)) | x \in X\}$ be a hesitant fuzzy set on a self distributive BE-algebra X in which h_X is injective. Then $h_X[a^k; b]$ is a filter of X for all $a, b \in X$ and $k \in \mathbb{N}$.*

Proof. Assume that X is a self distributive BE-algebra and h_X is injective. Obviously, $1 \in h_X[a^k; b]$. Let $a, b, x, y \in X$ and $k \in \mathbb{N}$ be such that $x * y \in h_X[a^k; b]$ and $x \in h_X[a^k; b]$. Then $h_X(a^k * (b * x)) = h_X(1)$ which implies that $a^k * (b * x) = 1$, since h_X is injective. Using (2.7), we have

$$\begin{aligned} h_X(1) &= h_X(a^k * (b * (x * y))) \\ &= h_X(a^{k-1} * (a * (b * (x * y)))) \\ &= h_X(a^{k-1} * (a * ((b * x) * (b * y)))) \\ &= \cdots \\ &= h_X((a^k * (b * x)) * (a^k * (b * y))) \\ &= h_X(1 * (a^k * (b * y)) \\ &= h_X(a^k * (b * y)) \end{aligned}$$

which imply that $y \in h_X[a^k; b]$. Therefore $h_X[a^k; b]$ is a filter of X for all $a, b \in X$ and $k \in \mathbb{N}$. □

Theorem 3.25. *$H_X := \{(x, h_X(x)) | x \in X\}$ be a hesitant fuzzy set of a self distributive BE-algebra X satisfying the condition (3.3) and $h_X(x * y) = h_X(x) \cap h_X(y)$, for all $x, y \in X$. Then $h_X[a^k; b]$ is a filter of X for all $a, b \in X$ and $k \in \mathbb{N}$.*

Proof. Let $a, b \in X$ and $k \in \mathbb{N}$. Obviously, $1 \in h_X[a^k; b]$. Let $x, y \in X$ be such that $x*y \in h_X[a^k; b]$ and $x \in h_X[a^k; b]$. Then $h_X(a^k * (b * (x * y))) = h_X(1)$ and $h_X(a^k * (b * x)) = h_X(1)$, which implies

from the hypothesis that

$$\begin{aligned}h_X(1) &= h_X(a^k * (b * (x * y)))\\ &= h_X(a^{k-1} * (a * (b * (x * y))))\\ &= h_X(a^{k-1} * (a * ((b * x) * (b * y))))\\ &= \cdots\\ &= h_X((a^k * (b * x)) * (a^k * (b * y)))\\ &= h_X(a^k * (b * x)) \cap h_X(a^k * (b * y))\\ &= h_X(1) \cap h_X(a^k * (b * y))\\ &= h_X(a^k * (b * y)).\end{aligned}$$

Hence $y \in h_X[a^k; b]$ *and therefore* $h_X[a^k; b]$ *is a filter of* X *for all* $a, b \in X$ *and* $k \in \mathbb{N}$. □

Proposition 3.26. $H_X := \{(x, h_X(x)) | x \in X\}$ *be a hesitant fuzzy set of a* BE*-algebra* X *in which* h_X *is injective. If* F *is a filter of* X*, then the following holds.*

$$(\forall a, b \in F)(\forall k \in \mathbb{N})(h_X[a^k; b] \subseteq F). \tag{3.7}$$

Proof. Assume that F is a filter of X and let $a, b \in F$ and $k \in \mathbb{N}$. If $x \in h_X[a^k; b]$, then $h_X(a * (a^{k-1} * (b * x))) = h_X(a^k * (b * x)) = h_X(1)$ and so $a * (a^{k-1} * (b * x)) = 1 \in F$ since h_X is injective. Since F is a filter of X, it follows from (F2) that $a^{k-1} * (b * x) \in F$. Continuing this process, we obtain $b * x \in F$ and so $x \in F$. Therefore $h_X[a^k; b] \subseteq F$ for all $a, b \in F$ and $k \in \mathbb{N}$. □

Theorem 2.27. $H_X := \{(x, h_X(x)) | x \in X\}$ *be a hesitant fuzzy set of a* BE*-algebra* X*. For any subset* F *of* X*, if the condition (3.7) holds, then* F *is a filter of* X*.*

Proof. Suppose that the condition (3.7) holds. Obviously, $1 \in h_X[a^k; b] \subseteq F$. Let $x, y \in X$ be such that $x * y \in F$ and $x \in F$. Then

$$\begin{aligned}h_X(x^k * ((x * y) * y)) &= h_X(x^{k-1} * (x * ((x * y) * y)))\\ &= h_X(x^{k-1} * ((x * y) * (x * y)))\\ &= h_X(x^{k-1} * 1) = h_X(1)\end{aligned}$$

and hence $y \in h_X[a^k; b] \subseteq F$, where $b = x * y$. Therefore F is a filter of X. □

Theorem 3.28. $H_X := \{(x, h_X(x)) | x \in X\}$ *be a hesitant fuzzy set of a* BE*-algebra* X*. If* F *is a filter of* X*, then*

$$(\forall k \in \mathbb{N})(F = \cup\{h_X[a^k; b] | a, b \in F\}).$$

Proof. *Let F is a filter of X. By Proposition 3.26, the inclusion $\cup\{h_X[a^k;b]|a,b\in F\}\subseteq F$ holds. Let $x\in F$. Since $x\in h_X[1^k;x]$ for all $k\in\mathbb{N}$, it follows that*

$$\begin{aligned} F &\subseteq \cup\{h_X[1^k;x]|x\in F\} \\ &\subseteq \cup\{h_X[a^k;b]|a,b\in F\}. \end{aligned}$$

This completes the proof. □

Theorem 3.29. *If $H_X := \{(x,h_X(x))|x\in X\}$ is a hesitant filter of X, then the set*

$$H_a := \{x\in X|h_X(a)\subseteq h_X(x)\}$$

is a filter of X for all $a\in X$.

Proof. *Let $x,y\in X$ be such that $x*y\in H_a$ and $x\in H_a$. Then $h_X(a)\subseteq h_X(x*y)$ and $h_X(a)\subseteq h_X(y)$. By (3.3) and (3.4), we have $h_X(a)\subseteq h_X(x*y)\cap h_X(x)\subseteq H_X(y)\subseteq h_X(1)$ and so $1\in H_a$ and $y\in H_a$. Therefore H_a is a filter of X.* □

Theorem 3.30. *Let $a\in X$ and $H_X := \{(x,h_X(x))|x\in X\}$ be a hesitant fuzzy set on X. Then the following properties are valid:*

(i) *if H_a is a filter of X, then $H_X := \{(x,h_X(x))|x\in X\}$ satisfies:*

$$(\forall x,y\in X)(h_X(a)\subseteq h_X(x*y)\cap h_X(x)\Rightarrow h_X(a)\subseteq h_X(y)). \tag{3.8}$$

(ii) *if $H_X := \{(x,h_X(x))|x\in X\}$ satisfies the condition (3.3) and (3.8), then H_a is a filter of X.*

Proof. *(i) Assume that H_a is a filter of X and let $x,y\in X$ be such that $h_X(a)\subseteq H_X(x*y)\cap H_X(x)$. Then $x*y\in H_a$ and $y\in H_a$. Since H_a is a filter of X, we obtain $x\in H_a$. Therefore $h_X(a)\subseteq h_X(y)$.*

(ii) Let $H_X := \{(x,h_X(x))|x\in X\}$ be a hesitant fuzzy set on X in which the conditions (3.3) and (3.8) hold. Then $1\in H_a$. Let $x,y\in X$ be such that $x*y\in H_a$ and $x\in H_a$. Then $h_X(a)\subseteq h_X(x*y)$ and $h_X(a)\subseteq h_X(x)$. Hence $H_X(a)\subseteq h_X(x*y)\cap h_X(x)$. Using (3.8), we have $h_X(a)\subseteq h_X(y)$, i.e., $y\in H_a$. Thus H_a is a filter of X. □

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A fixed point approach to the stability of nonic functional equation in non-Archimedean spaces*

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Abstract In this paper, a new nonic functional equation is introduced. The solution of this functional equation can also be determined in certain type of groups using two important results due to Székelyhidi. Using the fixed point theorems due to Brzdęk and Ciepliński, we give some Ulam–Hyers stability results for the nonic functional equation in non-Archimedean spaces.

Keywords Ulam–Hyers stability; nonic functional equation; non-Archimedean space; fixed point method.

Mathematics Subject Classification(2010) 39B82; 39B52; 46H25.

1 Introduction and preliminaries

In this paper $\mathbb{R}$ and $\mathbb{N}$ denote the sets of reals and positive integers, respectively. Moreover, $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

A valuation is a function $|\cdot|$ from a field $\mathbb{K}$ into $\mathbb{R}_+$ such that 0 is the unique element having the 0 valuation, $|rs| = |r| \cdot |s|$ and the triangle inequality holds, i.e.,

$$|r + s| \leq |r| + |s|, \quad \forall r, s \in \mathbb{K}.$$

A field $\mathbb{K}$ is called a valued field if $\mathbb{K}$ carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality.

Let $\mathbb{K}$ be a field. A non-Archimedean valuation on $\mathbb{K}$ is a function $|\cdot| : \mathbb{K} \to \mathbb{R}$ such that

(1) $|r| \geq 0$ and equality holds if and only if $r = 0$.

(2) $|rs| = |r||s|, \quad r, s \in \mathbb{K}$.

(3) $|r + s| \leq \max\{|r|, |s|\}, \quad r, s \in \mathbb{K}$.

Any field endowed with a non-Archimedean valuation is said to be a non-Archimedean field. In any such field we have $|\mathbf{1}| = |-\mathbf{1}| = 1$ and $|n \times \mathbf{1}| \leq 1$ for all $n \in \mathbb{N}$, where $\mathbf{1}$ is the neutral element of the semigroup $(\mathbb{K}, \cdot)$, $1 \times \mathbf{1} = \mathbf{1}$ and $(n + 1) \times \mathbf{1} = (n \times \mathbf{1}) + \mathbf{1}$ for $n \in \mathbb{N}$.

Let X be a linear space over a field $\mathbb{K}$ with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \to \mathbb{R}_+$ is a non-Archimedean norm if it satisfies the following conditions:

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(1) $\|x\| = 0$ if and only if $x = 0$;

(2) $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{K}$ and $x \in X$;

(3) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in X$. If $\|\cdot\| : X \to \mathbb{R}_+$ is a non-Archimedean norm in X, then the pair $(X, \|\cdot\|)$ is called a non-Archimedean normed space. Then $(X, \|\cdot\|)$ is called a non-Archimedean normed space.

Let X be a non-Archimedean normed space. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. In that case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $\lim_{n\to\infty} x_n = x$. A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim_{n\to\infty} \|x_{n+p} - x_n\| = 0$ for all $p = 1, 2, \ldots$. Due to the fact that

$$\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| : \quad m \leq j \leq n - 1\} \quad (n > m)$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space.

The most important examples of non-Archimedean spaces are p-adic numbers. The p-adic numbers have gained the interest of physicists because of their connections with some problems coming from quantum physics, p-adic strings and superstrings (see [15]).

In this paper, we first introduce the following new nonic functional equation

$$\begin{aligned} f(x+5y) - 9f(x+4y) + 36f(x+3y) - 84f(x+2y) + 126f(x+y) - 126f(x) + \\ 84f(x-y) - 36f(x-2y) + 9f(x-3y) - f(x-4y) = 9!f(y). \end{aligned} \tag{1}$$

It is easy to see that the function $f(x) = ax^9$ is a solution of the functional equation (1). Every solution of the functional equation (1) is said to be a nonic mapping.

The study of stability problems for functional equations is related to a question of Ulam [20] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [10]. The result of Hyers was generalized by Aoki [2] for approximate additive mappings and by Rassias [17] for approximate linear mappings by allowing the Cauchy difference operator $CDf(x, y) = f(x + y) - [f(x) + f(y)]$ to be controlled by $\epsilon(\|x\|^p + \|y\|^p)$. In 1994, a further generalization was obtained by Găvruţa [7], who replaced $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. We refer the reader to (see for instance [1, 3–6, 8, 11–14, 16, 18, 21, 22]) and references therein for more information on Ulam's problem during the last seventy years.

From now on S denotes a nonempty set and X stands for a complete non-Archimedean normed space. Given a set $Z \neq \emptyset$ and functions $\varphi : S \to S$ and $F : S \times Z \to Z$, we define an operator $\mathcal{L}_\varphi^F : Z^S \to Z^S$ (Z^S denotes the family of all functions mapping a set S into a set Z) by

$$\mathcal{L}_\varphi^F(\alpha)(t) := F(t, \alpha(\varphi(t))), \quad \alpha \in Z^S, t \in S.$$

Moreover, if $\Lambda : S \times \mathbb{R}_+ \to \mathbb{R}_+$, then we write $\Lambda_t := \Lambda(t, \cdot), t \in S$.

For explicitly later use, we recall the following results by Brzdęk and Ciepliński [4].

Theorem 1 *Let $\Lambda : S \times \mathbb{R}_+ \to \mathbb{R}_+, f : S \to X, \mathcal{T} : X^S \to X^S, \varphi : S \to S, \varepsilon : S \to \mathbb{R}_+$ and*

$$\|\mathcal{T}(\alpha)(t) - \mathcal{T}(\beta)(t)\| \leq \Lambda(t, \|\alpha(\varphi(t)) - \beta(\varphi(t))\|), \quad \alpha, \beta \in X^S, t \in S. \tag{2}$$

Assume also that Λ_t is nondecreasing for every $t \in S$, $\lim_{n\to\infty} (\mathcal{L}_\varphi^\Lambda)^n(\varepsilon)(t) = 0 (t \in S)$ holds and

$$\|\mathcal{T}(f)(t) - f(t)\| \leq \varepsilon(t), \qquad t \in S. \tag{3}$$

Then for each $t \in S$ the limit

$$\lim_{n\to\infty} \mathcal{T}^n(f)(t) =: A(t) \tag{4}$$

exists and the function $A \in X^S$ is the unique fixed point of $\mathcal{T}$ with

$$\|f(t) - A(t)\| \leq \sup_{n \in \mathbb{N}_0} (\mathcal{L}_\varphi^\Lambda)^n(\varepsilon)(t) =: h(t), \quad t \in S. \tag{5}$$

Corollary 1 *Let $F : S \times X \to X$, $\varphi : S \to S$, $\Lambda : S \times \mathbb{R}_+ \to \mathbb{R}_+$, $f : S \to X$, $\varepsilon : S \to \mathbb{R}_+$ and*

$$\|F(t, x) - F(t, y)\| \leq \Lambda(t, \|x - y\|), \quad t \in S, x, y \in X. \tag{6}$$

Assume also that, for every $t \in S$, Λ_t is nondecreasing, $\lim_{n\to\infty} (\mathcal{L}_\varphi^\Lambda)^n(\varepsilon)(t) = 0 (t \in S)$ holds and

$$\|f(t) - F(t, f(\varphi(t)))\| \leq \varepsilon(t), \qquad t \in S. \tag{7}$$

Then for each $t \in S$ the limit

$$\lim_{n\to\infty} (\mathcal{L}_\varphi^F)^n(f)(t) =: A(t) \tag{8}$$

exists and the function $A \in X^S$ is the unique solution of the functional equation

$$A(t) = F(t, A(\varphi(t))) \tag{9}$$

such that (5) holds.

We end this section with two corollaries, which are immediate consequences of Corollary 1.

Corollary 2 *Let $a : S \to \mathbb{K} \backslash \{0\}$, $\varphi : S \to S$, $f : S \to X$, $\delta : S \to \mathbb{R}_+$,*

$$\|f(\varphi(t)) - a(t)f(t)\| \leq \delta(t), \qquad t \in S \tag{10}$$

and

$$\lim_{n\to\infty} \frac{\delta(\varphi^n(t))}{|\prod_{i=0}^n a(\varphi^i(t))|} = 0, \qquad t \in S. \tag{11}$$

Then there exists a unique solution $A \in X^S$ of the functional equation

$$A(\varphi(t)) = a(t)A(t) \tag{12}$$

such that

$$\|f(t) - A(t)\| \leq \sup_{n \in \mathbb{N}_0} \frac{\delta(\varphi^n(t))}{|\prod_{i=0}^n a(\varphi^i(t))|}, \qquad t \in S. \tag{13}$$

Corollary 3 *Let $b : S \to \mathbb{K}$, $\psi : S \to S$, $f : S \to X$, $\varepsilon : S \to \mathbb{R}_+$,*

$$\|f(t) - b(t)f(\psi(t))\| \leq \varepsilon(t), \qquad t \in S \tag{14}$$

and

$$\lim_{n\to\infty} \left| \prod_{i=0}^n b(\psi^i(t)) \right| \varepsilon(\psi^{n+1}(t)) = 0, \qquad t \in S. \tag{15}$$

Then there exists a unique solution $B \in X^S$ of the functional equation

$$B(t) = b(t)B(\psi(t)) \tag{16}$$

such that

$$\|f(t) - B(t)\| \leq \max\left\{ \varepsilon(t), \sup_{n \in \mathbb{N}_0} \left| \prod_{i=0}^n b(\psi^i(t)) \right| \varepsilon(\psi^{n+1}(t)) \right\}, \qquad t \in S. \tag{17}$$

2 Solution of the nonic functional equation on commutative groups

In this section, we solve the functional equation (1) on commutative groups with some additional requirements.

A group S is said to be divisible if for every element $b \in S$ and every $n \in \mathbb{N}$, there exists an element $a \in S$ such that $na = b$. If this element a is unique, then S is said to be uniquely divisible. In a uniquely divisible group, this unique element a is denoted by $\frac{b}{n}$. That the equation $na = b$ has a solution is equivalent to saying that the multiplication by n is surjective. Similarly, that the equation $na = b$ has a unique solution is equivalent to saying that the multiplication by n is bijective.

The following two important results due to Székelyhidi (see [19] for the details).

Theorem 2 *Let G be a commutative semigroup with identity, S a commutative group and n a nonnegative integer. Let the multiplication by $n!$ be bijective in S. The function $f : G \to S$ is a solution of Fréchet functional equation*

$$\triangle_{x_1,\dots,x_{n+1}} f(x_0) = 0 \tag{18}$$

for all $x_0, x_1, \dots, x_{n+1} \in G$ if and only if f is a polynomial of degree at most n, i.e., f is given by

$$f(x) = A^n(x) + \cdots + A^1(x) + A^0(x), \qquad x \in G, \tag{19}$$

where $A^0(x) = A^0$ is an arbitrary element of S and $A^n(x)$ is the diagonal of an n-additive symmetric function $A_n : G^n \to S$.

Theorem 3 *Let G and S be commutative groups, n a nonnegative integer, φ_i, ψ_i additive functions from G into G and $\varphi_i(G) \subseteq \psi_i(G)(i = 1, 2, \dots, n+1)$. If the functions $f, f_i : G \to S(i = 1, 2, \dots, n+1)$ satisfy*

$$f(x) + \sum_{i=1}^{n+1} f_i(\varphi_i(x) + \psi_i(y)) = 0, \tag{20}$$

then f satisfies Fréchet functional equation $\triangle_{x_1,\dots,x_{n+1}} f(x_0) = 0$.

Using the results, we have the following theorem.

Theorem 4 *Let S be a commutative group and V be a linear space. Then the function $f : S \to V$ satisfies the functional equation (1) for all $x, y \in S$, if and only if f is of the form*

$$f(x) = A^9(x), \qquad x \in S,$$

where $A^9(x)$ is the diagonal of the 9-additive symmetric map $A_9 : S^9 \to V$.

Proof. Assume that f satisfies the functional equation (1). We can rewrite the functional equation (1) in the form

$$\begin{aligned} f(x) - \frac{1}{126}f(x+5y) + \frac{1}{14}f(x+4y) - \frac{2}{7}f(x+3y) + \frac{2}{3}f(x+2y) - f(x+y) \\ -\frac{2}{3}f(x-y) + \frac{2}{7}f(x-2y) - \frac{1}{14}f(x-3y) + \frac{1}{126}f(x-4y) + 2880f(y) = 0. \end{aligned} \tag{21}$$

Thus by Theorems 2 and 3, f is of the form

$$f(x) = \sum_{i=0}^{9} A^i(x), \qquad x \in S, \tag{22}$$

where $A^0(x) = A^0$ is an arbitrary element of V, and $A^i(x)$ is the diagonal of the i-additive symmetric map $A_i : S^i \to V$ for $i = 1, 2, \dots, 9$. Replacing $x = 0, y = 0$ in (1), one finds $f(0) = 0$. Hence $A^0(x) = A^0 = 0$.

Replacing $x=0, y=x$ and $x=x, y=-x$ in (1) and adding the two resulting equations, we get $f(-x)=-f(x)$ for all $x \in S$. So the function f is odd. Thus we have $A^8(x)=A^6(x)=A^4(x)=A^2(x)=0$ for all $x \in S$. It follows that $f(x)=A^9(x)+A^7(x)+A^5(x)+A^3(x)+A^1(x)$. Replacing (x,y) with $(0,2x)$ in (1), one obtains

$$f(10x)-8f(8x)+27f(6x)-48f(4x)-(9!-42)f(2x)=0. \tag{23}$$

Replacing (x,y) with $(5x,x)$, one gets

$$\begin{aligned} f(10x)-9f(9x)+36f(8x)-84f(7x)+126f(6x)-126f(5x)+84f(4x) \\ -36f(3x)+9f(2x)-(9!+1)f(x)=0. \end{aligned} \tag{24}$$

Subtracting equations (23) and (24), we find

$$\begin{aligned} 9f(9x)-44f(8x)+84f(7x)-99f(6x)+126f(5x)-132f(4x) \\ +36f(3x)-(9!-33)f(2x)+(9!+1)f(x)=0. \end{aligned} \tag{25}$$

Replacing (x,y) with $(4x,x)$, and multiplying the resulting equation by 9, one obtains

$$\begin{aligned} 9f(9x)-81f(8x)+324f(7x)-756f(6x)+1134f(5x)-1134f(4x) \\ +756f(3x)-324f(2x)-9(9!-9)f(x)=0. \end{aligned} \tag{26}$$

Subtracting equations (25) and (26), we get

$$\begin{aligned} 37f(8x)-240f(7x)+657f(6x)-1008f(5x)+1002f(4x)-720f(3x) \\ -(9!-357)f(2x)+(10!-80)f(x)=0. \end{aligned} \tag{27}$$

Replacing (x,y) with $(3x,x)$, and multiplying the resulting equation by 37, one finds

$$\begin{aligned} 37f(8x)-333f(7x)+1332f(6x)-3108f(5x)+4662f(4x)-4662f(3x) \\ +3108f(2x)-37(9!+35)f(x)=0. \end{aligned} \tag{28}$$

Subtracting equations (27) and (28), we arrive at

$$\begin{aligned} 93f(7x)-675f(8x)+2100f(5x)-3660f(4x)+3942f(3x)-(9!+2751)f(2x) \\ +(47\cdot 9!+1215)f(x)=0. \end{aligned} \tag{29}$$

Replacing (x,y) with $(2x,x)$, and multiplying the resulting equation by 93, one finds

$$\begin{aligned} 93f(7x)-837f(6x)+3348f(5x)-7812f(4x)+11718f(3x)-11625f(2x) \\ -93(9!-75)f(x)=0. \end{aligned} \tag{30}$$

Subtracting equations (29) and (30) and then dividing by 2, we arrive at

$$\begin{aligned} 81f(6x)-624f(5x)+2076f(4x)-3888f(3x)-\frac{1}{2}(9!-8874)f(2x) \\ +(70\cdot 9!-2880)f(x)=0. \end{aligned} \tag{31}$$

Replacing (x,y) with (x,x), and multiplying the resulting equation by 81, one finds

$$81f(6x)-729f(5x)+2916f(4x)-6723f(3x)+9477f(2x)-81(9!+90)f(x)=0. \tag{32}$$

Subtracting equations (31) and (32), we arrive at

$$105f(5x)-840f(4x)+2835f(3x)-\frac{1}{2}(9!+10080)f(2x)+(151\cdot 9!+4410)f(x)=0. \tag{33}$$

Replacing (x,y) with $(0,x)$, and multiplying the resulting equation by 105, one finds

$$105(5x)-840f(4x)+2835f(3x)-5040f(2x)-105(9!-42)f(x)=0. \tag{34}$$

Subtracting equations (33) and (34), we arrive at

$$f(2x)=2^9 f(x). \tag{35}$$

By (35) and $A^n(rx)=r^nA^n(x)$ whenever $x \in S$ and $r \in \mathbb{Q}$, we obtain $2^9(A^9(x)+A^7(x)+A^5(x)+A^3(x)+A^1(x)) = 2^9A^9(x)+2^7A^7(x)+2^5A^5(x)+2^3A^3(x)+2A^1(x)$. It follows that $A^7(x)=A^5(x)=A^3(x)=A^1(x)=0$ for all $x \in S$. Hence $f(x)=A^9(x)$. The converse is easily verified. $\square$

3 Stability results

Throughout this section, we assume that S is a commutative group and X is a complete non-Archimedean normed space. For a given mapping $f : S \to X$, we define the difference operators

$$\begin{aligned} Df(x,y) := & f(x+5y) - 9f(x+4y) + 36f(x+3y) - 84f(x+2y) + 126f(x+y) \\ & -126f(x) + 84f(x-y) - 36f(x-2y) + 9f(x-3y) - f(x-4y) - 9!f(y) \end{aligned}$$

for all $x, y \in S$.

Theorem 5 *Let $\varphi : S^2 \to \mathbb{R}_+$ be a function such that*

$$\lim_{n\to\infty} |2|^{-9n} \varphi(2^n x, 2^n y) = 0, \qquad x, y \in S. \tag{36}$$

Assume also that $f : S \to X$ be a mapping such that

$$\|Df(x,y)\| \le \varphi(x,y), \qquad x, y \in S. \tag{37}$$

Then there exists a unique nonic mapping $T : S \to X$ such that

$$\|f(x) - T(x)\| \le \sup_{n\in\mathbb{N}_0} |2|^{-9(n+1)} \delta(2^n x), \qquad x \in S, \tag{38}$$

where

$$\begin{aligned} \delta(x) \quad := \quad & \frac{1}{|9!|} \max\left\{ |210|\varphi(0,x), \frac{|210|}{|8!|}\varphi(0,3x), \frac{|210|}{|8!|}\varphi(3x,-3x), \frac{|15|}{|6!|}\varphi(2x,-2x), \right. \\ & \frac{|35|}{|6!|}\varphi(x,-x), \frac{|2940|}{|8!|}\varphi(0,0), \frac{|210|}{|9!|}\varphi(0,4x), \frac{|210|}{|9!|}\varphi(4x,-4x), |162|\varphi(x,x), \\ & \frac{|18|}{|8!|}\varphi(3x,-3x), |93|\varphi(2x,x), |37|\varphi(3x,x), |9|\varphi(4x,x), \varphi(5x,x), \varphi(0,2x), \\ & \left. \frac{1}{|9!|}\varphi(0,8x), \frac{1}{|9!|}\varphi(8x,-8x), \frac{1}{|8!|}\varphi(0,6x), \frac{1}{|8!|}\varphi(6x,-6x) \right\}. \end{aligned}$$

Proof. Replacing $x = y = 0$ in (37), we get

$$\|f(0)\| \le \frac{1}{|9!|}\varphi(0,0). \tag{39}$$

Replacing x and y by 0 and x in (37), respectively, we get

$$\begin{aligned} &\|f(5x) - 9f(4x) + 36f(3x) - 84f(2x) + 126f(x) - 126f(0) + 84f(-x) \\ &\quad -36f(-2x) + 9f(-3x) - f(-4x) - 9!f(x)\| \le \varphi(0,x) \end{aligned} \tag{40}$$

for all $x \in S$. Replacing x and y by x and $-x$ in (37), respectively, we have

$$\begin{aligned} &\|f(-4x) - 9f(-3x) + 36f(-2x) - 84f(-x) + 126f(0) - 126f(x) + 84f(2x) \\ &\quad -36f(3x) + 9f(4x) - f(5x) - 9!f(-x)\| \le \varphi(-x,x) \end{aligned} \tag{41}$$

for all $x \in S$. By (40) and (41), we obtain

$$\|f(x) + f(-x)\| \le \frac{1}{|9!|}\max\{\varphi(0,x), \varphi(x,-x)\} \tag{42}$$

for all $x \in S$. Replacing x and y by 0 and $2x$ in (37), respectively, and using (39) and (42), we find

$$\begin{aligned} &\|f(10x) - 8f(8x) + 27f(6x) - 48f(4x) - (9! - 42)f(2x)\| \\ &\quad \le \max\left\{ \varphi(0,2x), \frac{1}{|9!|}\varphi(0,8x), \frac{1}{|9!|}\varphi(8x,-8x), \frac{1}{|8!|}\varphi(0,6x), \frac{1}{|8!|}\varphi(6x,-6x), \right. \\ &\qquad \left. \frac{|4|}{|8!|}\varphi(0,4x), \frac{|4|}{|8!|}\varphi(4x,-4x), \frac{|84|}{|9!|}\varphi(2x,-2x) \right\} \end{aligned} \tag{43}$$

for all $x \in S$. Replacing x and y by $5x$ and x in (37), respectively, we get

$$\begin{aligned}\|f(10x) - 9f(9x) + 36f(8x) - 84f(7x) + 126f(6x) - 126f(5x) + 84f(4x) \\ -36f(3x) + 9f(2x) - (9! + 1)f(x)\| \le \varphi(5x, x)\end{aligned} \tag{44}$$

for all $x \in S$. By (43) and (44), we obtain

$$\begin{aligned}&\|9f(9x) - 44f(8x) + 84f(7x) - 99f(6x) + 126f(5x) - 132f(4x) \\ &\quad +36f(3x) - (9! - 33)f(2x) + (9! + 1)f(x)\| \\ &\le \max\Big\{\varphi(5x, x), \varphi(0, 2x), \frac{1}{|9!|}\varphi(0, 8x), \frac{1}{|9!|}\varphi(8x, -8x), \frac{1}{|8!|}\varphi(0, 6x), \\ &\quad \frac{1}{|8!|}\varphi(6x, -6x), \frac{|4|}{|8!|}\varphi(0, 4x), \frac{|4|}{|8!|}\varphi(4x, -4x), \frac{|84|}{|9!|}\varphi(2x, -2x)\Big\}\end{aligned} \tag{45}$$

for all $x \in S$. Replacing x and y by $4x$ and x in (37), respectively, and using (39) we have

$$\begin{aligned}&\|f(9x) - 9f(8x) + 36f(7x) - 84f(6x) + 126f(5x) - 126f(4x) \\ &\quad +84f(3x) - 36f(2x) - (9! - 9)f(x)\| \le \max\Big\{\varphi(4x, x), \frac{1}{|9!|}\varphi(0, 0)\Big\}\end{aligned} \tag{46}$$

for all $x \in S$. By (45) and (46), we get

$$\begin{aligned}&\|37f(8x) - 240f(7x) + 657f(6x) - 1008f(5x) + 1002f(4x) \\ &\quad -720f(3x) - (9! - 357)f(2x) + (10! - 80)f(x)\| \\ &\le \max\Big\{|9|\varphi(4x, x), \frac{1}{|8!|}\varphi(0, 0), \varphi(5x, x), \varphi(0, 2x), \frac{1}{|9!|}\varphi(0, 8x), \frac{1}{|9!|}\varphi(8x, -8x), \\ &\quad \frac{1}{|8!|}\varphi(0, 6x), \frac{1}{|8!|}\varphi(6x, -6x), \frac{|4|}{|8!|}\varphi(0, 4x), \frac{|4|}{|8!|}\varphi(4x, -4x), \frac{|84|}{|9!|}\varphi(2x, -2x)\Big\}\end{aligned} \tag{47}$$

for all $x \in S$. Replacing x and y by $3x$ and x in (37), respectively, then using (39) and (42), we have

$$\begin{aligned}&\|f(8x) - 9f(7x) + 36f(6x) - 84f(5x) + 126f(4x) - 126f(3x) + 84f(2x) \\ &\quad -(9! + 35)f(x)\| \le \max\Big\{\varphi(3x, x), \frac{1}{|8!|}\varphi(0, 0), \frac{1}{|9!|}\varphi(0, x), \frac{1}{|9!|}\varphi(x, -x)\Big\}\end{aligned} \tag{48}$$

for all $x \in S$. By (47) and (48), we get

$$\begin{aligned}&\|93f(7x) - 675f(6x) + 2100f(5x) - 3660f(4x) \\ &\quad +3942f(3x) - (9! + 2751)f(2x) + (47 \cdot 9! + 1215)f(x)\| \\ &\le \max\Big\{|37|\varphi(3x, x), \frac{|37|}{|8!|}\varphi(0, 0), \frac{|37|}{|9!|}\varphi(0, x), \frac{|37|}{|9!|}\varphi(x, -x), |9|\varphi(4x, x), \\ &\quad \varphi(5x, x), \varphi(0, 2x), \frac{1}{|9!|}\varphi(0, 8x), \frac{1}{|9!|}\varphi(8x, -8x), \frac{1}{|8!|}\varphi(0, 6x), \\ &\quad \frac{1}{|8!|}\varphi(6x, -6x), \frac{|4|}{|8!|}\varphi(0, 4x), \frac{|4|}{|8!|}\varphi(4x, -4x), \frac{|84|}{|9!|}\varphi(2x, -2x)\Big\}\end{aligned} \tag{49}$$

for all $x \in S$. Replacing x and y by $2x$ and x in (37), respectively, then using (39) and (42), we have

$$\begin{aligned}&\|f(7x) - 9f(6x) + 36f(5x) - 84f(4x) + 126f(3x) - 125f(2x) - (9! - 75)f(x)\| \\ &\le \max\Big\{\varphi(2x, x), \frac{1}{|9!|}\varphi(0, 2x), \frac{1}{|9!|}\varphi(2x, -2x), \frac{1}{|8!|}\varphi(0, x), \frac{1}{|8!|}\varphi(x, -x), \frac{|4|}{|8!|}\varphi(0, 0)\Big\}\end{aligned} \tag{50}$$

for all $x \in S$. By (49) and (50), we get

$$\begin{aligned}&\|81f(6x) - 624f(5x) + 2076f(4x) - 3888f(3x) - \frac{1}{2}(9! - 8874)f(2x) \\ &\quad +(70 \cdot 9! - 2880)f(x)\| \\ &\le \frac{1}{|2|}\max\Big\{|93|\varphi(2x, x), \frac{|93|}{|9!|}\varphi(2x, -2x), \frac{|93|}{|8!|}\varphi(0, x), \frac{|93|}{|8!|}\varphi(x, -x), \frac{|372|}{|8!|}\varphi(0, 0), \\ &\quad |37|\varphi(3x, x), |9|\varphi(4x, x), \varphi(5x, x), \varphi(0, 2x), \frac{1}{|9!|}\varphi(0, 8x), \frac{1}{|9!|}\varphi(8x, -8x), \\ &\quad \frac{1}{|8!|}\varphi(0, 6x), \frac{1}{|8!|}\varphi(6x, -6x), \frac{|4|}{|8!|}\varphi(0, 4x), \frac{|4|}{|8!|}\varphi(4x, -4x)\Big\}\end{aligned} \tag{51}$$

for all $x \in S$. Replacing x and y by x and x in (37), respectively, then using (39) and (42), we have

$$\begin{aligned} &\|f(6x) - 9f(5x) + 36f(4x) - 83f(3x) + 117f(2x) - (9! + 90)f(x)\| \\ &\quad \le \max\Big\{\varphi(x,x), \frac{|84|}{|9!|}\varphi(0,0), \frac{|4|}{|8!|}\varphi(0,x), \frac{|4|}{|8!|}\varphi(x,-x), \frac{1}{|8!|}\varphi(0,2x), \\ &\qquad \frac{1}{|8!|}\varphi(2x,-2x), \frac{1}{|9!|}\varphi(0,3x), \frac{1}{|9!|}\varphi(3x,-3x)\Big\} \end{aligned} \tag{52}$$

for all $x \in S$. By (51) and (52), we get

$$\begin{aligned} &\|105f(5x) - 840f(4x) + 2835f(3x) - (\frac{9!}{2} + 5040)f(2x) + (151 \cdot 9! + 4410)f(x)\| \\ &\quad \le \max\Big\{|81|\varphi(x,x), \frac{|756|}{|8!|}\varphi(0,0), \frac{|324|}{|8!|}\varphi(0,x), \frac{|324|}{|8!|}\varphi(x,-x), \frac{|81|}{|8!|}\varphi(2x,-2x), \\ &\qquad \frac{|9|}{|8!|}\varphi(0,3x), \frac{|9|}{|8!|}\varphi(3x,-3x), \frac{|93|}{|2|}\varphi(2x,x), \frac{|37|}{|2|}\varphi(3x,x), \frac{|9|}{|2|}\varphi(4x,x), \\ &\qquad \frac{1}{|2|}\varphi(5x,x), \frac{1}{|2|}\varphi(0,2x), \frac{1}{|2\cdot 9!|}\varphi(0,8x), \frac{1}{|2\cdot 9!|}\varphi(8x,-8x), \\ &\qquad \frac{1}{|2\cdot 8!|}\varphi(0,6x), \frac{1}{|2\cdot 8!|}\varphi(6x,-6x), \frac{|2|}{|8!|}\varphi(0,4x), \frac{|2|}{|8!|}\varphi(4x,-4x)\Big\} \end{aligned} \tag{53}$$

for all $x \in S$. Replacing x and y by 0 and x in (37), respectively, then using (39) and (42), we have

$$\begin{aligned} &\|f(5x) - 8f(4x) + 27f(3x) - 48f(2x) - (9! - 42)f(x)\| \\ &\quad \le \max\Big\{\varphi(0,x), \frac{1}{|8!|}\varphi(0,3x), \frac{1}{|8!|}\varphi(3x,-3x), \frac{|4|}{|8!|}\varphi(0,2x), \frac{|4|}{|8!|}\varphi(2x,-2x), \\ &\qquad \frac{|84|}{|9!|}\varphi(x,-x), \frac{|14|}{|8!|}\varphi(0,0), \frac{1}{|9!|}\varphi(0,4x), \frac{1}{|9!|}\varphi(4x,-4x)\Big\} \end{aligned} \tag{54}$$

for all $x \in S$. By (53) and (54), we get

$$\begin{aligned} \|f(2x) - 2^9 f(x)\| \quad \le \quad & \frac{1}{|9!|}\max\Big\{|210|\varphi(0,x), \frac{|210|}{|8!|}\varphi(0,3x), \frac{|210|}{|8!|}\varphi(3x,-3x), \\ & \frac{|15|}{|6!|}\varphi(2x,-2x), \frac{|35|}{|6!|}\varphi(x,-x), \frac{|2940|}{|8!|}\varphi(0,0), \frac{|210|}{|9!|}\varphi(0,4x), \\ & \frac{|210|}{|9!|}\varphi(4x,-4x), |162|\varphi(x,x), \frac{|18|}{|8!|}\varphi(3x,-3x), |93|\varphi(2x,x), \\ & |37|\varphi(3x,x), |9|\varphi(4x,x), \varphi(5x,x), \varphi(0,2x), \frac{1}{|9!|}\varphi(0,8x), \\ & \frac{1}{|9!|}\varphi(8x,-8x), \frac{1}{|8!|}\varphi(0,6x), \frac{1}{|8!|}\varphi(6x,-6x)\Big\} \\ =: \quad & \delta(x), \qquad x \in S. \end{aligned} \tag{55}$$

By Corollary 2, there exists a unique mapping $T : S \to X$ such that $T(2x) = 2^9 T(x)$ and (38) holds. By (8) in Corollary 1,

$$T(x) := \lim_{n\to\infty} (\mathcal{L}_\varphi^F)^n(f)(x) = \lim_{n\to\infty} 2^{-9n} f(2^n x), \qquad x \in S. \tag{56}$$

It remains to show that T is a nonic map. By (37), we have

$$\|Df(2^n x, 2^n y)/2^{9n}\| \le |2|^{-9n}\varphi(2^n x, 2^n y) \tag{57}$$

for all $x, y \in S$ and $n \in \mathbb{N}$. So, by (36), (54) and (55), $\|DT(x,y)\| = 0$ for all $x, y \in S$. Thus the mapping $T : S \to X$ is nonic. □

Similar to Theorem 5, one can prove the following result.

Theorem 6 *Assume that the multiplication by $2^n (n \in \mathbb{N})$ be bijective in S. Let $\varphi : S^2 \to \mathbb{R}_+$ be a function such that*

$$\lim_{n\to\infty} |2|^{9n}\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0, \qquad x, y \in S. \tag{58}$$

Assume also that $f : S \to X$ *be a mapping such that*

$$\|Df(x,y)\| \le \varphi(x,y), \qquad x,y \in S. \tag{59}$$

Then there exists a unique nonic mapping $T : S \to X$ *such that*

$$\|f(x) - T(x)\| \le \max\left\{\delta\left(\frac{x}{2}\right), \sup_{n\in\mathbb{N}_0} |2|^{9(n+1)}\delta\left(\frac{x}{2^{n+2}}\right)\right\}, \qquad x \in S, \tag{60}$$

where $\delta(x)$ *is defined as in Theorem* 5.

Proof. From (55), we have

$$\left\|f(x) - 2^9 f\left(\frac{x}{2}\right)\right\| \le \delta\left(\frac{x}{2}\right), \qquad x \in S. \tag{61}$$

By Corollary 3, there exists a unique mapping $T : S \to X$ such that $T(x) = 2^9 T(\frac{x}{2})$ and (60) holds. By (8) in Corollary 1,

$$T(x) := \lim_{n\to\infty} (\mathcal{L}_\varphi^F)^n(f)(t) = \lim_{n\to\infty} 2^{9n} f\left(\frac{t}{2^n}\right), \qquad x \in S. \tag{62}$$

The rest of the proof is similar to the proof of Theorem 5. □

Corollary 4 *Let* S *be a non-Archimedean normed space and* X *be a complete non-Archimedean normed space with* $|2| < 1$. *Let* ϵ, λ *be positive numbers with* $\lambda \neq 9$, *and* $f : S \to X$ *be a mapping satisfying*

$$\|Df(x,y)\| \le \epsilon(\|x\|^\lambda + \|y\|^\lambda), \qquad x,y \in S.$$

Then there exists a unique nonic mapping $T : S \to X$ *such that*

$$\|f(x) - T(x)\| \le \begin{cases} \dfrac{2\epsilon\|x\|^\lambda}{|9!|^2 \cdot |2|^9}, & \lambda > 9, x \in S; \\ \dfrac{2\epsilon\|x\|^\lambda}{|9!|^2 \cdot |2|^\lambda}, & \lambda < 9, x \in S. \end{cases}$$

Proof. Let $\varphi : S^2 \to \mathbb{R}_+$ be defined by $\varphi(x,y) = \epsilon(\|x\|^\lambda + \|y\|^\lambda)$ for all $x,y \in S$. Then the corollary is followed from Theorems 5 and 6. □

Similar to Corollary 4, one can obtain the following corollary.

Corollary 5 *Let* S *be a non-Archimedean normed space and* X *be a complete non-Archimedean normed space with* $|2| < 1$. *Let* ϵ, λ, μ *be positive numbers with* $\lambda + \mu \neq 9$, *and* $f : S \to X$ *be a mapping satisfying*

$$\|Df(x,y)\| \le \epsilon\|x\|^\lambda \cdot \|y\|^\mu, \qquad x,y \in S.$$

Then there exists a unique nonic mapping $T : S \to X$ *such that*

$$\|f(x) - T(x)\| \le \begin{cases} \dfrac{\epsilon\|x\|^{\lambda+\mu}}{|9!|^2 \cdot |2|^9}, & \lambda + \mu > 9, x \in S; \\ \dfrac{\epsilon\|x\|^{\lambda+\mu}}{|9!|^2 \cdot |2|^{\lambda+\mu}}, & \lambda + \mu < 9, x \in S. \end{cases}$$

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Global Attractivity and Periodicity Behavior of a Recursive Sequence

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ABSTRACT

Our aim in this paper is to study the global stability character and the periodic nature of the solutions of the difference equation

$$x_{n+1} = ax_{n-l} + \frac{bx_{n-k} + cx_{n-s}}{d + ex_{n-t}}, \quad n = 0, 1, ...,$$

where the initial conditions $x_{-r}, x_{-r+1}, x_{-r+2}, ..., x_0$ are arbitrary positive real numbers, $r = \max\{l, k, s, t\}$ is nonnegative integer and a, b, c, d, e are positive constants.

Keywords: stability, periodic solutions, global attractor, difference equations.

Mathematics Subject Classification: 39A10

1. INTRODUCTION

Our goal in this paper is to investigate the global stability character and the periodicity of the solutions of the difference equation

$$x_{n+1} = ax_{n-l} + \frac{bx_{n-k} + cx_{n-s}}{d + ex_{n-t}}, \quad n = 0, 1, ..., \tag{1}$$

where the initial conditions $x_{-r}, x_{-r+1}, x_{-r+2}, ..., x_0$ are arbitrary positive real numbers, $r = \max\{l, k, s, t\}$ is nonnegative integer and a, b, c, d, e are positive constants.

Recently there has been a lot of interest in studying the global attractivity, the boundedness character and the periodicity nature of nonlinear difference equations see for example [1-20].

The study of the nonlinear rational difference equations is interesting and attractive to many researchers working in this field It is quite challenging and rewarding, many real life phenomena are modelling using these equations. Examples from economy, biology,etc. may be obtained in [3,7,11,12] The study of some properties of these equations via the global attractivity, the boundedness and the periodicity of these equations is of great interest. For examples in the articles [11,12,15]. Recently, many researchers have investigated the behavior of the solution of difference equations for example: In [1] Ahmed investigated the behavior of the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-2k+1}}{\pm 1 \pm \prod_{i=1}^{k} x_{n-2i+1}}.$$

Elabbasy et al. [8] studied the boundedness, global stability, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}.$$

Elabbasy et al. [9] investigated the global stability, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = a + \frac{dx_{n-l}x_{n-k}}{cx_{n-s} - b}.$$

Yalçınkaya [32] has studied the following difference equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}.$$

For some related work see [21–35].

Here, we recall some basic definitions and some theorems that we need in the sequel.

Let I be some interval of real numbers and let

$$F : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

A point $\overline{x} \in I$ is called an equilibrium point of Eq.(2) if

$$\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Eq.(2), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 1.1. *(Periodicity)*

A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

Definition 1.2. *(Stability)*

(i) The equilibrium point $\overline{x}$ of Eq.(2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Eq.(2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of Eq.(2) is global attractor if for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of Eq.(2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(2).

(v) The equilibrium point $\overline{x}$ of Eq.(2) is unstable if $\overline{x}$ is not locally stable.

The linearized equation of Eq.(2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\overline{x},\overline{x},...,\overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{3}$$

Theorem A [26] Assume that $p, q \in R$ and $k \in \{0, 1, 2, ...\}$. Then

$$|p| + |q| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, \quad n = 0, 1, ...$$

Remark 1. *Theorem A can be easily extended to a general linear equations of the form*

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \;\; n = 0, 1, ..., \tag{4}$$

where $p_1, p_2, ..., p_k \in R$ and $k \in \{1, 2, ...\}$. Then Eq.(4) is asymptotically stable provided that

$$\sum_{i=1}^{k} |p_i| < 1.$$

Consider the following equation

$$x_{n+1} = g(x_n, x_{n-1}, ...x_{n-K}) \qquad n = 0, 1, 2... \tag{5}$$

The following theorem will be useful for the proof of our results in this paper.

Theorem B [27]: Let $[\alpha, \beta]$ be an interval of real numbers and assume that

$$g : [\alpha, \beta]^{k+1} \to [\alpha, \beta],$$

is a continuous function satisfying the following properties :

(a) $g(x_1, x_2, ..., x_{k+1})$ is non-increasing in one component (for example x_σ) for each $x_r (r \neq \sigma)$ in $[\alpha, \beta]$, and is non-increasing in the remaining components for each $x_\sigma \in [\alpha, \beta]$;

(b) If $(m, M) \in [\alpha, \beta] \times [\alpha, \beta]$ is a solution of the system

$$M = g(m, m, ..., m, M, m, ...m, m) \;\; and \;\; m = g(M, M, ..., M, m, M, ..., M, M),$$

then

$$m = M.$$

Then Eq.(5) has a unique equilibrium $\overline{x} \in [\alpha, \beta]$ and every solution of Eq.(5) converges to $\overline{x}$.

2. LOCAL STABILITY OF THE EQUILIBRIUM POINT OF EQ.(1)

In this section we study the local stability character of the solutions of Eq.(1). The equilibrium points of Eq.(1) are given by the relation

$$\overline{x} = a\overline{x} + \frac{b\overline{x} + c\overline{x}}{cd + de\overline{x}}.$$

If $a \neq 1$, then the equilibrium points of Eq.(1) is given by

$$\overline{x} = 0 \qquad and \qquad \overline{x} = \frac{b + c + d(a-1)}{e(1-a)}.$$

Let $f : (0,\infty)^4 \longrightarrow (0,\infty)$ be a function defined by

$$f(u_0,u_1,u_2,u_3) = au_0 + \frac{bu_1 + cu_2}{d + eu_3}.$$

Therefore at $\overline{x} = \frac{b+c+d(a-1)}{e(1-a)}$

$$\begin{aligned}
\frac{\partial f(\overline{x},\overline{x},\overline{x},\overline{x})}{\partial u_0} &= a = -c_0, \quad \frac{\partial f(\overline{x},\overline{x},\overline{x},\overline{x})}{\partial u_1} = -\frac{b(a-1)}{(b+c)} = -c_1, \\
\frac{\partial f(\overline{x},\overline{x},\overline{x},\overline{x})}{\partial u_2} &= -\frac{c(a-1)}{(b+c)} = -c_2, \quad \frac{\partial f(\overline{x},\overline{x},\overline{x},\overline{x})}{\partial u_3} = \frac{(a-1)(b+c-d+ad)}{(b+c)} = -c_3
\end{aligned}$$

Then we see that at $\overline{x} = 0$

$$\begin{aligned}
\frac{\partial f(\overline{x},\overline{x},\overline{x},\overline{x})}{\partial u_0} &= a = -c_0, \quad \frac{\partial f(\overline{x},\overline{x},\overline{x},\overline{x})}{\partial u_1} = \frac{b}{d} = -c_1 \\
\frac{\partial f(\overline{x},\overline{x},\overline{x},\overline{x})}{\partial u_1} &= \frac{c}{d} = -c_2, \quad \frac{\partial f(\overline{x},\overline{x},\overline{x},\overline{x})}{\partial u_1} = 0 = -c_3
\end{aligned}$$

Then the linearized equation of Eq.(1) about $\overline{x}$ is

$$y_{n+1} + c_0 y_{n-l} + c_1 y_{n-k} + c_2 y_{n-s} + c_3 y_{n-t} = 0. \tag{6}$$

Theorem 2.1. *Assume that*

$$1 < \frac{d(1-a)}{(b+c)}.$$

Then the positive equilibrium point $\overline{x} = 0$ of Eq.(1) is locally asymptotically stable.

Proof: It is follows by Theorem A that, Eq.(6) is asymptotically stable if

$$|c_3| + |c_2| + |c_1| + |c_0| < 1.$$

$$|0| + \left|\frac{c}{d}\right| + \left|\frac{b}{d}\right| + |a| < 1,$$

and so

$$1 < \frac{d(1-a)}{(b+c)}.$$

This completes the proof.

Theorem 2.2. *Assume that*

$$3 < \frac{d(1-a)}{(b+c)}, \qquad a < 1$$

Then the positive equilibrium point $\overline{x} = \frac{b+c+d(a-1)}{e(1-a)}$ of Eq.(1) is locally asymptotically stable.

Proof: It is follows by Theorem A that, Eq.(6) is asymptotically stable if

$$\left|\frac{(a-1)(b+c-d+ad)}{(b+c)}\right| + \left|-\frac{c(a-1)}{(b+c)}\right| + \left|-\frac{b(a-1)}{(b+c)}\right| + |a| < 1.$$

$$3a - \frac{d(a-1)^2}{(b+c)} < 3,$$

and so

$$1 < \frac{d(1-a)}{(b+c)}.$$

This completes the proof.

3. EXISTENCE OF PERIODIC SOLUTIONS

In this section we study the existence of periodic solutions of Eq.(1.1).

THEOREM 3.1. *Eq.(1) has a prime period two solutions if and only if*

$$e^2(b+c+d+ad)^2(a+1)^2 - 4ade^2(a+1)(b+c+d+ad) > 0 \ , \quad , \quad k,l,s,t-\text{even}. \tag{7}$$

Proof: First suppose that there exists a prime period two solution

$$..., p, q, p, q, ...,$$

of Eq.(1). We will prove that Condition (7) holds.

We see from Eq.(1) (when $k,l,s,t-$even) that

$$p = aq + \frac{bq+cq}{d+eq}, \qquad q = ap + \frac{bp+cp}{d+ep}.$$

Then

$$dp + epq = adq + aeq^2 + bq + cq, \tag{8}$$

and

$$dq + epq = adp + aep^2 + bp + cp. \tag{9}$$

Subtracting (8) from (9) gives

$$d(p-q) = ad(q-p) + ae(q^2-p^2) + b(q-p) + c(q-p).$$

Since $p \neq q$, it follows that

$$p + q = -\frac{(b+c+d+ad)}{ae}. \tag{10}$$

Again, adding (8) and (9) yields

$$2epq + d(p+q) = ad(p+q) + ad(p+q)^2 + 2aepq + b(p+q) + c(p+q). \tag{11}$$

It follows by (10), (11) and the relation $p^2+q^2 = (p+q)^2 - 2pq$ for all $p, q \in R$, that

$$pq = \frac{d(a+b+c+d+ad)}{ae^2(a+1)}. \tag{12}$$

Now it is clear from Eq.(10) and Eq.(12) that p and q are the two positive distinct roots of the quadratic equation

$$t^2 + \left(\tfrac{(b+c+d+ad)}{ae}\right)t + \left(\tfrac{d(a+b+c+d+ad)}{ae^2(a+1)}\right) = 0, \tag{13}$$

$$ae^2(a+1)t^2 + e(a+1)(b+c+d+ad)t + d(a+b+c+d+ad) = 0,$$

and so

$$((a+1)(b+c+d+ad))^2 > 4ad(a+1)(b+c+d+ad),$$

thus

$$(a+1)(b+c+d+ad) > 4ad.$$

Therefore Inequality (7) holds.

Second suppose that Inequality (7) is true. We will show that Eq.(1) has a prime period two solution. Assume that

$$p = \tfrac{-e(a+1)(b+c+d+ad)+\sqrt{\xi}}{2ae^2(a+1)} = \frac{-eAB+\sqrt{\xi}}{2ae^2A},$$

and

$$q = \frac{-eAB-\sqrt{\xi}}{2ae^2A}, \text{where} \quad A = (a+1), \quad B = (a+b+c+d+ad)$$

where $\xi = e^2(a+1)^2(b+c+d+ad)^2 - 4ade^2(a+1)(b+c+d+ad)$.
We see from Inequality (7) that

$$e^2(a+1)^2(b+c+d+ad)^2 - 4ade^2(a+1)(b+c+d+ad) > 0,$$

then after dividing by $e^2(a+1)(b+c+d+ad)$ we see that

$$\Rightarrow \qquad (a+1)(b+c+d+ad) > 4ad,$$

Therefore p and q are distinct real numbers.
Set

$$\begin{aligned} x_{-l} &= p, \ x_{-l+1} = q, \quad , x_{-k} = q, \ x_{-k+1} = p, \\ x_{-s} &= p, \ x_{-s+1} = q, \quad x_{-t} = p, \ x_{-t+1} = q, \ \text{and} \quad x_0 = q. \end{aligned}$$

We wish to show that

$$x_1 = x_{-1} = p \quad \text{and} \quad x_2 = x_0 = q.$$

It follows from Eq.(1.1) that

$$\begin{aligned} x_1 = ax_{-l} + \frac{bx_{-k}+cx_{-s}}{d+ex_{-t}} &= ap + \frac{bp+cp}{d+ep} = ap + \frac{(b+c)p}{d+ep} \\ &= ap + \frac{(b+c)\left(\frac{-eAB+\sqrt{\xi}}{2ae^2A}\right)}{d+e\left(\frac{-eAB+\sqrt{\xi}}{2ae^2A}\right)}. \end{aligned}$$

Multiplying the denominator and numerator of the right side by $2ae^2A$ gives

$$x_1 = ap + \frac{(b+c)\left(-eAB+\sqrt{\xi}\right)}{2ae^2Ad-e^2AB+e\sqrt{\xi}},$$

Multiplying the denominator and numerator of the right side by $\{2ae^2Ad - e^2AB - e\sqrt{\xi}\}$

$$\begin{aligned} x_1 &= ap + \frac{(b+c)\left(-eAB+\sqrt{\xi}\right)\left(2ae^2Ad-e^2AB-e\sqrt{\xi}\right)}{\left(2ae^2Ad-e^2AB+e\sqrt{\xi}\right)\left(2ae^2Ad-e^2AB-e\sqrt{\xi}\right)}, \\ &= ap + \frac{(b+c)\left[-2ade^3A^2B+e^3A^2B^2-e(e^2A^2B^2-4ade^2AB)+2ade^2A\sqrt{\xi}\right]}{(2ade^2A-e^2AB)^2-\left(e\sqrt{\xi}\right)^2}, \\ &= ap + \frac{(b+c)\left[2ade^3AB(2-A)+2ade^2A\sqrt{\xi}\right]}{e^4A^2(4a^2d^2+B^2-4adB)-e^4A^2B^2+4ade^4AB}, \end{aligned}$$

Replacing $A = (a+1)$ and $B = (b+c+d+ad)$ in denominator of above equation gives

$$\begin{aligned} x_1 &= ap + \frac{(b+c)\left[2ade^3AB(1-a)+2ade^2A\sqrt{\xi}\right]}{4a^2d^2e^4(a+1)^2-4ade^4(a+1)^2(b+c+d+ad)+4ade^4(1+a)(b+c+d+ad)} \\ &= ap + \frac{(b+c)\left[2ade^3AB(1-a)+2ade^2A\sqrt{\xi}\right]}{4a^2d^2e^4(a+1)^2-4ade^4(a+1)^2(b+c+d+ad)+4ade^4(1+a)(b+c+d+ad)} \\ &= ap - \frac{(b+c)\left[2ade^3AB(1-a)+2ade^2A\sqrt{\xi}\right]}{4a^2de^2(a+1)(b+c)} \end{aligned}$$

Dividing numerator and denominator by$(b+c)$ we get

$$x_1 = ap - \frac{2ade^3AB(1-a)+2ade^2A\sqrt{\xi}}{4a^2de^2(a+1)} = ap - \frac{eB(1-a)+\sqrt{\xi}}{2ae^2}$$

Now inserting the value of p we get

$$\begin{aligned} x_1 &= \frac{-eAB+\sqrt{\xi}}{2e^2A} - \frac{eB(1-a)+\sqrt{\xi}}{2ae^2} \\ &= \frac{1}{2e^2}\left(\frac{-eAB+\sqrt{\xi}}{A} - \frac{eB(1-a)+\sqrt{\xi}}{a}\right) = \frac{-eAB+a\sqrt{\xi}-eB(1-a)(1+a)-(a+1)\sqrt{\xi}}{2ae^2(a+1)} \\ &= \frac{-eaB(a+1)-eB+eBa^2-\sqrt{\xi}}{2ae^2(a+1)} \end{aligned}$$

putting the value of $B = \ (b+c+d+ad)$ we get

$$x_1 = \frac{-e(b+c+d+ad)(a+1)-\sqrt{\xi}}{2ae^2(a+1)} = q$$

Similarly as before one can easily show that

$$x_2 = p.$$

Then it follows by induction that

$$x_{2n} = p \quad \text{and} \quad x_{2n+1} = q \quad \text{for all} \quad n \geq -1.$$

Thus Eq.(1) has the positive prime period two solution

$$...,p,q,p,q,...,$$

where p and q are the distinct roots of the quadratic equation (13) and the proof is complete.

The following Theorems can be proved similarly.

THEOREM 3.2. *Eq.(1) has a prime period two solutions if and only if*

$$e^2(a+1)^2(d+ad+b+c)^2 - 4e^2(ad+b+c)(a+1)(d+ad+b+c) > 0,\ t-odd\ , l,k,s-even.$$

THEOREM 3.3. *Eq.(1) has a prime period two solutions if and only if*

$$e^2(a+1)^2(d+ad-b-c)^2 - 4e^2ad(a+1)(d+ad-b-c) > 0\ ,\ l-even\ , s,k,t-odd.$$

THEOREM 3.4. *Eq.(1) has a prime period two solutions if and only if*

$$e^2(d-ad+b+c)^2(a-1)^2 - 4e^2(a-1)^2(b+c)(d-ad+b+c) > 0,\ l,t-odd, s,k-even.$$

THEOREM 3.5. *Eq.(1) has a prime period two solutions if and only if*

$$e^2(a+1)^2(b+c-d-ad)^2 + 4ae^2(a+1)(b+c-d-ad)(d-b-c) > 0,\ l,t-even, s,k-odd.$$

4. GLOBAL ATTRACTIVITY OF THE EQUILIBRIUM POINT OF EQ.(1)

In this section we investigate the global attractivity character of solutions of Eq.(1).

THEOREM 4.1. *The equilibrium point $\overline{x}$ of Eq.(1) is global attractor.*

Proof: Let p, q are a real numbers and assume that $f : [p, q]^4 \longrightarrow [p, q]$ be a function defined by

$$f(u_0, u_1, u_2, u_3) = au_0 + \frac{bu_1 + cu_2}{d + eu_3}.$$

We can easily see that the function $f(u_0, u_1, u_2, u_3)$ increasing in u_0, u_1, u_2 and decreasing in u_3.

Suppose that (m, M) is a solution of the system

$$m = f(m, m, m, M) \quad and \quad M = f(M, M, M, m).$$

Then from Eq.(1), we see that

$$m = am + \frac{(b + c)m}{d + eM}, \quad M = aM + \frac{(b + c)M}{d + em},$$

That is

$$1 - a = \frac{b + c}{d + eM}, \quad 1 - a = \frac{b + c}{d + em},$$

or,

$$\frac{b + c}{d + eM} = \frac{b + c}{d + em},$$

then $d + eM = d + em$. Thus $M = m$. It follows by the Theorem B that $\overline{x}$ is a global attractor of Eq.(1) and then the proof is complete.

5. NUMERICAL EXAMPLES

For confirming the results of this paper, we consider numerical examples which represent different types of solutions to Eq. (1).

Example 1. We assume $l = 3,\ k = 2,\ s = 3,\ t = 2\ x_{-3} = 7,\ x_{-2} = 2,\ x_{-1} = 1,\ x_0 = 9,\ a = 0.1,\ b = 0.2,\ c = 0.9,\ d = 0.6\ \ e = 0.3$**.** See Fig. 1.

Example 2. See Fig. 2, since $l = 4,\ k = 3,\ x_{-4} = 12,\ x_{-3} = 7,\ x_{-2} = 9,\ x_{-1} = 10,\ x_0 = 5,\ a = 0.9,\ b = 2,\ c = 7,\ d = 3$.

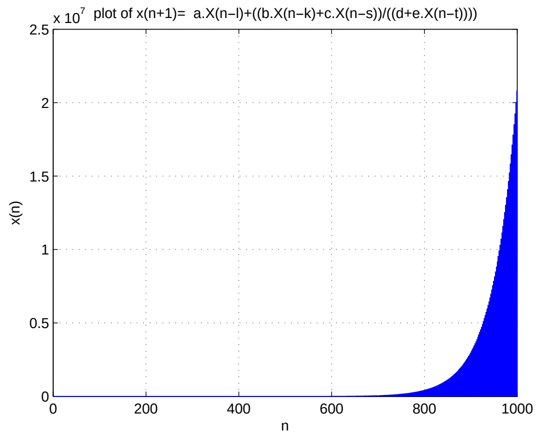

Figure 1.

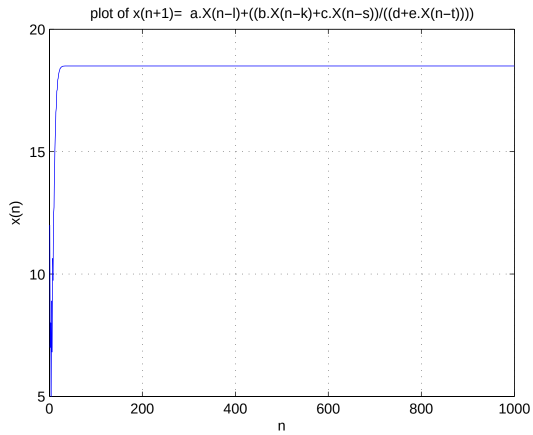

Figure 2.

Example 3. We consider $l = 3,\ k = 2,\ x_{-3} = 12, x_{-2} = 7,\ x_{-1} = 9,\ x_0 = 10,\ a = 0.3,\ b = 1.5,\ c = 11,\ d = 8$. See Fig. 3.

Example 4. See Fig. 4, since $l = 3,\ k = 4,\ x_{-4} = 12,\ x_{-3} = 7,\ x_{-2} = 9,\ x_{-1} = 10,\ x_0 = 5,\ a = 0.6,\ b = 2,\ c = 7,\ d = 4.$

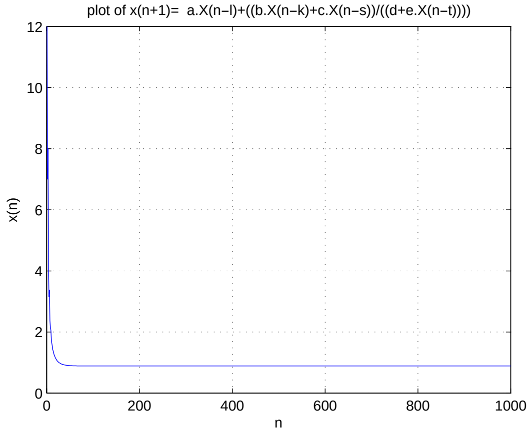

Figure 3.

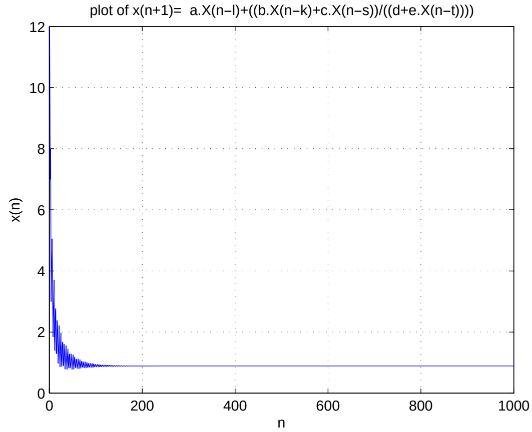

Figure 4.

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PERIODIC ORBITS OF SINGULAR RADIALLY SYMMETRIC SYSTEMS

SHENGJUN LI [1,2], WULAN LI [3] AND YIPING FU [1]

Abstract. We study the existence of periodic orbits of planar radially symmetric systems with a singularity. These orbits have periods which are large integer multiples of the period of the forcing, and rotate exactly once around the origin in their period time. The proof is based on the use of topological degree theory and a fixed point theorem in cones.

1. Introduction

In the paper [11], Fonda and J.Ureña have studied the periodic, subharmonic and quasi-periodic orbits for the radially symmetric system

$$\ddot{x} + f(t, |x|)\frac{x}{|x|} = 0, \quad x \in \mathbb{R}^2 \setminus \{0\}, \tag{1.1}$$

where $f \in C((\mathbb{R}/T\mathbb{Z}) \times (0, \infty), \mathbb{R})$ may be singular at the origin. As mentioned in [10], many phenomena of the nature obey to laws of (1.1), such as the Newtonian equation for the motion of a particle subjected to the gravitational attraction of a sun which lies at the origin. Setting $\rho(t) = |x(t)|$, they proved the following result:

Theorem 1.1 Suppose that $f(t, \rho) > 0$ for $t \in [0, T], \rho > 0$ and satisfies the following conditions:

(A_1) $\lim_{\rho\to\infty} f(t, \rho)/\rho = 0$, for *a.e.* $t \in \mathbb{R}$.

(A_2) There exists some function $h \in L^1_{loc}(\mathbb{R})$ and some number $r_0 > 0$ such that

$$|f(t, \rho)| \leq h(t)\rho, \text{ on } \mathbb{R} \times [r_0, +\infty].$$

Then, there exists a connect set $\mathcal{C}$ of T-radially periodic solutions of (1.1) which goes from zero to infinity.

We look for solutions $x(t) \in \mathbb{R}^2$ which never attain the singularity, in the sense that

$$x(t) \neq 0, \quad \text{for every } t \in \mathbb{R}.$$

Using the same idea in [8], we may write the solutions of (1.1) in polar coordinates

$$x(t) = \rho(t)(\cos\varphi(t), \sin\varphi(t)).$$

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Then we have the collisionless orbits if $\rho(t) > 0$ for every t. Moreover, equation (1.1) is equivalent to the following system

$$\begin{cases} \ddot{\rho} + f(t,\rho) - \frac{\mu^2}{\rho^3} = 0, \\ \rho^2\dot{\varphi} = \mu, \end{cases} \tag{1.2}$$

where μ is the angular momentum of $x(t)$. Recall that μ is constant in time along any solution.

If x is a T-radially periodic, then ρ must be T-periodic. We will prove the existence of a T-periodic solution ρ of the first equation in (1.2). We thus consider the boundary value problem

$$\begin{cases} \ddot{\rho} + f(t,\rho) = \frac{\mu^2}{\rho^3}, \\ \rho(0) = \rho(T), \quad \dot{\rho}(0) = \dot{\rho}(T). \end{cases} \tag{1.3}$$

Let $\mu = 0$, (1.3) can be written the singular T-periodic problem

$$\ddot{\rho} + f(t,\rho) = 0. \tag{1.4}$$

The question about the existence of non-collision periodic orbits for scalar e-quations and dynamical systems with singularities has attracted much attention of many researchers over many years. See[5, 7, 12, 13, 15, 24]. Usually, the proof is based on variational approach [1, 2, 6, 16, 22], the method of upper and lower solutions [3, 21], some fixed point theorems [19, 26, 27, 28, 29] or the topological degree theory [17, 18, 23, 30]. In particular, several existence results for the following scalar differential equation

$$\ddot{x} + a(t)x = f(t,x) \tag{1.5}$$

has been established in [23, 25, 27]. Note that (1.5) is a nonlinear perturbation of Hill equation

$$\ddot{x} + a(t)x = 0.$$

Moreover, it has been found that a particular case of (1.5), the Ermakov–Pinney equation, plays an important role in studying the Lyapunov stability of periodic solutions of Lagrangian equations [20].

Our main motivation is to obtain by the above papers [9, 17, 27], by the use of topological degree theory and a well-known fixed point theorem in cones, we prove the existence of large-amplitude periodic orbits whose minimal period is an integer multiple of T, and rotate exactly once around the origin in their period time.

The rest of this paper is organized as follows. In Section 2, some preliminaries are given. In Section 3, , we give the main results.

2. Preliminaries

We say that the scale linear equation

$$\ddot{x} + a(t)x = 0 \tag{2.1}$$

is nonresonant if its unique T-periodic solution is the trivial one. When (2.1) is nonresonant, as a consequence of Fredholm's alternative, the nonhomogeneous equation

$$\ddot{x} + a(t)x = h(t)$$

admits a unique T-periodic solution which can be written as

$$x(t) = \int_0^T G(t,s)h(s)ds,$$

where $G(t,s)$ is the Green's function of (2.1), associated with periodic boundary conditions

$$x(0) = x(T), \quad \dot{x}(0) = \dot{x}(T). \tag{2.2}$$

Throughout this paper, we always assume that the following standing hypothesis is satisfied:

(H) $a(t)$ is a continuous T-function and the Green's function of (2.1) is positive for all $(t,s) \in [0,T] \times [0,T]$.

In other words, the strict anti-maximum principle holds for (2.1)-(2.2). It is prove in [25] that if $a(t)$ satisfies $a \succ 0$ and $\underline{\lambda}_1(a) > 0$, then condition (H) is satisfied; here the notation $a \succ 0$ means that $a(t) \geq 0$ for all $t \in [0,T]$ and $a(t) > 0$ for t in a subset of positive measure, $\underline{\lambda}_1(a)$ denotes the first anti-periodic eigenvalue of

$$x'' + (\lambda + a(t))x = 0$$

subject to the anti-periodic boundary conditions

$$x(0) + x(T) = 0, \quad \dot{x}(0) + \dot{x}(T) = 0.$$

Now we make condition (H) clear. When $a(t) \equiv k^2$, condition (H) is equivalent to saying that $0 < k^2 \leq \lambda_1 = (\pi/T)^2$, where λ_1 is the first eigenvalue of the homogeneous equation $x'' + k^2 x = 0$ with Dirichlet boundary conditions $x(0) = x(T) = 0$. For a non-constant function $a(t)$, there is an L^p-criterion proved in [25]. To describe these, we use $\|\cdot\|_q$ to denote the usual L^q-norm over $(0,T)$ for any given exponent $q \in [1,\infty]$. The conjugate exponent of q is denoted by $p : \frac{1}{p} + \frac{1}{q} = 1$. Let $\mathbf{M}(q)$ denote the best Sobolev constant in the following inequality

$$C\|u\|_q^2 \leq \|u'\|_2^2 \qquad \text{for all } u \in H_0^1(0,T).$$

The explicit formula for $\mathbf{M}(q)$ is

$$\mathbf{M}(q) = \begin{cases} \frac{2\pi}{qT^{1+2/q}} \left(\frac{2}{q+2}\right)^{1-2/q} \left(\frac{\Gamma(1/q)}{\Gamma(1/2+1/q)}\right)^2, & \text{for } 1 \leq q < \infty, \\ \frac{4}{T}, & \text{for } q = \infty, \end{cases}$$

where $\Gamma(\cdot)$ is the Gamma function of Euler. Let us define

$$\mathcal{A} = \{a \in L^p[0,T] : a \succ 0, \|a\|_p < \mathbf{M}(2q) \ \text{ for some } 1 \leq p \leq +\infty\} \tag{2.3}$$

Lemma 2.1[25] Assume that $a(t) \in \mathcal{A}$, then (2.1) satisfies the standing hypothesis (H), i.e, $G(t,s) > 0$ for all $(t,s) \in [0,T] \times [0,T]$.

Remark 2.2 If $p = 1$, condition $\|a\|_p < \mathbf{M}(2q)$ can be weakened to $\|a\|_1 \leq \mathbf{M}(\infty) = 4$ by the celebrated stability criterion of Lyapunov. In case $p = \infty$, condition $\|a\|_p < \mathbf{M}(2q)$ reads as $\|a\|_\infty < \mathbf{M}(2) = \pi^2$, which is a well known criterion for the anti-maximum principle used in related literature. In this case,$\|a\|_p < \mathbf{M}(2q)$ can be weakened to $a(t) \prec \pi^2$.

Under hypothesis (H), we always denote

$$M = \max_{0 \leq s,t \leq T} G(t,s), \qquad m = \min_{0 \leq s,t \leq T} G(t,s), \qquad \sigma = \frac{m}{M}. \tag{2.4}$$

Thus $M > m > 0$ and $0 < \sigma < 1$.

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In order to prove our results, we need two preliminary results. The first one is a well-known fixed point theorem in cones, which can be found in [14].

Theorem 2.3 Let X be a Banach space and K a cone in X. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1, \bar{\Omega}_2 \backslash \Omega_2$. Let

$$T : K \cap (\bar{\Omega}_2 \backslash \Omega_1) \to K$$

be a continuous and completely continuous operator such that

(i) $\|Tx\| \leq \|x\|$ for $x \in K \cap \partial\Omega_1$;

(ii) There exist $\psi \in K$ $\{0\}$ such that $x \neq Tx + \lambda\psi$ for $x \in K \cap \partial\Omega_2$ and $\lambda > 0$.

Then T has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$. The same conclusion remains valid if (i) holds on $K \cap \partial\Omega_2$ and (ii) holds $K \cap \partial\Omega_1$.

In applications below, we take $X = C[0, T]$ with the supremum norm $\| \cdot \|$ and define

$$K = \{x \in X : x(t) \geq 0 \text{ for all } t \in [0, T] \text{ and } \min_{0 \leq t \leq T} \geq \sigma\|x\|\}.$$

where σ is as in (2.4).

One can readily verify that K is a cone in X. Define an operator $T : X \to X$ by

$$(Tx)(t) = \int_0^T G(t, s)F(s, x(s))ds$$

for $x \in X$ and $t \in [0, T]$, where $F : [0, T] \times \mathbb{R} \to [0, \infty)$ is continuous and $G(t, s)$ is the Green's function of (2.1).

Lemma 2.4 T is well defined and maps X into K. Moreover, T is continuous and completely continuous.

Proof It is easy to see that T is continuous and completely continuous since F is a continuous function. Thus, we only need to show that $T(X) \subset K$. Let $x \in X$, then we have

$$\begin{aligned}
\min_{0 \leq x \leq T} (Tx)(t) &= \min_{0 \leq x \leq T} \int_0^T G(t, s)F(s, x(s))ds \\
&\geq m \int_0^T F(s, x(s))ds \\
&= \sigma M \int_0^T F(s, x(s))ds \\
&\geq \sigma \max_{0 \leq x \leq T} \int_0^T G(t, s)F(s, x(s))ds \\
&= \sigma\|Tx\|.
\end{aligned}$$

This implies that $T(X) \subset K$ and the proof is completed.

To state the second preliminary result, we recall some notation and terminology from [9]. Let X be a Banach space of functions, such that $C^1([0, T]) \subseteq X \subseteq C([0, T])$, with continuous immersions, and set $X_* = \{\rho \in X : \min \rho > 0\}$.

Define the following two operators:

$$D(L) = \{\rho \in W^{2,1}(0, T) : \rho(0) = \rho(T), \dot{\rho}(0) = \dot{\rho}(T)\},$$

$$L : D(L) \subset X \to L^1(0, T), \quad L\rho = \ddot{\rho}, \tag{2.5}$$

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and

$$N : X_* \to L^1(0,T), \quad (N\rho)(t) = -f(t,\rho(t))$$

Taking $\sigma \in \mathbb{R}$ not belonging to the spectrum of L, (1.5) can be translated to the fixed problem

$$\rho = (L - \sigma I)^{-1}(N - \sigma I)\rho.$$

We will say that a set $\Omega \subseteq X$ is uniformly positively bounded below if there is a constant $\delta > 0$ such that $\min \rho \geq \delta$ for every $\rho \in \Omega$. we need the following theorem, which has been proved in [9].

Theorem 2.5 Let Ω be an open bounded subset of X, uniformly positively bounded below. Assume that there is no solution of (1.5), on the boundary $\partial\Omega$, and that

$$deg(I - (L - \sigma I)^{-1}(N - \sigma I), \Omega, 0) \neq 0.$$

Then, there exists a $k_1 \geq 1$ such that, for every integer $k \geq k_1$, systems (1.1) has a periodic solution $x_k(t)$ with minimal period kT, which makes exactly one revolution around the origin in the period time kT. The function $|x_k(t)|$ is T-periodic and, when restricted to $[0,T]$, it belongs to Ω. Moreover, if μ_k denotes the angular momentum associated to $x_k(t)$, then

$$\lim_{k\to\infty} \mu_k = 0.$$

3. Main Results

In this section, we state and prove the main results. First we recall that $\mathcal{A}$ denotes the set defined by (2.3).

Theorem 3.1 Suppose that there exist $a(t) \in \mathcal{A}$ and $0 < r < R$ such that

(H_1) $-a(t)\rho \leq f(t,\rho) \leq \sigma/r - 1/\sigma r$, $\forall \rho \in [\sigma r, r]$,

(H_2) $f(t,\rho) \geq 0$, $\forall \rho \in [\sigma R, R]$.

Then, equation(1.4) has a T-periodic solution, and there exists a $k_1 \geq 1$ such that, for every integer $k \geq k_1$, system (1.1) has a periodic solution with minimal period kT, which makes exactly one revolution around the origin in the period time kT. Moreover, exist constant $C > 0$ (independent of μ and k) such that

$$\frac{1}{C} < |x_k(t)| < C, \quad \text{for every } t \in \mathbb{R} \text{ and every } k \geq k_1,$$

and, if μ_k denotes the angular momentum associated to $x_k(t)$ then

$$\lim_{k\to\infty} \mu_k = 0.$$

Now we begin by showing that Theorem 3.1, and use topological degree theory. To this end, we deform (1.4) to a simpler singular autonomous equation

$$\ddot{\rho} + \frac{1}{r^2}\rho - \frac{1}{\rho} = 0.$$

where r is as in Theorem 3.1.

In order to apply Theorem 2.5, we consider the $\mu = 0$ and study for $\tau \in [0,1]$, the following homotopy equation

$$\ddot{\rho} + f(t,\rho;\tau) = 0, \quad \tau \in [0,1], \tag{3.1}$$

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associated to periodic boundary conditions

$$\rho(0) = \rho(T), \qquad \dot{\rho}(0) = \dot{\rho}(T),$$

where

$$f(t, \rho; \tau) = \tau f(t, \rho) + (1 - \tau)(\frac{\rho}{r^2} - \frac{1}{\rho}).$$

Note that $f(t, \rho; \tau)$ satisfies the conditions:

(H_1') $f(t, \rho; \tau) + a(t)\rho \geq 0,\ \forall \rho \in [\sigma r, R]$,

(H_2') $f(t, \rho; \tau) \leq 0, \forall \rho \in [\sigma r, r]$ and $f(t, \rho; \tau) \geq 0,\ \forall \rho \in [\sigma R, R]$.

uniformly with respect to $\tau \in [0, 1]$. We need to find a priori estimates for the possible positive T-periodic solutions of (3.1). The important point to be proved is the following.

Proposition 3.2 Suppose that there exist $a \in \mathcal{A}$ and $0 < r < R$ such that $f(t, \rho; \tau)$ satisfies (H_1') and (H_2'). Then, equation (3.1) has at least one T-periodic solution.

Proof The existence is established using Theorem 2.3. To do so, let us write equation (3.1) as

$$\ddot{\rho} + a(t)\rho = f(t, \rho; \tau) + a(t)\rho.$$

Define the open sets

$$\Omega_1 = \{\rho \in X : \|\rho\| < r, \quad \Omega_2 = \{\rho \in X : \|\rho\| < R\}.$$

Let K be a cone defined by (2.5) and define an operator on K by

$$(\Phi\rho)(t) = \int_0^T G(t, s)\,[f(s, \rho(s); \tau) + a(t)\rho]\,ds.$$

Clearly, $\Phi : K \cap (\bar{\Omega}_R \backslash \Omega_r) \to C[0, T]$ is continuous and completely continuous since $f : [0, T] \times [\sigma r, R] \times [0, 1]$ is continuous. Also we have $\Phi(K) \subset K$.

By the first inequality of condition (H_1'), we have $f(t, \rho; \tau) + a(t)\rho \geq a(t)\rho,\ \forall \rho \in [\sigma r, r]$. Let $\psi \equiv 1$, so $\psi \in K$. Now we prove that

$$\rho \neq \Phi\rho + \lambda\rho,\ \forall \rho \in K \cap \partial\Omega_r \text{ and } \lambda > 0. \tag{3.2}$$

Suppose not, that is, suppose there exist $\rho_0 \in K \cap \partial\Omega_r$ and $\lambda_0 > 0$ such that $\rho_0 = \Phi\rho_0 + \lambda_0\psi$. Now since $\rho_0 \in K \cap \partial\Omega_r$, then $\rho_0(t) \geq \sigma\|\rho_0\| = \sigma r$. Let $\mu = \min_{t\in[0,T]} \rho_0(t)$. Then we have

$$\begin{aligned}
\rho_0(t) &= (\Phi\rho_0)(t) + \lambda_0 \\
&= \int_0^T G(t, s)[f(s, \rho_0(s); \tau) + a(s)\rho_0(s)]ds + \lambda_0 \\
&\geq \int_0^T G(t, s)a(s)\rho_0(s)ds + \lambda_0 \\
&\geq \mu \int_0^T G(t, s)a(s)ds + \lambda_0 = \mu + \lambda_0,
\end{aligned}$$

note $\int_0^T G(t, s)a(s)ds = 1$. This implies $\mu \geq \mu + \lambda_0$, a contradiction. Therefore, (3.2) holds.

On the other hand, by the second inequality of condition (H_2'), we have

$$f(t, \rho; \tau) + a(t)\rho \leq a(t)\rho, \quad \forall \rho \in [\sigma R, R].$$

Now we prove that

$$\|\Phi x\| \leq \|x\|, \quad x \in K \cap \partial\Omega_R. \tag{3.3}$$

In fact, for any $\rho \in K \cap \partial\Omega_R$, we have

$$\begin{aligned}(\Phi x)(t) &= \int_0^T G(t,s)\left[f(s,\rho_0(s);\tau) + a(s)\rho(s)\right] ds \\ &\leq \int_0^T G(t,s)a(s)\rho(s)ds \\ &\leq \int_0^T G(t,s)a(s)ds \cdot \max_{t\in[0,T]} \rho(t) = \|\rho\|.\end{aligned}$$

Therefore, $\|\Phi\rho\| \leq \|\rho\|$, that is, (3.3) holds.

It follows from Theorem 3.3, (3.2) and (3.3) that Φ has a fixed point $\rho \in K \cap (\bar{\Omega}_R \backslash \Omega_r)$, the proof is finished. □

Proof of Theorem 3.1 Now, from Proposition 3.2, this fixed point is a positive solution of (3.1) satisfying $r \leq \|x\| \leq R$.

Notice the boundary condition $\dot{\rho}(0) = \dot{\rho}(T)$. Integrate (3.1) from 0 to T, we get

$$\int_0^T \ddot{\rho}(t)dt = -\int_0^T f(t,\rho(t);\tau)dt = 0.$$

Thus $\|f(t,\rho(t);\tau)\|_1 = 2\|f^+(t,\rho(t);\tau)\|_1$. Since $\rho(0) = \rho(T)$, there exists $t_1 \in [0,T]$ such that $\dot{\rho}(t_1) = 0$. Therefore

$$\begin{aligned}\|\dot{\rho}\| &= \max_{0\leq t\leq T} |\dot{\rho}(t)| = \max_{0\leq t\leq T} \left|\int_{t_1}^t \ddot{\rho}(s)ds\right| \\ &\leq \int_0^T |f(s,\rho(s);\tau)|\, ds = 2\int_0^T \left|f^+(s,\rho(s);\tau)\right| ds \\ &\leq 2\int_0^T |a(s)\rho(s)|\, ds \\ &\leq 2R\|a\|_1 := H.\end{aligned}$$

where $f^+(t,\rho(t);\tau) = \max\{f(t,\rho(t);\tau), 0\}$.

Define the linear operator L as in (2.5) and the Nemytzkii operator

$$N_\tau : X_* \to L^1(0,T),$$

$$(N_\tau \rho)(t) = -f(t,\rho(t);\tau),$$

(3.1) also can be translated to the fixed problem

$$\rho = (L-\sigma I)^{-1}(N_\tau - \sigma I)\rho, \tag{3.4}$$

since $L - \sigma I$ is invertible.

Take $C = \max\{1/r, R, H\}$ and let the open bounded in X be

$$\Omega = \{\rho \in X : \frac{1}{C} < \rho(t) < C \quad \text{and} \quad |\dot{\rho}(t)| < C \quad \text{for all } t \in [0,T]\}.$$

Obviously, Ω is an open subset of $C^1[0,T]$, and equation (3.4) has no solutions on $\partial\Omega$.

In order to compute the degree, we consider equation (3.1). By homotopy invariance of degree, the degree has to be the same for every $\tau \in [0,1]$. Therefore, we consider the equation (3.1) with $\tau = 0$, that is the equation

$$\ddot{\rho} + \frac{1}{r^2}\rho - \frac{1}{\rho} = 0,$$

which is equivalent to the system

$$\dot{Y} = F(Y) = \left(u, \frac{\rho}{r^2} - \frac{1}{\rho}\right),$$

where $Y = (\rho, u)$.

It is easy to know that F has a unique zero (ρ_0, u_0) and the determinant of Jacobian matrix satisfies $|J_F(\rho_0, u_0)| > 0$. By Lemma the result of Capietto, Mawhin and Zanolin [4], the Leray-Schauder degree of $I - L^{-1}N(\mu, \cdot)$ is equal to the Brouwer degree of F, i.e.,

$$d_L(I - L^{-1}N(\mu, \cdot,), \Omega, 0) = d_B(F, (\frac{1}{C}, C) \times (-C, C)) = 1.$$

By Theorem 2.1, the proof of Theorem 3.1 is thus completed.

It is a direct consequence of Theorem 3.1 taking r and R small and big enough, respectively. We obtain

Corollary 3.3 Assume that the following two conditions hold:

(H_3) $\lim\limits_{\rho \to 0^+} f(t, \rho)/\rho = -\infty$, uniformly for $t \in [0, T]$,

(H_4) $\lim\limits_{\rho \to +\infty} f(t, \rho)/\rho = +\infty$, uniformly for $t \in [0, T]$

Then problem (1.1) has the same conclusion of Theorem 3.1.

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Approximate ternary Jordan ring homomorphisms in ternary Banach algebras

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Abstract. Let A and B be real ternary Banach algebras. An additive mapping $\Im : (A, [\ ]_A) \longrightarrow (B, [\ ]_B)$ is called a ternary Jordan homomorphism if $\Im([x, x, x]_A) = [\Im(x), \Im(x), \Im(x)]_B$ for all $x \in A$.

In this paper, we investigate the stability and superstability of ternary Jordan ring homomorphisms in ternary Banach algebras by using the fixed point method.

1. Introduction

Ternary algebraic operations were considered in the 19th century by several mathematicians. Cayley [5] who introduced the notion of cubic matrix which in turn was generalized by Kapranov, Gelfand and Zelevinskii in 1990 [13]. As an application in physics, the quark model inspired a particular brand of ternary algebraic systems. There are also some applications, although still hypothetical, in the fractional quantum Hall effect, the non-standard statistics (the anyons), supersymmetric theories, Yang-Baxter equation, etc, (cf. [1, 26]).

The comments on physical applications of ternary structures can be found in [2, 8, 9, 21, 22, 23, 26].

Let A and B be ternary Banach algebras. An additive mapping $\Im : (A, [\ ]_A) \to (B, [\ ]_B)$ is called a ternary ring homomorphism if

$$\Im([x, y, z]_A) = [\Im(x), \Im(y), \Im(z)]_B$$

for all $x, y, z \in A$. An additive mapping $\Im : (A, [\ ]_A) \to (B, [\ ]_B)$ is called a ternary Jordan ring homomorphism if

$$\Im([x, x, x]_A) = [\Im(x), \Im(x), \Im(x)]_B$$

for all $x \in A$.

We say that a functional equation (Q) is stable if any function g satisfying the equation (Q) approximately is near to true solution of (Q). Also, we say that a functional equation is superstable if every approximately solution is an exact solution of it.

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Approximate ternary Jordan ring homomorphisms

The study of stability problems originated from a famous talk given by Ulam [25] in 1940: "Under what condition does there exist a homomorphism near an approximate homomorphism?" In 1941, Hyers [12] answered affirmatively the question of Ulam for additive mappings between Banach spaces. A generalized version of the theorem of Hyers for approximately additive maps was given by Rassias [20] in 1978. For more details about various results concerning such problems the reader is referred to [3, 6, 7, 10, 11, 15, 16, 17, 18, 19, 24].

We need the following fixed point theorem.

Theorem 1.1. [14] *Suppose that (Ω, d) is a complete generalized metric space and $T : \Omega \to \Omega$ is a strictly contractive mapping with the Lipschitz constant L. Then, for any $x \in \Omega$, either*

$$d(T^n x, T^{n+1}x) = \infty, \quad \forall n \geq 0,$$

or there exists a positive integer n_0 such that

(1) *$d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;*

(2) *the sequence $\{T^n x\}$ is convergent to a fixed point y^* of T;*

(3) *y^* is the unique fixed point of T in $\Lambda = \{y \in \Omega : d(T^{n_0}x, y) < \infty\}$;*

(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Lambda$.*

In this paper, we prove the stability and superstability of ternary Jordan ring homomorphisms in ternary Banach algebras by using the fixed point method.

2. Stability of ternary Jordan ring homomorphisms

In this section, we establish the stability ternary Jordan ring homomorphisms in ternary Banach algebras.

Throughout this section, assume that A and B are ternary Banach algebras.

Lemma 2.1. [9] *Let $f : A \to B$ be an additive mapping. Then the following assertions are equivalent*

$$f([a, a, a]) = [f(a), f(a), f(a)] \tag{2.1}$$

for all $a \in A$, and

$$f([a, b, c] + [b, c, a] + [c, a, b]) = [f(a), f(b), f(c)] + [f(b), f(c), f(a)] + [f(c), f(a), f(b)] \tag{2.2}$$

for all $a, b, c \in A$.

Theorem 2.2. *Let $f : A \to B$ be a mapping for which there exist functions $\varphi : A \times A \to [0, \infty)$ and $\psi : A \times A \times A \to [0, \infty)$ such that*

$$\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y), \tag{2.3}$$

$$\|f([x, y, z] + [y, z, x] + [z, x, y]) - [f(x), f(y), f(z)] - [f(y), f(z), f(x)] - [f(z), f(x), f(y)]\| \leq \psi(x, y, z) \tag{2.4}$$

for all $x, y, z \in A$. If there exists a constant $0 < L < 1$ such that

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \leq \frac{L}{2}\varphi(x, y), \tag{2.5}$$

$$\psi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \frac{L}{2^3}\psi(x, y, z) \tag{2.6}$$

for all $x, y, z \in A$, then there exists a unique ternary Jordan ring homomorphism $\Im : A \to B$

$$\|f(x) - \Im(x)\| \leq \frac{L}{2(1-L)}\varphi(x, x) \tag{2.7}$$

for all $x \in A$.

Proof. It follows from (2.5) and (2.6) that

$$\lim_{n\to\infty} 2^n \varphi\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big) = 0, \tag{2.8}$$

$$\lim_{n\to\infty} 2^{3n} \psi\Big(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\Big) = 0 \tag{2.9}$$

for all $x, y, z \in A$. By (2.5), $\lim_{n\to\infty} 2^n\varphi(0, 0) = 0$ and so $\varphi(0, 0) = 0$. Letting $x = y = 0$ in (2.3), we get $f(0) \leq \varphi(0, 0) = 0$ and so $f(0) = 0$.

Let $\Omega = \{g : A \to B,\ g(0) = 0\}$. We introduce a generalized metric on Ω as follows:

$$d(g, h) = d_\varphi(g, h) = \inf\{K \in (0, \infty) :\ \|g(x) - h(x)\| \leq K\varphi(x, x),\ \forall x \in A\} \quad .$$

It is easy to show that (Ω, d) is a generalized complete metric space.

Now, we consider the mapping $T : \Omega \to \Omega$ defined by $Tg(x) = 2g(\frac{x}{2})$ for all $x \in A$ and $g \in \Omega$. Note that, for all $g, h \in \Omega$ and $x \in A$,

$$\begin{aligned} d(g, h) < K \ &\Rightarrow\ \|g(x) - h(x)\| \leq K\varphi(x, x) \\ &\Rightarrow\ \|2g(\frac{x}{2}) - 2h(\frac{x}{2})\| \leq 2\ K\ \varphi(\frac{x}{2}, \frac{x}{2}) \\ &\Rightarrow\ \|2g(\frac{x}{2}) - 2h(\frac{x}{2})\| \leq L\ K\ \varphi(x, x) \\ &\Rightarrow\ d(Tg, Th) \leq L\ K. \end{aligned}$$

Hence we show that

$$d(Tg, Th) \leq L\ d(g, h)$$

for all $g, h \in \Omega$, that is, T is a strictly contractive mapping of Ω with the Lipschitz constant L.

Putting $y = x$ in (2.3), we get

$$\|f(2x) - 2f(x)\| \leq \varphi(x, x)$$

for all $x \in A$. So

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \leq \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \leq \frac{L}{2}\ \varphi(x, x)$$

for all $x \in A$, that is, $d(f, Tf) \leq \frac{L}{2} < \infty$.

Let us denote

$$\Im(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in A$ since $\lim_{n\to\infty} d(T^n f, \Im) = 0$. By the result in [4] , $\Im$ is a ternary Jordan mapping and so it follows from the definition of $\Im$, (2.4) and (2.9) that

$$\begin{aligned}
&\|\Im([x,y,z]+[y,z,x]+[z,x,y]) - [\Im(x),\Im(y),\Im(z)] - [\Im(y),\Im(z),\Im(x)] - [\Im(z),\Im(x),\Im(y)]\| \\
&= \lim_{n\to\infty} \Big\| f\left(2^{3n}[\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}] + 2^{3n}[\frac{z}{2^n},\frac{y}{2^n},\frac{x}{2^n}] + 2^{3n}[\frac{z}{2^n},\frac{x}{2^n},\frac{y}{2^n}]\right) \\
&- 2^{3n}[f\left(\frac{x}{2^n}\right), f\left(\frac{y}{2^n}\right), f\left(\frac{z}{2^n}\right)] - 2^{3n}[f\left(\frac{y}{2^n}\right), f\left(\frac{z}{2^n}\right), f\left(\frac{x}{2^n}\right)] - 2^{3n}[f\left(\frac{z}{2^n}\right), f\left(\frac{x}{2^n}\right), f\left(\frac{y}{2^n}\right)]\Big\| \\
&\le \lim_{n\to\infty} 2^{3n}\psi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right) = 0
\end{aligned}$$

and so

$$\Im([x,y,z]+[y,z,x]+[z,x,y]) = [\Im(x),\Im(y),\Im(z)] + [\Im(y),\Im(z),\Im(x)] + [\Im(z),\Im(x),\Im(y)]$$

for all $x \in A$.

According to Theorem 1.1, since $\mathfrak{F}$ is the unique fixed point of T in the set $\Lambda = \{g \in \Omega : d(f,g) < \infty\}$, $\mathfrak{F}$ is the unique mapping such that

$$\|f(x) - \mathfrak{F}(x)\| \le K\ \varphi(x,x)$$

for all $x \in A$ and $K > 0$. Again, using Theorem 1.1, we have

$$d(f,\mathfrak{F}) \le \frac{1}{1-L} d(f,Tf) \le \frac{L}{2(1-L)}$$

and so

$$\|f(x) - \mathfrak{F}(x)\| \le \frac{L}{2(1-L)}\ \varphi(x,x)$$

This completes the proof. □

Corollary 2.3. *Let θ, p be nonnegative real numbers with $r,p > 1$ and $\frac{r-3p}{2} \ge 1$. Suppose that $f : A \to B$ is a mapping such that*

$$\|f(x+y) - f(x) - f(y)\| \le \theta(\|x\|^r + \|y\|^r),$$

$$\|f([x,y,z]+[y,z,x]+[z,x,y]) - [f(x),f(y),f(z)] - [f(y),f(z),f(x)] - [f(z),f(x),f(y)]\| \le \theta(\|x\|^p.\|y\|^p.\|z\|^p)$$

for all $x,y,z \in A$. Then there exists a unique ternary Jordan ring homomorphism $\mathfrak{F} : A \to B$ satisfying

$$\|f(x) - \mathfrak{F}(x)\| \le \frac{\theta}{(2^r-2)}\|x\|^r$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.2 by taking

$$\varphi(x,y) := \theta(\|x\|^r + \|y\|^r), \quad \psi(x,y,z) := \theta(\|x\|^p.\|y\|^p.\|z\|^p)$$

for all $x,y \in A$. Then we can choose $L = 2^{1-r}$ and so the desired conclusion follows. □

Remark 2.4. Let $f : A \to B$ be a mapping with $f(0) = 0$ such that there exist functions $\varphi : A \times A \to [0, \infty)$ and $\psi : A \times A \times A \to [0, \infty)$ satisfying (2.3) and (2.4). Let $0 < L < 1$ be a constant such that

$$\varphi(2x, 2y) \leq 2L\varphi(x, y), \quad \psi(2x, 2y, 2z) \leq 2^3 L\psi(x, y, z)$$

for all $x, y, z \in A$. By the similar method as in the proof of Theorem 2.2, one can show that there exists a unique ternary Jordan ring homomorphism $\mathfrak{F} : A \to X$ satisfying

$$\|f(x) - \mathfrak{F}(x)\| \leq \frac{1}{2(1 - L)}\varphi(x, x)$$

for all $x \in A$. For the case

$$\varphi(x, y) := \delta + \theta(\|x\|^r + \|y\|^r), \quad \psi(x, y, z) := \delta + \theta(\|x\|^p \cdot \|y\|^p \cdot \|z\|^p)$$

(where θ, δ are nonnegative real numbers and $r > 0, p < 1$ and $\frac{r-3p}{2} \geq 1$), there exists a unique ternary Jordan ring homomorphism $\Im : A \to X$ satisfying

$$\|f(x) - \mathfrak{F}(x)\| \leq \frac{\delta}{(2 - 2^r)} + \frac{\theta}{(2 - 2^r)}\|x\|^r$$

for all $x \in A$.

3. **Superstability of ternary Jordan ring homomorphisms**

In this section, we formulate and prove the superstability of ternary Jordan ring homomorphisms.

Theorem 3.1. *Suppose that there exist function $\psi : A \times A \times A \to [0, \infty)$ and a constant $0 < L < 1$ such that*

$$\psi\Big(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\Big) \leq \frac{L}{2^3}\psi(x, y, z)$$

for all $x, y, z \in A$. Moreover, if $f : A \to B$ is an additive mapping such that

$$\|f([x, y, z] + [y, z, x] + [z, x, y]) - [f(x), f(y), f(z)] - [f(y), f(z), f(x)] - [f(z), f(x), f(y)]\| \leq \psi(x, y, z)$$

for all $x, y, z \in A$, then f is a ternary Jordan ring homomorphism.

Proof. The proof of this theorem is omitted as similar to the proof of Theorem 2.2. □

Corollary 3.2. *Let θ, r, s be nonnegative real numbers with $r > 1$ and $s > 3$. If $f : A \to B$ is an additive mapping such that*

$$\|f([x, y, z] + [y, z, x] + [z, x, y]) - [f(x), f(y), f(z)] - [f(y), f(z), f(x)] - [f(z), f(x), f(y)]\| \leq \theta(\|x\|^s + \|y\|^s + \|z\|^s)$$

for all $x, y, z \in A$, then f is a ternary Jordan ring homomorphism.

Remark 3.3. Let θ, r be nonnegative real numbers with $r < 1$. Suppose that there exists a function $\psi : A \times A \times A \to [0, \infty)$ and a constant $0 < L < 1$ such that

$$\psi(2x, 2y, 2z) \leq 2^3 L\psi(x, y, z)$$

for all $x, y, z \in A$. If $f : A \to B$ is an additive mapping such that

$$\|f([x, y, z] + [y, z, x] + [z, x, y]) - [f(x), f(y), f(z)] - [f(y), f(z), f(x)] - [f(z), f(x), f(y)]\| \leq \psi(x, y, z)$$

for all $x, y, z \in A$, then f is a ternary Jordan ring homomorphism.

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Approximate controllability of fractional impulsive stochastic functional differential inclusions with infinite delay and fractional sectorial operators

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Abstract: In this paper, the approximate controllability of fractional impulsive stochastic functional differential inclusions with infinite delay and fractional sectorial operators is considered. By using the stochastic analysis, the fractional sectorial operators and a fixed-point theorem for multi-valued maps, a new set of necessary and sufficient conditions are formulated which guarantees the approximate controllability of the fractional impulsive stochastic system. The results are obtained under the assumption that the associated linear system is approximately controllable. Finally, an example is also given to illustrate the obtained theory.

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Keywords: Approximate controllability; Fractional impulsive stochastic functional differential inclusions; Infinite delay; Fractional sectorial operators; Fixed point theorem

1 Introduction

The notion of controllability has played a central role throughout the history of modern control theory. Moreover, approximate controllable systems are more prevalent and very often approximate controllability is completely adequate in applications; see [1-3]. Therefore, various approximate controllability problems for different kinds of dynamical systems have been investigated in many publications; see [4,5] and references therein. The fractional differential equations has received a great deal of attention, and they play an important role in many applied fields, including viscoelasticity, electrochemistry, control, porous media, electromagnetic and so on. In recent years, several papers have studied the approximate controllability of semilinear fractional differential systems without delay and infinite delay (see [6-9]). As a result of its widespread use, the approximate controllability of stochastic systems have received extensive attention. More recently, there are very few contributions regarding the approximate

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controllability of fractional stochastic control system. For example, Sakthivel et al. [10], Kerboua et al. [11], Muthukumar and Rajivganthi [12], Farahi and Guendouzi [13].

Impulsive partial functional differential equations or inclusions have become an active area of investigation due to their applications in fields such as mechanics, electrical engineering, medicine biology (see [14]). Recently, the approximate controllability for some fractional impulsive semilinear differential systems have been studied in several papers. For example, Liu and Bin [15] studied the approximate controllability for a class of Riemann-Liouville fractional impulsive differential inclusions. Balasubramaniam et al. [16] derived sufficient conditions for the approximate controllability of impulsive fractional integro-differential systems with nonlocal conditions in Hilbert space. Chalishajar et al. [17] discussed the approximate controllability of abstract impulsive fractional neutral evolution equations with infinite delay in Banach spaces. However, besides impulse effects and delays, stochastic effects likewise exist in real systems. For semilinear impulsive stochastic control systems in Hilbert spaces, there are several papers devoted to the approximate controllability (see [18,19]). Zang and Li [20] obtained the approximate controllability of fractional impulsive neutral stochastic differential equations with nonlocal conditions by using Krasnoselskii-Schaefer-type fixed point theorem.

Motivated by the researches mentioned previously, in this paper we consider the approximate controllability of a class of fractional impulsive stochastic functional differential inclusions with infinite delay in Hilbert spaces of the form

$$^cD^\alpha N(x_t) \in AN(x_t) + Bu(t) + F(t, x_t)\frac{dw(t)}{dt}, \qquad (1)$$
$$\alpha \in (0,1), t \in J = [0,b], t \neq t_k,$$

$$x_0 = \varphi \in \mathcal{B}, \qquad (2)$$

$$\Delta x(t_k) = I_k(x_{t_k}), \quad k = 1, \dots, m, \qquad (3)$$

where the state $x(\cdot)$ takes values in a separable real Hilbert space H with inner product $\langle\cdot,\cdot\rangle_H$ and norm $\|\cdot\|_H$. Here $^cD^\alpha$ is the Caputo fractional derivative of the order $\alpha \in (0,1)$ with the lower limit zero, A is a fractional sectorial operator defined on $(H, \|\cdot\|_H)$. Let K be another separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_K$ and norm $\|\cdot\|_K$. Suppose $\{w(t) : t \geq 0\}$ is a given K-valued Wiener process with a covariance operator $Q > 0$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$ equipped with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$, which is generated by the Wiener process w. The control function $u \in L^p_{\mathcal{F}}(J, U)$, a Hilbert space of admissible control functions, $p \geq 2$ be an integer, and B is a bounded linear operator from a Banach space U to H. The time history $x_t : (-\infty, 0] \to H$ given by $x_t(\theta) = x(t+\theta)$ belongs to some abstract phase space $\mathcal{B}$ defined axiomatically; $F, G, I_k(k = 1, \dots, m), N(\psi) = \psi(0) - G(t, \psi), \psi \in \mathcal{B}$, are given functions to be specified later. Moreover, let $0 < t_1 < \cdots < t_m < b$, are prefixed points and the symbol $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, where $x(t_k^-)$ and $x(t_k^+)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. The

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initial data $\{\varphi(t) : -\infty < t \leq 0\}$ is an $\mathcal{F}_0$-adapted, $\mathcal{B}$-valued random variable independent of the Wiener process w with finite second moment.

To the best of our knowledge, the approximate controllability of fractional impulsive stochastic functional differential inclusions with infinite delay and fractional sectorial operators and the form (1)-(3) is an untreated topic in the literature. To close the gap in this paper, we study this interesting problem. Sufficient conditions for the approximate controllability results are derived by a fixed-point theorem for multi-valued maps combined with the stochastic analysis and the fractional sectorial operators. The known results appeared in [15-20] are generalized to the fractional impulsive stochastic systems settings and the case of infinite delay.

The rest of this paper is organized as follows. In Section 2, we introduce some notations and necessary preliminaries. In Section 3, we give our main results. In Section 4, an example is given to illustrate our results.

2 Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with probability measure P on Ω and a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Let H, K be two real separable Hilbert spaces and we denote by $\langle\cdot,\cdot\rangle_H, \langle\cdot,\cdot\rangle_K$ their inner products and by $\|\cdot\|_H, \|\cdot\|_K$ their vector norms, respectively. $L(H, K)$ be the space of bounded linear operators mapping K into H equipped with the usual norm $\|\cdot\|_H$ and $L(H)$ denotes the Banach space of bounded linear operators from H to H. Let $\{w(t) : t \geq 0\}$ denote an K-valued Wiener process defined on the probability space $(\Omega, \mathcal{F}, P)$ with covariance operator Q. We assume that there exists a complete orthonormal system $\{e_n\}_{n=1}^{\infty}$ in K, a bounded sequence of nonnegative real numbers $\{\lambda_n\}_{n=1}^{\infty}$ such that $Qe_n = \lambda_n e_n, n = 1, 2, \ldots$, and a sequence β_n of independent Brownian motions such that

$$\langle w(t), e\rangle = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\langle e_n, e\rangle\beta_n(t),\ e \in K, t \in J,$$

and $\mathcal{F}_t = \mathcal{F}_t^w$, where $\mathcal{F}_t^w$ is the σ-algebra generated by $\{w(s) : 0 \leq s \leq t\}$. Let $L_2^0 = L_2(K_0, H)$ be the space of all Hilbert-Schmidt operators from K_0 to H with the norm $\|\psi\|_{L_2^0}^2 = \mathrm{Tr}((\psi Q^{1/2})(\psi Q^{1/2})^*)$ for any $\psi \in L_2^0$. Clearly for any bounded operators $\psi \in L(K, H)$ this norm reduces to $\|\psi\|_{L_2^0}^2 = \mathrm{Tr}(\psi Q\psi^*)$. Let $L^p(\mathcal{F}_b, H)$ be the Banach space of all $\mathcal{F}_b$-measurable pth power integrable random variables with values in the Hilbert space H. Let $C([0, b]; L^p(\mathcal{F}, H))$ be the Banach space of continuous maps from $[0, b]$ into $L^p(\mathcal{F}, H)$ satisfying the condition $\sup_{t\in J} E \|x(t)\|_H^p < \infty$.

We use the notations $\mathcal{P}(H)$ is the family of all subsets of H. Let us introduce the following notations:

$$\mathcal{P}_{cl}(H) = \{x \in \mathcal{P}(H) : x \text{ is closed}\}, \quad \mathcal{P}_{bd}(H) = \{x \in \mathcal{P}(H) : x \text{ is bounded}\},$$

$$\mathcal{P}_{cv}(H) = \{x \in \mathcal{P}(H) : x \text{ is convex}\}, \quad \mathcal{P}_{cp}(H) = \{x \in \mathcal{P}(H) : x \text{ is compact}\}.$$

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Consider $H_d : \mathcal{P}(H) \times \mathcal{P}(H) \to R^+ \cup \{\infty\}$ given by

$$H_d(\widetilde{A}, \widetilde{B}) = \max\left\{ \sup_{\tilde{a}\in\widetilde{A}} d(\tilde{a}, \widetilde{B}), \sup_{\tilde{b}\in\widetilde{B}} d(\widetilde{A}, \tilde{a}) \right\},$$

where $d(\widetilde{A}, \tilde{b}) = \inf_{\tilde{a}\in\widetilde{A}} d(\tilde{a}, \tilde{b}), d(\tilde{a}, \widetilde{B}) = \inf_{\tilde{b}\in\widetilde{B}} d(\tilde{a}, \tilde{b})$. Then, $(\mathcal{P}_{bd,cl}(H), H_d)$ is a metric space and $(\mathcal{P}_{cl}(H), H_d)$ is a generalized metric space. In what follows, we briefly introduce some facts on multi-valued analysis. For more details, one can see [21,22].

A multi-valued map $\Phi : J \to \mathcal{P}_{cl}(H)$ is said to be measurable if for each $x \in H$, the function $Y : J \to R^+$ defined by $Y(t) = d(x, \Phi(t)) = \inf\{d(x, z) : z \in \Phi(t)\}$ is measurable.

Φ has a fixed point if there is $x \in H$ such that $x \in \Phi(x)$. The set of fixed points of the multi-valued operator Φ will be denoted by FixΦ.

Definition 2.1. A multi-valued operator $\Phi : H \to P_{cl}(H)$ is called:

(a) γ-Lipschitz if there exists $\gamma > 0$ such that

$$H_d(\Phi(x), \Phi(y)) \leq \gamma d(x, y), \ \ x, y \in H.$$

(b) a contraction if it is γ-Lipschitz with $\gamma < 1$.

In this paper, we assume that the phase space $(\mathcal{B}, \| \cdot \|_{\mathcal{B}})$ is a seminormed linear space of $\mathcal{F}_0$-measurable functions mapping $(-\infty, 0]$ into H, and satisfying the following fundamental axioms due to Hale and Kato (see e.g., in [23]).

(A) If $x : (-\infty, \sigma + b] \to H$, $b > 0$, is such that $x|_{[\sigma,\sigma+b]} \in C([\sigma, \sigma + b], H)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + b]$ the following conditions hold:

  (i) x_t is in $\mathcal{B}$;

  (ii) $\| x(t) \|_H \leq \tilde{H} \| x_t \|_{\mathcal{B}}$;

  (iii) $\| x_t \|_{\mathcal{B}} \leq K(t - \sigma) \sup\{\| x(s) \|_H : \sigma \leq s \leq t\} + M(t - \sigma) \| x_\sigma \|_{\mathcal{B}}$, where $\tilde{H} \geq 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, K is continuous and M is locally bounded, and $\tilde{H}, K, M$ are independent of $x(\cdot)$.

(B) For the function $x(\cdot)$ in (A), the function $t \to x_t$ is continuous from $[\sigma, \sigma+b]$ into $\mathcal{B}$.

(C) The space $\mathcal{B}$ is complete.

The next result is a consequence of the phase space axioms.

Lemma 2.1. Let $x : (-\infty, b] \to H$ be an $\mathcal{F}_t$-adapted measurable process such that the $\mathcal{F}_0$-adapted process $x_0 = \varphi(t) \in L_2^0(\Omega, \mathcal{B})$ and $x|_{[0,b]} \in \mathcal{PC}([0, b], H)$, then

$$\| x_s \|_{\mathcal{B}} \leq M_b E \| \varphi \|_{\mathcal{B}} + K_b \sup_{0\leq s\leq b} E \| x(s) \|_H,$$

where $K_b = \sup\{K(t) : 0 \leq t \leq b\}, M_b = \sup\{M(t) : 0 \leq t \leq b\}$.

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We introduce the space $\mathcal{PC}$ formed by all $\mathcal{F}_t$-adapted measurable, H-valued stochastic processes $\{x(t) : t \in [0, b]\}$ such that x is continuous at $t \neq t_k$, $x(t_k) = x(t_k^-)$ and $x(t_k^+)$ exists for all $k = 1, ..., m$. In this paper, we always assume that $\mathcal{PC}$ is endowed with the norm

$$\| x \|_{\mathcal{PC}} = (\sup_{0 \leq t \leq b} E \| x(t) \|_H^p)^{\frac{1}{p}}.$$

Then $(\mathcal{PC}, \| \cdot \|_{\mathcal{PC}})$ is a Banach space.

Definition 2.2 ([24]). The fractional integral of order γ with the lower limit zero for a function $h \in L^1(J, H)$ is defined as

$$I_t^\gamma h(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{h(s)}{(t-s)^{1-\gamma}} ds, \quad t > 0, \gamma > 0$$

provided the right side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.3 ([24]). The Riemann-Liouville derivative of order γ with the lower limit zero for a function $h \in L^1(J, H)$ can be written as

$$D_t^\gamma h(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{h(s)}{(t-s)^{\gamma+1-n}} ds, \quad t > 0, n-1 < \gamma < n.$$

Definition 2.4 ([24]). The Caputo derivative of order γ for a function $h \in L^1(J, H)$ can be written as

$$D_t^\gamma h(t) = D_t^\gamma (h(t) - h(0)), \quad t > 0, 0 < \gamma < 1.$$

Next, we are ready to recall some facts of fractional Cauchy problem.

$$^cD_t^\alpha x(t) = Ax(t), \quad t \geq 0, \tag{4}$$

$$x_0 = \varphi \in \mathcal{B}, \tag{5}$$

where A is linear closed and $D(A)$ is dense.

Definition 2.5 ([25]). A family $\{S_\alpha(t) : t \geq 0\} \subset L(H)$ is called a solution operator for (4)-(5) if the following conditions are satisfied:

(a) $S_\alpha(t)$ is strongly continuous for $t \geq 0$ and $S_\alpha(0) = I$.

(b) $S_\alpha(t)D(A) \subset D(A)$ and $AS_\alpha(t)\varphi = S_\alpha(t)A\varphi$ for all $\varphi \in D(A), t \geq 0$.

(c) $S_\alpha(t)\varphi$ is a solution of (4)-(5) for all $\varphi \in D(A), t \geq 0$.

Definition 2.6 ([24]). An operator A is said to be belong to $e^\alpha(M, \omega)$ if the solution operator $S_\alpha(\cdot)$ of (4)-(5) satisfies

$$\| S_\alpha(t) \|_{L(H)} \leq Me^{\omega t}, \quad t \geq 0$$

for some constants $M \geq 1$ and $\omega \geq 0$.

Definition 2.7 ([24]). A solution operator $S_\alpha(\cdot)$ of (4)-(5) is called analytic if it admits an analytic extension to a sector $\Sigma_{\theta_0} = \{\lambda \in \mathbf{C} - \{0\} : \arg \lambda < \theta_0\}$ for some $\theta_0 \in (0, \frac{\pi}{2}]$. An analytic solution operator is said to be of analyticity type (θ_0, ω_0) if for each $\theta < \theta_0$ and $\omega > \omega_0$ there is an $M = M(\theta, \omega)$ such that

$$\| S_\alpha(t) \|_{L(H)} \leq M e^{\omega \mathrm{Re} t}, \quad t \in \Sigma_\theta.$$

Set

$$e^\alpha(\omega) = \bigcup \{e^\alpha(M, \omega) : M \geq 1\}, e^\alpha := \bigcup \{e^\alpha(\omega) : \omega \geq 0\},$$

and $A^\alpha(\theta_0, \omega_0) = \{A \in e^\alpha : A$ generates an analytic solution operator S_α of type $(\theta_0, \omega_0)\}$.

Remark 2.3 ([25, Theorem 2.14]). Let $\alpha \in (0, 2)$. A linear closed densely defined operator A belongs to $A^\alpha(\theta_0, \omega_0)$ if and only if $\lambda^\alpha \in \rho(A)$ for each $\lambda \in \Sigma_{\theta_0 + \frac{\pi}{2}}(\omega_0) = \{\mathbf{C} - \{0\} : |\arg(\lambda - \omega_0)| < \theta_0 + \frac{\pi}{2}\}$ and for any $\omega > \omega_0, \theta < \theta_0$ there is a constant $C = C(\theta, \omega)$ such that

$$\| \lambda^{\alpha-1} R(\lambda^\alpha, A) \|_{L(H)} \leq \frac{C}{|\lambda - \omega|}$$

for $\lambda \in \Sigma_{\theta_0 + \frac{\pi}{2}}$.

According to the proof of Theorem 2.14 in [25], if $A \in A^\alpha(\theta_0, \omega_0)$ for some $\theta_0 \in (0, \pi)$ and $\omega_0 \in R$, the solution operator for the Eq. (4)-(5) is given by

$$S_\alpha(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \lambda^{\alpha-1} R(\lambda^\alpha, A) d\lambda$$

for a suitable path Γ. Next, a mild solution of the Cauchy problem

$${}^c D_t^\alpha x(t) = Ax(t) + f(t), \quad t \in J,$$

$$x_0 = \varphi \in \mathcal{B},$$

can be defined by

$$x(t) = S_\alpha(t)\varphi + \int_0^t T_\alpha(t-s) f(s) ds,$$

where

$$T_\alpha(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} R(\lambda^\alpha, A) d\lambda$$

for a suitable path Γ and $f : J \to H$ is continuous.

Lemma 2.2 ([25]). If $A \in A^\alpha(\theta_0, \omega_0)$ then

$$\| S_\alpha(t) \|_{L(H)} \leq M e^{\omega t}, \quad \| T_\alpha(t) \|_{L(H)} \leq C e^{\omega t}(1 + t^{\alpha-1})$$

for every $t > 0, \omega > \omega_0$. So putting

$$\tilde{M}_S := \sup_{0 \leq t \leq b} \| S_\alpha(t) \|_{L(H)}, \quad \tilde{M}_T := \sup_{0 \leq t \leq b} C e^{\omega t}(1 + t^{\alpha-1}),$$

we get

$$\| S_\alpha(t) \|_{L(H)} \leq \tilde{M}_S, \quad \| T_\alpha(t) \|_{L(H)} \leq t^{\alpha-1}\tilde{M}_T.$$

Based on the above consideration, we introduce the definition of mild solution for (1)-(3).

Definition 2.8. Let $A \in A^\alpha(\theta_0, \omega_0)$ with $\theta_0 \in (0, \frac{\pi}{2}]$ and $\omega_0 \in R$. An $\mathcal{F}_t$-adapted stochastic process $x : (-\infty, b] \to H$ is called a mild solution of the system (1)-(3) if $x_0 = \varphi \in \mathcal{B}$ satisfying $x_0 \in L_2^0(\Omega, H), x|_{[0,b]} \in \mathcal{PC}$, and

$$x(t) = \begin{cases} S_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t, x_t) \\ \quad + \int_0^t T_\alpha(t-s)Bu(s)ds \\ \quad + \int_0^t T_\alpha(t-s)f(s)dw(s), & t \in [0, t_1], \\ S_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t, x_t) \\ \quad + S_\alpha(t-t_1)I_1(x_{t_1}) \\ \quad + \int_0^t \mathcal{S}_\alpha(t-s)Bu(s)ds \\ \quad + \int_0^t \mathcal{S}_\alpha(t-s)f(s)dw(s), & t \in (t_1, t_2], \\ \vdots \\ S_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t, x_t) \\ \quad + \sum_{k=1}^m S_\alpha(t-t_k)I_k(x_{t_k}) \\ \quad + \int_0^t \mathcal{S}_\alpha(t-s)Bu(s)ds \\ \quad + \int_0^t \mathcal{S}_\alpha(t-s)f(s)dw(s), & t \in (t_m, b], \end{cases}$$

where $f \in S_{F,x} = \{f \in L^p(J, L_2^0) : f(t) \in F(t, x_t) \text{ a.e. } t \in J\}$.

Let $x(t; \varphi, u)$ denotes state value of the system (1)-(3) at time t corresponding to the control $u \in L^p_{\mathcal{F}}(J, U)$. In particular, the state of system (1)-(3) at $t = b, x(b; \varphi, u)$ is called the terminal state with control u and the initial value φ. Introduce the set $\mathcal{B}(b; \varphi, u) = \{x(b; \varphi, u), u(\cdot) \in L^p_{\mathcal{F}}(J, U)\}$ is called the reachable set of the system (1)-(3), where $L^p_{\mathcal{F}}(J, U)$ is the closed subspace of $L^p_{\mathcal{F}}(J \times \Omega, U)$, consisting of all $\mathcal{F}_t$-adapted, U-valued stochastic processes.

Definition 2.9. The system (1)-(3) is said to be approximately controllable on the interval J if $\overline{\mathcal{B}(b; \varphi, u)} = L^p(\mathcal{F}_b, H)$, where $\overline{\mathcal{B}(b; \varphi, u)}$ is the closure of the reachable set.

It is convenient at this point to define operators

$$\Gamma_\tau^b = \int_\tau^b \mathcal{S}_\alpha(b-s)BB^*\mathcal{S}_\alpha^*(b-s)ds, \quad 0 \leq \tau < b,$$

$$\Gamma_0^b = \int_0^b \mathcal{S}_\alpha(b-s)BB^*\mathcal{S}_\alpha^*(b-s)ds,$$

$$R(a, \Gamma_\tau^b) = (aI + \Gamma_\tau^b)^{-1}, R(a, \Gamma_0^b) = (aI + \Gamma_0^b)^{-1} \text{ for } a > 0,$$

where B^* denotes the adjoint of B and $\mathcal{S}_\alpha^*(t)$ is the adjoint of $\mathcal{S}_\alpha(t)$. It is straightforward that the operator Γ_τ^b is a linear bounded operator.

Lemma 2.4 ([3]). For any $\tilde{x}_b \in L^p(\mathcal{F}_b, H)$ there exists $\tilde{\phi} \in L^p_{\mathcal{F}}(\Omega; L^2(0, b; L_2^0))$ such that $\tilde{x}_b = E\tilde{x}_b + \int_0^b \tilde{\phi}(s)dw(s)$.

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Now for any $a > 0$ and $\tilde{x}_b \in L^p(\mathcal{F}_b, H)$ we define the control function

$$u_x^a(t) = \begin{cases} S^*T_\alpha^*(b-t)(aI+\Gamma_0^b)^{-1}\Big[E\tilde{x}_b + \int_0^b \tilde{\phi}(s)dw(s) \\ \quad -S_\alpha(b)[\varphi(0) - G(0,\varphi)] - G(b, x_b)\Big] \\ \quad -B^*T_\alpha^*(b-t)\int_0^b (aI+\Gamma_s^b)^{-1}\mathcal{S}_\alpha(b-s)f(s)dw(s), \quad t \in [0, t_1], \\ S^*T_\alpha^*(b-t)(aI+\Gamma_0^b)^{-1}\Big[E\tilde{x}_b + \int_0^b \tilde{\phi}(s)dw(s) \\ \quad -S_\alpha(b)[\varphi(0) - G(0,\varphi)] - G(b, x_b)\Big] \\ \quad -B^*T_\alpha^*(b-t)(aI+\Gamma_s^b)^{-1}S_\alpha(t-t_1)I_1(x_{t_1}) \\ \quad -B^*T_\alpha^*(b-t)\int_0^b (aI+\Gamma_s^b)^{-1}\mathcal{S}_\alpha(b-s)f(s)dw(s), \quad t \in (t_1, t_2], \\ \vdots \\ S^*T_\alpha^*(b-t)(aI+\Gamma_0^b)^{-1}\Big[E\tilde{x}_b + \int_0^b \tilde{\phi}(s)dw(s) \\ \quad -S_\alpha(b)[\varphi(0) - G(0,\varphi)] - G(b, x_b)\Big] \\ \quad -B^*T_\alpha^*(b-t)\sum_{k=1}^m (aI+\Gamma_s^b)^{-1}S_\alpha(t-t_k)I_k(x_{t_k}) \\ \quad -B^*T_\alpha^*(b-t)\int_0^b (aI+\Gamma_s^b)^{-1}\mathcal{S}_\alpha(b-s)f(s)dw(s), \quad t \in (t_m, b], \end{cases}$$

where $f \in S_{F,x} = \{f \in L^p(J, L_2^0) : f(t) \in F(t, x_t) \text{ a.e. } t \in J\}$.
Lemma 2.5 ([26]). For any $p \geq 1$ and for arbitrary L_2^0-valued predictable process $\phi(\cdot)$ such that

$$\sup_{s\in[0,t]} E\left\| \int_0^s \phi(v)dw(v) \right\|_H^{2p} \leq (p(2p-1))^p \left(\int_0^t (E \parallel \phi(s) \parallel_{L_2^0}^{2p})^{1/p} ds \right)^p, t \in [0, \infty).$$

In the rest of this paper, we denote by $M_1 = \parallel B \parallel_H, C_p = (p(p-1)/2)^{p/2}$.

Our main results are based on the following lemma.
Lemma 2.6 ([27]). Let (H, d) be a complete metric space. If $\Phi : H \to P_{cl}(H)$ is a contraction, then Fix $\Phi \neq \emptyset$.

3 Main results

In this section we shall present and prove our main results. Let us list the following hypotheses.

(H1) The function $G : J \times \mathcal{B} \to H$ is continuous, and there exists a positive constant L_G such that

$$E \parallel G(t, \psi_1) - G(t, \psi_2) \parallel_H^p \leq L_G \parallel \psi_1 - \psi_2 \parallel_{\mathcal{B}}^p$$

for $t \in J, \psi_1, \psi_2 \in \mathcal{B}$.

(H2) The function $F : J \times \mathcal{B} \to \mathcal{P}_{cp}(L_2^0)$ is a multifunction such that $(\cdot, \phi) \to F(t, \phi)$ is measurable for each $\phi \in \mathcal{B}$.

(H3) There exists a function $l(t) \in L^{\frac{1}{q}}(J, R^+), q \in (0, \alpha)$ such that

$$EH_d^p(F(t, \phi_1), F(t, \phi_2)) \leq l(t) \parallel \phi_1 - \phi_2 \parallel_{\mathcal{B}}^p$$

for $t \in J, \phi_1, \phi_2 \in \mathcal{B}$, and

$$d^p(0, F(t, 0)) \leq l(t)$$

for a.e. $t \in J$.

(H4) The functions $I_k : \mathcal{B} \to H$ are continuous and there exist constants c_k such that

$$E \parallel I_k(\psi_1) - I_k(\psi_2) \parallel_H^p \leq c_k \parallel \psi_1 - \psi_2 \parallel_{\mathcal{B}}^p$$

for $\psi_1, \psi_2 \in \mathcal{B}, k = 1, \dots, m$.

(H5) For each $0 \leq t < b$, the operator $aR(a, \Gamma_\tau^b) \to 0$ in the strong operator topology as $a \to 0^+$ i.e., the linear differential Cauchy problem corresponding to system (1)-(3) is approximately controllable on J.

Theorem 3.1. *Let $A \in A^\alpha(\theta_0, \omega_0)$ with $\theta_0 \in (0, \frac{\pi}{2}]$ and $\omega_0 \in R$. If the assumptions* (H1)-(H4) *are satisfied, then the system* (1)-(3) *has at least one mild solution on J, provided that*

$$4^{p-1}K_b^p\Bigg[L_G + m^{p-1}\tilde{M}_S^p\sum_{i=1}^m c_i + C_p\tilde{M}_T^p\left(\frac{1-q}{p(1-\alpha)+1-q}\right)^{1-q}$$
$$\times b^{p(\alpha-1/2)-q} \parallel l \parallel_{L^{\frac{1}{q}}(J,R^+)}\Bigg]\left[1 + 3^{p-1}\tilde{M}_T^{2p}M_1^{2p}\frac{1}{a^p}\frac{b^{p(2\alpha-1)}}{2p(\alpha-1)+1}\right] < 1.$$

Proof. We introduce the space $\mathcal{B}_b$ of all functions $x : (-\infty, b] \to H$ such that $x_0 \in \mathcal{B}$ and the the restriction $x|_{[0,b]} \in \mathcal{PC}$. Let$\| \cdot \|_b$ be a seminorm in $\mathcal{B}_b$ defined by

$$\parallel x \parallel_b = \parallel x_0 \parallel_{\mathcal{B}} + (\sup_{0 \leq s \leq b} \parallel x(s) \parallel_H^p)^{\frac{1}{p}}, \quad x \in \mathcal{B}_b.$$

We consider the multi-valued map $\Phi : \mathcal{B}_b \to \mathcal{P}(\mathcal{B}_b)$ by Φx the set of $\rho \in \mathcal{B}_b$ such that

$$\rho(t) = \begin{cases} S_\alpha(t)[\varphi(0) - G(0, \varphi)] + G(t, x_t) \\ \quad + \int_0^t T_\alpha(t-s)Bu_x^a(s)ds + \int_0^t T_\alpha(t-s)f(s)dw(s), \quad t \in [0, t_1], \\ S_\alpha(t)[\varphi(0) - G(0, \varphi)] + G(t, x_t) \\ \quad + S_\alpha(t - t_1)I_1(x_{t_1}) \\ \quad + \int_0^t \mathcal{S}_\alpha(t-s)Bu_x^a(s)ds + \int_0^t \mathcal{S}_\alpha(t-s)f(s)dw(s), \quad t \in (t_1, t_2], \\ \vdots \\ S_\alpha(t)[\varphi(0) - G(0, \varphi)] + G(t, x_t) \\ \quad + \sum_{k=1}^m S_\alpha(t - t_k)I_k(x_{t_k}) \\ \quad + \int_0^t \mathcal{S}_\alpha(t-s)Bu_x^a(s)ds + \int_0^t \mathcal{S}_\alpha(t-s)f(s)dw(s), \quad t \in (t_m, b], \end{cases}$$

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where $f \in S_{F,x} = \{f \in L^p(J, L_2^0) : f(t) \in F(t, x_t) \text{ a.e. } t \in J\}$.

For $\varphi \in \mathcal{B}$, we define $\tilde{\varphi}$ by

$$\tilde{\varphi}(t) = \begin{cases} \varphi(t), & -\infty < t \leq 0, \\ S_\alpha(t)\varphi(0), & 0 \leq t \leq b, \end{cases}$$

then $\tilde{\varphi} \in \mathcal{B}_b$. Set $x(t) = y(t) + \tilde{\varphi}(t), -\infty < t \leq b$. It is clear to see that x satisfies Definition 2.8 if and only if y satisfies $y_0 = 0$ and

$$y(t) = \begin{cases} S_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t, y_t + \tilde{\varphi}_t) \\ \quad + \int_0^t T_\alpha(t-s)Bu_y^a(s)ds + \int_0^t T_\alpha(t-s)f(s)dw(s), & t \in [0, t_1], \\ S_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t, y_t + \tilde{\varphi}_t) \\ \quad + S_\alpha(t-t_1)I_1(y_{t_1} + \tilde{\varphi}_{t_1}) \\ \quad + \int_0^t \mathcal{S}_\alpha(t-s)Bu_y^a(s)ds + \int_0^t \mathcal{S}_\alpha(t-s)f(s)dw(s), & t \in (t_1, t_2], \\ \vdots \\ S_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t, y_t + \tilde{\varphi}_t) \\ \quad + \sum_{k=1}^m S_\alpha(t-t_k)I_k(y_{t_k} + \tilde{\varphi}_{t_k}) \\ \quad + \int_0^t \mathcal{S}_\alpha(t-s)Bu_y^a(s)ds + \int_0^t \mathcal{S}_\alpha(t-s)f(s)dw(s), & t \in (t_m, b], \end{cases}$$

where

$$u_y^a(t) = \begin{cases} S^*T_\alpha^*(b-t)(aI + \Gamma_0^b)^{-1}\Big[E\tilde{x}_b + \int_0^b \tilde{\phi}(s)dw(s) \\ \quad -S_\alpha(b)[\varphi(0) - G(0,\varphi)] - G(b, x_b)\Big] \\ \quad -B^*T_\alpha^*(b-t)\int_0^b (aI + \Gamma_s^b)^{-1}T_\alpha(b-s)f(s)dw(s), & t \in [0, t_1], \\ S^*T_\alpha^*(b-t)(aI + \Gamma_0^b)^{-1}\Big[E\tilde{x}_b + \int_0^b \tilde{\phi}(s)dw(s) \\ \quad -S_\alpha(b)[\varphi(0) - G(0,\varphi)] - G(b, x_b)\Big] \\ \quad -B^*T_\alpha^*(b-t)(aI + \Gamma_s^b)^{-1}S_\alpha(t-t_1)I_1(x_{t_1}) \\ \quad -B^*T_\alpha^*(b-t)\int_0^b (aI + \Gamma_s^b)^{-1}T_\alpha(b-s)f(s)dw(s), & t \in (t_1, t_2], \\ \vdots \\ S^*T_\alpha^*(b-t)(aI + \Gamma_0^b)^{-1}\Big[E\tilde{x}_b + \int_0^b \tilde{\phi}(s)dw(s) \\ \quad -S_\alpha(b)[\varphi(0) - G(0,\varphi)] - G(b, x_b)\Big] \\ \quad -B^*T_\alpha^*(b-t)\sum_{k=1}^m (aI + \Gamma_s^b)^{-1}S_\alpha(t-t_k)I_k(x_{t_k}) \\ \quad -B^*T_\alpha^*(b-t)\int_0^b (aI + \Gamma_s^b)^{-1}T_\alpha(b-s)f(s)dw(s), & t \in (t_m, b], \end{cases}$$

and $f \in S_{F,y} = \{f \in L^p(J, L_2^0) : f(t) \in F(t, y_s + \tilde{\varphi}_s) \text{ a.e. } t \in J\}$.

Let $\mathcal{B}_b^0 = \{y \in \mathcal{B}_b : y_0 = 0 \in \mathcal{B}\}$. For any $y \in \mathcal{B}_b^0$,

$$\| y \|_b = \| y_0 \|_{\mathcal{B}} + (\sup_{0 \leq s \leq b} \| y(s) \|_H^p)^{\frac{1}{p}} = (\sup_{0 \leq s \leq b} \| y(s) \|_H^p)^{\frac{1}{p}},$$

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thus $(\mathcal{B}_b^0, \|\cdot\|_b)$ is a Banach space. Define the multi-valued map $\bar{\Phi} : \mathcal{B}_b^0 \to \mathcal{P}(\mathcal{B}_b^0)$ by $\bar{\Phi}y$ the set of $\bar{\rho} \in \mathcal{B}_b^0$ such that $\bar{\rho}(t) = 0, t \in [-\infty, 0]$ and

$$\bar{\rho}(t) = \begin{cases} -S_\alpha(t)G(0,\varphi) + G(t, y_t + \tilde{\varphi}_t) \\ \quad + \int_0^t T_\alpha(t-s)Bu_y^a(s)ds + \int_0^t T_\alpha(t-s)f(s)dw(s), \quad t \in [0, t_1], \\ -S_\alpha(t)\varphi(0) + G(t, y_t + \tilde{\varphi}_t) \\ \quad + S_\alpha(t-t_1)I_1(y_{t_1} + \tilde{\varphi}_{t_1}) \\ \quad + \int_0^t \mathcal{S}_\alpha(t-s)Bu_y^a(s)ds + \int_0^t T_\alpha(t-s)f(s)dw(s), \quad t \in (t_1, t_2], \\ \vdots \\ -S_\alpha(t)\varphi(0) + G(t, y_t + \tilde{\varphi}_t) \\ \quad + \sum_{k=1}^m S_\alpha(t-t_k)I_k(y_{t_k} + \tilde{\varphi}_{t_k}) \\ \quad + \int_0^t \mathcal{S}_\alpha(t-s)Bu_y^a(s)ds + \int_0^t T_\alpha(t-s)f(s)dw(s), \quad t \in (t_m, b], \end{cases}$$

where $f \in S_{F,y}$. Obviously, the operator Φ has a fixed point if and only if operator $\bar{\Phi}$ has a fixed point, to prove which we shall employ Lemma 2.6. For better readability, we break the proof into a sequence of steps.

Step 1. We show that $(\bar{\Phi}y)(t) \in \mathcal{P}_{cl}(\mathcal{B}_b^0)$.

Indeed, let $y^{(n)}(t) \to y^*(t)$, $(\bar{\rho}_n)_{n\geq 0} \in (\bar{\Phi}y)(t)$ such that $\bar{\rho}_n(t) \to \bar{\rho}_*(t)$ in $\mathcal{B}_b^0$. Then $\bar{\rho}_*(t) \in \mathcal{B}_b^0$ and there exists $f_n \in S_{F,y^{(n)}}$ such that, for each $t \in [0, t_1]$,

$$\begin{aligned} \bar{\rho}_n(t) = &-S_\alpha(t)G(0,\varphi) + G(t, y_t^{(n)} + \tilde{\varphi}_t) \\ &+ \int_0^t T_\alpha(t-s)Bu_{y^{(n)}}^a(s)ds + \int_0^t T_\alpha(t-s)f_n(s)dw(s), \end{aligned}$$

where

$$\begin{aligned} u_{y^{(n)}}^a(t) = B^* T_\alpha^*(b-t)(aI + \Gamma_0^b)^{-1}\Big[&E\tilde{x}_b + \int_0^b \tilde{\phi}(s)dw(s) \\ &-S_\alpha(b)[\varphi(0) - G(0,\varphi)] - G(b, y_b^{(n)} + \tilde{\varphi}_b)\Big] \\ &-B^* T_\alpha^*(b-t)\int_0^b (aI + \Gamma_s^b)^{-1} T_\alpha(b-s)f_n(s)dw(s). \end{aligned}$$

Using the fact that F has compact values and (H3) holds, we may pass to a subsequence if necessary to obtain that f_n converges to f_* in $L^p([0, t_1], L_2^0)$, hence, $f_* \in S_{F,y^*}$. Then, for each $t \in [0, t_1]$,

$$\begin{aligned} \bar{\rho}_n(t) \to \bar{\rho}_*(t) = &-S_\alpha(t)G(0,\varphi) + G(t, y_t^* + \tilde{\varphi}_t) \\ &+ \int_0^t T_\alpha(t-s)Bu_{y^*}^a(s)ds + \int_0^t T_\alpha(t-s)f_n(s)dw(s), \end{aligned}$$

where

$$u_{y^*}^a(t) = S^* T_\alpha^*(b-t)(aI + \Gamma_0^b)^{-1}\Big[E\tilde{x}_b + \int_0^b \tilde{\phi}(s)dw(s)$$

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$$-S_\alpha(b)[\varphi(0) - G(0,\varphi)] - G(b, y_b^* + \tilde{\varphi}_b)\Big]$$
$$-B^* T_\alpha^*(b-t)\int_0^b (aI + \Gamma_s^b)^{-1} \mathcal{S}_\alpha(b-s) f_*(s)dw(s).$$

Similarly, for each $t \in (t_k, t_{k+1}], k = 1, \dots, m$, we have

$$\bar{\rho}_n(t) = -S_\alpha(t)G(0,\varphi) + G(t, y_t^{(n)} + \tilde{\varphi}_t) + \sum_{i=1}^{k} S_\alpha(t-t_i) I_i(y_{t_i}^{(n)} + \tilde{\varphi}_{t_i})$$
$$+ \int_0^t T_\alpha(t-s) B u_{y^{(n)}}^a(s)ds + \int_0^t T_\alpha(t-s) f_n(s)dw(s),$$

where

$$u_{y^{(n)}}^a(t) = S^* T_\alpha^*(b-t)(aI + \Gamma_0^b)^{-1}\Big[E\tilde{x}_b + \int_0^b \tilde{\phi}(s)dw(s)$$
$$-S_\alpha(b)[\varphi(0) - G(0,\varphi)] - G(b, y_b^{(n)} + \tilde{\varphi}_b)\Big]$$
$$-B^* T_\alpha^*(b-t)(aI + \Gamma_s^b)^{-1} \sum_{i=1}^{k} S_\alpha(t-t_i) I_i(y_{t_i}^{(n)} + \tilde{\varphi}_{t_i})$$
$$-B^* T_\alpha^*(b-t)\int_0^b (aI + \Gamma_s^b)^{-1} T_\alpha(b-s) f_n(s)dw(s).$$

Using the fact that F has compact values and (H3) holds, we may pass to a subsequence if necessary to obtain that f_n converges to f_* in $L^p([t_k, t_{k+1}], L_2^0)$, hence, $f_* \in S_{F,y^*}$. Then, for each $t \in [t_k, t_{k+1}], k = 1, \dots, m$,

$$\bar{\rho}_n(t) \to \bar{\rho}_*(t) = -S_\alpha(t)G(0,\varphi) + G(t, y_t^* + \tilde{\varphi}_t)$$
$$+ \sum_{i=1}^{k} S_\alpha(t-t_i) I_i(y_{t_i}^* + \tilde{\varphi}_{t_i})$$
$$+ \int_0^t T_\alpha(t-s) B u_{y^*}^a(s)ds + \int_0^t T_\alpha(t-s) f_n(s)dw(s),$$

where

$$u_{y^*}^a(t) = S^* T_\alpha^*(b-t)(aI + \Gamma_0^b)^{-1}\Big[E\tilde{x}_b + \int_0^b \tilde{\phi}(s)dw(s)$$
$$-S_\alpha(b)[\varphi(0) - G(0,\varphi)] - G(b, y_b^* + \tilde{\varphi}_b)\Big]$$
$$-B^* T_\alpha^*(b-t)(aI + \Gamma_s^b)^{-1} \sum_{i=1}^{k} S_\alpha(t-t_i) I_i(y_{t_i}^* + \tilde{\varphi}_{t_i})$$
$$-B^* T_\alpha^*(b-t)\int_0^b (aI + \Gamma_s^b)^{-1} T_\alpha(b-s) f_*(s)dw(s).$$

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Therefore, $\bar{\rho}_*(t) \in (\bar{\Phi}y)(t)$ and $(\bar{\Phi}y)(t) \in \mathcal{P}_{cl}(\mathcal{B}_b^0)$.

Step 2. We show that $(\bar{\Phi}y)(t)$ is a contractive multi-valued map for each $y(t) \in \mathcal{B}_b^0$.

Let $t \in [0, t_1]$ and $y(t), \hat{y}(t) \in \mathcal{B}_b^0$ and let $\bar{\rho}(t) \in (\bar{\Phi}y)(t)$. Then there exists $f \in S_{F,y}$ such that

$$\begin{aligned}\bar{\rho}(t) &= -S_\alpha(t)G(0,\varphi) + G(t, y_t + \tilde{\varphi}_t)\\ &\quad + \int_0^t T_\alpha(t-s)Bu_y^a(s)ds + \int_0^t T_\alpha(t-s)f(s)dw(s).\end{aligned}$$

From (H3), there exists $v(t) \in F(t, \hat{y}_t + \tilde{\varphi}_t)$ such that

$$E \parallel f(t) - v(t) \parallel_{L_2^0}^p \leq l(t) \parallel y_t - \hat{y}_t \parallel_{\mathcal{B}}^p .$$

Consider $\Lambda : [0, t_1] \to \mathcal{P}(L_2^0)$, given by

$$\Lambda(t) = \{v(t) \in H : E \parallel f(t) - v(t) \parallel_{L_2^0}^p \leq l(t) \parallel y_t - \hat{y}_t \parallel_{\mathcal{B}}^p\}.$$

Since the multi-valued operator $W(t) = \Lambda(t) \cap F(t, \hat{y}_t + \tilde{\varphi}_t)$ is measurable (see [28], Proposition III.4), there exists a function $\hat{f}(t)$, which is a measurable selection for W. So, $\hat{f}(t) \in F(t, \hat{y}_t + \tilde{\varphi}_t)$ and

$$E \parallel f(t) - \hat{f}(t) \parallel_{L_2^0}^p \leq l(t) \parallel y_t - \hat{y}_t \parallel_{\mathcal{B}}^p .$$

For each $t \in [0, t_1]$, we define

$$\begin{aligned}\hat{\rho}(t) &= -S_\alpha(t)G(0,\varphi) + G(t, \hat{y}_t + \tilde{\varphi}_t)\\ &\quad + \int_0^t T_\alpha(t-s)Bu_{\hat{y}}^a(s)ds + \int_0^t T_\alpha(t-s)\hat{f}(s)dw(s).\end{aligned}$$

Then, for each $t \in [0, t_1]$, we have

$$\begin{aligned}&E \parallel \bar{\rho}(t) - \hat{\rho}(t) \parallel_H^p\\ &\quad \leq 3^{p-1}E \parallel G(t, y_t + \tilde{\varphi}_t) - G(t, \hat{y}_t + \tilde{\varphi}_t) \parallel_H^p\\ &\qquad +3^{p-1}E\left\| \int_0^t T_\alpha(t-s)B[u_y^a(s) - u_{\hat{y}}^a(s)]ds \right\|_H^p\\ &\qquad +3^{p-1}E\left\| \int_0^t T_\alpha(t-s)[f(s) - \hat{f}(s)]dw(s) \right\|_H^p\\ &\quad \leq 3^{p-1}L_G \parallel y_t - \hat{y}_t \parallel_{\mathcal{B}}^p\\ &\qquad +3^{p-1}\tilde{M}_T^p t_1^{p-1} \int_0^t (t-s)^{p(\alpha-1)} E \parallel B[u_y^a(s) - u_{\hat{y}}^a(s)] \parallel_H^p ds\\ &\qquad +3^{p-1}C_p\tilde{M}_T^p\left[\int_0^t \left[(t-s)^{p(\alpha-1)} E \parallel f(s) - \hat{f}(s) \parallel_{L_2^0}^p\right]^{2/p} ds\right]^{p/2}\\ &\quad \leq 3^{p-1}L \parallel y_t - \hat{y}_t \parallel_{\mathcal{B}}^p\end{aligned}$$

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$$
\begin{aligned}
&+6^{p-1}\tilde{M}_T^{2p}M_1^{2p}\frac{1}{a^p}t_1^{p-1}\int_0^t [(t-s)(b-s)]^{p(\alpha-1)}\Bigg[L \parallel y_t-\hat{y}_t \parallel_{\mathcal{B}}^p \\
&+C_p t_1^{p/2-1}\tilde{M}_T^p \int_0^b (b-\tau)^{p(\alpha-1)} l(\tau) \parallel y_\tau - \hat{y}_\tau \parallel_{\mathcal{B}}^p d\tau\Bigg] ds \\
&+3^{p-1}C_p t_1^{p/2-1}\tilde{M}_T^p \int_0^t (t-s)^{p(\alpha-1)} l(s) \parallel y_s-\hat{y}_s \parallel_{\mathcal{B}}^p ds \\
\le\ & 3^{p-1}K_b^p L_G \parallel y-\hat{y} \parallel_b^p \\
&+6^{p-1}K_b^p\tilde{M}_T^{2p}M_1^{2p}\frac{1}{a^p}t_1^{p-1}\int_0^t [(t-s)(b-s)]^{p(\alpha-1)}\Bigg[L_G \\
&+C_p t_1^{p/2-1}\tilde{M}_T^p \left(\int_0^b (b-\tau)^{\frac{p(\alpha-1)}{1-q}} d\tau\right)^{1-q} \parallel l \parallel_{L^{\frac{1}{q}}(J,R^+)}\Bigg] ds \parallel y-\hat{y} \parallel_b^p \\
&+3^{p-1}K_b^p C_p t_1^{p/2-1}\tilde{M}_T^p \left(\int_0^t (t-s)^{\frac{p(\alpha-1)}{1-q}} ds\right)^{1-q} \\
&\times \parallel l \parallel_{L^{\frac{1}{q}}(J,R^+)} \parallel y-\hat{y} \parallel_b^p \\
\le\ & 3^{p-1}K_b^p\Bigg(L_G + 2^{p-1}\tilde{M}_T^{2p}M_1^{2p}\frac{1}{a^p}t_1^{p-1}\frac{1}{2p(\alpha-1)+1}b^{2p(\alpha-1)+1}\Bigg[L_G \\
&+C_p t_1^{p/2-1}\tilde{M}_T^p\left(\frac{1-q}{p(1-\alpha)+1-q}\right)^{1-q} b^{p(\alpha-1)+1-q} \parallel l \parallel_{L^{\frac{1}{q}}(J,R^+)}\Bigg] \\
&+3^{p-1}K_b^p C_p t_1^{p/2-1}\tilde{M}_T^p\left(\frac{1-q}{p(1-\alpha)+1-q}\right)^{1-q} \\
&\times t_1^{p(\alpha-1)+1-q} \parallel l \parallel_{L^{\frac{1}{q}}(J,R^+)}\Bigg) \parallel y-\hat{y} \parallel_b^p .
\end{aligned}
$$

Similarly, for each $t \in (t_k, t_{k+1}], k = 1, \ldots, m$. Let $y(t), \hat{y}(t) \in \mathcal{B}_b^0$ and let $\bar{\rho}(t) \in (\bar{\Phi}y)(t)$. Then there exists $f \in S_{F,y}$ such that

$$
\begin{aligned}
\bar{\rho}(t) = & -S_\alpha(t)G(0,\varphi) + G(t, y_t+\tilde{\varphi}_t) + \sum_{i=1}^k S_\alpha(t-t_i)I_i(y_{t_i}+\tilde{\varphi}_{t_i}) \\
& + \int_0^t T_\alpha(t-s)Bu_y^a(s)ds + \int_0^t T_\alpha(t-s)f(s)dw(s).
\end{aligned}
$$

From (H3), there exists $v(t) \in F(t, \hat{y}_t+\tilde{\varphi}_t)$ such that

$$E \parallel f(t)-v(t) \parallel_{L_2^0}^p \le l(t) \parallel y_t-\hat{y}_t \parallel_{\mathcal{B}}^p .$$

Consider $\Lambda : (t_k, t_{k+1}] \to \mathcal{P}(L_2^0)$, given by

$$\Lambda(t) = \{v(t) \in H : E \parallel f(t)-v(t) \parallel_{L_2^0}^p \le l(t) \parallel y_t-\hat{y}_t \parallel_{\mathcal{B}}^p\}.$$

Since the multi-valued operator $W(t) = \Lambda(t) \cap F(t, \hat{y}_t+\tilde{\varphi}_t)$ is measurable (see [28], Proposition III.4), there exists a function $\hat{f}(t)$, which is a measurable se-

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lection for W. So, $\hat{f}(t) \in F(t, \hat{y}_t + \tilde{\varphi}_t)$ and

$$E \parallel f(t) - \hat{f}(t) \parallel_{L_2^0}^p \leq l(t) \parallel y_t - \hat{y}_t \parallel_{\mathcal{B}}^p .$$

For each $t \in (t_k, t_{k+1}], k = 1, \ldots, m$, we define

$$\begin{aligned}
\hat{\rho}(t) &= -S_\alpha(t)G(0,\varphi) + G(t, \hat{y}_t + \tilde{\varphi}_t) + \sum_{i=1}^{k} S_\alpha(t-t_i)I_i(\hat{y}_{t_i} + \tilde{\varphi}_{t_i}) \\
&\quad + \int_0^t T_\alpha(t-s)Bu_{\hat{y}}^a(s)ds + \int_0^t T_\alpha(t-s)\hat{f}(s)dw(s).
\end{aligned}$$

Then, for each $t \in (t_k, t_{k+1}], k = 1, \ldots, m$, we have

$$\begin{aligned}
&E \parallel \bar{\rho}(t) - \hat{\rho}(t) \parallel_H^p \\
&\leq 4^{p-1} E \parallel G(t, y_t + \tilde{\varphi}_t) - G(t, \hat{y}_t + \tilde{\varphi}_t) \parallel_H^p \\
&\quad +4^{p-1} E \left\| \sum_{i=1}^{k} S_\alpha(t-t_i)[I_i(y_{t_i} + \tilde{\varphi}_{t_i}) - I_i(\hat{y}_{t_i} + \tilde{\varphi}_{t_i})] \right\|_H^p \\
&\quad +4^{p-1} E \left\| \int_0^t T_\alpha(t-s)B[u_y^a(s) - u_{\hat{y}}^a(s)]ds \right\|_H^p \\
&\quad +4^{p-1} E \left\| \int_0^t T_\alpha(t-s)[f(s) - \hat{f}(s)]dw(s) \right\|_H^p \\
&\leq 4^{p-1} L_G \parallel y_t - \hat{y}_t \parallel_{\mathcal{B}}^p \\
&\quad +4^{p-1} k^{p-1} \tilde{M}_S^p \sum_{i=1}^{k} E \parallel I_i(y_{t_i} + \tilde{\varphi}_{t_i}) - I_i(\hat{y}_{t_i} + \tilde{\varphi}_{t_i}) \parallel_H^p \\
&\quad +3^{p-1} \tilde{M}_T^p (t_{k+1} - t_k)^{p-1} \int_0^t (t-s)^{p(\alpha-1)} E \parallel B[u_y^a(s) - u_{\hat{y}}^a(s)] \parallel_H^p ds \\
&\quad +4^{p-1} C_p \tilde{M}_T^p \left[\int_0^t \left[(t-s)^{p(\alpha-1)} E \parallel f(s) - \hat{f}(s) \parallel_{L_2^0}^p \right]^{2/p} ds \right]^{p/2} \\
&\leq 4^{p-1} L_G \parallel y_t - \hat{y}_t \parallel_{\mathcal{B}}^p + 4^{p-1} k^{p-1} \tilde{M}_S^p \sum_{i=1}^{k} c_i \parallel y_{t_i} - \hat{y}_{t_i} \parallel_{\mathcal{B}}^p \\
&\quad +12^{p-1} \tilde{M}_T^{2p} M_1^{2p} \frac{1}{a^p} (t_{k+1} - t_k)^{p-1} \int_0^t [(t-s)(b-s)]^{p(\alpha-1)} \\
&\quad \times \Bigg[L_G \parallel y_t - \hat{y}_t \parallel_{\mathcal{B}}^p + k^{p-1} \tilde{M}_S^p \sum_{i=1}^{k} c_i \parallel y_{t_i} - \hat{y}_{t_i} \parallel_{\mathcal{B}}^p \\
&\quad + C_p (t_{k+1} - t_k)^{p/2-1} \tilde{M}_T^p \int_0^b (b-\tau)^{p(\alpha-1)} l(\tau) \parallel y_\tau - \hat{y}_\tau \parallel_{\mathcal{B}}^p d\tau \Bigg] ds \\
&\quad +4^{p-1} C_p (t_{k+1} - t_k)^{p/2-1} \tilde{M}_T^p \int_0^t (t-s)^{p(\alpha-1)} l(s) \parallel y_s - \hat{y}_s \parallel_{\mathcal{B}}^p ds
\end{aligned}$$

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$$
\begin{aligned}
&\leq 4^{p-1}K_b^p L_G \parallel y-\hat{y}\parallel_b^p + 4^{p-1}K_b^p k^{p-1}\tilde{M}_S^p\sum_{i=1}^k c_i \parallel y-\hat{y}\parallel_b^p \\
&\quad +12^{p-1}K_b^p\tilde{M}_T^{2p}M_1^{2p}\frac{1}{a^p}(t_{k+1}-t_k)^{p-1}\int_0^t[(t-s)(b-s)]^{p(\alpha-1)} \\
&\quad \times\Bigg[L_G+k^{p-1}\tilde{M}_S^p\sum_{i=1}^k c_i + C_p(t_{k+1}-t_k)^{p/2-1}\tilde{M}_T^p \\
&\quad \times\bigg(\int_0^b(b-\tau)^{\frac{p(\alpha-1)}{1-q}}d\tau\bigg)^{1-q}\parallel l\parallel_{L^{\frac{1}{q}}(J,R^+)}\Bigg]ds\parallel y-\hat{y}\parallel_b^p \\
&\quad +4^{p-1}K_b^p C_p(t_{k+1}-t_k)^{p/2-1}\tilde{M}_T^p\bigg(\int_0^t(t-s)^{\frac{p(\alpha-1)}{1-q}}ds\bigg)^{1-q} \\
&\quad \times\parallel l\parallel_{L^{\frac{1}{q}}(J,R^+)}\parallel y-\hat{y}\parallel_b^p \\
&\leq 4^{p-1}K_b^p\Bigg(L_G+k^{p-1}\tilde{M}_S^p\sum_{i=1}^k c_i + 3^{p-1}\tilde{M}_T^{2p}M_1^{2p}\frac{1}{a^p}(t_{k+1}-t_k)^{p-1} \\
&\quad \times\frac{1}{2p(\alpha-1)+1}b^{2p(\alpha-1)+1}\Bigg[L_G+k^{p-1}\tilde{M}_S^p\sum_{i=1}^k c_i \\
&\quad +C_p(t_{k+1}-t_k)^{p/2-1}\tilde{M}_T^p\bigg(\frac{1-q}{p(1-\alpha)+1-q}\bigg)^{1-q} \\
&\quad \times b^{p(\alpha-1)+1-q}\parallel l\parallel_{L^{\frac{1}{q}}(J,R^+)}\Bigg] \\
&\quad +K_b^p C_p(t_{k+1}-t_k)^{p/2-1}\tilde{M}_T^p\bigg(\frac{1-q}{p(1-\alpha)+1-q}\bigg)^{1-q} \\
&\quad \times(t_{k+1}-t_k)^{p(\alpha-1)+1-q}\parallel l\parallel_{L^{\frac{1}{q}}(J,R^+)}\Bigg)\parallel y-\hat{y}\parallel_b^p .
\end{aligned}
$$

Thus, for all $t\in[0,b]$, we have

$$\parallel\bar{\rho}-\hat{\rho}\parallel_b^p\leq\tilde{L}\parallel y-\hat{y}\parallel_b^p,$$

and

$$H_d^p(\bar{\Phi}y,\bar{\Phi}\hat{y})\leq\tilde{L}\parallel y-\hat{y}\parallel_b^p,$$

where

$$
\begin{aligned}
\tilde{L}&=4^{p-1}K_b^p\Bigg[L_G+m^{p-1}\tilde{M}_S^p\sum_{i=1}^m c_i + C_p\tilde{M}_T^p\bigg(\frac{1-q}{p(1-\alpha)+1-q}\bigg)^{1-q} \\
&\quad \times b^{p(\alpha-1/2)-q}\parallel l\parallel_{L^{\frac{1}{q}}(J,R^+)}\Bigg]\Bigg[1+3^{p-1}\tilde{M}_T^{2p}M_1^{2p}\frac{1}{a^p}\frac{b^{p(2\alpha-1)}}{2p(\alpha-1)+1}\Bigg]<1.
\end{aligned}
$$

Hence, $\bar{\Phi}$ is a contraction on $\mathcal{B}_b^0$. In view of Lemma 2.6, we conclude that $\bar{\Phi}$ has at least one fixed point $y^*\in\mathcal{B}_b^0$. Let $x(t)=y^*(t)+\tilde{\varphi}(t), t\in(-\infty,b]$. Then, x

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is a fixed point of the operator Φ, which implies that x is a mild solution of the problem (1)-(3) and the proof of Theorem 3.1 is complete.

Theorem 3.2. *Assume that assumptions of Theorem* 3.1 *and* (H5) *are satisfied and* $\{T_\alpha(t): t \geq 0\}$ *is compact. Moreover, if* F *is uniformly-bounded, then the system* (1)-(3) *is approximately controllable on* J.

Proof. Let $x^a(\cdot)$ be a fixed point of Φ in $\mathcal{B}_b$. By Theorem 3.1, any fixed point of Φ is a mild solution of the system (1)-(3). This means that there is $x^a \in \Phi(x^a)$, that is, there is $f \in S_{F,x^a}$ such that

$$x^a(t) = \begin{cases} S_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t, x_t^a) \\ \quad + \int_0^t T_\alpha(t-s)Bu_{x^a}^a(s)ds + \int_0^t T_\alpha(t-s)f(s)dw(s), \quad t \in [0, t_1], \\ S_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t, x_t^a) + S_\alpha(t-t_1)I_1(x_{t_1}^a) \\ \quad + \int_0^t \mathcal{S}_\alpha(t-s)Bu_{x^a}^a(s)ds + \int_0^t \mathcal{S}_\alpha(t-s)f(s)dw(s), \quad t \in (t_1, t_2], \\ \vdots \\ S_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t, x_t^a) \\ \quad + \sum_{k=1}^m S_\alpha(t-t_k)I_k(x_{t_k}^a) \\ \quad + \int_0^t \mathcal{S}_\alpha(t-s)Bu_{x^a}^a(s)ds + \int_0^t \mathcal{S}_\alpha(t-s)f(s)dw(s), \quad t \in (t_m, b], \end{cases}$$

where

$$u_x^a(t) = \begin{cases} S^* T_\alpha^*(b-t)(aI + \Gamma_0^b)^{-1}\Big[E\tilde{x}_b + \int_0^b \tilde{\phi}(s)dw(s) \\ \quad -S_\alpha(b)[\varphi(0) - G(0,\varphi)] - G(b, x_b)\Big] \\ \quad -B^* T_\alpha^*(b-t)\int_0^b (aI + \Gamma_s^b)^{-1}\mathcal{S}_\alpha(b-s)f(s)dw(s), \quad t \in [0, t_1], \\ S^* T_\alpha^*(b-t)(aI + \Gamma_0^b)^{-1}\Big[E\tilde{x}_b + \int_0^b \tilde{\phi}(s)dw(s) \\ \quad -S_\alpha(b)\varphi(0) - G(0,\varphi)] - G(b, x_b)\Big] \\ \quad -B^* T_\alpha^*(b-t)(aI + \Gamma_s^b)^{-1}S_\alpha(t-t_1)I_1(x_{t_1}) \\ \quad -B^* T_\alpha^*(b-t)\int_0^b (aI + \Gamma_s^b)^{-1}\mathcal{S}_\alpha(b-s)f(s)dw(s), \quad t \in (t_1, t_2], \\ \vdots \\ S^* T_\alpha^*(b-t)(aI + \Gamma_0^b)^{-1}\Big[E\tilde{x}_b + \int_0^b \tilde{\phi}(s)dw(s) \\ \quad -S_\alpha(b)\varphi(0) - G(0,\varphi)] - G(b, x_b)\Big] \\ \quad -B^* T_\alpha^*(b-t)\sum_{k=1}^m (aI + \Gamma_s^b)^{-1}S_\alpha(t-t_k)I_k(x_{t_k}) \\ \quad -B^* T_\alpha^*(b-t)\int_0^b (aI + \Gamma_s^b)^{-1}\mathcal{S}_\alpha(b-s)f(s)dw(s), \quad t \in (t_m, b]. \end{cases}$$

By using the stochastic Fubini theorem, it is easy to see that

$$x^a(b) = S_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(b, x_b^a) + \sum_{k=1}^m S_\alpha(b-t_k)I_k(x_{t_k}^a)$$
$$+ \int_0^b \mathcal{S}_\alpha(b-s)Bu_{x^a}^a(s)ds + \int_0^b \mathcal{S}_\alpha(t-s)f(s)dw(s)$$

$$
\begin{aligned}
&= \tilde{x}_b - a(aI + \Gamma_0^b)^{-1}\Big[E\tilde{x}_b + \int_0^b \tilde{\phi}(s)dw(s) - S_\alpha(b)[\varphi(0) - G(0,\varphi)] \\
&\quad -G(b,\bar{x}_b^a)\Big] - a\sum_{k=1}^m (aI + \Gamma_s^b)^{-1} S_\alpha(b-t_k) I_k(x_{t_k}) \\
&\quad -a\int_0^b (aI + \Gamma_s^b)^{-1} \mathcal{S}_\alpha(b-s) f(s) dw(s).
\end{aligned}
$$

By the assumption that the sequences $\{f(s)\}$ is uniformly bounded on J. Thus there is a subsequence, still denoted by $\{f(s)\}$ that converge weakly to say $f^{**}(s)$ in L_2^0. Now, the compactness of $T_\alpha(t), t > 0$ which implies that $T_\alpha(b - s)[f(s) - f^{**}(s)] \to 0$. Also, by (H5), for all $t \in J$, $a(aI + \Gamma_s^b)^{-1} \to 0$ strongly as $a \to 0^+$ and $\| a(aI+\Gamma_s^b)^{-1} \| \le 1$. Thus, for $t \in [0, b]$, by the Lebesque dominated convergence theorem it follows that

$$
\begin{aligned}
&E \parallel x^a(b) - \tilde{x}_b \parallel_H^p \\
&\quad \le 5^{p-1} E \parallel a(aI + \Gamma_0^b)^{-1}[E\tilde{x}_b - S_\alpha(b)[\varphi(0) - G(0,\varphi,0)] - G(b,\bar{x}_b^a)] \parallel_H^p \\
&\quad +5^{p-1} E \bigg\| \sum_{k=1}^m a(aI + \Gamma_s^b)^{-1} S_\alpha(b-t_k) I_k(x_{t_k}) \bigg\|_H^p \\
&\quad +5^{p-1} E\bigg(\int_0^b \parallel a(aI + \Gamma_0^b)^{-1} \tilde{\phi}(s) \parallel_H^2 ds\bigg)^{p/2} \\
&\quad +5^{p-1} E\bigg(\int_0^b \parallel a(aI + \Gamma_s^b)^{-1} T_\alpha(b-s)[f(s) - f^{**}(s)] \parallel_H^2 ds\bigg)^{p/2} \\
&\quad +5^{p-1} E\bigg(\int_0^b \parallel a(aI + \Gamma_s^b)^{-1} T_\alpha(b-s) f^{**}(s) \parallel_H^2 ds\bigg)^{p/2} \\
&\quad \to 0 \quad \text{as } a \to 0^+.
\end{aligned}
$$

So $x^a(b) \to \tilde{x}_b$ holds, which shows that the system (1)-(3) is approximately controllable and the proof is complete.

4 Application

Consider the fractional impulsive partial stochastic neutral functional differential inclusions in the following form

$$
\begin{aligned}
D_t^\alpha N(z_t)(x) \in{}& \frac{\partial^2}{\partial x^2} N(z_t)(x) + \tilde{u}(t,x) \\
&+ \int_{-\infty}^t \tilde{b}_1(t, s-t, x, z(s,x)) ds \frac{w(t)}{dt}, \qquad (6) \\
&\qquad 0 \le t \le b, 0 \le x \le \pi,
\end{aligned}
$$

$$
z(t,0) = z(t,\pi) = 0, \quad 0 \le t \le b, \qquad (7)
$$

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$$z(\tau,x)=\varphi(\tau,x), \quad \tau\leq 0, 0\leq x\leq \pi, \tag{8}$$

$$\triangle z(t_k,x)=\int_{-\infty}^{t_k}\eta_k(s-t_k)z(s,x)ds, \quad k=1,2,\dots,m, \tag{9}$$

where D_t^{α} is a Caputo fractional partial derivative of order $0<\alpha<1$, and $\tilde{u}(\cdot)$ is a real function of bounded variation on $[0,b]$. $w(t)$ denotes a standard cylindrical Wiener process in H defined on a stochastic space $(\Omega,\mathcal{F},P)$. In this system,

$$N(z_t)(x)=z(t,x)-\int_{-\infty}^{t}b_1(s-t)z(s,x)ds.$$

Let $H=L^2([0,\pi])$ with the norm $\|\cdot\|$ and define the operators $A:D(A)\subseteq H\to H$ by $A\omega=\omega''$ with the domain

$$D(A):=\{\omega\in H:\omega,\omega' \text{ are absolutely continuous}, \omega''\in H, \omega(0)=\omega(\pi)=0\}.$$

It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ in H. Furthermore, A has a discrete spectrum with eigenvalues of the form $-n^2, n\in\mathbf{N}$ and corresponding normalized eigenfunctions are given by $x_n(z)=\sqrt{\frac{2}{\pi}}\sin(nz)$. In addition $\{x_n:n\in\mathbf{N}\}$ is an orthonormal basis for H, $T(t)y=\sum_{n=1}^{\infty}e^{-n^2t}(y,x_n)x_n$ for all $y\in H$, and every $t>0$. From these expressions it follows that $(T(t))_{t\geq 0}$ is a uniformly bounded compact semigroup, so that $R(\lambda,A)=(\lambda-A)^{-1}$ is a compact operator for all $\lambda\in\rho(A)$ i.e. $A\in A^{\alpha}(\theta^0,\omega^0)$.

Let $r\geq 0, 1\leq p<\infty$ and let $\tilde{h}:(-\infty,-r]\to R$ be a nonnegative measurable function which satisfies the conditions (h-5), (h-6) in the terminology of Hino et al. [29]. Briefly, this means that $\tilde{h}$ is locally integrable and there is a nonnegative, locally bounded function γ on $(-\infty,0]$ such that $\tilde{h}(\xi+\tau)\leq\gamma(\xi)\tilde{h}(\tau)$ for all $\xi\leq 0$ and $\theta\in(-\infty,-r)\setminus N_{\xi}$, where $N_{\xi}\subseteq(-\infty,-r)$ is a set whose Lebesgue measure zero. We denote by $\mathcal{PC}_r\times L^p(\tilde{h},H)$ the set consists of all classes of functions $\varphi:(-\infty,0]\to H$ such that $\varphi_{|_{[-r,0]}}\in\mathcal{PC}([-r,0],H)$, $\varphi(\cdot)$ is Lebesgue measurable on $(-\infty,-r)$, and $\tilde{h}\parallel\varphi\parallel^p$ is Lebesgue integrable on $(-\infty,-r)$. The seminorm is given by

$$\parallel\varphi\parallel_{\mathcal{B}}=\sup_{-r\leq\tau\leq 0}\parallel\varphi(\tau)\parallel+\bigg(\int_{-\infty}^{-r}\tilde{h}(\tau)\parallel\varphi\parallel^p d\tau\bigg)^{1/p}.$$

The space $\mathcal{B}=\mathcal{PC}_r\times L^p(\tilde{h},H)$ satisfies axioms (A)-(C). Moreover, when $r=0$ and $p=2$, we can take $\tilde{H}=1$, $M(t)=\gamma(-t)^{1/2}$ and $K(t)=1+(\int_{-t}^{0}\tilde{h}(\tau)d\tau)^{1/2}$, for $t\geq 0$ (see [29, Theorem 1.3.8] for details).

Additionally, we will assume that

(i) The function $b_1:R\to R$, is continuous, and $\tilde{L}_1=(\int_{-\infty}^{0}\frac{(b_1(s))^2}{\tilde{h}(s)}ds)^{\frac{1}{2}}<\infty$,

(ii) The function $\tilde{b}_1:R^4\to R$, is continuous and there exist continuous functions $a_j:R\to R, j=1,2,$ such that

$$|\tilde{b}_1(t,s,x,y)|\leq a_1(t)a_2(s)|y|, \quad (t,s,x,y)\in R^4,$$

and

$$|\tilde{b}_1(t,s,x,y_1) - \tilde{b}_1(t,s,x,y_2)| \\ \leq a_1(t)a_2(s)|y_1 - y_2|, \quad (t,s,x,y_1),(t,s,x,y_2) \in R^4$$

with $\hat{L}_1 = (\int_{-\infty}^{0} \frac{(a_2(s))^2}{\tilde{h}(s)} ds)^{\frac{1}{2}} < \infty$.

(iii) The functions $\eta_k : R \to R, k = 1,2,\ldots,m$, are continuous, and $L_k = (\int_{-\infty}^{0} \frac{(\eta_k(s))^2}{\tilde{h}(s)} ds)^{\frac{1}{2}} < \infty$ for every $k = 1,2,\ldots,m$,

Take $\varphi \in \mathcal{B} = \mathcal{PC}_0 \times L^2(\tilde{h}, H)$ with $\varphi(\theta)(x) = \varphi(\theta,x), (\theta,x) \in (-\infty,0] \times \mathcal{B}$. Let $G : [0,b] \times \mathcal{B} \to H$, $F : [0,b] \times \mathcal{B} \to \mathcal{P}(H)$ be the operators defined by

$$N(\psi)(x) = \psi(0,x) - G(t,\psi)(x),$$

$$G(t,\psi)(x) = \int_{-\infty}^{0} b_1(s)\psi(s,x)ds,$$

$$F(t,\psi)(x) = \int_{-\infty}^{0} \tilde{b}_1(t,s,x,\psi(s,x))ds.$$

Also defining the maps I_k and B by

$$I_k(\psi)(x) = \int_{-\infty}^{0} \eta_k(s)\psi(s,x)ds, \ (Bu)(t)(x) = \tilde{u}(t,x).$$

Using these definitions, we can represent the system (6)-(9) in the abstract form (1)-(3). Moreover, for any $t \in [0,b], \psi, \psi_1 \in \mathcal{B}$, we have that $E \parallel G(t,\psi) - G(t,\psi_1) \parallel^p \leq L_G \parallel \psi - \psi_1 \parallel_{\mathcal{B}}^p$, $E \parallel F(t,\psi) - F(t,\psi_1) \parallel^p \leq L_F \parallel \psi - \psi_1 \parallel_{\mathcal{B}}^p$, $E \parallel I_k(\psi) - I_k(\psi_1) \parallel^p \leq (L_k)^p \parallel \psi - \psi_1 \parallel_{\mathcal{B}}^p, k = 1,2,\ldots,m$, and F is bounded linear operators with $E \parallel F \parallel_{L(\mathcal{B},H)}^p \leq L_F$, where $L_G = (\tilde{L}_1)^p, L_F = (\parallel a_1 \parallel_\infty \hat{L}_1)^p$. Further, we can impose some suitable conditions on the above-defined functions to verify the assumptions on Theorem 3.2. Hence by Theorems 3.2, the system (6)-(9) is approximately controllable on $[0,b]$.

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HYERS-ULAM STABILITY OF GENERAL ADDITIVE MAPPINGS IN C^*-ALGEBRA

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Abstract. In this paper, we prove that the generalized Hyers-Ulam stability of homomorphisms in C^*-algebras and Lie C^*-algebras and also of derivations on C^*-algebras and Lie C^*-algebras for an 4-variable additive functional equation

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [28] concerning the stability of group homomorphisms. Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [19] for linear mappings by considering an unbounded Cauchy difference. The paper of Rassias has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [9] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach. The stability problems for several functional equations or inequations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2]–[8],[10], [12]–[14], [18]–[21],[22]-[27],[29]).

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) d(x,y)=0 if and only if x=y;
(2) d(x,y)=d(y,x) for all $x, y \in X$;
(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.1 (see[6],[7]). *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for*

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each given element $x \in X$, either

$$d(J^n x, J^{n+1}x) = \infty \tag{1.1}$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) *$d(J^n x, J^{n+1}x) < \infty$, fir all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X | d(J^{N_0}x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

By the using fixed point method, the stability problems of several functional equations have been extensively investigated by a number of authors(see[5][6][16][17]).

This paper, using the fixed point method, we prove the generalized Hyers-Ulam stability of homomorphisms in C^*-algebras and of derivations on C^*-algebras for the general Jensen-type functional equation. And, we prove that the generalized Hyers-Ulam stability of homomorphisms in Lie C^*-algebras and of derivations on Lie C^*-algebras for the following additive functional equation:

$$\begin{aligned} &f(dx_1 + ax_2 + bx_3 + cx_4) + f(ax_1 + dx_2 + cx_3 + bx_4) \\ &+ f(bx_1 + cx_2 + dx_3 + ax_4) + f(cx_1 + bx_2 + ax_3 + dx_4) \\ &= (d + a + b + c)f(x_1 + x_2 + x_3 + x_4) \end{aligned} \tag{1.2}$$

Here a, b, c and d are real numbers with $a+b+c+d \neq 0$. Throughout the paper, assume that k is $a + b + c + d$.

2. Stability of Homomorphisms and Derivations in C^*-Algebras

Throughout this section, assume that X is a C^*-algebras with norm $\|\cdot\|_X$ and that Y is a C^*-algebra with norm $\|\cdot\|_Y$.

For a given mapping $f : X \to Y$, we define

$$\begin{aligned} &F_\mu f(x_1, x_2, x_3, x_4) := \\ &\mu f(dx_1 + ax_2 + bx_3 + cx_4) + \mu f(ax_1 + dx_2 + cx_3 + bx_4) \\ &+ \mu f(bx_1 + cx_2 + dx_3 + ax_4) + \mu f(cx_1 + bx_2 + ax_3 + dx_4) \\ &- (d + a + b + c)f(\mu(x_1 + x_2 + x_3 + x_4)) \end{aligned} \tag{2.1}$$

for all $\mu \in \mathbb{T}^1 := \{v \in \mathbb{C} : |v| = 1\}$ and $x_1, \cdots, x_m \in X$.

Note that a $\mathbb{C}$-linear mapping $H : X \to Y$ is called a *homomorphism* in C^*-algebras if H satisfies $H(xy) = H(x)H(y)$ and $H(x^*) = H(x)^*$ for all $x, y \in X$. Now we prove the Hyers-Ulam stability of homomorphisms in C^*-algebras for the functional equation $F_\mu f(x, y) = 0$.

Theorem 2.1. *Let a, b, c and d be fixed nonzero real numbers. Let $f : X \to Y$ be an mapping for which there exists a function $\varphi : X^4 \to [0, \infty)$, such that*

$$\|F_\mu f(x_1, x_2, x_3, x_4)\|_Y \leq \varphi(x_1, x_2, x_3, x_4), \tag{2.2}$$

$$\|f(xy) - f(x)f(y)\|_Y \leq \varphi(x, x, y, y), \tag{2.3}$$

$$\|f(x^*) - f(x)^*\|_Y \leq \varphi(x, x, x, x) \tag{2.4}$$

for all $\mu \in \mathbb{T}^1 := \{v \in \mathbb{C} : |v| = 1\}$ and all $x_1, x_2, x_3, x_4, x, y \in X$. If there exists an $0 < L < 1$ such that $\varphi(x_1, x_2, x_3, x_4) \leq \frac{|k|}{4} L\varphi\left(\frac{4}{k}x_1, \frac{4}{k}x_2, \frac{4}{k}x_3, \frac{4}{k}x_4\right)$ for all $x_1, x_2, x_3, x_4 \in X$, $d, a, b, c, \alpha_4 \in \mathbb{R}$ with $4 < |k|$, then there exists a unique C^*-algebra homomorphism $H : X \to Y$ such that

$$\|f(x) - H(x)\|_Y \leq \frac{4}{|k|(1-L)}\varphi\left(\frac{x}{4}, \frac{x}{4}, \frac{x}{4}, \frac{x}{4}\right) \tag{2.5}$$

for all $x \in X$.

Proof. It follows that $\varphi(x_1, x_2, x_3, x_4) \leq \frac{|k|}{4} L\varphi\left(\frac{4}{k}x_1, \frac{4}{k}x_2, \frac{4}{k}x_3, \frac{4}{k}x_4\right)$ that

$$\lim_{j\to\infty} \frac{4^j}{|k|^j}\varphi\left(\frac{(k)^j}{4^j}x_1, \frac{(k)^j}{4^j}x_2, \frac{(k)^j}{4^j}x_3, \frac{(k)^j}{4^j}x_4\right) = 0 \tag{2.6}$$

for all $x, y \in X$.

Consider the set

$$A := \{g : X \to Y\} \tag{2.7}$$

and introduce the *generalized metric* on A:

$$d(g, h) = \inf\{C \in \mathbb{R}_+ : \|g(x) - h(x)\|_Y \leq C\varphi\left(\frac{x}{4}, \frac{x}{4}, \frac{x}{4}, \frac{x}{4}\right), \forall x \in X\}. \tag{2.8}$$

It is easy to show that (A, d) is complete.

Now we consider the linear mapping $J : A \to A$ such that

$$Jg(x) := \frac{4}{|k|}g\left(\frac{k}{4}x\right) \tag{2.9}$$

for all $x \in X$.

By Theorem 3.1 of [6]

$$d(Jg, Jh) \leq Ld(g, h) \tag{2.10}$$

for all $g, h \in A$.

Letting $\mu = 1$ and $x_1 = x_2 = x_3 = x_4 = x$ in (2.2), we get

$$\left\|\frac{4}{k}f\left(\frac{k}{4}x\right) - f(x)\right\| \leq \frac{1}{|k|}\varphi(\frac{x}{4}, \frac{x}{4}, \frac{x}{4}, \frac{x}{4}) \leq \frac{4}{|k|}\varphi(\frac{x}{4}, \frac{x}{4}, \frac{x}{4}, \frac{x}{4})$$

for all $x \in X$.

Hence $d(f, Jf) \leq \frac{4}{|k|}$.

By Theorem 1.1, there exists a mapping $H : X \to Y$ such that

(1) H is a fixed point of J, that is,

$$\frac{4}{|k|}H\left(\frac{k}{4}x\right) = H(x) \tag{2.11}$$

for all $x \in X$. The mapping H is a unique fixed point of J in the set

$$B = \{g \in A : d(f, g) < \infty\}. \tag{2.12}$$

This implies that H is a unique mapping satisfying (2.24) such that there exists $C \in (0, \infty)$ satisfying

$$\|H(x) - f(x)\|_Y \le C\varphi\left(\frac{x}{4}, \frac{x}{4}, \frac{x}{4}, \frac{x}{4}\right) \tag{2.13}$$

for all $x \in X$.

(2) $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the inequality

$$\lim_{n\to\infty} \frac{4^n}{|k|^n} f\left(\frac{(k)^n x}{4^n}\right) = H(x) \tag{2.14}$$

for all $x \in X$.

(3) $d(f, H) \le \frac{1}{1-L} d(f, Jf)$, which implies the ineqality

$$d(f, H) \le \frac{4}{|k|(1 - L)}. \tag{2.15}$$

This implies that the inequality (2.5) holds.

Next, we show that $H(x)$ is additive map.

$$\begin{aligned}
&\|H(dx_1 + ax_2 + bx_3 + cx_4) + H(ax_1 + dx_2 + cx_3 + bx_4)\\
&+H(bx_1 + cx_2 + dx_3 + ax_4)\\
&+H(cx_1 + bx_2 + ax_3 + dx_4) - (d + a + b + c)H(x_1 + x_2 + x_3 + x_4)\|\\
&= \lim_{l\to\infty} \left\| \frac{4^l}{|k|^l} f\left(\frac{(k)^l}{4^l}(dx_1 + ax_2 + bx_3 + cx_4)\right)\right.\\
&+\frac{4^l}{|k|^l} f\left(\frac{(k)^l}{4^l}(ax_1 + dx_2 + cx_3 + bx_4)\right)\\
&+\frac{4^l}{|k|^l} f\left(\frac{(k)^l}{4^l}(bx_1 + cx_2 + dx_3 + ax_4)\right)\\
&+\frac{4^l}{|k|^l} f\left(\frac{(k)^l}{4^l}(cx_1 + bx_2 + ax_3 + dx_4)\right)\\
&\left.-(d + a + b + c)\frac{4^l}{|k|^l} f\left(\frac{(k)^l}{4^l}(x_1 + x_2 + x_3 + x_4)\right)\right\|\\
&\le \lim_{l\to\infty} \frac{4^l}{|k|^l}\varphi\left(\frac{(k)^l}{4^l}x_1, \frac{(k)^l}{4^l}x_2, \frac{(k)^l}{4^l}x_3, \frac{(k)^l}{4^l}x_4\right) = 0
\end{aligned}$$

Therefore, the mapping $H : X \to Y$ is Cauchy additive.

By a similar method with above, we may get

$$\mu H(x) = H(\mu x)$$

for all $\mu \in \mathbb{T}^1$ and all $x \in X$. Thus one can show that the mapping $H : X \to Y$ is $\mathbb{C}$-linear.

It follows from (2.3)that

$$\begin{aligned}
&\|H(xy) - H(x)H(y)\|_Y \\
&= \lim_{n\to\infty} \left|\frac{4}{k}\right|^{2n} \left\| f\left(\frac{(k)^{2n}xy}{4^{2n}}\right) - f\left(\frac{(k)^n x}{4^n}\right) f\left(\frac{(k)^n y}{4^n}\right)\right\| \\
&\leq \lim_{n\to\infty} \left|\frac{4}{k}\right|^{n} \left\| f\left(\frac{(k)^{2n}xy}{4^{2n}}\right) - f\left(\frac{(k)^n x}{4^n}\right) f\left(\frac{(k)^n y}{4^n}\right)\right\| \\
&\leq \lim_{n\to\infty} \left|\frac{4}{k}\right|^{n} \varphi\left(\frac{(k)^n x}{4^n}, \frac{(k)^n x}{4^n}, \frac{(k)^n y}{4^n}, \frac{(k)^n y}{4^n}\right) = 0
\end{aligned}$$

for all $x, y \in X$. So

$$H(xy) = H(x)H(y) \tag{2.16}$$

for all $x, y \in X$.

It follows from (2.4) that

$$\begin{aligned}
\|H(x^*) - H(x)^*\|_Y &= \lim_{n\to\infty} \left|\frac{4}{k}\right|^{n} \left\| f\left(\frac{(k)^n x^*}{4^n}\right) - f\left(\frac{(k)^n x}{4^n}\right)^*\right\|_Y \\
&\leq \lim_{n\to\infty} \left|\frac{4}{k}\right|^{n} \varphi\left(\frac{(k)^n x}{4^n}, \frac{(k)^n x}{4^n}, \frac{(k)^n x}{4^n}, \frac{(k)^n x}{4^n}\right) = 0
\end{aligned}$$

for all $x \in X$. So

$$H(x^*) = H(x)^*$$

for all $x \in X$.

Thus $H : X \to Y$ is C^*-algebra homomorphism satisfying (2.5), as desired.

□

Theorem 2.2. *Let a, b, c and d be fixed nonzero real numbers. Let $f : X \to Y$ be a mapping for which there exists a function $\varphi : X^4 \to [0, \infty)$ satisfying (2.2),(2.3) and (2.4). If there exists an $L < 1$ such $\varphi(x_1, x_2, x_3, x_4) \leq \frac{4}{|k|} L\varphi\left(\frac{k}{4}x_1, \frac{k}{4}x_2, \frac{k}{4}x_3, \frac{k}{4}x_4\right)$ for all $x_1, x_2, x_3, x_4 \in X$ with $|k| < 4$, then there exists a unique C^*-algebra homomorphism $H : X \to Y$ such that*

$$\|f(x) - H(x)\| \leq \frac{1}{4 - 4L}\varphi\left(\frac{x}{k}, \frac{x}{k}, \frac{x}{k}, \frac{x}{k}\right)$$

for all $x \in X$.

Proof. Consider the set

$$A := \{g : X \to Y\}$$

and introduce the generalized metric on A:

$$d(g,h) = \inf\{C \in \mathbb{R}_+ \|g(x) - h(x)\|_Y \leq C\varphi\left(\frac{x}{k}, \frac{x}{k}, \frac{x}{k}, \frac{x}{k}\right), \forall x \in X\} \tag{2.17}$$

We consider the linear mapping $J : A \to A$ such that

$$Jg(x) := \frac{|k|}{4} g\left(\frac{4}{k}\right) \tag{2.18}$$

for all $x \in X$.

It follow from (2.2) that

$$\left\| f(x) - \frac{k}{4}\left(\frac{4}{k}x\right)\right\| \leq \frac{1}{4}\varphi\left(\frac{x}{k}, \frac{x}{k}, \frac{x}{k}, \frac{x}{k}\right) \tag{2.19}$$

for all $x \in X$. Hence $d(f, Jf) \leq \frac{1}{4}$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Recall that a $\mathbb{C}$-linear mapping $\delta : X \to Y$ is called a *derivation* on X satisfies $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in X$.

Theorem 2.3. *Let a, b, c, d be the fixed real numbers. Let $f : X \to Y$ be an mapping for which there exists a function $\varphi : X^4 \to [0, \infty)$, such that*

$$\|F_\mu f(x_1, x_2, x_3, x_4)\|_Y \leq \varphi(x_1, x_2, x_3, x_4), \tag{2.20}$$

$$\|f(xy) - f(x)y - xf(y)\|_Y \leq \varphi(x, x, y, y), \tag{2.21}$$

for all $\mu \in \mathbb{T}^1 := \{v \in \mathbb{C} : |v| = 1\}$ and all $x_1, x_2, x_3, x_4, x, y \in X$. If there exists an $L < 1$ such that $\varphi(x_1, x_2, x_3, x_4) \leq \frac{|k|}{4}L\varphi\left(\frac{4}{k}x_1, \frac{4}{k}x_2, \frac{4}{k}x_3, \frac{4}{k}x_4\right)$ for all $x_1, x_2, x_3, x_4 \in X$ with $|k| > 4$, then there exists a unique derivation $\delta : X \to X$ such that

$$\|f(x) - \delta(x)\|_Y \leq \frac{1}{|k|(1-L)}\varphi\left(\frac{x}{4}, \frac{x}{4}, \frac{x}{4}, \frac{x}{4}\right) \tag{2.22}$$

for all $x \in X$.

Proof. It follows from $\varphi(x_1, x_2, x_3, x_4) \leq \frac{|k|}{4}L\varphi\left(\frac{4}{k}x_1, \frac{4}{k}x_2, \frac{4}{k}x_3, \frac{4}{k}x_4\right)$ that

$$\lim_{j\to\infty}\left|\frac{4}{|k|}\right|^j \varphi\left(\left(\frac{k}{4}\right)^j x_1, \left(\frac{k}{4}\right)^j x_2, \left(\frac{k}{4}\right)^j x_3, \left(\frac{k}{4}\right)^j x_4\right) = 0$$

for all $x_1, x_2, x_3, x_4 \in X$.

Consider the set

$$A := \{g : X \to X\}$$

and introduce the generalized metric on A:

$$d(g,h) = \inf\{C \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq C\varphi\left(\frac{x}{4}, \frac{x}{4}, \frac{x}{4}, \frac{x}{4}\right), \forall x \in X\}.$$

It is easy to show that (A, d) is complete.

Now we consider the linear mapping $J : A \to A$ such that

$$Jg(x) := \frac{4}{k} g\left(\frac{k}{4} x\right)$$

for all $x \in X$.

By Theorem 3.1 of [6]

$$d(Jg, Jh) \leq L d(g, h) \tag{2.23}$$

for all $g, h \in A$.

Letting $\mu = 1$ and $x_1 = x_2 = x_3 = x_4 = x$ in (2.2), we get

$$\left\| \frac{4}{k} f\left(\frac{k}{4} x\right) - f(x) \right\| \leq \frac{1}{|k|} \varphi(\frac{x}{4}, \frac{x}{4}, \frac{x}{4}, \frac{x}{4})$$

for all $x \in X$.

Hence $d(f, Jf) \leq \frac{1}{|k|}$.

By Theorem 1.1, there exists a mapping $\delta : X \to Y$ such that

(1) δ is a fixed point of J, that is,

$$\frac{4}{k} \delta\left(\frac{k}{4} x\right) = \delta(x) \tag{2.24}$$

for all $x \in X$. The mapping δ is a unique fixed point of J in the set

$$B = \{g \in A : d(f, g) < \infty\}. \tag{2.25}$$

This implies that δ is a unique mapping satisfying (2.24) such that there exists $C \in (0, \infty)$ satisfying

$$\|\delta(x) - f(x)\|_Y \leq C \varphi\left(\frac{x}{4}, \frac{x}{4}, \frac{x}{4}, \frac{x}{4}\right) \tag{2.26}$$

for all $x \in X$.

(2) $d(J^n f, \delta) \to 0$ as $n \to \infty$. This implies the inequality

$$\lim_{n\to\infty} \frac{4^n}{|k|^n} f\left(\frac{(k)^n x}{4^n}\right) = \delta(x) \tag{2.27}$$

for all $x \in X$.

(3) $d(f, \delta) \leq \frac{1}{1-L} d(f, Jf)$, which implies the ineqality

$$d(f, H) \leq \frac{1}{|k|(1 - L)}. \tag{2.28}$$

This implies that the inequality (2.22) holds.

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 2.4. *Let a, b, c, d be the fixed real numbers. Let $f : X \to Y$ be an mapping for which there exists a function $\varphi : X^4 \to [0, \infty)$, such that*

$$\|F_\mu f(x_1, x_2, x_3, x_4)\|_Y \leq \varphi(x_1, x_2, x_3, x_4), \tag{2.29}$$

$$\|f(xy) - f(x)y - xf(y)\|_Y \leq \varphi(x, x, y, y), \tag{2.30}$$

for all $\mu \in \mathbb{T}^1 := \{v \in \mathbb{C} : |v| = 1\}$ and all $x_1, x_2, x_3, x_4, x, y \in X$. If there exists an $L < 1$ such that $\varphi(x_1, x_2, x_3, x_4) \leq \frac{4}{|k|} L\varphi\left(\frac{k}{4}x_1, \frac{k}{4}x_2, \frac{k}{4}x_3, \frac{k}{4}x_4\right)$ for all $x_1, x_2, x_3, x_4 \in X$ with $|k| < 4$, then there exists a unique derivation $\delta : X \to X$ such that

$$\|f(x) - \delta(x)\|_Y \leq \frac{1}{4(1-L)}\varphi\left(\frac{x}{k}, \frac{x}{k}, \frac{x}{k}, \frac{x}{k}\right) \tag{2.31}$$

for all $x \in X$.

Proof. The proof is similar to the proof of 2.3. □

3. Stability of homomorphisms in Lie C^*-algebras

A C^*-algebra $\mathcal{C}$, endowed with the Lie product

$$[x, y] := \frac{xy - yx}{2}$$

on $\mathcal{C}$, is called a Lie C^*-algebras(see[5],[15]).

Definition 3.1. Let X and Y be Lie C^*-algebras. A $\mathbb{C}$-linear mapping $H : X \to Y$ is called a Lie C^**-algebras homomorphism* if $H([x, y]) = [H(x), H(y)]$ for all $x, y \in X$.

Throughout this section, assume that X is a Lie C^*-algebras with a norm $\|\cdot\|_X$ and B is a Lie C^*-algebras with a norm $\|\cdot\|_Y$.

Now, we prove the generalized Hyers-Ulam stability of homomorphisms in Lie C^*-algebras for the functional equation $D_\mu f(xx_1, x_2, x_3, x_4) = 0$.

Theorem 3.2. *Let a, b, c, d be the fixed real numbers. Let $f : X \to Y$ be an mapping for which there exists a function $\varphi : X^4 \to [0, \infty)$, such that*

$$\|F_\mu f(x_1, x_2, x_3, x_4)\|_Y \leq \varphi(x_1, x_2, x_3, x_4), \tag{3.1}$$

$$\|f([x, y]) - [f(x), f(y)]\|_Y \leq \varphi(x, x, y, y), \tag{3.2}$$

for all $\mu \in \mathbb{T}^1 := \{v \in \mathbb{C} : |v| = 1\}$ and all $x_1, x_2, x_3, x_4, x, y \in X$. If there exists an $0 < L < 1$ such that $\varphi(x_1, x_2, x_3, x_4) \leq \frac{|k|}{4} L\varphi\left(\frac{4}{k}x_1, \frac{4}{k}x_2, \frac{4}{k}x_3, \frac{4}{k}x_4\right)$ for all $x_1, x_2, x_3, x_4 \in X$ with $|k| > 4$, then there exists a unique derivation $H : X \to X$ such that

$$\|f(x) - H(x)\|_Y \leq \frac{1}{|k|(1-L)}\varphi\left(\frac{x}{4}, \frac{x}{4}, \frac{x}{4}, \frac{x}{4}\right) \tag{3.3}$$

for all $x \in X$.

Proof. By the same method as in the proof of Theorem 2.1, we can get the mapping $H : X \to Y$ given by

$$H(x) = \lim_{n\to\infty} \frac{4^n}{|k|^n} f\left(\frac{k^n}{4^n}x\right)$$

for all $x \in X$. Thus it follows from 3.2that

$$\begin{aligned}\|H([x,y]) - [H(x), H(y)]\|_Y &= \lim_{n\to\infty} \frac{4^{2n}}{|k|^{2n}} \left\| f\left(\frac{k^{2n}}{4^{2n}}[x,y]\right) - \left[f\left(\frac{k^n}{4^n}x\right), f\left(\frac{k^n}{4^n}x\right)\right]\right\|_Y \\ &\le \lim_{n\to\infty} \frac{4^{2n}}{|k|^{2n}} \varphi\left(\frac{k^n}{4^n}x, \frac{k^n}{4^n}y\right) = 0\end{aligned}$$

for all $x, y \in X$, and so

$$H([x,y]) = [H(x), H(y)]$$

for all $x, y \in X$. Therefore, $H : X \to Y$ is a Lie C^*-algebras homomorphism satisfying 3.3. This completes the proof. □

Theorem 3.3. *Let a, b, c, d be the fixed real numbers. Let $f : X \to Y$ be an mapping for which there exists a function $\varphi : X^4 \to [0, \infty)$, such that*

$$\|F_\mu f(x_1, x_2, x_3, x_4)\|_Y \le \varphi(x_1, x_2, x_3, x_4), \tag{3.4}$$

$$\|f([x,y]) - [f(x), f(y)]\|_Y \le \varphi(x, x, y, y), \tag{3.5}$$

for all $\mu \in \mathbb{T}^1 := \{v \in \mathbb{C} : |v| = 1\}$ and all $x_1, x_2, x_3, x_4, x, y \in X$. If there exists an $0 < L < 1$ such that $\varphi(x_1, x_2, x_3, x_4) \le \frac{4}{|k|} L\varphi\left(\frac{k}{4}x_1, \frac{k}{4}x_2, \frac{k}{4}x_3, \frac{k}{4}x_4\right)$ for all $x_1, x_2, x_3, x_4 \in X$ with $|k| < 4$, then there exists a unique derivation $H : X \to X$ such that

$$\|f(x) - H(x)\|_Y \le \frac{1}{4(1-L)} \varphi\left(\frac{x}{k}, \frac{x}{k}, \frac{x}{k}, \frac{x}{k}\right) \tag{3.6}$$

for all $x \in X$.

Proof. By the same method as in the proof of Theorem 2.1, we can get the mapping $H : X \to Y$ given by

$$H(x) = \lim_{n\to\infty} \frac{|k|^n}{4^n} f\left(\frac{4^n}{k^n}x\right)$$

for all $x \in X$. Thus it follows from 3.5that

$$\begin{aligned}\|H([x,y]) - [H(x), H(y)]\|_Y &= \lim_{n\to\infty} \frac{|k|^{2n}}{4^{2n}} \left\| f\left(\frac{4^{2n}}{k^{2n}}[x,y]\right) - \left[f\left(\frac{4^n}{k^n}x\right), f\left(\frac{4^n}{k^n}x\right)\right]\right\|_Y \\ &\le \lim_{n\to\infty} \frac{|k|^{2n}}{4^{2n}} \varphi\left(\frac{4^n}{k^n}x, \frac{4^n}{k^n}y\right) = 0\end{aligned}$$

for all $x, y \in X$, and so

$$H([x,y]) = [H(x), H(y)]$$

for all $x, y \in X$. Therefore, $H : X \to Y$ is a Lie C^*-algebras homomorphism satisfying 3.6. This completes the proof. □

4. Stability of derivations in Lie C^*-algebras

Definition 4.1. Let X be a Lie C^*-algebra. A $\mathcal{C}$-linear mapping $\delta : X \to X$ is called a *Lie derivation* if $\delta([x,y]) = [\delta(x), y] + [x, \delta(y)]$ for all $x, y \in X$.

Throughout this section, assume that X is a Lie C^*-algebra with a norm $\|\cdot\|_X$.

Finally, we prove the generalized Hyers-Ulam stability of derivations on Lie C^*-algebras for the functional equation $D_\mu f(x_1, x_2, x_3, x_4) = 0$.

Theorem 4.2. *Let a, b, c, d be the fixed real numbers. Let $f : X \to Y$ be an mapping for which there exists a function $\varphi : X^4 \to [0, \infty)$, such that*

$$\|F_\mu f(x_1, x_2, x_3, x_4)\|_Y \le \varphi(x_1, x_2, x_3, x_4), \tag{4.1}$$

$$\|f([x,y]) - [f(x), y] - [x, f(y)]\|_Y \le \varphi(x, x, y, y), \tag{4.2}$$

for all $\mu \in \mathbb{T}^1 := \{v \in \mathbb{C} : |v| = 1\}$ and all $x_1, x_2, x_3, x_4, x, y \in X$. If there exists an $0 < L < 1$ such that $\varphi(x_1, x_2, x_3, x_4) \le \frac{|k|}{4} L \varphi\left(\frac{4}{k}x_1, \frac{4}{k}x_2, \frac{4}{k}x_3, \frac{4}{k}x_4\right)$ for all $x_1, x_2, x_3, x_4 \in X$ with $|k| > 4$, then there exists a unique derivation $\delta : X \to X$ such that

$$\|f(x) - \delta(x)\|_Y \le \frac{1}{|k|(1-L)} \varphi\left(\frac{x}{4}, \frac{x}{4}, \frac{x}{4}, \frac{x}{4}\right) \tag{4.3}$$

for all $x \in X$.

Proof. By the same method as in the proof of Theorem 2.1, we can get the mapping $\delta : X \to Y$ given by

$$\delta(x) = \lim_{n\to\infty} \frac{4^n}{|k|^n} f\left(\frac{k^n}{4^n} x\right)$$

for all $x \in X$. Thus it follows from 4.2that

$$\begin{aligned}
&\|\delta([x,y]) - [\delta(x), y] - [x, \delta(y)]\|_Y \\
&= \lim_{n\to\infty} \frac{4^{2n}}{|k|^{2n}} \left\| f\left(\frac{k^{2n}}{4^{2n}}[x,y]\right) - \left[f\left(\frac{k^n}{4^n}x\right), \frac{k^n}{4^n}y\right] - \left[\frac{k^n}{4^n}x, f\left(\frac{k^n}{4^n}y\right)\right]\right\|_Y \\
&\le \lim_{n\to\infty} \frac{4^{2n}}{|k|^{2n}} \varphi\left(\frac{k^n}{4^n}x, \frac{k^n}{4^n}y\right) = 0
\end{aligned}$$

for all $x, y \in X$, and so

$$\delta([x,y]) = [\delta(x), y] + [x, \delta(y)]$$

for all $x, y \in X$. Therefore, $\delta : X \to X$ is a Lie C^*-algebras homomorphism satisfying 4.3. This completes the proof. □

Theorem 4.3. *Let a, b, c, d be the fixed real numbers. Let $f : X \to Y$ be an mapping for which there exists a function $\varphi : X^4 \to [0, \infty)$, such that*

$$\|F_\mu f(x_1, x_2, x_3, x_4)\|_Y \le \varphi(x_1, x_2, x_3, x_4), \tag{4.4}$$

$$\|f([x,y]) - [f(x), y] - [x, f(y)]\|_Y \le \varphi(x, x, y, y), \tag{4.5}$$

for all $\mu \in \mathbb{T}^1 := \{v \in \mathbb{C} : |v| = 1\}$ *and all* $x_1, x_2, x_3, x_4, x, y \in X$. *If there exists an* $0 < L < 1$ *such that* $\varphi(x_1, x_2, x_3, x_4) \leq \frac{4}{|k|} L\varphi\left(\frac{k}{4}x_1, \frac{k}{4}x_2, \frac{k}{4}x_3, \frac{k}{4}x_4\right)$ *for all* $x_1, x_2, x_3, x_4 \in X$ *with* $|k| < 4$, *then there exists a unique derivation* $\delta : X \to X$ *such that*

$$\|f(x) - \delta(x)\|_Y \leq \frac{1}{4(1-L)}\varphi\left(\frac{x}{k}, \frac{x}{k}, \frac{x}{k}, \frac{x}{k}\right) \tag{4.6}$$

for all $x \in X$.

Proof. By the same method as in the proof of Theorem 2.1, we can get the mapping $\delta : X \to Y$ given by

$$\delta(x) = \lim_{n\to\infty} \frac{|k|^n}{4^n} f\left(\frac{4^n}{k^n}x\right)$$

for all $x \in X$. Thus it follows from 4.5 that

$$\begin{aligned}
&\|\delta([x, y]) - [H(x), y] - [x, H(y)]\|_Y \\
&= \lim_{n\to\infty} \frac{|k|^{2n}}{4^{2n}} \left\| \left(\frac{4^{2n}}{k^{2n}}[x, y]\right) - \left[f\left(\frac{4^n}{k^n}x\right), \frac{4^n}{k^n}y\right] - \left[\frac{4^n}{k^n}x, f\left(\frac{4^n}{k^n}x\right)\right] \right\|_Y \\
&\leq \lim_{n\to\infty} \frac{|k|^{2n}}{4^{2n}} \varphi\left(\frac{4^n}{k^n}x, \frac{4^n}{k^n}y\right) = 0
\end{aligned}$$

for all $x, y \in X$, and so

$$\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$$

for all $x, y \in X$. Therefore, $\delta : X \to X$ is a Lie C^*-algebras homomorphism satisfying 4.6. This completes the proof. $\square$

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A Higher Order Multi-step Iterative Method for Computing the Numerical Solution of Systems of Nonlinear Equations Associated with Nonlinear PDEs and ODEs

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Abstract

The main focus of research in the current article is to address the construction of an efficient higher order multi-step iterative methods to solve systems of nonlinear equations associated with nonlinear partial differential equations (PDEs) and ordinary differential equations (ODEs). The construction includes second order Frechet derivatives. The proposed multi-step iterative method uses two Jacobian evaluations at different points and requires only one inversion (in the sense of LU-factorization) of Jacobian. The enhancement of convergence-order (CO) is hidden in the formation of matrix polynomial. The cost of matrix vector multiplication is expensive computationally. We developed a matrix polynomial of degree two for base method and degree one to perform multi-steps so we need just one matrix vector multiplication to perform each further step. The base method has convergence order four and each additional step enhance the CO by three. The general formula for CO is $3s-2$ for $s \geq 2$ and 2 for $s = 1$ where s is the step number. The number of function evaluations including Jacobian are $s+2$ and number of matrix vectors multiplications are s. For s-step iterative method we solve s upper and lower triangular systems when right hand side is a vector and 1 pair of triangular systems when right hand side is a matrix. It is shown that the computational cost is almost same for Jacobian and second order Frechet derivative associated with systems of nonlinear equations due to PDEs and ODEs. The accuracy and validity of proposed multi-step iterative method is checked with different PDEs and ODEs.

Keywords: Multi-step, Iterative methods, Systems of nonlinear equations, Nonlinear partial differential equations, Nonlinear ordinary differential equations

1. Introduction

A valuable discussion can be found about Frechet derivatives in [1]. We will show that why higher order Frechet derivatives are avoided in the construction of iterative methods for general systems of nonlinear equations and why there are suitable with for a particular class of systems of nonlinear equations associated with ODEs and PDEs. To make things simpler, consider a system of three nonlinear equations

$$\mathbf{F}(\mathbf{y}) = [f_1(\mathbf{y}), f_2(\mathbf{y}), f_3(\mathbf{y})]^T = 0, \tag{1}$$

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where $\mathbf{y} = [y_1,\ y_2,\ y_3]^T$. The first order Frechet derivative (Jacobian) of (1) is

$$\mathbf{F}'(\mathbf{y}) = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \frac{\partial f_1}{\partial y_3} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \frac{\partial f_2}{\partial y_3} \\ \frac{\partial f_3}{\partial y_1} & \frac{\partial f_3}{\partial y_2} & \frac{\partial f_3}{\partial y_3} \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \tag{2}$$

Next we proceed for the calculation of second-order Frechet derivative. Suppose $\mathbf{h} = [h_1,\ h_2,\ h_3]^T$ is a constant vector.

$$\mathbf{F}'(\mathbf{y})\mathbf{h} = \begin{bmatrix} h_1 f_{11} + h_2 f_{12} + h_3 f_{13} \\ h_1 f_{21} + h_2 f_{22} + h_3 f_{23} \\ h_1 f_{31} + h_2 f_{32} + h_3 f_{33} \end{bmatrix}, \tag{3}$$

$$\mathbf{F}''(\mathbf{y})\mathbf{h}^2 = \begin{bmatrix} f_{111} & f_{122} & f_{133} \\ f_{211} & f_{222} & f_{233} \\ f_{311} & f_{322} & f_{333} \end{bmatrix} \begin{bmatrix} h_1^2 \\ h_2^2 \\ h_3^2 \end{bmatrix} + 2 \begin{bmatrix} f_{121} & f_{113} & f_{123} \\ f_{212} & f_{213} & f_{223} \\ f_{312} & f_{313} & f_{323} \end{bmatrix} \begin{bmatrix} h_1 h_2 \\ h_1 h_3 \\ h_2 h_3 \end{bmatrix}. \tag{4}$$

Clearly the computational cost for second-order Frechet derivative is high in the case of general systems of nonlinear equations. Many systems of nonlinear equations associated with PDEs and ODEs can be written as

$$\begin{cases} \mathbf{F}(y) = L(y) + f(y) + w = 0, \\ \mathbf{F}(\mathbf{y}) = \mathbf{A}\mathbf{y} + f(\mathbf{y}) + \mathbf{w} = 0, \end{cases} \tag{5}$$

where A is the discrete approximation to linear differential operator $L(\cdot)$ and $f(\cdot)$ is the nonlinear function. If we write down the second-order Frechet derivative of (5) by using (4) we get

$$\mathbf{F}''(\mathbf{y})\mathbf{h}^2 = \begin{bmatrix} f''(y_1) & 0 & 0 & \cdots & 0 \\ 0 & f''(y_2) & 0 & \cdots & 0 \\ 0 & 0 & f''(y_3) & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & f''(y_n) \end{bmatrix} \begin{bmatrix} h_1^2 \\ h_2^2 \\ h_3^2 \\ \vdots \\ h_n^2 \end{bmatrix} \tag{6}$$

For the further analysis , we introduce some notation. If $\boldsymbol{a} = [a_1,\ a_2,\ ,\cdots, a_n]^T$ and $\boldsymbol{b} = [b_1,\ b_2,\ ,\cdots, b_n]^T$ are vectors then the diagonal matrix of a vector and point-wise product we define as

$$\mathrm{diag}(\boldsymbol{a}) = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix}, \ \boldsymbol{a} \odot \boldsymbol{b} = \mathrm{diag}(\boldsymbol{a})\, \boldsymbol{b} = [a_1 b_1,\ a_2 b_2, \cdots, a_n b_n]^T. \tag{7}$$

For the motivation of readers we list some famous nonlinear ODEs and PDEs and their first- and second-order derivatives in scalar and vectorial forms (Frechet derivatives). Let $\mathbf{D}_x$ and $\mathbf{D}_t$ are the discrete approximations of differential operators in spatial and temporal dimensions and u is the function of spatial variables and in some cases temporal variable is also taken. We also introduce a function h which is independent from u and I_t, I_x are identity matrices of the size number of nodes in temporal and spatial dimensions respectively.

1.1. Bratu problem

The Bratu problem is discussed in [2] and it is stated as

$$\begin{cases} f(u) = u'' + \lambda e^u = 0, \qquad u(0) = u(1) = 0, \\ \dfrac{df(u)}{du}h = h'' + \lambda e^u h, \\ \dfrac{d^2 f(u)}{du^2}h^2 = \lambda e^u h^2, \\ \mathbf{F}(\mathbf{u}) = \mathbf{D}_x^2\mathbf{u} + \lambda e^{\mathbf{u}} = \mathbf{0}, \\ \mathbf{F}'\mathbf{h} = \mathbf{D}_x^2\mathbf{h} + \lambda e^{\mathbf{u}} \odot \mathbf{h}, \\ \mathbf{F}' = \mathbf{D}_x^2 + \lambda\, \mathrm{diag}(e^{\mathbf{u}}), \\ \mathbf{F}''\mathbf{h}^2 = \lambda e^{\mathbf{u}} \odot \mathbf{h}^2. \end{cases} \tag{8}$$

The closed form solution of Bratu problem can be written as

$$\begin{cases} u(x) = -2log\left(\dfrac{cosh((x-0.5)(0.5\theta))}{cosh(0.25\theta)}\right), \\ \theta = \sqrt{2\lambda}cosh(0.25\theta). \end{cases} \tag{9}$$

The critical value of λ satisfies $4 = \sqrt{4\lambda_c}sinh(0.25\theta_c)$. The Bratu problem has two solution, unique solution and no solution if $\lambda < \lambda_c$, $\lambda = \lambda_c$ and $\lambda > \lambda_c$ respectively. The critical value $\lambda_c = 3.51383071912516$.

1.2. Frank-Kamenetzkii problem

The Frank-Kamenetzkii problem [3] is written as

$$\begin{cases} u'' + \dfrac{1}{x}u' + \lambda e^u = 0, \qquad\qquad u'(0) = u(1) = 0, \\ \mathbf{F}(\mathbf{u}) = \mathbf{D}_x^2\mathbf{u} + \dfrac{1}{\mathbf{x}} \odot \mathbf{D}_x\mathbf{u} + \lambda e^{\mathbf{u}} = \mathbf{0}, \\ \mathbf{F}'\mathbf{h} = \mathbf{D}_x^2\mathbf{h} + \dfrac{1}{\mathbf{x}} \odot \mathbf{D}_x\mathbf{h} + \lambda e^{\mathbf{u}} \odot \mathbf{h}, \\ \mathbf{F}' = \mathbf{D}_x^2 + \mathrm{diag}\left(\dfrac{1}{\mathbf{x}}\right)\mathbf{D}_x + \lambda\, \mathrm{diag}(e^{\mathbf{u}}), \\ \mathbf{F}''\mathbf{h}^2 = \lambda e^{\mathbf{u}} \odot \mathbf{h}^2. \end{cases} \tag{10}$$

The Frank-Kamenetzkii problem has no solution ($\lambda > 2$), ($\lambda = 2$) and two solution ($\lambda < 2$). The closed form solution of (10) is given as

$$\begin{cases} c_1 = log\left(2(4-\lambda) \pm 4\sqrt{2(2-\lambda)}\right), \\ c_2 = log\left(\dfrac{4-\lambda) \pm 2\sqrt{2(2-\lambda)}}{2\lambda^2}\right), \\ u(x) = log\left(\dfrac{16e^{c_1}}{(2\lambda + e^{c_1}x^2)^2}\right), \\ u(x) = log\left(\dfrac{16e^{c_1}}{(1 + 2\lambda e^{c_2}x^2)^2}\right). \end{cases} \tag{11}$$

1.3. Lane-Emden equation

The Lane-Emden equation is classical equation [4] which is introduced in 1870 by Lane and later Emden (1907) studied it. Lane-Emden equation deals with mass density distribution inside a spherical star when it is in hydrostatic

equilibrium. The lane-Emden equation for index $n = 5$ can be written as

$$\begin{cases} u'' + \dfrac{2}{x}u' + u^5 = 0, & u(0) = 1,\ u'(0) = 0, \\ \mathbf{F}(\mathbf{u}) = \mathbf{D}_x^2\mathbf{u} + \dfrac{1}{\mathbf{x}} \odot \mathbf{D}_x\mathbf{u} + \mathbf{u}^5, \\ \mathbf{F}'\mathbf{h} = \mathbf{D}_x^2\mathbf{h} + \dfrac{1}{\mathbf{x}} \odot \mathbf{D}_x\mathbf{h} + 5\mathbf{u}^4 \odot \mathbf{h}, \\ \mathbf{F}' = \mathbf{D}_x^2 + \operatorname{diag}\left(\dfrac{1}{\mathbf{x}}\right)\mathbf{D}_x + 5\operatorname{diag}(\mathbf{u}^4), \\ \mathbf{F}''\mathbf{h}^2 = 20\,\mathbf{u}^3 \odot \mathbf{h}^2. \end{cases} \tag{12}$$

The closed form solution of (12) can be written as

$$u(x) = \left(1 + \frac{x^2}{3}\right)^{-\frac{1}{2}}. \tag{13}$$

1.4. Klien-Gordan equation

Klien-Gordan equation is discussed and solved in [5].

$$\begin{cases} u_{tt} - c^2u_{xx} + f(u) = p, & -\infty < x < \infty,\ t > 0 \\ \mathbf{F}(\mathbf{u}) = (\mathbf{D}_t^2 - c^2\mathbf{D}_x^2)\mathbf{u} + f(\mathbf{u}) - \boldsymbol{p}, \\ \mathbf{F}'\mathbf{h} = (\mathbf{D}_t^2 - c^2\mathbf{D}_x^2)\mathbf{h} + f'(\mathbf{u}) \odot \boldsymbol{h}, \\ \mathbf{F}' = \mathbf{D}_t^2 - c^2\mathbf{D}_x^2 + \operatorname{diag}(f'(\mathbf{u})), \\ \mathbf{F}''\mathbf{h}^2 = f''(\mathbf{u}) \odot \mathbf{h}^2, \end{cases} \tag{14}$$

where $f(u)$ is the odd function of u and initial conditions are

$$\begin{cases} u(x,0) = g_1(x), \\ u_t(x,0) = g_2(x). \end{cases} \tag{15}$$

We have calculated the second-order Frechet derivatives of four different nonlinear ODEs and PDEs. Clearly the computational cost of second-order Frechet derivatives are not higher than first-order Frechet derivatives or Jacobians. So we insist that the second-order Frechet derivatives for particular class of ODEs and PDEs are not expensive as they are in the case of general systems of nonlinear equations. The main source of information about iterative methods is the manuscript written by J. F. Traub [6] in 1964. Recently many researchers have contributed in the area of iterative method for systems of nonlinear equations [7–16]. The major part of work is devoted for the construction iterative methods for the single variable nonlinear equations[17]. According to Traub's conjecture if we use n function evaluations, then the maximum CO is 2^n in the case of single variable nonlinear equation but for multi-variable case we do not have such claim. In the case of systems of nonlinear equations the multi-steps iterative methods are interesting because with minimum computational cost we are aimed to construct higher-order convergence iterative methods. For the better understanding we can divide multi-steps iterative methods in two parts one is called base method and second part is called multi-steps. In the base method we construct an iterative method in way that it provides maximum enhancement in the convergence-order with minimum computational cost when we perform multi-steps. Malik et. al.

[18] proposed the following multi-step iterative method (MZ_1) :

$$\mathrm{MZ}_1 = \begin{cases} \text{Number of steps} & = m \geq 2 \\ \text{CO} & = 2m \\ \text{Function evaluations} & = m+1 \\ \text{Inverses} & = 2 \\ \text{Matrix vector multiplications} & = 1 \\ \text{Number of solutions of systems} & \\ \text{of linear equations} & \\ \text{when right hand side is matrix} & = 1 \\ \text{when right hand side is vector} & = m-1 \end{cases} \quad \begin{cases} \text{Base-Method} \rightarrow \begin{cases} \mathbf{F}'(\mathbf{x})\boldsymbol{\phi}_1 = \mathbf{F}(\mathbf{x}) \\ \mathbf{y}_1 = \mathbf{x} - \frac{2}{3}\boldsymbol{\phi}_1 \\ \mathbf{W} = \frac{1}{2}\left(3\mathbf{F}'(\mathbf{y}_1) - \mathbf{F}'(\mathbf{x})\right) \\ \mathbf{W}\mathbf{T} = 3\mathbf{F}'(\mathbf{y}_1) + \mathbf{F}'(\mathbf{x}) \\ \mathbf{y}_2 = \mathbf{x} - \frac{1}{4}T\boldsymbol{\phi}_1 \end{cases} \\ (m-2)\text{-steps} \rightarrow \begin{cases} \text{for } s = 1, m-2 \\ \quad \mathbf{W}\boldsymbol{\phi}_{s+1} = \mathbf{F}(\mathbf{y}_{s+1}), \\ \quad \mathbf{y}_{s+2} = \mathbf{y}_{s+1} - \boldsymbol{\phi}_{s+1}, \\ \text{end} \end{cases} \end{cases}$$

In [19] F. Soleymani and co-researchers constructed an other multi-step iterative method (FS):

$$\mathrm{FS} = \begin{cases} \text{Number of steps} & = m \geq 2 \\ \text{CO} & = 2m \\ \text{Function evaluations} & = m+1 \\ \text{Inverses} & = 2 \\ \text{Matrix vector multiplications} & = 2m-3 \\ \text{Number of solutions of systems} & \\ \text{of linear equations} & \\ \text{when right hand side is matrix} & = 1 \\ \text{when right hand side is vector} & = m-1 \end{cases} \quad \begin{cases} \text{Base-Method} \rightarrow \begin{cases} \mathbf{F}'(\mathbf{x})\boldsymbol{\phi}_1 = \mathbf{F}(\mathbf{x}) \\ \mathbf{y}_1 = \mathbf{x} - \frac{2}{3}\boldsymbol{\phi}_1 \\ \mathbf{W} = \frac{1}{2}\left(3\mathbf{F}'(\mathbf{y}_1) - \mathbf{F}'(\mathbf{x})\right) \\ \mathbf{W}\mathbf{T} = 3\mathbf{F}'(\mathbf{y}_1) + \mathbf{F}'(\mathbf{x}) \\ \mathbf{y}_2 = \mathbf{x} - \mathbf{T}\boldsymbol{\phi}_1 \end{cases} \\ (m-2)\text{-steps} \rightarrow \begin{cases} \text{for } s = 1, m-2 \\ \quad \mathbf{F}'(\mathbf{x})\boldsymbol{\phi}_{s+1} = \mathbf{F}(\mathbf{y}_{s+1}), \\ \quad \mathbf{y}_{s+2} = \mathbf{y}_{s+1} - \mathbf{T}^2\boldsymbol{\phi}_{s+1}, \\ \text{end} \end{cases} \end{cases}$$

H. Montazeri et. al. [20] developed the more efficient multi-step iterative methods (HM):

$$\mathrm{HM} = \begin{cases} \text{Number of steps} & = m \geq 2 \\ \text{CO} & = 2m \\ \text{Function evaluations} & = m+1 \\ \text{Inverses} & = 1 \\ \text{Matrix vector multiplications} & = m \\ \text{Number of solutions of systems} & \\ \text{of linear equations} & \\ \text{when right hand side is matrix} & = 1 \\ \text{when right hand side is vector} & = m-1 \end{cases} \quad \begin{cases} \text{Base-Method} \rightarrow \begin{cases} \mathbf{F}'(\mathbf{x})\boldsymbol{\phi}_1 = \mathbf{F}(\mathbf{x}) \\ \mathbf{y}_1 = \mathbf{x} - \frac{2}{3}\boldsymbol{\phi}_1 \\ \mathbf{F}'(\mathbf{x})\mathbf{T} = \mathbf{F}'(\mathbf{y}_1) \\ \mathbf{y}_2 = \mathbf{x} - \left(\frac{23}{8}\mathbf{I} - 3\mathbf{T} + \frac{9}{8}\mathbf{T}^2\right)\boldsymbol{\phi}_1 \end{cases} \\ (m-2)\text{-steps} \rightarrow \begin{cases} \text{for } s = 1, m-2 \\ \quad \mathbf{F}'(\mathbf{x})\boldsymbol{\phi}_{s+1} = \mathbf{F}(\mathbf{y}_{s+1}), \\ \quad \mathbf{y}_{s+2} = \mathbf{y}_{s+1} - \left(\frac{5}{2}\mathbf{I} - \frac{3}{2}\mathbf{T}\right)\boldsymbol{\phi}_{s+1}, \\ \text{end} \end{cases} \end{cases}$$

2. The proposed new multi-step iterative method

We proposed a new multi-step iterative method (MZ_2):

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$$\mathrm{MZ}_2 = \begin{cases} \text{Number of steps} & = m \geq 2 \\ \text{CO} & = 3m-2 \\ \text{Function evaluations} & = m+2 \\ \text{Inverses} & = 1 \\ \text{Matrix vector} & \\ \text{multiplications} & = m \\ \text{Number of solutions} & \\ \text{of systems of linear} & \\ \text{ equations when} & \\ \text{right hand side is matrix} & = 1 \\ \text{right hand side is vector} & = m \end{cases} \quad \begin{cases} \text{Base-Method} \rightarrow \begin{cases} \mathbf{F}'(\mathbf{x})\boldsymbol{\phi}_1 = \mathbf{F}(\mathbf{x}) \\ \mathbf{F}'(\mathbf{x})\boldsymbol{\phi}_2 = \mathbf{F}''\left(\mathbf{x} - \frac{4}{9}\boldsymbol{\phi}_1\right)\boldsymbol{\phi}_1^2 \\ \mathbf{y}_1 = \mathbf{x} - \left(\boldsymbol{\phi}_1 + \frac{3}{2}\boldsymbol{\phi}_2\right) \\ \mathbf{F}'(\mathbf{x})\mathbf{T} = \mathbf{F}'(\mathbf{y}_1) \\ \mathbf{y}_2 = \mathbf{x} - \left(\frac{7}{2}\mathbf{I} - 6\mathbf{T} + \frac{7}{2}\mathbf{T}^2\right)\left(\boldsymbol{\phi}_1 + \frac{3}{2}\boldsymbol{\phi}_2\right) \end{cases} \\ (m-2)\text{-steps} \rightarrow \begin{cases} \text{for } s = 1, m-2 \\ \quad \mathbf{F}'(\mathbf{x})\boldsymbol{\phi}_{s+2} = \mathbf{F}(\mathbf{y}_{s+1}), \\ \quad \mathbf{y}_{s+2} = \mathbf{y}_{s+1} - \left(2\mathbf{I} - \mathbf{T}\right)\boldsymbol{\phi}_{s+2}, \\ \text{end} \end{cases} \end{cases}$$

We claim that the convergence-order of our proposed multi-step iterative method is

$$\mathrm{CO} = \begin{cases} 2 & m = 1, \\ 3m-2 & m \geq 2, \end{cases} \tag{16}$$

where m is the number of steps of MZ_2. The computational costs of MZ_1 and FS are high because both methods use two inversions of matrices. The multi-step iterative method HM use only one inversion of Jacobian and hence is a good candidate for the performance comparison. For further discussion we will not consider MZ_1 and FS methods. We presented comparison between MZ_2 and HM in Table1 and 2. The Table 1 tells us if the number of function evaluations and number of solutions of system of linear equations are equal then the performance of MZ_2 in terms of convergence-order is better than HM when number of step of MZ_2 are grater or equal to four. When the convergence-orders of both iterative methods are equal then we can see from Table 2 that the computation effort of HM is always more than that of MZ_2 for $m \geq 2$. The performance index to measure the efficiency of an iterative method to solve systems of nonlinear equation is defined as

$$\rho = CO^{\frac{1}{flops}}. \tag{17}$$

In Table 3 we provided the computational cost of different operation and Table 4 shows the performance index as defined in (20) for a particular case when HM and MZ_2 have the same convergence-order. Clearly the performance index of MZ_2 is better than that of HM.

Table 1: Comparison between multi-steps iterative method MZ_2 and HM if number of function evaluations and solutions of system of linear equations are equal.

	MZ_2 ($m \geq 2$)	HM ($m \geq 2$)	MZ_2 ($m = 2$)	HM ($m = 3$)	MZ_2 ($m = m_1$)	HM ($m = m_1 + 1$)	Difference $\text{MZ}_2 - HM$
Number of steps	m	m	2	3	m_1	$m_1 + 1$	1
Convergence-order	$3m - 2$	$2m$	4	6	$3m_1 - 2$	$2(m_1 + 1)$	$m_1 - 4$
Function evaluations	$m + 2$	$m + 1$	4	4	$m_1 + 2$	$m_1 + 2$	0
Solution of system of linear equations when right hand side is vector	m	$m - 1$	2	2	m_1	m_1	0
Solution of system of linear equations when right hand side is matrix	1	1	1	1	1	1	0
Matrix vector multiplications	m	m	2	3	m_1	$m_1 + 1$	-1

Table 2: Comparison between multi-steps iterative method MZ_2 and HM if convergence-orders are equal.

	MZ_2 ($m \geq 1$)	HM ($m \geq 1$)	Difference $HM - \text{MZ}_2$
Number of steps	$2m$	$3m - 1$	$m - 1$
Convergence-order	$6m - 2$	$6m - 2$	0
Function evaluations	$2m + 2$	$3m$	$m - 2$
Solution of system of linear equations when right hand side is vector	$2m$	$3m$	m
Solution of system of linear equations when right hand side is matrix	1	1	0
Matrix vector multiplications	$2m$	$3m - 1$	$m - 1$

Table 3: Computational cost of different operations (the computational cost of a division is three times to multiplication).

LU decomposition		
Multiplications $\frac{n(n-1)(2n-1)}{6}$	Divisions $\frac{n(n-1)}{2}$	Total cost $\frac{n(n-1)(2n-1)}{6} + 3\frac{n(n-1)}{2}$
Two triangular systems (if right hand side is a vector)		
Multiplications $n(n-1)$	Divisions n	Total cost $n(n-1) + 3n$
Two triangular systems (if right hand side is a matrix)		
Multiplications $n^2(n-1)$	Divisions n^2	Total cost $n^2(n-1) + 3n^2$
Matrix vector multiplication		
n^2		

Table 4: Comparison of performance index between multi-steps iterative methods MZ_2 and HM.

Iterative methods	HM	MZ_2
Number of steps	5	4
Rate of convergence	10	10
Number of functional evaluations	$6n$	$6n$
The classical efficiency index	$2^{1/(6n)}$	$2^{1/(6n)}$
Number of Lu factorizations	1	1
Cost of Lu factorizations	$\frac{n(n-1)(2n-1)}{6} + 3\frac{n(n-1)}{2}$	$\frac{n(n-1)(2n-1)}{6} + 3\frac{n(n-1)}{2}$
Cost of linear systems	$4(n(n-1) + 3n) + n^2(n-1) + 3n^2$	$4(n(n-1) + 3n) + n^2(n-1) + 3n^2$
Matrix vector multiplications	$5n^2$	$4n^2$
Flops-like efficiency index	$10^{1/\left(\frac{4n^3}{3}+12n^2+\frac{38}{3}n\right)}$	$10^{1/\left(\frac{4n^3}{3}+11n^2+\frac{38}{3}n\right)}$

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3. Convergence Analysis

In this section, we will prove that the local convergence-order of MZ_2 is seven for $m = 3$ and later we will establish a proof for the convergence-order of multi-step iterative scheme MZ_2, by using mathematical induction.

Theorem 3.1. *Let* $\mathbf{F} : \Gamma \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ *be sufficiently Frechet differentiable on an open convex neighborhood* Γ *of* $\mathbf{x}^* \in \mathbb{R}^n$ *with* $\mathbf{F}(\mathbf{x}^*) = 0$ *and* $det(\mathbf{F}'(\mathbf{x}^*)) \neq 0$*. Then the sequence* $\{\mathbf{x}_k\}$ *generated by the iterative scheme* MZ_2 *converges to* $\mathbf{x}^*$ *with local order of convergence seven, and produces the following error equation*

$$\mathbf{e}_{k+1} = \mathbf{L}\mathbf{e}_k{}^7 + O(\mathbf{e}_k{}^8), \tag{18}$$

where $\mathbf{e}_k = \mathbf{x}_k - \mathbf{x}^*$, $\mathbf{e}_k{}^p = \overbrace{(\mathbf{e}_k, \mathbf{e}_k, \cdots, \mathbf{e}_k)}^{p\text{-times}}$ *and* $\mathbf{L} = -2060\mathbf{C}_2^6 - 618\mathbf{C}_3\mathbf{C}_2^4 + 260/9\mathbf{C}_2^3\mathbf{C}_4 + 26/3\mathbf{C}_3\mathbf{C}_2\mathbf{C}_4 - 30\mathbf{C}_3\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2 - 6\mathbf{C}_3\mathbf{C}_2^2\mathbf{C}_3 - 100\mathbf{C}_2^3\mathbf{C}_3\mathbf{C}_2 - 20\mathbf{C}_2^4\mathbf{C}_3$ *is a p-linear function i.e.* $\mathbf{L} \in \mathbb{L}\,\overbrace{(\mathbb{R}^n, \mathbb{R}^n, \cdots, \mathbb{R}^n)}^{p\text{-times}}$ *and* $L\mathbf{e}_k{}^p \in \mathbb{R}^n$.

Proof. Let $\mathbf{F} : \Gamma \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ be sufficiently Frechet differentiable function in Γ. The qth Frechet derivative of $\mathbf{F}$ at $v \in \mathbb{R}^n$, $q \geq 1$, is the $q-linear$ function $\mathbf{F}^{(q)}(v) : \overbrace{\mathbb{R}^n\mathbb{R}^n \cdots \mathbb{R}^n}^{\text{q-times}}$ such that $\mathbf{F}^{(q)}(v)(u_1, u_2, \cdots, u_q) \in \mathbb{R}^n$. The Taylor's series expansion of $\mathbf{F}(\mathbf{x}_k)$ around $\mathbf{x}^*$ can be written as:

$$\mathbf{F}(\mathbf{x}_k) = \mathbf{F}(\mathbf{x}^* + \mathbf{x}_k - \mathbf{x}^*) = \mathbf{F}(\mathbf{x}^* + \mathbf{e}_k), \tag{19}$$

$$= \mathbf{F}(\mathbf{x}^*) + \mathbf{F}'(\mathbf{x}^*)\mathbf{e}_k + \frac{1}{2!}\mathbf{F}''(\mathbf{x}^*)\mathbf{e}_k{}^2 + \frac{1}{3!}F^{(3)}(\mathbf{x}^*)\mathbf{e}_k{}^3 + O\left(\mathbf{e}_k^4\right), \tag{20}$$

$$= \mathbf{F}'(\mathbf{x}^*)\Big(\mathbf{e}_k + \frac{1}{2!}\mathbf{F}'(\mathbf{x}^*)^{-1}\mathbf{F}''(\mathbf{x}^*)\mathbf{e}_k{}^2 + \frac{1}{3!}\mathbf{F}'(\mathbf{x}^*)^{-1}F^{(3)}(\mathbf{x}^*)\mathbf{e}_k{}^3 + O\left(\mathbf{e}_k^4\right)\Big), \tag{21}$$

$$= \mathbf{C}_1\Big(\mathbf{e}_k + \mathbf{C}_2\mathbf{e}_k{}^2 + \mathbf{C}_3\mathbf{e}_k{}^3 + O\left(\mathbf{e}_k^4\right)\Big), \tag{22}$$

where $\mathbf{C}_1 = \mathbf{F}'(\mathbf{x}^*)$ and $\mathbf{C}_s = \dfrac{1}{s!}\mathbf{F}'(\mathbf{x}^*)^{-1}F^{(s)}(\mathbf{x}^*)$ for $s \geq 2$. From (22), we can calculate the Frechet derivative of $\mathbf{F}$:

$$\mathbf{F}'(\mathbf{x}_k) = \mathbf{C}_1\Big(\mathbf{I} + 2\mathbf{C}_2\mathbf{e}_k + 3\mathbf{C}_3\mathbf{e}_k{}^2 + 4\mathbf{C}_3\mathbf{e}_k{}^3 + O\left(\mathbf{e}_k^4\right)\Big), \tag{23}$$

where $\mathbf{I}$ is the identity matrix. Furthermore, we calculate the inverse of the Jacobian matrix

$$\begin{aligned}\mathbf{F}'(\mathbf{x}_k)^{-1} = \Big(&\mathbf{I} - 2\mathbf{C}_2\mathbf{e}_k + \left(4\mathbf{C}_2^2 - 3\mathbf{C}_3\right)\mathbf{e}_k^2 + \left(6\mathbf{C}_3\mathbf{C}_2 + 6\mathbf{C}_2\mathbf{C}_3 - 8\mathbf{C}_2^3 - 4\mathbf{C}_4\right)\mathbf{e}_k^3 + \big(8\mathbf{C}_4\mathbf{C}_2 + 9\mathbf{C}_3^2 + 8\mathbf{C}_2\mathbf{C}_4 - 5\mathbf{C}_5 -\\ &12\mathbf{C}_3\mathbf{C}_2^2 - 12\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2 - 12\mathbf{C}_2^2\mathbf{C}_3 + 16\mathbf{C}_2^4\big)\mathbf{e}_k^4 + \big(24\mathbf{C}_3\mathbf{C}_2^3 + 24\mathbf{C}_2^3\mathbf{C}_3 + 24\mathbf{C}_2^2\mathbf{C}_3\mathbf{C}_2 + 24\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2^2 +\\ &10\mathbf{C}_5\mathbf{C}_2 + 12\mathbf{C}_4\mathbf{C}_3 + 12\mathbf{C}_3\mathbf{C}_4 + 10\mathbf{C}_2\mathbf{C}_5 - 6\mathbf{C}_6 - 16\mathbf{C}_4\mathbf{C}_2^2 - 18\mathbf{C}_3^2\mathbf{C}_2 - 18\mathbf{C}_3\mathbf{C}_2\mathbf{C}_3 - 16\mathbf{C}_2\mathbf{C}_4\mathbf{C}_2 -\\ &18\mathbf{C}_2\mathbf{C}_3^2 - 16\mathbf{C}_2^2\mathbf{C}_4 - 32\mathbf{C}_2^5\big)\mathbf{e}_k^5 + \big(32\mathbf{C}_4\mathbf{C}_2^3 + 64\mathbf{C}_2^6 - 48\mathbf{C}_3\mathbf{C}_2^4 + 12\mathbf{C}_2\mathbf{C}_6 + 16\mathbf{C}_4^2 + 15\mathbf{C}_3\mathbf{C}_5 +\\ &15\mathbf{C}_5\mathbf{C}_3 + 12\mathbf{C}_6\mathbf{C}_2 - 24\mathbf{C}_4\mathbf{C}_2\mathbf{C}_3 - 24\mathbf{C}_4\mathbf{C}_3\mathbf{C}_2 - 20\mathbf{C}_2^2\mathbf{C}_5 - 24\mathbf{C}_2\mathbf{C}_3\mathbf{C}_4 - 24\mathbf{C}_2\mathbf{C}_4\mathbf{C}_3 + 32\mathbf{C}_2^3\mathbf{C}_4 -\\ &20\mathbf{C}_2\mathbf{C}_5\mathbf{C}_2 + 36\mathbf{C}_2^2\mathbf{C}_3^2 - 20\mathbf{C}_5\mathbf{C}_2^2 + 32\mathbf{C}_2^2\mathbf{C}_4\mathbf{C}_2 + 32\mathbf{C}_2\mathbf{C}_4\mathbf{C}_2^2 + 36\mathbf{C}_2\mathbf{C}_3^2\mathbf{C}_2 + 36\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2\mathbf{C}_3 +\\ &36\mathbf{C}_3^2\mathbf{C}_2^2 - 7\mathbf{C}_7 - 24\mathbf{C}_3\mathbf{C}_2\mathbf{C}_4 - 27\mathbf{C}_3^3 - 24\mathbf{C}_3\mathbf{C}_4\mathbf{C}_2 + 36\mathbf{C}_3\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2 + 36\mathbf{C}_3\mathbf{C}_2^2\mathbf{C}_3 - 48\mathbf{C}_2^2\mathbf{C}_3\mathbf{C}_2^2 -\\ &48\mathbf{C}_2^3\mathbf{C}_3\mathbf{C}_2 - 48\mathbf{C}_2^4\mathbf{C}_3 - 48\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2^3\big)\mathbf{e}_k^6 + O\left(\mathbf{e}_k^7\right)\Big)\mathbf{C}_1^{-1}\end{aligned} \tag{24}$$

By multiplying $\mathbf{F}'(\mathbf{x}_k)^{-1}$ and $\mathbf{F}(\mathbf{x}_k)$, we obtain $\boldsymbol{\phi}_1$:

$$\begin{aligned}\boldsymbol{\phi}_1 = &\mathbf{e}_k - \mathbf{C}_2\mathbf{e}_k^2 + \left(2\mathbf{C}_2^2 - 2\mathbf{C}_3\right)\mathbf{e}_k^3 + \left(-3\mathbf{C}_4 - 4\mathbf{C}_2^3 + 3\mathbf{C}_3\mathbf{C}_2 + 4\mathbf{C}_2\mathbf{C}_3\right)\mathbf{e}_k^4 + \big(-4\mathbf{C}_5 - 6\mathbf{C}_3\mathbf{C}_2^2 - 6\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2 -\\ &8\mathbf{C}_2^2\mathbf{C}_3 + 8\mathbf{C}_2^4 + 4\mathbf{C}_4\mathbf{C}_2 + 6\mathbf{C}_3^2 + 6\mathbf{C}_2\mathbf{C}_4\big)\mathbf{e}_k^5 + \big(-5\mathbf{C}_6 + 12\mathbf{C}_3\mathbf{C}_2^3 + 16\mathbf{C}_2^3\mathbf{C}_3 + 12\mathbf{C}_2^2\mathbf{C}_3\mathbf{C}_2 +\\ &12\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2^2 - 8\mathbf{C}_4\mathbf{C}_2^2 - 9\mathbf{C}_3^2\mathbf{C}_2 - 12\mathbf{C}_3\mathbf{C}_2\mathbf{C}_3 - 8\mathbf{C}_2\mathbf{C}_4\mathbf{C}_2 - 12\mathbf{C}_2\mathbf{C}_3^2 - 12\mathbf{C}_2^2\mathbf{C}_4 - 16\mathbf{C}_2^5 + 5\mathbf{C}_5\mathbf{C}_2 +\\ &8\mathbf{C}_4\mathbf{C}_3 + 9\mathbf{C}_3\mathbf{C}_4 + 8\mathbf{C}_2\mathbf{C}_5\big)\mathbf{e}_k^6 + O\left(\mathbf{e}_k^7\right).\end{aligned} \tag{25}$$

The expression for $\boldsymbol{\phi}_2$ is the following:

$$\begin{aligned}\boldsymbol{\phi}_2 = {} & 2\mathbf{C}_2\mathbf{e}_k^2 + \Big(-8\mathbf{C}_2^2 + 10/3\mathbf{C}_3\Big)\mathbf{e}_k^3 + \Big(26\mathbf{C}_2^3 - 38/3\mathbf{C}_3\mathbf{C}_2 - 12\mathbf{C}_2\mathbf{C}_3 + 100/27\mathbf{C}_4\Big)\mathbf{e}_k^4 + \\ & \Big(-364/27\mathbf{C}_2\mathbf{C}_4 - 18\mathbf{C}_3^2 - 416/27\mathbf{C}_4\mathbf{C}_2 + 116/3\mathbf{C}_2^2\mathbf{C}_3 + 36\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2 + 122/3\mathbf{C}_3\mathbf{C}_2^2 + \\ & 2500/729\mathbf{C}_5 - 76\mathbf{C}_2^4\Big)\mathbf{e}_k^5 + \Big(-106\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2^2 - 298/3\mathbf{C}_2^2\mathbf{C}_3\mathbf{C}_2 - 344/3\mathbf{C}_2^3\mathbf{C}_3 + 1282/27\mathbf{C}_2^2\mathbf{C}_4 + \\ & 140/3\mathbf{C}_2\mathbf{C}_3^2 + 1106/27\mathbf{C}_2\mathbf{C}_4\mathbf{C}_2 - 118\mathbf{C}_3\mathbf{C}_2^3 + 1364/27\mathbf{C}_4\mathbf{C}_2^2 - 10664/729\mathbf{C}_2\mathbf{C}_5 - \\ & 520/27\mathbf{C}_3\mathbf{C}_4 - 544/27\mathbf{C}_4\mathbf{C}_3 - 12290/729\mathbf{C}_5\mathbf{C}_2 + 54\mathbf{C}_3^2\mathbf{C}_2 + 184/3\mathbf{C}_3\mathbf{C}_2\mathbf{C}_3 + 6250/2187\mathbf{C}_6 + \\ & 208\mathbf{C}_2^5\Big)\mathbf{e}_k^6 + O\Big(\mathbf{e}_k^7\Big).\end{aligned} \tag{26}$$

The expressions for $\mathbf{y}_1$, T, $\mathbf{y}_2$ and $\mathbf{y}_3$ in order are

$$\begin{aligned}\mathbf{y}_1 - \mathbf{x}^* = {} & -2\mathbf{C}_2\mathbf{e}_k^2 + \Big(10\mathbf{C}_2^2 - 3\mathbf{C}_3\Big)\mathbf{e}_k^3 + \Big(-23/9\mathbf{C}_4 - 35\mathbf{C}_2^3 + 16\mathbf{C}_3\mathbf{C}_2 + 14\mathbf{C}_2\mathbf{C}_3\Big)\mathbf{e}_k^4 + (-278/243\mathbf{C}_5 - \\ & 55\mathbf{C}_3\mathbf{C}_2^2 - 48\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2 - 50\mathbf{C}_2^2\mathbf{C}_3 + 106\mathbf{C}_2^4 + 172/9\mathbf{C}_4\mathbf{C}_2 + 21/9\mathbf{C}_3^2 + 128\mathbf{C}_2\mathbf{C}_4\Big)\mathbf{e}_k^5 + \Big(147\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2^2 + \\ & 137\mathbf{C}_2^2\mathbf{C}_3\mathbf{C}_2 + 156\mathbf{C}_2^3\mathbf{C}_3 - 533/9\mathbf{C}_2^2\mathbf{C}_4 - 58\mathbf{C}_2\mathbf{C}_3^2 - 481/9\mathbf{C}_2\mathbf{C}_4\mathbf{C}_2 + 165\mathbf{C}_3\mathbf{C}_2^3 - 610/9\mathbf{C}_4\mathbf{C}_2^2 + \\ & 3388/243\mathbf{C}_2\mathbf{C}_5 + 179/9\mathbf{C}_3\mathbf{C}_4 + 200/9\mathbf{C}_4\mathbf{C}_3 + 4930/243\mathbf{C}_5\mathbf{C}_2 - 72\mathbf{C}_3^2\mathbf{C}_2 - 80\mathbf{C}_3\mathbf{C}_2\mathbf{C}_3 + \\ & 520/729\mathbf{C}_6 - 296\mathbf{C}_2^5\Big)\mathbf{e}_k^6 + O\Big(\mathbf{e}_k^7\Big).\end{aligned} \tag{27}$$

$$\begin{aligned}\mathbf{T} = {} & \mathbf{I} - 2\mathbf{C}_2\mathbf{e}_k - 3\mathbf{C}_3\mathbf{e}_k^2 + \Big(6\mathbf{C}_3\mathbf{C}_2 - 4\mathbf{C}_4 + 20\mathbf{C}_2^3\Big)\mathbf{e}_k^3 + \Big(12\mathbf{C}_3\mathbf{C}_2^2 + 20\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2 + 28\mathbf{C}_2^2\mathbf{C}_3 - 110\mathbf{C}_2^4 + \\ & 8\mathbf{C}_4\mathbf{C}_2 + 9\mathbf{C}_3^2 + 26/9\mathbf{C}_2\mathbf{C}_4 - 5\mathbf{C}_5\Big)\mathbf{e}_k^4 + \Big(-180\mathbf{C}_3\mathbf{C}_2^3 - 156\mathbf{C}_2^3\mathbf{C}_3 - 136\mathbf{C}_2^2\mathbf{C}_3\mathbf{C}_2 - 134\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2^2 + \\ & 18\mathbf{C}_3\mathbf{C}_2\mathbf{C}_3 + 200/9\mathbf{C}_2\mathbf{C}_4\mathbf{C}_2) + 24\mathbf{C}_2\mathbf{C}_3^2 + 68/3\mathbf{C}_2^2\mathbf{C}_4 + 432\mathbf{C}_2^5 + 10\mathbf{C}_5\mathbf{C}_2 + 12\mathbf{C}_4\mathbf{C}_3 + 12\mathbf{C}_3\mathbf{C}_4 + \\ & 1874/243\mathbf{C}_2\mathbf{C}_5 - 6\mathbf{C}_6\Big)\mathbf{e}_k^5 + \Big(-112\mathbf{C}_4\mathbf{C}_2^3 - 1456\mathbf{C}_2^6 + 1050\mathbf{C}_3\mathbf{C}_2^4 + 9788/729\mathbf{C}_2\mathbf{C}_6 + 16\mathbf{C}_4^2 + \\ & 15\mathbf{C}_3\mathbf{C}_5 + 15\mathbf{C}_5\mathbf{C}_3 + 12\mathbf{C}_6\mathbf{C}_2 - 24\mathbf{C}_4\mathbf{C}_3\mathbf{C}_2 + 3028/243\mathbf{C}_2^2\mathbf{C}_5 + 142/9\mathbf{C}_2\mathbf{C}_3\mathbf{C}_4 + 184/9\mathbf{C}_2\mathbf{C}_4\mathbf{C}_3 - \\ & 1474/9\mathbf{C}_2^3\mathbf{C}_4 + 5000/243\mathbf{C}_2\mathbf{C}_5\mathbf{C}_2 - 164\mathbf{C}_2^2\mathbf{C}_3^2 - 454/3\mathbf{C}_2^2\mathbf{C}_4\mathbf{C}_2 - 1220/9\mathbf{C}_2\mathbf{C}_4\mathbf{C}_2^2 - 144\mathbf{C}_2\mathbf{C}_3^2\mathbf{C}_2 - \\ & 196\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2\mathbf{C}_3 - 222\mathbf{C}_3^2\mathbf{C}_2^2 - 7\mathbf{C}_7 + 20/3\mathbf{C}_3\mathbf{C}_2\mathbf{C}_4 - 26/3\mathbf{C}_3\mathbf{C}_4\mathbf{C}_2 - 240\mathbf{C}_3\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2 - 258\mathbf{C}_3\mathbf{C}_2^2\mathbf{C}_3 + \\ & 562\mathbf{C}_2^2\mathbf{C}_3\mathbf{C}_2^2 + 546\mathbf{C}_2^3\mathbf{C}_3\mathbf{C}_2 + 624\mathbf{C}_2^4\mathbf{C}_3 + 690\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2^3\Big)\mathbf{e}_k^6 + O\Big(\mathbf{e}_k^7\Big).\end{aligned} \tag{28}$$

$$\begin{aligned}\mathbf{y}_2 - \mathbf{x}^* = {} & \Big(-5\mathbf{C}_3\mathbf{C}_2 + 13/9\mathbf{C}_4 - 103\mathbf{C}_2^3 - \mathbf{C}_2\mathbf{C}_3\Big)\mathbf{e}_k^4 + \Big(-104/9\mathbf{C}_2\mathbf{C}_4 - 21/2\mathbf{C}_3^2 - 80/9\mathbf{C}_4\mathbf{C}_2 - \\ & 148\mathbf{C}_2^2\mathbf{C}_3 - 100\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2 - 109\mathbf{C}_3\mathbf{C}_2^2 + 937/243\mathbf{C}_5 + 666\mathbf{C}_2^4\Big)\mathbf{e}_k^5 + \Big(869\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2^2 + 873\mathbf{C}_2^2\mathbf{C}_3\mathbf{C}_2 + \\ & 954\mathbf{C}_2^3\mathbf{C}_3 - 1133/9\mathbf{C}_2^2\mathbf{C}_4 - 124\mathbf{C}_2(\mathbf{C}_3^2) - 895/9\mathbf{C}_2\mathbf{C}_4\mathbf{C}_2 + 1074\mathbf{C}_3\mathbf{C}_2^3 - 1114/9\mathbf{C}_4\mathbf{C}_2^2 - \\ & 715/27\mathbf{C}_2\mathbf{C}_5 - 238/9\mathbf{C}_3\mathbf{C}_4 - 178/9\mathbf{C}_4\mathbf{C}_3 - 3575/243\mathbf{C}_5\mathbf{C}_2 - 75\mathbf{C}_3^2\mathbf{C}_2 - 158\mathbf{C}_3\mathbf{C}_2\mathbf{C}_3 + \\ & 4894/729\mathbf{C}_6 - 1990\mathbf{C}_2^5\Big)\mathbf{e}_k^6 + \Big(3632/3\mathbf{C}_4\mathbf{C}_2^3 + 420\mathbf{C}_2^6 - 4958\mathbf{C}_3\mathbf{C}_2^4 - 30616/729\mathbf{C}_2\mathbf{C}_6 - \\ & 404/9\mathbf{C}_4^2 - 7343/162\mathbf{C}_3\mathbf{C}_5 - 16001/486\mathbf{C}_5\mathbf{C}_3 - 15620/729\mathbf{C}_6\mathbf{C}_2 - 1580/9\mathbf{C}_4\mathbf{C}_2\mathbf{C}_3 - \\ & 580/9\mathbf{C}_4\mathbf{C}_3\mathbf{C}_2 - 18334/243\mathbf{C}_2^2\mathbf{C}_5 - 761/9\mathbf{C}_2\mathbf{C}_3\mathbf{C}_4 - 847/9\mathbf{C}_2\mathbf{C}_4\mathbf{C}_3 + 1074\mathbf{C}_2^3\mathbf{C}_4 - \\ & 19556/243\mathbf{C}_2\mathbf{C}_5\mathbf{C}_2 + 1118\mathbf{C}_2^2\mathbf{C}_3^2 - 35410/243\mathbf{C}_5\mathbf{C}_2^2) + 8924/9\mathbf{C}_2^2\mathbf{C}_4\mathbf{C}_2 + 3038/3\mathbf{C}_2\mathbf{C}_4\mathbf{C}_2^2 + \\ & 1040\mathbf{C}_2\mathbf{C}_3^2\mathbf{C}_2 + 1262\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2\mathbf{C}_3 + 1390\mathbf{C}_3^2\mathbf{C}_2^2 + 63418/6561\mathbf{C}_7 - 919/9\mathbf{C}_3\mathbf{C}_2\mathbf{C}_4 - \\ & 165/2\mathbf{C}_3^3 - 589/9\mathbf{C}_3\mathbf{C}_4\mathbf{C}_2 + 1331\mathbf{C}_3\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2 + 1542\mathbf{C}_3\mathbf{C}_2^2\mathbf{C}_3 - 2678\mathbf{C}_2^2\mathbf{C}_3\mathbf{C}_2^2 - 2886\mathbf{C}_2^3\mathbf{C}_3\mathbf{C}_2 - \\ & 2881\mathbf{C}_2^4\mathbf{C}_3 - 3871\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2^3\Big)\mathbf{e}_k^7 + O\Big(\mathbf{e}_k^8\Big).\end{aligned} \tag{29}$$

$$\begin{aligned}\mathbf{y}_3 - \mathbf{x}^* = {} & \Big(-2060\mathbf{C}_2^6 - 618\mathbf{C}_3\mathbf{C}_2^4 + 260/9\mathbf{C}_2^3\mathbf{C}_4 + 26/3\mathbf{C}_3\mathbf{C}_2\mathbf{C}_4 - 30\mathbf{C}_3\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2 - 6\mathbf{C}_3\mathbf{C}_2^2\mathbf{C}_3 - 100\mathbf{C}_2^3\mathbf{C}_3\mathbf{C}_2 - \\ & 20\mathbf{C}_2^4\mathbf{C}_3\Big)\mathbf{e}_k^7 + O\Big(\mathbf{e}_k^8\Big).\end{aligned} \tag{30}$$

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□

Theorem 3.2. *The multi-step iterative scheme MZ_2 has the local convergence-order $3m-2$, using $m(\geq 2)$ evaluations of a sufficiently differentiable function* $\mathbf{F}$*, two first-order Frechet derivatives* $\mathbf{F}'$ *and one second-order Frechet derivate* $\mathbf{F}''$ *per full-cycle.*

Proof. The proof is established from mathematical induction. For $m = 1,\ 2,\ 3$ the convergence-orders are two, four and seven from (27), (29) and (30) respectively. Consequently our claim concerning the convergence-order $3m-2$ is true for $m = 2,\ 3$.

We assume that our claim is true for $m = q > 3$, i.e., the convergence-order of MZ_2 is $3q-2$. The qth-step and $(q-1)th$-step of iterative scheme MZ_2 can be written as:

$$\text{Frozen-factor} = (2\mathbf{I} - \mathbf{T})\mathbf{F}'(\mathbf{x})^{-1}, \tag{31}$$

$$\mathbf{y}_{q-1} = \mathbf{y}_{q-2} - (\text{Frozen-factor})\,\mathbf{F}(\mathbf{y}_{q-2}), \tag{32}$$

$$\mathbf{y}_{q} = \mathbf{y}_{q-1} - (\text{Frozen-factor})\,\mathbf{F}(\mathbf{y}_{q-1}). \tag{33}$$

The enhancement in the convergence-order of MZ_2 from $(q-1)th$-step to qth-step is $(3q-2) - (3(q-1)-2) = 3$. Now we write the $(q+1)th$-step of MZ_2:

$$\mathbf{y}_{q+1} = \mathbf{y}_{q} - (\text{Frozen-factor})\,\mathbf{F}(\mathbf{y}_{q}). \tag{34}$$

The increment in the convergence-order of MZ_2, due to $(q+1)th$-step, is exactly three, because the use of the Frozen-factor adds an additive constant in the convergence-order[19]. Finally the convergence-order after the addition of the $(q+1)th$-step is $3q-2+3 = 3q+1 = 3(q+1)-2$, which completes the proof. □

4. Numerical Testing

For the verification of convergence-order, we use the following definition for the computational convergence-order (COC):

$$\text{COC} \approx \frac{log\Big(\|\mathbf{x}_{q+2} - \mathbf{x}^*\|_\infty / \|\mathbf{x}_{q+1} - \mathbf{x}^*\|_\infty\Big)}{log\Big(\|\mathbf{x}_{q+1} - \mathbf{x}^*\|_\infty / \|\mathbf{x}_{q} - \mathbf{x}^*\|_\infty\Big)}, \tag{35}$$

where $Max(|\mathbf{x}_{q+2} - \mathbf{x}^*|)$ is the maximum absolute error. The number of solutions of systems of linear equations are same in both iterative methods when right hand side is a matrix so we will not mention it in comparison tables. The main benefit of multi-step iterative methods is that we invert Jacobian once and then use it again and again in multi-steps part to get better convergence-order for a single cycle of iterative method. We have conducted numerical tests for four different problems to show the accuracy and validity of our proposed multi-step iterative method MZ_2. For the purpose of comparison we adopt two ways (i) when both iterative methods have same number of function evaluations and solution of systems of linear equations (ii) when both schemes have same convergence order. Tables 5, 7 and 8 show that when we number of function evaluations and solutions of systems of linear equation are equal and the convergence order of MZ_2 is higher than ten then our proposed scheme show better accuracy in less execution time. On the other hand if convergence-order of MZ_2 is less than ten then the performance of HM is relatively better. For the second cases when we equate the convergence-orders the execution time of MZ_2 are always less than that of HM because HM performs more steps to achieve the same convergence-order. Tables 6, 9 and 10 shows that MZ_2 achieve better or almost equal accuracy with less execution time. We have also simulated one PDE Klein-Gordon and results are depicted in Table 11. As we have commented if the convergence-order is less ten the performance of HM is better and it is clearly evident in Table 11 but the accuracy of MZ_2 is comparable with HM. The numerical error in solution due to MZ_2 is shown in Figure 1 and Figure 2 corresponds to numerical solution of Klein-Gordon PDE. In the case of Klein-Gordon equation by keeping the mesh size fix, if we increase the number of iterations or either number of steps both iterative method can not improve the accuracy.

Table 5: Comparison of performances for different multi-step methods in the case of the Bratu problem when number of function evaluations and number of solutions of systems of linear equations are equal in both iterative methods.

Iterative methods		MZ_2	HM
Number of iterations		1	1
Size of problem		200	200
Number of steps		32	33
Theoretical convergence-order(CO)		94	66
Number of function evaluations per iteration		34	34
Solutions of system of linear equations per iteration		32	32
Number of matrix vector multiplication per iteration		32	33
	λ		
$\|\|\mathbf{x}_q - \mathbf{x}^*\|\|_\infty$	1	$3.62e-156$	$7.55e-110$
	2	$4.78e-142$	$2.31e-98$
	3	$3.91e-50$	$4.05e-35$
Execution time		23.48	24.0

Table 6: Comparison of performances for different multi-step methods in the case of the Bratu problem when convergence orders are equal in both itrative methods.

Iterative methods	MZ_2	HM
Number of iterations	1	1
Size of problem	250	250
Number of steps	120	179
Theoretical convergence-order(CO)	358	358
Number of function evaluations per iteration	122	180
Solutions of system of linear equations per iteration	120	178
Number of matrix vector multiplication per iteration	120	179
$\|\|\mathbf{x}_q - \mathbf{x}^*\|\|_\infty$, $(\lambda = 1)$	$3.98e-235$	$3.98e-235$
Execution time	59.67	70.22

Table 7: Comparison of performances for different multi-step methods in the case of the Bratu problem when number of function evaluations and number of solutions of systems of linear equations are equal in both iterative methods.

Iterative methods	MZ_2	HM
Number of iterations	3	3
Size of problem	250	250
Number of steps	3	4
Theoretical convergence-order(CO)	7	8
Computational convergence-order(COC)	6.75	7.81
Number of function evaluations per iteration	5	5
Solutions of system of linear equations per iteration	3	3
Number of matrix vector multiplication per iteration	3	4
$\|\|\mathbf{x}_q - \mathbf{x}^*\|\|_\infty$	$8.44e-150$	$3.92e-161$
Execution time	63.75	64.66

Table 8: Comparison of performances for different multi-step methods in the case of the Frank Kamenetzkii problem when number of function evaluations and number of solutions of systems of linear equations are equal in both iterative methods.

Iterative methods	MZ_2	HM
Number of iterations	3	3
Size of problem	150	150
Number of steps	3	4
Theoretical convergence-order(CO)	7	8
Computational convergence-order(COC)	7.39	8.64
Number of function evaluations per iteration	5	5
Solutions of system of linear equations per iteration	3	3
Number of matrix vector multiplication per iteration	3	4
$\|\|\mathbf{x}_q - \mathbf{x}^*\|\|_\infty$	$4.21e-126$	$3.21e-149$
Execution time	16.10	16.68

Table 9: Comparison of performances for different multi-step methods in the case of the Frank Kamenetzkii problem when convergence orders are equal in both iterative methods.

Iterative methods	MZ_2	HM
Number of iterations	1	1
Size of problem	150	150
Number of steps	80	119
Theoretical convergence-order(CO)	238	238
Number of function evaluations per iteration	82	120
Solutions of system of linear equations per iteration	80	118
Number of matrix vector multiplication per iteration	80	119
$\|\|\mathbf{x}_k - \mathbf{x}^*\|\|_\infty, (\lambda = 1)$	$6.46e-116$	$3.95e-99$
Execution time	19.89	28.21

Table 10: Comparison of performances for different multi-step methods in the case of the Lane-Emden equation when convergence orders are equal.

Iterative methods	MZ_2	HM
Number of iterations	1	1
Size of problem	100	100
Number of steps	30	44
Theoretical convergence-order(CO)	88	88
Number of function evaluations per iteration	32	45
Solutions of system of linear equations per iteration	30	43
Number of matrix vector multiplication per iteration	30	44
$\|\mathbf{x}_q - \mathbf{x}^*\|_\infty$	$1.95e-34$	$2.64e-37$
Execution time	3.01	3.53

Table 11: Comparison of performances for different multi-step methods in the case of the Klien Gordon equation , initial guess $u(x_i, t_j) = 0$, $u(x,t) = \delta sech(\kappa(x - \nu t))$, $\kappa = \sqrt{\dfrac{k}{c^2 - \nu^2}}$, $\delta = \sqrt{\dfrac{2k}{\gamma}}$, $c = 1$, $\gamma = 1$, $\nu = 0.5$, $k = 0.5$, $n_x = 170$, $n_t = 26$, $x \in [-22,\ 22]$, $t \in [0,\ 0.5]$.

Iterative methods		MZ_2	HM
Number of iterations		1	1
Size of problem		4420	4420
Number of steps		4	4
Theoretical convergence-order(CO)		10	8
Number of function evaluations per iteration		6	5
Solutions of system of linear equations per iteration		4	3
Number of matrix vector multiplication per iteration		4	4
	Steps		
$\|\mathbf{x}_q - \mathbf{x}^*\|_\infty$	1	$3.24e-1$	$4.11e-1$
	2	$7.51e-3$	$2.62e-3$
	3	$2.70e-5$	$2.63e-5$
	4	$5.59e-7$	$4.39e-7$
Execution time		94.13	80.18

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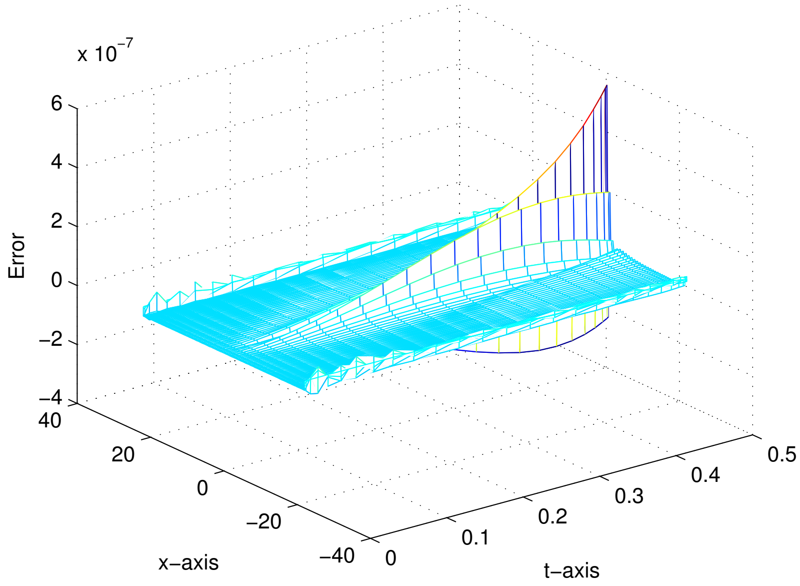

Figure 1: Absolute error plot for multi-step method MZ_2 in the case of the Klien Gordon equation , initial guess $u(x_i, t_j) = 0$, $u(x,t) = \delta sech(\kappa(x - \nu t)$, $\kappa = \sqrt{\frac{k}{c^2 - \nu^2}}$, $\delta = \sqrt{\frac{2k}{\gamma}}$, $c = 1$, $\gamma = 1$, $\nu = 0.5$, $k = 0.5$, $n_x = 170$, $n_t = 26$, $x \in [-22,\ 22]$, $t \in [0,\ 0.5]$.

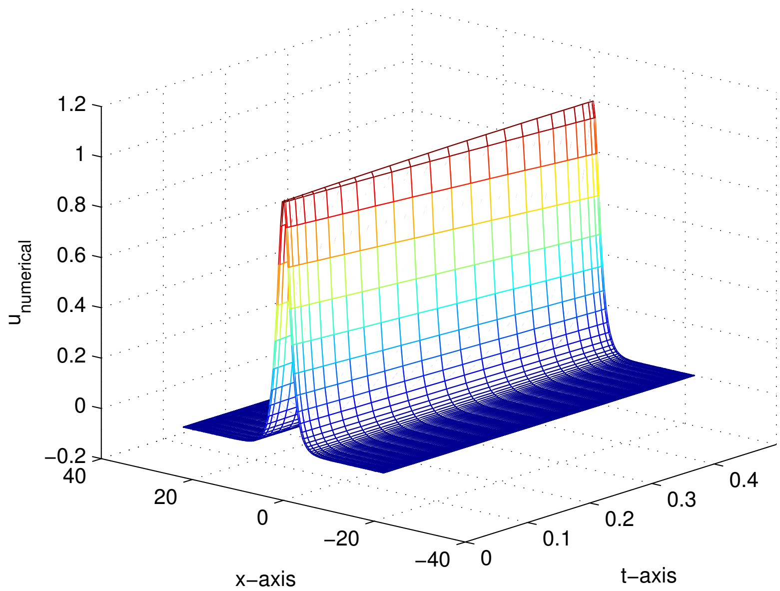

Figure 2: Numerical solution of the Klien Gordon equation , $x \in [-22,\ 22]$, $t \in [0,\ 0.5]$.

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5. Conclusions

The inversion of Jacobian is computationally expensive and multi-step iterative methods can provide remedy to it by offering good convergence-order with relatively less computational cost. The best way to construct a multi-step method is to reduce the number of Jacobian and function evaluations, inversion of Jacobian, matrix-vector and vector-vector multiplications. Higher-order Frechet derivatives are computationally expensive when use them for the solution of systems of nonlinear equations but for a particular of ODEs and PDEs we could use them because they are just diagonal matrices. Our proposed scheme MZ_2 shows good accuracy when we perform more and more multi-steps and it also depends on the nature of problem sometime. The computational convergence-order of MZ_2 is also calculated in some examples and it agrees with theoretical proved convergence-order.

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QUADRATIC ρ-FUNCTIONAL INEQUALITIES IN FUZZY NORMED SPACES

JI-HYE KIM AND CHOONKIL PARK*

Abstract. In this paper, we solve the following quadratic ρ-functional inequalities

$$\begin{aligned} & N(f(x+y)+f(x-y)-2f(x)-2f(y),t) \\ & \geq N\left(\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right),t\right) \end{aligned} \tag{0.1}$$

where ρ is a fixed real number with $|\rho|<1$, and

$$\begin{aligned} & N\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y),t\right) \\ & \geq N\left(\rho\left(f(x+y)+f(x-y)-2f(x)-2f(y)\right),t\right) \end{aligned} \tag{0.2}$$

where ρ is a fixed real number with $|\rho|<\frac{1}{2}$.

Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic ρ-functional inequalities (0.1) and (0.2) in fuzzy Banach spaces.

1. Introduction and preliminaries

Katsaras [21] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [13, 24, 52]. In particular, Bag and Samanta [2], following Cheng and Mordeson [8], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [23]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [3].

We use the definition of fuzzy normed spaces given in [2, 28, 29] to investigate the Hyers-Ulam stability of quadratic ρ-functional inequalities in fuzzy Banach spaces.

Definition 1.1. [2, 28, 29, 30] Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N_1) $N(x,t)=0$ for $t \leq 0$;
(N_2) $x=0$ if and only if $N(x,t)=1$ for all $t>0$;
(N_3) $N(cx,t)=N(x,\frac{t}{|c|})$ if $c \neq 0$;
(N_4) $N(x+y,s+t) \geq \min\{N(x,s),N(y,t)\}$;
(N_5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t)=1$.
(N_6) for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X,N) is called a *fuzzy normed vector space.*

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [27, 28].

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Definition 1.2. [2, 28, 29, 30] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by N-$\lim_{n\to\infty} x_n = x$.

Definition 1.3. [2, 28, 29, 30] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [3]).

The stability problem of functional equations originated from a question of Ulam [51] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [17] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [40] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [14] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

The functional equation $f(x+y)+f(x-y) = 2f(x)+2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [50] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [9] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. Czerwik [10] proved the Hyers-Ulam stability of the quadratic functional equation. The functional equation $f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$ is called a *Jensen type quadratic equation*. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [4, 18, 20, 25, 36, 37, 38, 41, 42, 44, 45, 46, 47, 48, 49]).

Gilányi [15] showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(x-y)\| \leq \|f(x+y)\| \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(x+y) + f(x-y).$$

See also [43]. Fechner [12] and Gilányi [16] proved the Hyers-Ulam stability of the functional inequality (1.1). Park, Cho and Han [35] investigated the Cauchy additive functional inequality

$$\|f(x) + f(y) + f(z)\| \leq \|f(x+y+z)\| \tag{1.2}$$

and the Cauchy-Jensen additive functional inequality

$$\|f(x) + f(y) + 2f(z)\| \leq \left\|2f\left(\frac{x+y}{2} + z\right)\right\| \tag{1.3}$$

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and proved the Hyers-Ulam stability of the functional inequalities (1.2) and (1.3) in Banach spaces.

Park [33, 34] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces.

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.4. [5, 11] Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \ge n_0$;
(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;
(4) $d(y, y^*) \le \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

In 1996, G. Isac and Th.M. Rassias [19] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [6, 7, 22, 27, 31, 32, 38, 39]).

In Section 2, we solve the quadratic ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.1) in fuzzy Banach spaces by using the fixed point method.

In Section 3, we solve the quadratic ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.2) in fuzzy Banach spaces by using the fixed point method.

Throughout this paper, assume that X is a real vector space and (Y, N) is a fuzzy Banach space.

2. Quadratic ρ-functional inequality (0.1)

In this section, we prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.1) in fuzzy Banach spaces. Let ρ be a real number with $|\rho| < 1$. We need the following lemma to prove the main results.

Lemma 2.1. *Let $f : X \to Y$ be a mapping such that*

$$\begin{aligned} N(f(x+y) + f(x-y) - 2f(x) - 2f(y), t) \\ \ge N\left(\rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right), t\right) \end{aligned} \tag{2.1}$$

for all $x, y \in X$ and all $t > 0$. Then f is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $N(2f(0), t) \ge N(\rho(2f(0)), t) = N\left(2f(0), \frac{t}{|\rho|}\right)$ for all $t > 0$. By (N_5) and (N_6), $N(f(0), t) = 1$ for all $t > 0$.
It follows from (N_2) that $f(0) = 0$.

Letting $y = x$ in (2.1), we get $N(f(2x) - 4f(x), t) \geq N(0, t) = 1$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} &N(f(x+y) + f(x-y) - 2f(x) - 2f(y), t) \\ &\quad \geq N\left(\rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right), t\right) \\ &\quad = N\left(\frac{1}{2}\rho\left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right), t\right) \\ &\quad = N\left(f(x+y) + f(x-y) - 2f(x) - 2f(y), \frac{2t}{|\rho|}\right) \end{aligned}$$

for all $t > 0$. By (N_5) and (N_6), $N(f(x+y) + f(x-y) - 2f(x) - 2f(y), t) = 1$ for all $t > 0$. It follows from (N_2) that $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in X$. $\square$

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{4}\varphi(2x, 2y)$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and

$$\begin{aligned} &N(f(x+y) + f(x-y) - 2f(x) - 2f(y), t) \\ &\geq \min\left\{N\left(\rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right), t\right), \frac{t}{t + \varphi(x, y)}\right\} \end{aligned} \tag{2.3}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{(4 - 4L)t}{(4 - 4L)t + L\varphi(x, x)} \tag{2.4}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $y = x$ in (2.3), we get

$$N(f(2x) - 4f(x), t) \geq \frac{t}{t + \varphi(x, x)} \tag{2.5}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf\left\{\mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t + \varphi(x, x)}, \ \forall x \in X, \forall t > 0\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [26, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right), L\varepsilon t\right) = N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{4}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)} \geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \frac{L}{4}\varphi(x,x)} = \frac{t}{t + \varphi(x,x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq L d(g,h)$$

for all $g, h \in S$.

It follows from (2.5) that $N\left(f(x) - 4f\left(\frac{x}{2}\right), \frac{L}{4}t\right) \geq \frac{t}{t+\varphi(x,x)}$ for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{4}$.

By Theorem 1.4, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(\frac{x}{2}\right) = \frac{1}{4}Q(x) \tag{2.6}$$

for all $x \in X$. Since $f : X \to Y$ is even, $Q : X \to Y$ is a even mapping. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - Q(x), \mu t) \geq \frac{t}{t + \varphi(x,x)}$$

for all $x \in X$;

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f, Q) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f,Q) \leq \frac{L}{4 - 4L}.$$

This implies that the inequality (2.4) holds.

By (2.3),

$$\begin{aligned} &N\left(4^n\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right), 4^n t\right) \\ &\geq \min\left\{N\left(\rho\left(4^n\left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right), 4^n t\right), \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}\right\} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$N\left(4^n\left(f\left(\frac{x+y}{2^n}\right)+f\left(\frac{x-y}{2^n}\right)-2f\left(\frac{x}{2^n}\right)-2f\left(\frac{y}{2^n}\right)\right),t\right)$$
$$\geq \min\left\{N\left(\rho\left(4^n\left(2f\left(\frac{x+y}{2^{n+1}}\right)+2f\left(\frac{x-y}{2^{n+1}}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right)\right),t\right),\frac{\frac{t}{4^n}}{\frac{t}{4^n}+\frac{L^n}{4^n}\varphi(x,y)}\right\}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty}\frac{\frac{t}{4^n}}{\frac{t}{4^n}+\frac{L^n}{4^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$N\left(Q(x+y)+Q(x-y)-2Q(x)-2Q(y),t\right)$$
$$\geq N\left(\rho\left(2Q\left(\frac{x+y}{2}\right)+2Q\left(\frac{x-y}{2}\right)-Q(x)-Q(y)\right),t\right)$$

for all $x, y \in X$ and all $t > 0$. By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic, as desired. □

Corollary 2.3. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying*

$$N(f(x+y)+f(x-y)-2f(x)-2f(y),t)$$
$$\geq \min\left\{N\left(\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right),t\right),\frac{t}{t+\theta(\|x\|^p+\|y\|^p)}\right\}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x)-Q(x),t\right) \geq \frac{(2^p-4)t}{(2^p-4)t+2\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x,y) := \theta(\|x\|^p+\|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{2-p}$, and we get the desired result. □

Theorem 2.4. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq 4L\varphi\left(\frac{x}{2},\frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.3). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x)-Q(x),t\right) \geq \frac{(4-4L)t}{(4-4L)t+\varphi(x,x)} \tag{2.7}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S,d) be the generalized metric space defined in the proof of Theorem 2.2.

It follows from (2.5) that

$$N\left(f(x)-\frac{1}{4}f(2x),\frac{1}{4}t\right) \geq \frac{t}{t+\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. So $d(f,Jf) \leq \frac{1}{4}$. Hence $d(f,Q) \leq \frac{1}{4-4L}$, which implies that the inequality (2.7) holds.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $\theta \ge 0$ and let p be a real number with $0 < p < 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying*

$$\begin{aligned}&N(f(x+y)+f(x-y)-2f(x)-2f(y),t)\\ &\ge \min\left\{N\left(\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right),t\right),\frac{t}{t+\theta(\|x\|^p+\|y\|^p)}\right\}\end{aligned}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x)-Q(x),t\right) \ge \frac{(4-2^p)t}{(4-2^p)t+2\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{p-2}$, and we get the desired result. □

3. Quadratic ρ-functional inequality (0.2)

In this section, we prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.2) in fuzzy Banach spaces. Let ρ be a real number with $|\rho| < \frac{1}{2}$. We need the following lemma to prove the main results.

Lemma 3.1. *Let $f : X \to Y$ be a mapping such that*

$$\begin{aligned}N&\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y),t\right)\\ &\ge N\left(\rho\left(f(x+y)+f(x-y)-2f(x)-2f(y)\right),t\right)\end{aligned} \tag{3.1}$$

for all $x, y \in X$ and all $t > 0$. Then f is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $x = y = 0$ in (3.1), we get $N(2f(0),t) \ge N(\rho(2f(0)),t) = N\left(2f(0), \frac{t}{|\rho|}\right)$ for all $t > 0$. By (N_5) and (N_6), $N(f(0),t) = 1$ for all $t > 0$.
It follows from (N_2) that $f(0) = 0$.

Letting $y = 0$ in (3.1), we get $N\left(4f\left(\frac{x}{2}\right) - f(x), t\right) \ge N(0,t) = 1$ for all $t > 0$ and so

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{3.2}$$

for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned}N&\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y),t\right)\\ &= N\left(\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y),t\right)\\ &= N\left(f(x+y)+f(x-y)-2f(x)-2f(y),2t\right)\\ &\ge N\left(\rho(f(x+y)+f(x-y)-2f(x)-2f(y)),t\right)\\ &= N\left(f(x+y)+f(x-y)-2f(x)-2f(y),\frac{t}{|\rho|}\right)\end{aligned}$$

for all $t > 0$. By (N_5) and (N_6), $N(f(x+y)+f(x-y)-2f(x)-2f(y),t) = 1$ for all $t > 0$. It follows from (N_2) that $f(x+y)+f(x-y) = 2f(x)+2f(y)$ for all $x, y \in X$. □

Theorem 3.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq \frac{L}{4}\varphi(2x,2y)$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying

$$\begin{aligned} & N\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y), t\right) \\ & \geq \min\left\{N\left(\rho\left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right), t\right), \frac{t}{t+\varphi(x,y)}\right\} \end{aligned} \tag{3.3}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x) - Q(x), t\right) \geq \frac{(1-L)t}{(1-L)t + \varphi(x,0)} \tag{3.4}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $x = y = 0$ in (3.3), we get $N(2f(0), t) \geq N\left(\rho(2f(0)), t\right) = N\left(2f(0), \frac{t}{|\rho|}\right)$ for all $t > 0$. So $f(0) = 0$.

Letting $y = 0$ in (3.3), we get

$$N\left(4f\left(\frac{x}{2}\right) - f(x), t\right) \geq \frac{t}{t+\varphi(x,0)} \tag{3.5}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t+\varphi(x,0)},\ \forall x \in X, \forall t > 0\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [26, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t+\varphi(x,0)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right), L\varepsilon t\right) = N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{4}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \varphi\left(\frac{x}{2}, 0\right)} \geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \frac{L}{4}\varphi(x,0)} = \frac{t}{t+\varphi(x,0)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (3.5) that

$$N\left(f(x) - 4f\left(\frac{x}{2}\right), t\right) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq 1$.

By Theorem 1.4, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(\frac{x}{2}\right) = \frac{1}{4}Q(x) \tag{3.6}$$

for all $x \in X$. Since $f : X \to Y$ is even, $Q : X \to Y$ is an even mapping. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (3.6) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - Q(x), \mu t) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$;

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f, Q) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, Q) \leq \frac{1}{1 - L}.$$

This implies that the inequality (3.4) holds.

By (3.3),

$$\begin{aligned} &N\left(4^n\left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right), 4^n t\right) \\ &\geq \min\left\{N\left(\rho\left(4^n\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right)\right), 4^n t\right), \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}\right\} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\begin{aligned} &N\left(4^n\left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right), t\right) \\ &\geq \min\left\{N\left(\rho\left(4^n\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right)\right), t\right), \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n}\varphi(x, y)}\right\} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$\begin{aligned} &N\left(2Q\left(\frac{x+y}{2}\right) + 2\left(\frac{x-y}{2}\right) - Q(x) - Q(y), t\right) \\ &\geq N\left(\rho\left(Q\left(x+y\right) + Q\left(x-y\right) - 2Q(x) - 2Q(y)\right), t\right) \end{aligned}$$

for all $x, y \in X$ and all $t > 0$. By Lemma 3.1, the mapping $Q : X \to Y$ is quadratic, as desired. $\square$

Corollary 3.3. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\| \cdot \|$. Let $f : X \to Y$ be an even mapping satisfying*

$$N\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y),t\right)$$
$$\geq \min\left\{N\left(\rho\left(f(x+y)+f(x-y)-2f(x)-2f(y)\right),t\right), \frac{t}{t+\theta(\|x\|^p+\|y\|^p)}\right\}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x)-Q(x),t\right) \geq \frac{(2^p-4)t}{(2^p-4)t+2^p\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{2-p}$, and we get the desired result. □

Theorem 3.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq 4L\varphi\left(\frac{x}{2},\frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.3). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x)-Q(x),t\right) \geq \frac{(1-L)t}{(1-L)t+\varphi(x,0)} \tag{3.7}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 3.2.

It follows from (3.5) that

$$N\left(f(x)-\frac{1}{4}f(2x),Lt\right) \geq \frac{t}{t+\varphi(x,0)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq L$. Hence

$$d(f,Q) \leq \frac{1}{1-L},$$

which implies that the inequality (3.7) holds.

The rest of the proof is similar to the proof of Theorem 3.2. □

Corollary 3.5. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 2$. Let X be a normed vector space with norm $\| \cdot \|$. Let $f : X \to Y$ be an even mapping satisfying*

$$N\left(2f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)-f(x)-f(y),t\right)$$
$$\geq \min\left\{N\left(\rho\left(f\left(x+y\right)+f\left(x-y\right)-2f(x)-2f(y)\right),t\right), \frac{t}{t+\theta(\|x\|^p+\|y\|^p)}\right\}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x)-Q(x),t\right) \geq \frac{(4-2^p)t}{(4-2^p)t+2^p\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{p-2}$, and we get the desired result. □

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The Quadrature rules of the fuzzy Henstock – Stieltjes integral on a infinite interval†

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Abstract: In this paper, the calculating methods for the fuzzy Henstock-Stieltjes integral on a infinite interval are proposed. It includes quadrature rules and the error estimates such as the midpoint-type rule, trapezoidal-type rule, Simpson's formula, $\delta-$fine quadrature rules, their error estimates, and so on. Finally, an example is given to illuminate the effectiveness the methods proposed in this paper.

Keywords: Fuzzy numbers; Fuzzy Henstock-Stieltjes integral; calculating methods

AMS subject classifications. 26E50; 28E10.

1 Introduction

It is well known that the notion of the Stieltjes integral for fuzzy-number-valued functions was originally proposed by Nanda [1] in 1989. Many generalizations of the fuzzy Riemann-Stieltjes integral were considered by scholars [2, 3, 4]. In 1998, Wu [5] proposed the concept of fuzzy Riemann-Stieltjes integral by means of the representation theorem of fuzzy-number-valued functions, whose membership function could be obtained by solving a nonlinear programming problem, but it is difficult to calculate and extend to the higher-dimensional space. In 2006, Ren et al. introduced the concept of two kinds of fuzzy Riemann-Stieltjes integral for fuzzy-number-valued functions [3, 4] and showed that a continuous fuzzy-number-valued function was fuzzy Riemann-Stieltjes integrable with respect to a real-valued increasing function. To overcome the limitations of the existing studies and to characterize continuous linear functionals on the space of Henstock integrable fuzzy-number-valued functions, the concept of the Henstock-Stieltjes integral for fuzzy-number-valued functions was defined and discussed in 2012, and some useful results for this integral were shown, such as the integrability, the continuity and the differentiability of the primitive, numerical calculus of the integration, the convergence theorems, and so on. The integral for fuzzy-number-valued functions on a infinite interval, as a expectation of fuzzy random variable, was originally investigated by Puri and Ralescu in 1986 [6]. In their opinion, a fuzzy random variable as a fuzzy-number-valued function and the expectation $E(X)$ of a fuzzy random variable X equals to a fuzzy integral $E(X) = \int X$ or set-valued integral of X_λ. In 2007, the concept of the fuzzy Henstock integral on infinite interval was proposed and discussed in order to solve the expectation $E(X)$ of a fuzzy random variable X which distribution function has some kinds of discontinuity or non-integrability by Gong and Wang [7]. After that, the Henstock-Stieltjes integral for fuzzy-number-valued functions on infinite interval which is an extension of the usual fuzzy Riemann-Stieltjes integral on infinite interval was investigated by Duan in 2014 [8], and several necessary and sufficient conditions of the integrability for fuzzy-number-valued functions are given by means of the Henstock-Stieltjes integral of real-valued functions on infinite interval and Henstock integral of fuzzy-number-valued functions on infinite interval. In this paper, we shall discuss the calculating methods for the fuzzy Henstock-Stieltjes integral on a infinite interval: one is to calculate directly by the fuzzy Henstock-Stieltjes integral on a infinite interval, including quadrature rules and the error estimates such as the midpoint-type rule, trapezoidal-type rule, Simpson's formula, $\delta-$fine quadrature rules and their error estimates; another is to calculate by using the equivalent characteristic of fuzzy Henstock-Stieltjes integrability, whose membership function could be obtained by solving a nonlinear programming problem.

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2 Preliminaries

Fuzzy set $\tilde{u} \in E^1$ is called a fuzzy number if $\tilde{u}$ is a normal, convex fuzzy set, upper semi-continuous and supp $u = \overline{\{x \in \mathbb{R} \mid u(x) > 0\}}$ is compact. Here $\bar{A}$ denotes the closure of A. We use E^1 to denote the fuzzy number space [1-6].

Let $\tilde{u}, \tilde{v} \in E^1, k \in \mathbb{R}$, the addition and scalar multiplication are defined by

$$[\tilde{u} + \tilde{v}]_\lambda = [\tilde{u}]_\lambda + [\tilde{v}]_\lambda, \qquad [k\tilde{u}]_\lambda = k[\tilde{u}]_\lambda,$$

respectively, where $[\tilde{u}]_\lambda = \{x : u(x) \geqslant \lambda\} = [u_\lambda^-, u_\lambda^+]$, for any $\lambda \in [0,1]$.

We use the Hausdorff distance between fuzzy numbers given by $D : E^1 \times E^1 \to [0, +\infty)$ as follows [1-6]:

$$D(\tilde{u}, \tilde{v}) = \sup_{\lambda \in [0,1]} d([\tilde{u}]_\lambda, [\tilde{v}]_\lambda) = \sup_{\lambda \in [0,1]} \max\{|u_\lambda^- - v_\lambda^-|, |u_\lambda^+ - v_\lambda^+|\},$$

where d is the Hausdorff metric. $D(\tilde{u}, \tilde{v})$ is called the distance between $\tilde{u}$ and $\tilde{v}$.

Recall, also, that a function $\tilde{f} : [a,b] \to E^1$ is said to be bounded if there exists $M \in \mathbb{R}$ such that $\|\tilde{f}(x)\| = D(\tilde{f}(x), \tilde{0}) \leqslant M$ for any $x \in [a,b]$. Notice that here $\|\tilde{f}(x_0)\|$ does not stand for the norm of E^1.

Definition 2.1 [7,8,9]. $\overline{R}$ denote the generalized real line, for $\tilde{f}$ defined on $[a, +\infty]$, we define $\tilde{f}(+\infty) = \tilde{0}$, and $\tilde{0} \cdot (+\infty) = \tilde{0}$.

Let $\delta : [a, +\infty] \to R^+$ be a positive real function. A division $P = \{[x_{i-1}, x_i]; \xi_i\}$ is said to be δ-fine, if the following conditions are satisfied:
(1)$a = x_0 < x_1 < ... < x_{n-1} = b < x_n = +\infty$;
(2)$\xi_i \in [x_{i-1}, x_i] \subset O(\xi_i), i = 1, 2, ..., n$;
where $O(\xi_i) = (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for $i = 1, 2, ..., n-1$, and $O(\xi_n) = [b, +\infty)$.

For brevity, we write $T = \{[u,v]; \xi\}$, where $[u,v]$ denotes a typical interval in T and ξ is the associated point of $[u,v]$.

Definition 2.2 [8]. Let $\alpha : [a, +\infty] \to \mathbb{R}$ be an increasing function. A fuzzy-number-valued function $\tilde{f}(x)$ is said to be fuzzy Henstock-Stieltjes integrable with respect to α on $[a, +\infty]$ if there exists a fuzzy number $\tilde{H} \in E^1$ such that for every $\varepsilon > 0$, there is a function $\delta(x) > 0$ on $[a, +\infty]$ such that for any δ-fine division $T = \{[x_{i-1}, x_i]; \xi_i\}_{i=1}^n$, we have

$$D(\sum_{i=1}^n [\alpha(x_i) - \alpha(x_{i-1})]\tilde{f}(\xi_i), \tilde{H}) < \varepsilon.$$

We write $(FHS) \int_a^{+\infty} \tilde{f}(x) \mathrm{d}\alpha = \tilde{H}$ and $(\tilde{f}, \alpha) \in FHS[a, +\infty]$.

The definition of $\tilde{f} \in FHS(-\infty, a]$ is similar. Naturally, we define $\tilde{f} \in FHS(-\infty, +\infty)$ iff $\tilde{f} \in FHS(-\infty, a]$ and $\tilde{f} \in FHS[a, +\infty)$, and furthermore

$$(FHS) \int_{-\infty}^{+\infty} \tilde{f}(x) d\mathrm{d}\alpha = (FHS) \int_{-\infty}^{a} \tilde{f}(x) d\mathrm{d}\alpha + (FHS) \int_{a}^{+\infty} \tilde{f}(x) d\mathrm{d}\alpha.$$

For brevity, we always assume that $\alpha : [a, +\infty] \to \mathbb{R}$ is an increasing function.

Lemma 2.1[8]. Let $\alpha : [a, +\infty] \to \mathbb{R}$ be an increasing function and let $\tilde{f} : [a, +\infty] \to E^1$. Then the following statements are equivalent:

(1) $(\tilde{f}, \alpha) \in FHS[a, +\infty]$ and $(FHS) \int_a^{+\infty} \tilde{f}(x) \mathrm{d}\alpha = \tilde{A}$;

(2) for any $\lambda \in [0,1]$, f_λ^- and f_λ^+ are Henstock-Stieltjes integrable with respect to α on $[a, +\infty]$ for any $\lambda \in [0,1]$ uniformly ($\delta(x)$ is independent of $\lambda \in [0,1]$), and

$$[(FHS) \int_a^{+\infty} \tilde{f}(x) \mathrm{d}\alpha]_\lambda = [(HS) \int_a^{+\infty} f_\lambda^-(x) \mathrm{d}\alpha, (HS) \int_0^{+\infty} f_\lambda^+(x) \mathrm{d}\alpha].$$

(3) For any $b > a$, $\tilde{f} \in FHS[a,b]$, $\lim\limits_{b \to +\infty} \int_a^b \tilde{f}(x) \mathrm{d}\alpha$ as a fuzzy number exists and

$$\lim_{b \to +\infty} \int_a^b \tilde{f}(x) \mathrm{d}\alpha = \int_a^{+\infty} \tilde{f}(x) \mathrm{d}\alpha.$$

3 Quadrature rules of the Henstock-Stieltjes integral for fuzzy-number-valued functions on infinite interval

We shall use the modulus of oscillation for a fuzzy-valued function to discuss the quadrature rules of expectations for fuzzy random variables in this section. For the numerical calculus of fuzzy integral, there were some discussions by the fuzzy Riemann integral, improper fuzzy Riemann integral, using the probabilistic Monte Carlo method, and the quadrature rules for fuzzy Henstock integral on a finite interval[1, 3, 4]. However, the calculus above will be restricted when the distribution function of a random variable on $(-\infty, +\infty)$ or the distribution function of a random variable has some kind of discontinuity or non-integrability. Furthermore, fuzzy Henstock integral is convenient for numerical calculus since it is a Riemann-type integral. Since a fuzzy random variable is a measurable fuzzy-valued function $\tilde{f} : (-\infty, +\infty) \to E^1$, therefore without loss of the generality, we only discuss the quadrature rules of Henstock integrals for the measurable fuzzy-valued functions on $[a, +\infty)$. For a fuzzy-valued function, since its Henstock integrability implies measurability, for brevity we always assume that the fuzzy-valued functions discussed are measurable throughout this section.

Definition 3.1[7, 10]. Let $\tilde{f} : [a, +\infty) \to E^1$ be a bounded mapping. Then the function $\omega_{[a,+\infty)}(\tilde{f}, \cdot) : R^+ \bigcup\{0\} \to R^+$,

$$\omega_{[a,+\infty)}(\tilde{f}, \delta) = \sup\{D(\tilde{f}(x), \tilde{f}(y)) : x, y \in [a, +\infty), |x - y| \leq \delta\}$$

is called the modulus of oscillation of $\tilde{f}$ on $[a, +\infty)$.

Theorem 3.1[7] Obviously, the following statements hold:

(i) $D(\tilde{f}(x), \tilde{f}(y)) \leq \omega_{[a,+\infty)}(\tilde{f}, |x - y|), \forall x, y \in [a, +\infty)$ for any $x, y \in [a, +\infty)$;

(ii) $\omega_{[a,+\infty)}(\tilde{f}, \delta)$ is nondecreasing mapping in δ and nonincreasing in a;

(iii) $\omega_{[a,+\infty)}(\tilde{f}, 0) = 0$;

(iv) $\omega_{[a,+\infty)}(\tilde{f}, \delta_1 + \delta_2) \leq \omega_{[a,+\infty)}(\tilde{f}, \delta_1) + \omega_{[a,+\infty)}(\tilde{f}, \delta_2)$ for any $\delta_1, \delta_2 \geq 0$;

(v) $\omega_{[a,+\infty)}(\tilde{f}, n\delta) \leq n\omega_{[a,+\infty)}(\tilde{f}, \delta)$ for any $\delta \geq 0, n \in N$;

(vi) $\omega_{[a,+\infty)}(\tilde{f}, \lambda\delta) \leq (\lambda + 1)\omega_{[a,+\infty)}(\tilde{f}, \delta)$ for any $\delta \geq 0, \lambda \geq 0$.

Theorem 3.2 Let $\tilde{f} \in FHS[a, +\infty)$ be a bounded function, and $\alpha : [a, +\infty] \to \mathbb{R}$ an increasing function. Then for any division $T : a = x_0 < x_1 < ... < x_{n-1} = b < x_n = +\infty$ and any point $\xi_i \in [x_{i-1}, x_i], i = 1, 2, 3, ..., n - 1$, and $\xi_n = +\infty$, we have

$$D\left(\int_a^{+\infty} \tilde{f}(x)d\alpha, \sum_{i=1}^{n}[\alpha(x_i) - \alpha(x_{i-1})]\tilde{f}(\xi_i)\right) \leq \sum_{i=1}^{n-1}[\alpha(x_i) - \alpha(x_{i-1})]\omega_{[x_{i-1},x_i]}(\tilde{f}, x_i - x_{i-1}) + \alpha_b,$$

where α_b stands for $\| \int_b^{+\infty} \tilde{f}(x)d\alpha\|_{E^1}$ or $\int_b^{+\infty} \|\tilde{f}(x)\|_{E^1}d\alpha$, and $\alpha_b \to 0$ $(b \to +\infty)$.

Proof. The subinterval including $+\infty$ is denoted by $[b, +\infty](x_{n-1} = b, x_n = +\infty)$, according to the additivity of interval for fuzzy Henstock-Stieltjes integral, we have $\int_a^{+\infty} \tilde{f}(x)d\alpha = \int_a^b \tilde{f}(x)d\alpha +$

$\int_b^{+\infty} \tilde{f}(x)d\alpha$, and

$$\begin{aligned}
&D(\int_a^{+\infty} \tilde{f}(x)d\alpha, \sum_{i=1}^{n}[\alpha(x_i)-\alpha(x_{i-1})]\tilde{f}(\xi_i)) \\
&\le D(\int_a^{b} \tilde{f}(x)d\alpha, \sum_{i=1}^{n-1}[\alpha(x_i)-\alpha(x_{i-1})]\tilde{f}(\xi_i)) + D(\int_b^{+\infty} \tilde{f}(x)d\alpha, \int_b^{+\infty} \tilde{f}(\xi_n)d\alpha,) \\
&\le \sum_{i=1}^{n-1}[\alpha(x_i)-\alpha(x_{i-1})]\omega_{[x_{i-1},x_i]}(\tilde{f}, x_i - x_{i-1}) + D(\int_b^{+\infty} \tilde{f}(x), \tilde{0})d\alpha \\
&\le \sum_{i=1}^{n}[\alpha(x_i)-\alpha(x_{i-1})]\omega_{[x_{i-1},x_i]}(\tilde{f}, x_i - x_{i-1}) + \alpha_b,
\end{aligned}$$

where $\tilde{f}(\xi_n)=\tilde{0}$. By Lemma 2.1, $\alpha_b \to 0$ when $b \to +\infty$.
The proof is complete.

Taking in Theorem 3.2 $n=2, x_1=\xi_1=\xi_2=x$; $n=2, x_1=x, \xi_1=u, \xi_2=v$ and $n=4, x_1=\alpha, x_2=\beta, \xi_1=u, \xi_2=v, \xi_3=w$ respectively, we obtain the midpoint-type, trapezoidal-type and Simpson's inequalities in some sense with its error estimations as follows.

Corollary 3.1 Let $\tilde{f} \in FHS[a,+\infty)$ be a bounded function. Then
(i)

$$D(\int_a^{+\infty} \tilde{f}(x)d\alpha, [\alpha(b)-\alpha(a)]\tilde{f}(x)) \le [\alpha(x)-\alpha(a)]\omega_{[a,x]}(\tilde{f}, x-a) + [\alpha(b)-\alpha(x)]\omega_{[x,b]}(\tilde{f}, b-x) + \alpha_b$$

for any $b \ge a$ and $x \in [a,b]$;
(ii)

$$\begin{aligned}
&D(\int_a^{+\infty} \tilde{f}(x)d\alpha, [\alpha(x)-\alpha(a)]\tilde{f}(u) + [\alpha(b)-\alpha(x)]\tilde{f}(v)) \\
&\le [\alpha(x)-\alpha(a)]\omega_{[a,x]}(\tilde{f}, x-a) + [\alpha(b)-\alpha(x)]\omega_{[x,b]}(\tilde{f}, b-x) + \alpha_b
\end{aligned}$$

for any $b \ge a$ and $x \in [a,b], u \in [a,x], v \in [x,b]$;
(iii)

$$\begin{aligned}
&D(\int_a^{+\infty} \tilde{f}(x)d\alpha, [\alpha(\beta_1)-\alpha(a)]\tilde{f}(u) + [\alpha(\beta_2)-\alpha(\beta_1)]\tilde{f}(v) + [\alpha(b)-\alpha(\beta_2)]\tilde{f}(w)) \\
&\le [\alpha(\beta_1)-\alpha(a)]\omega_{[a,\alpha]}(\tilde{f}, \beta_1 - a) + [\alpha(\beta_2)-\alpha(\beta_1)]\omega_{[\beta_1,\beta_2]}(\tilde{f}, \beta_2-\beta_1) \\
&+ [\alpha(b)-\alpha(\beta_2)]\omega_{[\beta_2,b]}(\tilde{f}, b-\beta_2) + \alpha_b
\end{aligned}$$

for any $b \ge a$, $\alpha, \beta \in [a,b]$, and $u \in [a,\beta_1]$, $v \in [\beta_1,\beta_2]$, $w \in [\beta_2, b]$, where α_b stands for $\| \int_b^{+\infty} \tilde{f}(x)d\alpha\|_{E^1}$ or $\int_b^{+\infty} \|\tilde{f}(x)\|_{E^1} d\alpha$, and $\alpha_b \to 0$ $(b \to +\infty)$.

Corollary 3.2 Let $\tilde{f} \in FHS[a,+\infty)$ be a bounded function. Then
(i)

$$\begin{aligned}
&D(\int_a^{+\infty} \tilde{f}(x)d\alpha, [\alpha(b)-\alpha(a)]\tilde{f}(\frac{a+b}{2})) \\
&\le [\alpha(b)-\alpha(a)]\omega_{[a,b]}(\tilde{f}, \frac{b-a}{2}) + \alpha_b \\
&\le [\alpha(b)-\alpha(a)]\omega_{[a,+\infty)}(\tilde{f}, \frac{b-a}{2}) + \alpha_b;
\end{aligned}$$

(ii)

$$\begin{aligned}
&D(\int_a^{+\infty}\tilde{f}(x)d\alpha,[\alpha(\frac{b+a}{2})-\alpha(a)]\tilde{f}(a)+[\alpha(b)-\alpha(\frac{b+a}{2})]\tilde{f}(b))\\
&\leq[\alpha(\frac{b+a}{2})-\alpha(a)]\omega_{[a,\frac{b+a}{2}]}(\tilde{f},\frac{b-a}{2})+[\alpha(b)-\alpha(\frac{b+a}{2})]\omega_{[\frac{b+a}{2},b]}(\tilde{f},\frac{b-a}{2})+\alpha_b;
\end{aligned}$$

(iii)

$$\begin{aligned}
&D(\int_a^{+\infty}\tilde{f}(x)d\alpha,[\alpha(\frac{2a+b}{3})-\alpha(a)]\tilde{f}(a)+[\alpha(\frac{a+2b}{3})-\alpha(\frac{2a+b}{3})]\tilde{f}(\frac{a+b}{2})+[\alpha(b)-\alpha(\frac{a+2b}{3})]\tilde{f}(b))\\
&\leq[\alpha(\frac{2a+b}{3})-\alpha(a)]\omega_{[a,\frac{2a+b}{3}]}(\tilde{f},\frac{b-a}{3})+[\alpha(\frac{a+2b}{3})-\alpha(\frac{2a+b}{3})]\omega_{[\frac{2a+b}{3},[\frac{a+2b}{3}]}(\tilde{f},\frac{b-a}{3})\\
&+[\alpha(b)-\alpha(\frac{a+2b}{3})]\omega_{[\frac{a+2b}{3},b]}(\tilde{f},\frac{b-a}{3})+\alpha_b,
\end{aligned}$$

where α_b stands for $\|\int_b^{+\infty}\tilde{f}(x)d\alpha\|_{E^1}$ or $\int_b^{+\infty}\|\tilde{f}(x)\|_{E^1}d\alpha$, and $\alpha_b\to 0$ $(b\to+\infty)$.

Using Theorem 3.2, we can also obtain another numerical calculous of Henstock-Stieltjes integrals with error estimations.

Corollary 3.3 Let $\tilde{f}\in FHS[a,+\infty)$ be a bounded function. Then

(1)

$$\begin{aligned}
&D(\int_a^{+\infty}\tilde{f}(x)d\alpha,\sum_{i=1}^{n}[\alpha(x_i)-\alpha(x_{i-1})]\tilde{f}(\xi_i))\\
&\leq[\alpha(b)-\alpha(a)]\omega_{[a,b]}(\tilde{f},\|T\|)+\alpha_b\\
&\leq[\alpha(b)-\alpha(a)]\omega_{[a,+\infty)}(\tilde{f},\|T\|)+\alpha_b;
\end{aligned}$$

(2)

$$\begin{aligned}
&D(\int_a^{+\infty}\tilde{f}(x)d\alpha,\sum_{i=1}^{n}[\alpha(x_i)-\alpha(x_{i-1})]\tilde{f}(\xi_i))\\
&\leq\|\alpha(T)\|\sum_{i=1}^{n-1}\omega_{[a,b]}(\tilde{f},x_i-x_{i-1})+\alpha_b\\
&\leq\|\alpha(T)\|\sum_{i=1}^{n-1}\omega_{[a,+\infty)}(\tilde{f},x_i-x_{i-1})+\alpha_b;
\end{aligned}$$

(3) If $\alpha:[a,b]\to\mathbb{R}$ is an increasing function satisfying $\alpha\in C^1[a,+\infty]$, then

$$\begin{aligned}
&D(\int_a^{+\infty}\tilde{f}(x)d\alpha,\sum_{i=1}^{n}[\alpha(x_i)-\alpha(x_{i-1})]\tilde{f}(\xi_i))\\
&\leq M\|T\|\sum_{i=1}^{n-1}\omega_{[a,b]}(\tilde{f},x_i-x_{i-1})+\alpha_b
\end{aligned}$$

for any division $T: a=x_0<x_1<...<x_{n-1}=b<x_n=+\infty$ and any point $\xi_i\in[x_{i-1},x_i], i=1,2,...n-1$ $\xi_n=+\infty$, where α_b stands for $\|\int_b^{+\infty}\tilde{f}(x)d\alpha\|_{E^1}$ or $\int_b^{+\infty}\|\tilde{f}(x)\|_{E^1}d\alpha$, $\alpha_b\to 0$ $(b\to+\infty)$, $\|T\|=\max\{x_i-x_{i-1}:i=1,2,...n-1\}$ denotes the modulus of division T, $\|\alpha(T)\|=\max\{\alpha(x_i)-\alpha(x_{i-1}): i=1,2,...n-1\}$, and M is the bound of α on $[a,b]$.

Proof. By using Theorem 3.2 and Theorem 3.1, (1) and (2) are obvious. We only prove that (3) holds. In fact, we have

$$
\begin{aligned}
&D(\int_a^{+\infty} \tilde{f}(x)d\alpha, \sum_{i=1}^{n}[\alpha(x_i)-\alpha(x_{i-1})]\tilde{f}(\xi_i)) \\
&\leq \sum_{i=1}^{n-1}[\alpha(x_i-\alpha(x_{i-1})]\omega_{[x_{i-1},x_i]}(\tilde{f}, x_i-x_{i-1})+\alpha_b \\
&= \sum_{i=1}^{n-1}\alpha'(\zeta_i)(x_i-x_{i-1})\omega_{[x_{i-1},x_i]}(\tilde{f}, x_i-x_{i-1})+\alpha_b \\
&\leq M\|T\|\sum_{i=1}^{n-1}\omega_{[a,b]}(\tilde{f}, x_i-x_{i-1})+\alpha_b \\
&\leq M\|T\|\sum_{i=1}^{n-1}\omega_{[a,+\infty)}(\tilde{f}, x_i-x_{i-1})+\alpha_b.
\end{aligned}
$$

4. $\delta-$ fine quadrature rules of the Henstock-Stieltjes integral for fuzzy-number-valued functions on infinite interval

Definition 4.1 Let $S_n = \sum\limits_{i=1}^{n}[\alpha(x_i)-\alpha(x_{i-1})]\tilde{f}(\xi_i)$ be a quadrature rule and $\delta : [a,+\infty] \to R^+$. S_n is said to be a $\delta-$fine quadrature rule, if $\xi_i \in [x_{i-1}, x_i] \subset O(\xi_i), i = 1,2,...,n$, where $O(\xi_i) = (\xi_i-\delta(\xi_i), \xi_i+\delta(\xi_i))$ for $i = 1,2,...,n-1$, and $O(\xi_n) = [b,+\infty)$.

We can deduce expressions for the remainder of $\delta-$fine quadrature rules by using Theorem 3.2 and Theorem 3.1(ii,v) as follows.

Theorem 4.1 Let $\tilde{f} \in FHS[a,+\infty)$ be a bounded function. If $S_n = \sum\limits_{i=1}^{n}[\alpha(x_i)-\alpha(x_{i-1})]\tilde{f}(\xi_i)$ is a $\delta-$fine quadrature rule, then

$$D(\int_a^{+\infty}\tilde{f}(x)d\alpha, S_n) \leq 2\sum_{i=1}^{n-1}[\alpha(x_i)-\alpha(x_{i-1})]\delta(\xi_i)\omega_{[x_{i-1}x_i]}(\tilde{f}, \delta(\xi_i))+\alpha_b.$$

Here α_b stands for $\| \int_b^{+\infty}\tilde{f}(x)d\alpha\|_{E^1}$ or $\int_b^{+\infty}\|\tilde{f}(x)\|_{E^1}d\alpha$, and $\alpha_b \to 0$ $(b \to +\infty)$.

Theorem 4.2 Let $\tilde{f} \in FHS[a,+\infty)$ be a bounded function, and $\alpha : [a,b] \to \mathbb{R}$ be an increasing function such that $\alpha \in C^1[a,+\infty]$. If $S_n = \sum\limits_{i=1}^{n}[\alpha(x_i)-\alpha(x_{i-1})]\tilde{f}(\xi_i)$ is a $\delta-$fine quadrature rule, then

$$D(\int_a^{+\infty}\tilde{f}(x)d\alpha, S_n) \leq 4M\sum_{i=1}^{n-1}\delta(\xi_i)\omega_{[x_{i-1}x_i]}(\tilde{f}, \delta(\xi_i))+\alpha_b.$$

Here M is the bound of α' on $[a,b]$, α_b stands for $\| \int_b^{+\infty}\tilde{f}(x)d\alpha\|_{E^1}$ or $\int_b^{+\infty}\|\tilde{f}(x)\|_{E^1}d\alpha$, and $\alpha_b \to 0$ $(b \to +\infty)$.

Proof By using Theorem 3.2 and Theorem 3.1(ii,v), we have

$$
\begin{aligned}
&D(\int_a^{+\infty}\tilde{f}(x)d\alpha, S_n)\\
&\le 2\sum_{i=1}^{n-1}[\alpha(x_i)-\alpha(x_{i-1})]\delta(\xi_i)\omega_{[x_{i-1}x_i]}(\tilde{f},\delta(\xi_i))+\alpha_b\\
&\le 2\sum_{i=1}^{n-1}\alpha'(\zeta_i)(x_i-x_{i-1})\delta(\xi_i)\omega_{[x_{i-1}x_i]}(\tilde{f},\delta(\xi_i))+\alpha_b\\
&\le 4M\sum_{i=1}^{n-1}\delta(\xi_i)\omega_{[x_{i-1}x_i]}(\tilde{f},\delta(\xi_i))+\alpha_b.
\end{aligned}
$$

Corollary 4.1 Let $\tilde{f}\in FHS[a,+\infty)$ be a bounded function, $\alpha:[a,b]\to\mathbb{R}$ an increasing function such that $\alpha\in C^1[a,+\infty]$, M the bound of α' on $[a,b]$. Then
(i)

$$
\begin{aligned}
&D(\int_a^{+\infty}\tilde{f}(x)d\alpha,[\alpha(b)-\alpha(a)]\tilde{f}(x))\\
&\le 4M\delta(x)\omega_{[a,b]}(\tilde{f},\delta(x))+\alpha_b;
\end{aligned}
$$

for any $x\in[a,b]$ such that the quadrature rule $[\alpha(b)-\alpha(a)]\tilde{f}(x)$ is $\delta-$fine;
(ii)

$$
\begin{aligned}
&D(\int_a^{+\infty}\tilde{f}(x)d\alpha,[\alpha(x)-\alpha(a)]\tilde{f}(u)+[\alpha(b)-\alpha(x)]\tilde{f}(v))\\
&\le 4M[\delta(u)\omega_{[a,x]}(\tilde{f},\delta(u)+\delta(v)\omega_{[x,b]}(\tilde{f},\delta(v))]+\alpha_b;
\end{aligned}
$$

for any $x\in[a,b], u\in[a,x]$ and $v\in[x,b]$ such that the trapezoidal-type quadrature rule $[\alpha(x)-\alpha(a)]\tilde{f}(u)+[\alpha(b)-\alpha(x)]\tilde{f}(v))$ is $\delta-$fine;
(iii)

$$
\begin{aligned}
&D(\int_a^{+\infty}\tilde{f}(x)d\alpha,[\alpha(\beta_1)-\alpha(a)]\tilde{f}(u)+[\alpha(\beta_2)-\alpha(\beta_1)]\tilde{f}(v)+[\alpha(b)-\alpha(\beta_2)]\tilde{f}(w))\\
&\le 4M[\delta(u)\omega_{[a,b]}(\tilde{f},\delta(u))+\delta(v)\omega_{[a,b]}(\tilde{f},\delta(v))+\delta(w)\omega_{[a,b]}(\tilde{f},\delta(w))]+\alpha_b
\end{aligned}
$$

for any $\beta_1,\beta_2\in[a,b]$, and $u\in[a,\beta_1], v\in[\beta_1,\beta_2], w\in[\beta_2,b]$, such that Simpson's formula is $\delta-$fine. Here α_b stands for $\|\int_b^{+\infty}\tilde{f}(x)d\alpha\|_{E^1}$ or $\int_b^{+\infty}\|\tilde{f}(x)\|_{E^1}d\alpha$, and $\alpha_b\to 0$ $(b\to+\infty)$.

The following theorem shows that $\delta-$fine quadrature rules converge for the bounded Henstock-Stieltjes integrable functions.

Theorem 4.3 Let $\tilde{f}\in FHS[a,+\infty)$ be a bounded function. Then there exist functions δ_n : $[a,+\infty]\to R^+$ and a sequence of δ_n-fine quadrature rules $S_n=\sum\limits_{i=1}^{m_n}[\alpha(x_i)-\alpha(x_{i-1})]\tilde{f}(\xi_i)$ such that S_n converges to $\int_a^{+\infty}\tilde{f}(x)d\alpha$.

Proof. From the definition of Henstock-Stieltjes integrability on infinite interval for all $\varepsilon>0$ there exists a function δ such that for any $\delta-$fine division(which can be interpreted as a $\delta-$fine quadrature rule), we have

$$D(\int_a^{+\infty}\tilde{f}(x)d\alpha,S_n)<\varepsilon.$$

Taking $\varepsilon=\frac{1}{n}$ in the inequality we obtain that the statement of the theorem holds. The proof is complete.

Corollary 4.2 Let $\tilde{f} \in FHS[a,+\infty)$ be a bounded function. Then for any natural number n, there exist functions $\delta_n : [a,+\infty] \to R^+$, $b_n \geq a$, and a sequence of δ_n-fine quadrature rules $S_n = \sum_{i=1}^{m_n}[\alpha(x_i)-\alpha(x_{i-1})]\tilde{f}(\xi_i)$ such that

$$D(\int_a^{+\infty}\tilde{f}(x)d\alpha, S_n) < \varepsilon.$$

5. Examples

Example 5.1 Let $\tilde{f} : [1,+\infty] \to E^1$ be given by

$$\tilde{f}(x,s) = \begin{cases} s, & s \in [0,1], x \text{ is rational}, \\ 0, & s \in (-\infty,0)\cup(1,+\infty), x \text{ is rational}, \\ 1-\frac{s}{e^{-x^2}}, & s \in [0,e^{-x^2}], x \text{ is irrational}, \\ 0, & s \in (-\infty,0)\cup(e^{-x^2},+\infty), x \text{ is irrational}, \\ 1, & s = 0, x = +\infty, \\ 0, & s \in (-\infty,0)\cup(0,+\infty), x = +\infty, \end{cases}$$

and $\alpha(x) = x$.

We could prove that $\tilde{f}$ is (FHS) integrable on $[0,+\infty)$ according to the equivalence of fuzzy (HS) integrability and uniform (HS) integrability of f_λ^- and f_λ^+. Furthermore, $\int_0^{+\infty}\tilde{f}(x)d\alpha = \tilde{H}$, and $\delta-$fine quadrature rule $S_n = \sum_{i=1}^{n}[\alpha(x_i)-\alpha(x_{i-1})]\tilde{f}(\xi_i)$ converges to fuzzy number $\tilde{H}$ which relationship function is defined by

$$H(s) = \begin{cases} 1-\frac{2}{\sqrt{\pi}}s, & s \in [0,\frac{\sqrt{\pi}}{2}], \\ 0, & x \text{ is others}, \end{cases}$$

That is to say, $H_\lambda^- = 0$, $\ H_\lambda^+ = (1-\lambda)\frac{\sqrt{\pi}}{2}$. In fact, we note that

$$f_\lambda^-(x) = \begin{cases} \lambda, & x \text{ is rational}, \\ 0, & x = +\infty, \\ 0, & x \text{ is irrational}, \end{cases} \quad f_\lambda^+(x) = \begin{cases} 1, & x \text{ is rational}, \\ 0, & x = +\infty, \\ (1-\lambda)e^{-x^2}, & x \text{ is irrational}. \end{cases}$$

Since $f_\lambda^+(x) \leq e^{-x^2}$, f_λ^-, f_λ^+ are Henstock integrable uniformly for $\lambda \in [0,1]$ and

$$\int_0^{+\infty} f_\lambda^-(x)d\alpha = 0(+\infty) = 0, \int_0^{+\infty} f_\lambda^+(x)d\alpha = \lim_{b\to+\infty}\int_0^b f_\lambda^+(x)d\alpha = (1-\lambda)\frac{\sqrt{\pi}}{2}.$$

It follows that

$$\int_0^{+\infty}\tilde{f}(x)d\alpha = \tilde{H}.$$

For any $\varepsilon > 0$, we define

$$\delta(\xi) = \begin{cases} \frac{\varepsilon}{2^{i+2}}, & \xi = r_i, \\ \frac{\varepsilon}{4}\xi, & otherwise, \end{cases}$$

where $Q = \{r_1, r_2, r_3, ...\}$ stands for the set of all rational numbers on $[0,+\infty)$ and for any $\delta-$fine division $T : 1 = x_0 < x_1 < ... < x_{n-1} = b < x_n = +\infty(\xi_n = +\infty, \tilde{f}(\xi_n) = (0,0,0))$, then $S_n = \sum_{i=1}^{n}[\alpha(x_i)-\alpha(x_{i-1})]\tilde{f}(\xi_i)$ a any $\delta-$fine quadrature rule. Note that $\omega_{[x_{i-1}x_i]}(\tilde{f},\delta(\xi_i)) = 1$. Then we have the following results.

(1) According to Theorem 3.2, we have

$$
\begin{aligned}
&D(S_n, \tilde{H})\\
&\leq \sum_{i=1}^{n-1}[\alpha(x_i)-\alpha(x_{i-1})]\omega_{[x_{i-1},x_i]}(\tilde{f}, x_i - x_{i-1}) + \alpha_b\\
&= \sum_{i=1}^{n-1}[\alpha(x_i)-\alpha(x_{i-1})] + \alpha_b\\
&= b + \alpha_b,
\end{aligned}
$$

where $\alpha_b = \| \int_b^{+\infty} \tilde{f}(x)d\alpha\|_{E^1}$ or $\int_b^{+\infty} \|\tilde{f}(x)\|_{E^1}d\alpha$, and $\alpha_b \to 0$ $(b \to +\infty)$. Indeed,

$$\alpha_b = \int_b^{+\infty} \|\tilde{f}(x)\|_{E^1}d\alpha = \sup_{\lambda\in[0,1]}\{\int_b^{+\infty}(1-\lambda)e^{-x^2}\},$$

and $\alpha_b \to 0$.

(2) According to Theorem 4.2, we have

$$
\begin{aligned}
&D(S_n, \tilde{H})\\
&\leq \sum_{i=1}^{n-1}\delta(\xi_i)\omega_{[x_{i-1}x_i]}(\tilde{f}, \delta(\xi_i)) + \|\int_b^{+\infty}\tilde{f}(x)dx\|_{E^1}\\
&=\leq 4\sum_{i=1}^{n-1}\delta(\xi_i) + \alpha_b.
\end{aligned}
$$

(iii) According to Corollary 4.1, we have

(i)

$$D((b-0)\tilde{f}(x), \tilde{H}) \leq 4\delta(x) + \alpha_b$$

for any $x \in [0, b]$ such that the quadrature rule $(b-0)\tilde{f}(x)$ is $\delta-$fine;

(ii)

$$D((x-0)\tilde{f}(u) + (b-x)\tilde{f}(v), \tilde{H}) \leq 4(\delta(u) + \delta(v)) + \alpha_b$$

for any $x \in [0, b], u \in [0, x]$ and $v \in [x, b]$ such that the trapezoidal-type quadrature rule $(x-0)\tilde{f}(u) + (b-x)\tilde{f}(v)$ is $\delta-$fine;

(iii)

$$
\begin{aligned}
&D((\beta_1 - 0)\tilde{f}(u) + (\beta_2 - \beta_1)\tilde{f}(v) + (b-\beta_2)\tilde{f}(w), \tilde{H})\\
&\leq 4(\delta(u) + \delta(v) + \delta(w)) + \alpha_b
\end{aligned}
$$

for any $\alpha, \beta \in [0, b]$, and $u \in [0, \alpha], v \in [\alpha, \beta], w \in [\beta, b]$, such that Simpson's formula is $\delta-$fine, where $\alpha_b = \| \int_b^{+\infty} \tilde{f}(x)d\alpha\|_{E^1}$ or $\int_b^{+\infty} \|\tilde{f}(x)\|_{E^1}d\alpha$, and $\alpha_b \to 0$ $(b \to +\infty)$. Indeed,

$$\alpha_b = \int_b^{+\infty} \|\tilde{f}(x)\|_{E^1}d\alpha = \sup_{\lambda\in[0,1]}\{\int_b^{+\infty}(1-\lambda)e^{-x^2}\},$$

and $\alpha_b \to 0$.

5. Conclusion

We have discussed the numerical calculus of the fuzzy Henstock-Stieltjes integral for fuzzy-valued functions on $[a, +\infty)$. It is well known that the quadrature rules and numerical calculus are restricted when the distribution function of a random variable is unbounded, defined on $(-\infty, +\infty)$ or have some kind of non-integrability in the previous papers, however, applying the methods proposed in this paper, the problems mentioned above are solved. It includes quadrature rules and the error estimates, such as the midpoint-type rule, trapezoidal-type rule, Simpson's rule, $\delta-$ fine formula and their error estimates, and so on.

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CUBIC AND QUARTIC ρ-FUNCTIONAL INEQUALITIES IN FUZZY NORMED SPACES

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Abstract. In this paper, we solve the following cubic ρ-functional inequality

$$\begin{aligned} & N\left(f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x),t\right) \\ & \geq N\left(\rho\left(4f\left(x+\frac{y}{2}\right)+4f\left(x-\frac{y}{2}\right)-f(x+y)-f(x-y)-6f(x)\right),t\right) \end{aligned} \tag{0.1}$$

in fuzzy normed spaces, where ρ is a fixed real number with $|\rho|<2$, and the following quartic ρ-functional inequality

$$\begin{aligned} & N\left(f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y),t\right) \\ & \geq N\left(\rho\left(8f\left(x+\frac{y}{2}\right)+8f\left(x-\frac{y}{2}\right)-2f(x+y)-2f(x-y)-12f(x)+3f(y)\right),t\right) \end{aligned} \tag{0.2}$$

in fuzzy normed spaces, where ρ is a fixed real number with $|\rho|<2$.

Using the fixed point method, we prove the Hyers-Ulam stability of the cubic ρ-functional inequality (0.1) and the quartic ρ-functional inequality (0.2) in fuzzy Banach spaces.

1. Introduction and preliminaries

Katsaras [20] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [11, 24, 50]. In particular, Bag and Samanta [2], following Cheng and Mordeson [8], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [23]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [3].

We use the definition of fuzzy normed spaces given in [2, 29, 30] to investigate the Hyers-Ulam stability of cubic ρ-functional inequalities and quartic ρ-functional inequalities in fuzzy Banach spaces.

Definition 1.1. [2, 29, 30, 31] Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N_1) $N(x,t) = 0$ for $t \leq 0$;
(N_2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
(N_3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N_4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;
(N_5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.
(N_6) for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X,N) is called a *fuzzy normed vector space.*

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [28, 29].

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Definition 1.2. [2, 29, 30, 31] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by $N\text{-}\lim_{n\to\infty} x_n = x$.

Definition 1.3. [2, 29, 30, 31] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [3]).

The stability problem of functional equations originated from a question of Ulam [49] concerning the stability of group homomorphisms. Hyers [15] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [41] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [12] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [7, 16, 19, 21, 22, 25, 37, 38, 39, 43, 44, 45, 46, 47, 48]).

In [18], Jun and Kim considered the following cubic functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x). \tag{1.1}$$

It is easy to show that the function $f(x) = x^3$ satisfies the functional equation (1.1), which is called a *cubic functional equation* and every solution of the cubic functional equation is said to be a *cubic mapping*.

In [26], Lee et al. considered the following quartic functional equation

$$f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 24f(x) - 6f(y). \tag{1.2}$$

It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.2), which is called a *quartic functional equation* and every solution of the quartic functional equation is said to be a *quartic mapping*.

Gilányi [13] showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(x - y)\| \leq \|f(x + y)\| \tag{1.3}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(x + y) + f(x - y).$$

See also [42]. Fechner [10] and Gilányi [14] proved the Hyers-Ulam stability of the functional inequality (1.3). Park, Cho and Han [36] investigated the Cauchy additive functional inequality

$$\|f(x) + f(y) + f(z)\| \leq \|f(x + y + z)\| \tag{1.4}$$

and the Cauchy-Jensen additive functional inequality

$$\|f(x)+f(y)+2f(z)\| \le \left\|2f\left(\frac{x+y}{2}+z\right)\right\| \tag{1.5}$$

and proved the Hyers-Ulam stability of the functional inequalities (1.4) and (1.5) in Banach spaces.

Park [34, 35] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces.

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.4. [4, 9] Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either

$$d(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \ge n_0$;
(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;
(4) $d(y, y^*) \le \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

In 1996, G. Isac and Th.M. Rassias [17] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5, 6, 28, 32, 33, 39, 40]).

In Section 2, we solve the cubic ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the cubic ρ-functional inequality (0.1) in fuzzy Banach spaces by using the fixed point method.

In Section 3, we solve the quartic ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the quartic ρ-functional inequality (0.2) in fuzzy Banach spaces by using the fixed point method.

Throughout this paper, assume that ρ is a fixed real number with $|\rho| < 2$.

2. Cubic ρ-functional inequality (0.1)

In this section, we solve and investigate the cubic ρ-functional inequality (0.1) in fuzzy Banach spaces.

Lemma 2.1. *Let (Y, N) be a fuzzy normed vector space. Let $f : X \to Y$ be a mapping such that*

$$\begin{aligned} &N((f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x), t) \\ &\ge N\left(\rho\left(4f\left(x+\frac{y}{2}\right)+4f\left(x-\frac{y}{2}\right)-f(x+y)-f(x-y)-6f(x)\right), t\right) \end{aligned} \tag{2.1}$$

for all $x, y \in X$ and all $t > 0$. Then $f : X \to Y$ is cubic.

Proof. Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $N(-14f(0), t) \geq 1$. So $f(0) = 0$.

Letting $y = 0$ in (2.1), we get $N(2f(2x) - 16f(x), t) \geq 1$ and so $f(2x) = 8f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{8}f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} &N\left(f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x), t\right) \\ &\geq N\left(\rho\left(4f\left(x+\frac{y}{2}\right) + 4f\left(x-\frac{y}{2}\right) - f(x+y) - f(x-y) - 6f(x)\right), t\right) \\ &= N\left(f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x), \frac{2t}{|\rho|}\right) \end{aligned}$$

for all $t > 0$. By (N_5) and (N_6),

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x)$$

for all $x, y \in X$, since $|\rho| < 2$. So $f : X \to Y$ is cubic. □

We prove the Hyers-Ulam stability of the cubic ρ-functional inequality (2.1) in fuzzy Banach spaces.

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{8}\varphi(2x, 2y)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying

$$\begin{aligned} &N\left(f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x), t\right) \\ &\geq min(N\left(\rho\left(4f\left(x+\frac{y}{2}\right) + 4f\left(x-\frac{y}{2}\right) - f(x+y) - f(x-y) - 6f(x)\right), t\right), \frac{t}{t+\varphi(x,y)})) \end{aligned} \tag{2.3}$$

for all $x, y \in X$ and all $t > 0$. Then $C(x) := N\text{-}\lim_{n\to\infty} 8^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that

$$N\left(f(x) - C(x), t\right) \geq \frac{(16-16L)t}{(16-16L)t + L\varphi(x, 0)} \tag{2.4}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $y = 0$ in (2.3), we get

$$N(2f(2x) - 16f(x), t) \geq \frac{t}{t + \varphi(x, 0)} \tag{2.5}$$

and so $N\left(f(x) - 8f\left(\frac{x}{2}\right), \frac{t}{2}\right) \geq \frac{t}{t+\varphi\left(\frac{x}{2},0\right)}$ for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf\left\{\mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t + \varphi(x, 0)},\ \forall x \in X, \forall t > 0\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [27, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 8g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(8g\left(\frac{x}{2}\right) - 8h\left(\frac{x}{2}\right), L\varepsilon t\right) = N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{8}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{8}}{\frac{Lt}{8} + \varphi\left(\frac{x}{2}, 0\right)} \geq \frac{\frac{Lt}{8}}{\frac{Lt}{8} + \frac{L}{8}\varphi(x, 0)} = \frac{t}{t + \varphi(x, 0)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (2.5) that

$$N\left(f(x) - 8f\left(\frac{x}{2}\right), \frac{L}{16}t\right) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{16}$.

By Theorem 1.4, there exists a mapping $C : X \to Y$ satisfying the following:

(1) C is a fixed point of J, i.e.,

$$C\left(\frac{x}{2}\right) = \frac{1}{8}C(x) \tag{2.6}$$

for all $x \in X$. The mapping C is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that C is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - C(x), \mu t) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$;

(2) $d(J^n f, C) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 8^n f\left(\frac{x}{2^n}\right) = C(x)$$

for all $x \in X$;

(3) $d(f, C) \leq \frac{1}{1-L}d(f, Jf)$, which implies the inequality

$$d(f, C) \leq \frac{L}{16 - 16L}.$$

This implies that the inequality (2.4) holds.

By (2.3),

$$\begin{aligned}
&N\left(8^n\left(f\left(\frac{2x+y}{2^n}\right)+f\left(\frac{2x-y}{2^n}\right)-2f\left(\frac{x+y}{2^n}\right)-2f\left(\frac{x-y}{2^n}\right)-12f\left(\frac{x}{2^n}\right)\right),8^nt\right)\\
&\geq \min\left\{N\left(8^n\rho\left(4f\left(\frac{x+\frac{y}{2}}{2^n}\right)+4f\left(\frac{x-\frac{y}{2}}{2^n}\right)-f\left(\frac{x+y}{2^n}\right)-f\left(\frac{x-y}{2^n}\right)-6f\left(\frac{x}{2^n}\right)\right),8^nt\right),\right.\\
&\qquad\left.\frac{t}{t+\varphi(\frac{x}{2^n},\frac{y}{2^n})}\right\}
\end{aligned}$$

for all $x,y\in X$, all $t>0$ and all $n\in\mathbb{N}$. So

$$\begin{aligned}
&N\left(8^n\left(f\left(\frac{2x+y}{2^n}\right)+f\left(\frac{2x-y}{2^n}\right)-2f\left(\frac{x+y}{2^n}\right)-2f\left(\frac{x-y}{2^n}\right)-12f\left(\frac{x}{2^n}\right)\right),t\right)\\
&\geq \min\left\{N\left(8^n\rho\left(4f\left(\frac{x+\frac{y}{2}}{2^n}\right)+4f\left(\frac{x-\frac{y}{2}}{2^n}\right)-f\left(\frac{x+y}{2^n}\right)-f\left(\frac{x-y}{2^n}\right)-6f\left(\frac{x}{2^n}\right)\right),t\right),\right.\\
&\qquad\left.\frac{\frac{t}{8^n}}{\frac{t}{8^n}+\frac{L^n}{8^n}\varphi(x,y)}\right\}
\end{aligned}$$

Since $\lim_{n\to\infty}\frac{\frac{t}{8^n}}{\frac{t}{8^n}+\frac{L^n}{8^n}\varphi(x,y)}=1$ for all $x,y\in X$ and all $t>0$,

$$\begin{aligned}
&N(C(2x+y)+C(2x-y)-2C(x+y)-2C(x-y)-12C(x),t)\\
&\quad\geq N(\rho(4C(x+\frac{y}{2})+4C(x-\frac{y}{2})-C(x+y)-C(x-y)-6C(x),t)
\end{aligned}$$

for all $x,y\in X$ and all $t>0$. By Lemma 2.1, the mapping $C:X\to Y$ is cubic, as desired. □

Corollary 2.3. *Let $\theta\geq 0$ and let p be a real number with $p>3$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f:X\to Y$ be a mapping satisfying*

$$\begin{aligned}
&N\left(f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x),t\right) \qquad (2.7)\\
&\geq min\left\{N\left(\rho\left(4f\left(x+\frac{y}{2}\right)+4f\left(x-\frac{y}{2}\right)-f(x+y)-f(x-y)-6f(x)\right),t\right),\right.\\
&\qquad\left.\frac{t}{t+\theta(\|x\|^p+\|y\|^p)}\right\}
\end{aligned}$$

for all $x,y\in X$ and all $t>0$. Then $C(x):=N\text{-}\lim_{n\to\infty}8^nf(\frac{x}{2^n})$ exists for each $x\in X$ and defines a cubic mapping $C:X\to Y$ such that

$$N\left(f(x)-C(x),t\right)\geq\frac{2(2^p-8)t}{2(2^p-8)t+\theta\|x\|^p}$$

for all $x\in X$ and all $t>0$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x,y):=\theta(\|x\|^p+\|y\|^p)$ for all $x,y\in X$. Then we can choose $L=2^{3-p}$, and we get the desired result. □

Theorem 2.4. *Let $\varphi:X^2\to[0,\infty)$ be a function such that there exists an $L<1$ with*

$$\varphi(x,y)\leq 8L\varphi\left(\frac{x}{2},\frac{y}{2}\right)$$

for all $x, y \in X$. *Let* $f : X \to Y$ *be a mapping satisfying* (2.3). *Then* $C(x) := N\text{-}\lim_{n\to\infty} \frac{1}{8^n} f(2^n x)$ *exists for each* $x \in X$ *and defines a cubic mapping* $C : X \to Y$ *such that*

$$N(f(x) - C(x), t) \geq \frac{(16 - 16L)t}{(16 - 16L)t + \varphi(x, 0)} \tag{2.8}$$

for all $x \in X$ *and all* $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

It follows from (2.5) that

$$N\left(f(x) - \frac{1}{8}f(2x), \frac{1}{16}t\right) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$. Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{8}g(2x)$$

for all $x \in X$. Then $d(f, Jf) \leq \frac{1}{16}$. Hence

$$d(f, C) \leq \frac{1}{16 - 16L},$$

which implies that the inequality (2.8) holds.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let* $\theta \geq 0$ *and let* p *be a real number with* $0 < p < 3$. *Let* X *be a normed vector space with norm* $\|\cdot\|$. *Let* $f : X \to Y$ *be a mapping satisfying* (2.7). *Then* $C(x) := N\text{-}\lim_{n\to\infty} \frac{1}{8^n} f(2^n x)$ *exists for each* $x \in X$ *and defines a cubic mapping* $C : X \to Y$ *such that*

$$N(f(x) - C(x), t) \geq \frac{2(8 - 2^p)t}{2(8 - 2^p)t + \theta\|x\|^p}$$

for all $x \in X$ *and all* $t > 0$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{p-3}$, and we get the desired result. □

3. Quartic ρ-functional inequality (0.2)

In this section, we solve and investigate the quartic ρ-functional inequality (0.2) in fuzzy Banach spaces.

Lemma 3.1. *Let* (Y, N) *be a fuzzy normed vector space. A mapping* $f : X \to Y$ *satisfies* $f(0) = 0$ *and*

$$\begin{aligned} &N(f(2x + y) + f(2x - y) - 4f(x + y) - 4f(x - y) - 24f(x) + 6f(y), t) \qquad (3.1) \\ &\geq N\left(\rho\left(8f\left(x + \frac{y}{2}\right) + 8f\left(x - \frac{y}{2}\right) - 2f(x + y) - 2f(x - y) - 12f(x) + 3f(y)\right), t\right) \end{aligned}$$

for all $x, y \in X$ *and all* $t > 0$. *Then* f *is quartic.*

Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $y = 0$ in (3.1), we get $N(2f(2x) - 32f(x), t) \geq N(0, t) = 1$ and so

$$f\left(\frac{x}{2}\right) = \frac{1}{16}f(x) \tag{3.2}$$

for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} & N\left(f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y),t\right) \\ & \geq N\left(\rho\left(8f\left(x+\frac{y}{2}\right)+8f\left(x-\frac{y}{2}\right)-2f\left(x+y\right)-2f\left(x-y\right)-12f\left(x\right)+3f\left(y\right)\right),t\right) \\ & = N\left(\frac{\rho}{2}\left(f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y)\right),t\right) \\ & = N\left(f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y),\frac{2t}{|\rho|}\right) \end{aligned}$$

for all $t > 0$ and all $x, y \in X$. By (N_5) and (N_6),

$$N\left(f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y),t\right)=1$$

for all $t > 0$ and all $x, y \in X$. It follows from (N_2) that

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y)$$

for all $x, y \in X$. □

We prove the Hyers-Ulam stability of the quartic ρ-functional inequality (3.1) in fuzzy Banach spaces.

Theorem 3.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq \frac{L}{16}\varphi\left(2x,2y\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\begin{aligned} & N\left(f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y),t\right) \\ & \geq \min\left\{N\left(\rho\left(8f\left(x+\frac{y}{2}\right)+8f\left(x-\frac{y}{2}\right)-2f\left(x+y\right)-2f\left(x-y\right)-12f\left(x\right)+3f\left(y\right)\right),t\right),\right. \\ & \qquad \left.\frac{t}{t+\varphi(x,0)}\right\} \end{aligned} \tag{3.3}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 16^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that

$$N\left(f(x)-Q(x),t\right) \geq \frac{(32-32L)t}{(32-32L)t+L\varphi(x,0)} \tag{3.4}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Letting $y = 0$ in (3.3), we get

$$N\left(2f(2x)-32f\left(x\right),t\right)=N\left(32f\left(x\right)-2f(2x),t\right) \geq \frac{t}{t+\varphi(x,0)} \tag{3.5}$$

for all $x \in X$. Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 16g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x)-h(x),\varepsilon t) \geq \frac{t}{t+\varphi(x,0)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(16g\left(\frac{x}{2}\right) - 16h\left(\frac{x}{2}\right), L\varepsilon t\right) = N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{16}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{16}}{\frac{Lt}{16} + \varphi\left(\frac{x}{2}, 0\right)} \geq \frac{\frac{Lt}{16}}{\frac{Lt}{16} + \frac{L}{16}\varphi(x, 0)} = \frac{t}{t + \varphi(x, 0)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (3.5) that

$$N\left(f(x) - 16f\left(\frac{x}{2}\right), \frac{L}{32}t\right) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{32}$.

By Theorem 1.4, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(\frac{x}{2}\right) = \frac{1}{16}Q(x) \tag{3.6}$$

for all $x \in X$. Since $f : X \to Y$ is even, $Q : X \to Y$ is an even mapping. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (3.6) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - Q(x), \mu t) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$;

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 16^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f, Q) \leq \frac{1}{1-L}d(f, Jf)$, which implies the inequality

$$d(f, Q) \leq \frac{L}{32 - 32L}.$$

This implies that the inequality (3.4) holds.

By the same method as in the proof of Theorem 2.2, it follows from (3.3) that

$$\begin{aligned} &N\left(Q(2x + y) + Q(2x - y) - 4Q(x + y) - 4Q(x - y) - 24Q(x) + 6Q(y), t\right) \\ &\geq N\left(\rho\left(8Q\left(x + \frac{y}{2}\right) + 8Q\left(x - \frac{y}{2}\right) - 2Q\left(x + y\right) - 2Q\left(x - y\right) - 12Q\left(x\right) + 3Q\left(y\right)\right), t\right) \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. By Lemma 3.1, the mapping $Q : X \to Y$ is quartic. $\square$

Corollary 3.3. *Let $\theta \geq 0$ and let p be a real number with $p > 4$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} &N\left(f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y),t\right) \\ &\geq \min\left\{N\left(\rho\left(8f\left(x+\frac{y}{2}\right)+8f\left(x-\frac{y}{2}\right)-2f\left(x+y\right)-2f\left(x-y\right)-12f\left(x\right)+3f\left(y\right)\right),t\right),\right. \\ &\qquad\left.\frac{t}{t+\theta(\|x\|^p+\|y\|^p)}\right\} \end{aligned} \tag{3.7}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 16^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that

$$N\left(f(x)-Q(x),t\right) \geq \frac{2(2^p-16)t}{2(2^p-16)t+\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{4-p}$, and we get the desired result. □

Theorem 3.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq 16L\varphi\left(\frac{x}{2},\frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.3). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{16^n} f(2^n x)$ exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that

$$N\left(f(x)-Q(x),t\right) \geq \frac{(32-32L)t}{(32-32L)t+\varphi(x,0)} \tag{3.8}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

It follows from (3.5) that

$$N\left(f(x)-\frac{1}{16}f(2x),\frac{1}{32}t\right) \geq \frac{t}{t+\varphi(x,0)}$$

for all $x \in X$ and all $t > 0$. Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{16}g\left(2x\right)$$

for all $x \in X$. Then $d(f, Jf) \leq \frac{1}{32}$. Hence

$$d(f,Q) \leq \frac{1}{32-32L},$$

which implies that the inequality (3.8) holds.

The rest of the proof is similar to the proof of Theorem 3.2. □

Corollary 3.5. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 4$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.7). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{16^n} f(2^n x)$ exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that*

$$N\left(f(x)-Q(x),t\right) \geq \frac{2(16-2^p)t}{2(16-2^p)t+\theta\|x\|^p}$$

for all $x \in X$*.*

Proof. The proof follows from Theorem 3.4 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{p-4}$, and we get the desired result. $\square$

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A RIGHT PARALLELISM RELATION FOR MAPPINGS TO POSETS

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ABSTRACT. In this paper, we study mappings $f, g : X \to P$, where P is a poset and X is a set, under the relation $f \,||\, g$, of right parallelism, $f(a) \le f(b)$ implies $g(a) \le g(b)$, which is reflexive and transitive but not necessarily symmetric. We prove several results of the type: if f has property P and $f \,||\, g$, then g has property P as well, or of the converse type. Doing so permits us to observe several conditions on mappings and/or groupoids $(X, *)$, upon which mappings may act in particular ways, which are of interest in their own right also. The special case $f(x) = x$ with $f \,||\, g$ yielding increasing/non-decreasing mappings $g : X \to P$ brings into focus a number of well-known situations seen from a different perspective.

1. Introduction.

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([4, 5]).

In the study of groupoids $(X, *)$ defined on a set X, it has also proven useful to investigate the semigroups $(Bin(X), \Box)$ where $Bin(X)$ is the set of all binary systems (groupoids) $(X, *)$ along with an associative product operation $(X, *)\Box(X, \bullet) = (X, \Box)$ such that $x \Box y = (x * y) \bullet (y * x)$ for all $x, y \in X$. Thus, e.g., it becomes possible to recognize that the left-zero-semigroup $(X, *)$ with $x * y = x$ for all $x, y \in X$ acts as the identity of this semigroup ([2]). H. F. Fayoumi ([1]) introduced the notion of the center $ZBin(X)$ in the semigroup $Bin(X)$ of all binary systems on a set X, and showed that a groupoid $(X, \bullet) \in ZBin(X)$ if and only if it is a locally-zero groupoid.

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In this paper, we study mappings $f, g : X \to P$ where P is a poset and relations of the type $f \,||\, g$, i.e., f is right-(left-) parallel to g provided that $f(a) \le f(b)$ implies $g(a) \le g(b)$ as well, for all $a, b \in X$, where this condition applies. Since no assumptions about any order relation on X are made, this relation is a generalization of the special case $X = P$ and $f(x) = x$, where $a \le b$ implies $g(a) \le g(b)$, i.e., g is an order-preserving mapping. Even in this most general format it is possible to extract information concerning properties of $||$, i.e., $f \,||\, f$, and $f \,||\, g, g \,||\, h$ implies $f \,||\, h$ and the fact that $f \,||\, g$ does not imply $g \,||\, f$, to demonstrate the one-sided-ness of $f \,||\, g$. At the same time through the introduction of the groupoid structures $(X, *)$ as elements of $(Bin(X), \Box)$, the semigroup of binary systems (groupoids) on X, mappings f may acquire many different kinds of properties, such as $f(x * y) \le f(x)$ for all $x, y \in X$ (left shrinking), for example, which then implies $g(x * y) \le g(x)$ for all $x, y \in X$, so that this property is preserved by parallelism. If $P = [0, 1]$ with the usual order, then $f, g : X \to P$ yields the mappings f, g as fuzzy subsets of X and then the condition $f(x * y) \ge \min\{f(x), f(y)\}$ implies that if $f \,||\, g$, then $g(x * y) \ge \min\{g(x), g(y)\}$ as well, i.e., if f is a fuzzy subgroupoid of $(X, *)$ and $f \,||\, g$, then g is a fuzzy subgroupoid also. From these examples it should be clear that many other similar conclusions can be obtained in this setting, several of which we have provided in the following.

2. Preliminaries.

Let $(X, <)$ be a poset (partially ordered set), i.e., a set equipped with a relation $<$ where $x < y$ implies $y \not< x$ and $x < y, y < z$ implies $x < z$. The relation $\le$ as usual means $x = y$ or $x < y$. For details on the theory of posets we refer the reader to [3, 4]. In these texts further references are supplied as well.

Given a non-empty set X, we let $Bin(X)$ denote the collection of all groupoids $(X, *)$, where $* : X \times X \to X$ is a map and where $*(x, y)$ is written in the usual product form. Given elements $(X, *)$ and $(X, \bullet)$ of $Bin(X)$, define a product "$\Box$" on these groupoids as follows:

$$(X, *) \,\Box\, (X, \bullet) = (X, \Box)$$

where

$$x \,\Box\, y = (x * y) \bullet (y * x)$$

for any $x, y \in X$. Using that notion, H. S. Kim and J. Neggers proved the following theorem.

Theorem 2.1. ([2]) $(Bin(X), \Box)$ *is a semigroup, i.e., the operation "$\Box$" as defined in general is associative. Furthermore, the left- zero-semigroup is the identity for this operation.*

The notion of BCK/BCI-algebras was introduced by Y. Imai and K. Iséki. An algebra $(X, *, 0)$ of type $(2, 0)$ is called a BCI*-algebra* if for any $x, y, z \in X$, it satisfies the following conditions:

(I) $((x * y) * (x * z)) * (z * y) = 0$,
(II) $x * 0 = x$,
(III) $x * y = 0$ and $y * x = 0$ imply $x = y$.

A BCI-algebra $(X, *, 0)$ is said to be *p-semisimple* if $0 * (0 * x) = x$ for all $x \in X$.

Theorem 2.2. ([5]) *Let X be a BCI-algebra. Then the following are equivalent: for all $x, y, z \in X$,*

(1) *X is p-semisimple,*
(II) $(x * y) * (z * u) = (x * z) * (y * u)$,
(III) $0 * (x * y) = y * x$.

For further reference on BCK/BCI-algebras, we refer to [5].

3. Right(left)-parallelism.

Given a set X and a poset P, we shall consider the set P^X consisting of all functions $f : X \to P$, i.e.,

$$P^X := \{\varphi \mid \varphi : X \to P : \text{ a map}\}$$

If the order relation on P is denoted by $\leq$, then P^X has an induced order $f \leq g$, provided $f(x) \leq g(x)$ for all $x \in X$.

Let $f, g \in P^X$. A map g is said to be *right-parallel* to f if any $a, b \in X$ with $f(a) \leq f(b)$ it is true that $g(a) \leq g(b)$, and we denote this fact by $f \,||\, g$. In this case, f is said to be *left-parallel* to g.

The following hold for parallelism: for any $f, g, h \in P^X$,

(i) $f \,||\, f$,
(ii) if $f \,||\, g$ and $g \,||\, h$, then $f \,||\, h$

Thus, the relation of "right-parallelism" is both reflexive and transitive in all cases.

Remark. The relation "$||$" is not symmetric. If $P := \mathbf{R}$, the real numbers with the regular order, then $f(x) := x^3$ and $g(x) := \max\{0, x\}$ implies $f \,||\, g$, but not $g \,||\, f$, i.e., $a^3 \leq b^3$ implies $\max\{0, a\} \leq \max\{0, b\}$

but $a=-1, b=-2$ implies $\max\{0,-1\}=\max\{0,-2\}$ and $(-2)^3 < (-1)^3$.

Proposition 3.1. *Let $f(\in P^X)$ be a constant mapping. If g is right-parallel to f, then g is also a constant mapping.*

Proof. Given $a, b \in X$, since f is constant, we have $f(a) \leq f(b)$. It follows that $g(a) \leq g(b)$, since $f \,||\, g$. Similarly, $f(b) \leq f(a)$ implies $g(b) \leq g(a)$, proving the proposition. □

Proposition 3.2. *Any function f is left-parallel to a constant function $g : X \to P$.*

Proof. Straightforward. □

Proposition 3.3. *Let $f : P \to P$ be the identity function. If g is right-parallel to f, then $g : P \to P$ is monotonically increasing.*

Proof. Let $f : P \to P$ be an identity function and let g be right-parallel to f. Then, for any $a, b \in P$ with $a \leq b$, we have $f(a) \leq f(b)$. Since $f \,||\, g$, $g(a) \leq g(b)$. This proves that g is monotonically increasing. □

Proposition 3.4. *Let $f : P \to P$ be the identity function. If f is right-parallel to g, then $g(a) \leq g(b)$ implies $a \leq b$.*

Proof. Given $a, b \in P$, since f is right-parallel to g, $g(a) \leq g(b)$ implies $f(a) \leq f(b)$. Since f is an identity function, $g(a) \leq g(b)$ implies $a \leq b$, □

Proposition 3.5. *If g is right-parallel to f and $f(a) = f(b)$ for some $a, b \in X$, then $g(a) = g(b)$.*

Proof. $f(a) = f(b)$ implies $f(a) \leq f(b)$. Since g is right-parallel to f, we have $g(a) \leq g(b)$. Similarly, $f(b) \leq f(a)$ implies $g(b) \leq g(a)$, proving the proposition. □

Proposition 3.6. *Let $\varphi : P \to Q$ be an order-preserving mapping of posets satisfying the condition: for any $\alpha, \beta \in P$,*

$$\varphi(\alpha) \leq \varphi(\beta) \quad \textit{implies} \quad \alpha \leq \beta$$

If $f, g : X \to P$ are mappings with $f \,||\, g$, then $\varphi \circ f \,||\, \varphi \circ g$.

Proof. Assume that $\varphi(f(a)) \leq \varphi(f(b))$ for some $a, b \in X$. Then $f(a) \leq f(b)$. It follows from $f \,||\, g$ that $g(a) \,||\, g(b)$. Since φ is order-preserving, we obtain $\varphi(g(a)) \leq \varphi(g(b))$, proving the proposition. □

Proposition 3.7. *Let P, Q be posets, and let $f, k : X \to P$ be maps with $f \,||\, k$ and let $g, h : Y \to Q$ be maps with $g \,||\, h$. If we define $f \times g : X \times Y \to P \times Q$ by $(f \times g)(x, y) := (f(x), g(y))$ and*

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$k\times h : X\times Y \to P\times Q$ *by* $(k\times h)(x, y) := (k(x), h(y))$, *then* $f\times g \mid\mid k\times h$.

Proof. Suppose that $(f\times g)(a, b) \leq (f\times g)(a', b')$ for some $(a, b), (a', b') \in X \times Y$. Then $(f(a), g(b)) \leq (f(a'), g(b'))$ in $P \times Q$. It follows that $f(a) \leq f(a')$ and $g(b) \leq g(b')$. Since $f \mid\mid k$ and $g \mid\mid h$, we have $f \times g \mid\mid k \times h$. □

Proposition 3.8. *Let* $f, g : X \to P$ *be mappings with* $f \mid\mid g$. *If* f_A, g_A *are restrictions of* f *and* g *respectively, where* $A \subseteq X$, *then* $f_A \mid\mid g_A$.

4. Right-parallel-property.

Let $f : X \to P$ be a map. A property α is said to be a *right-parallel-property* for f if $f \mid\mid g$, then g also has the same property.

Proposition 4.1. *Constancy is a right-parallel-property.*

Proof. See Proposition 3.1. □

Let $X := \mathbf{R}$ be the set of all real numbers and let P be a poset. A map $f : X \to P$ is said to be *periodic of period* p if

$$f(x + p) = f(x)$$

for all $x \in X$.

Proposition 4.2. *Periodicity is a right-parallel-property.*

Proof. Assume that f is periodic of period p and $f \mid\mid g$. If $f(x+p) = f(x)$ for all $x \in X$, then $f(x + p) \leq f(x)$. Since $f \mid\mid g$, we have $g(x + p) \leq g(x)$. Similarly, $f(x) \leq f(x + p)$ implies $g(x) \leq g(x + p)$, proving that $g(x + p) = g(x)$ for all $x \in X$. □

Let $(X, *)$ be a groupoid. A map $f : X \to P$ is said to be a *rank subalgebra* of $(X, *)$ if for all $x, y \in X$,

$$f(x * y) \geq \min\{f(x), f(y)\}$$

In this case, $(X, *)$ is said to be a *rank-characteristic-groupoid* for the mapping $f : X \to P$. Note that $f(x)$ and $f(y)$ may not be "comparable" in a general poset. We need to also consider the case $f(x*y) \geq f(x)$ or $f(x * y) \geq f(y)$ for further investigation.

Proposition 4.3. *Rank-subalgebra is a right-parallel-property.*

Proof. Assume that f is a rank-subalgebra of a groupoid $(X, *)$ and $f \mid\mid g$. Then $f(x * y) \geq \min\{f(x), f(y)\}$ for all $x, y \in X$. Without loss of generality, we let $f(x * y) \geq f(x)$. Since $f \mid\mid g$, we have $g(x * y) \geq g(x)$, proving the proposition. □

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Proposition 4.4. *Let $(X,*)$ and $(X,\star)$ be rank-characteristic-groupoids for $f : X \to P$. If $(X,\Box) := (X,*)\Box(X,\star)$, then $(X,\Box)$ is also a rank-characteristic-groupoid for f.*

Proof. Since $(X,\Box) := (X,*)\Box(X,\star)$, $x\Box y = (x*y)\star(y*x)$ for all $x,y \in X$. It follows that

$$\begin{aligned} f(x\Box y) &= f((x*y)\star(y*x)) \\ &\geq \min\{f(x*y), f(y*x)\} \\ &\geq \min\{f(x), f(y)\}, \end{aligned}$$

showing that $(X,\Box)$ is also a rank-characteristic-groupoid for f. $\Box$

This shows that *the rank-characteristic-groupoids for $f : X \to P$ form a subsemigroup with respect to the product $\Box$ of the semigroup $(Bin(X),\Box)$.*

Proposition 4.5. *The left-zero-semigroup $(X,*)$ is a rank-characteristic-groupoid for any map $f : X \to P$.*

Proof. Given $x,y \in X$, we have

$$f(x*y) = f(x) \geq \min\{f(x), f(y)\},$$

for any map $f : X \to P$. $\Box$

A map $f : X \to P$ is said to be *strongly bounded above* if there exists an $x_1 \in X$ such that $f(x) \leq f(x_1)$ for all $x \in X$.

Proposition 4.6. *Strongly bounded above is a right-parallel-property.*

Proof. Let $f : X \to P$ be strongly bounded above and let $f \mid\mid g$. Then there exists an $x_1 \in X$ such that $f(x) \leq f(x_1)$ for all $x \in X$. Since $f \mid\mid g$, we obtain $g(x) \leq g(x_1)$ for all $x \in X$, proving that g is strongly bounded above. $\Box$

A map $f : X \to P$ is said to be *strongly bounded below* if there exists an $x_0 \in X$ such that $f(x_0) \leq f(x)$ for all $x \in X$.

Proposition 4.6′. *Strongly bounded below is a right-parallel-property.*

A map $f : X \to P$ is said to be *P-compact* if there exist $x_0, x_1 \in X$ such that $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in X$.

Proposition 4.7. *P-compact is a right-parallel-property.*

5. Left-shrinking.

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Given a groupoid $(X, *)$, a mapping $f : X \to P$ is said to be *left-shrinking* if, for all $x, y \in X$,

$$f(x * y) \le f(x)$$

Similarly, a mapping $f : X \to P$ is said to be *right-shrinking* if, for all $x, y \in X$, $f(x * y) \le f(y)$. Note that we need to consider the case $f(x * y) \le f(x)$ or $f(x * y) \le f(y)$ for all $x, y \in X$ for further investigation.

Proposition 5.1. *If $f : X \to P$ is left-shrinking and $f \mid\mid g$, then $g : X \to P$ is also left-shrinking.*

Proof. If $f : X \to P$ is left-shrinking, then $f(x * y) \le f(x)$ for all $x, y \in X$. Since $f \mid\mid g$, we have $g(x*y) \le g(x)$ for all $x, y \in X$, proving that g is left-shrinking. $\square$

Proposition 5.2. *Let $(X, *)$ and $(X, \star)$ be groupoids and let $(X, \Box) := (X, *)\Box(X, \star)$. If $f : X \to P$ is left-shrinking for $(X, *)$ and $(X, \star)$, then f is also left-shrinking for $(X, \Box)$.*

Proof. Given $x, y \in X$, we have

$$\begin{aligned} f(x\Box y) &= f((x * y) \star (y * x)) \\ &\le f(x * y) \\ &\le f(x), \end{aligned}$$

proving that f is left-shrinking for $(X, \Box)$. $\square$

Proposition 5.3. *Let $(P, \le), (Q, \le)$ be posets and let $(X, *), (Y, \bullet)$ be groupoids. Define a binary operation $\diamond$ on $X \times Y$ by*

$$(x, y) \diamond (x', y') := (x * x', y \bullet y').$$

If we define $f \times g$ as in Proposition 3.7 for any left-shrinking maps $f : X \to P$ and $g : Y \to Q$, then $f \times g$ is also left-shrinking for $(X \times Y, \diamond)$.

Proof. Given $(x, y), (x', y') \in X \times Y$, we have

$$\begin{aligned} (f \times g)((x, y) \diamond (x', y')) &= (f \times g)(x * x', y \bullet y')) \\ &= (f(x * x'), g(y \bullet y')) \\ &\le (f(x), g(y)) \\ &= (f \times g)(x, y), \end{aligned}$$

proving the proposition. $\square$

Proposition 5.4. *Let $(X, *)$ be a left-zero-semigroup. If $f : X \to P$ is right-shrinking, then f is a constant mapping.*

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Proof. Given $x, y \in X$, we have $f(x) = f(x*y) \leq f(y)$, proving that $f(x) = f(y)$ for all $x, y \in X$. □

Proposition 5.5. *Let $\varphi : (X, *) \to (Y, \bullet)$ be a homomorphism of groupoids and let $f : (Y, \bullet) \to P$ be left-shrinking. Then $f \circ \varphi : (X, *) \to P$ is also left-shrinking.*

Proof. For any $x, y \in X$, we have

$$\begin{aligned}(f \circ \varphi)(x * y) &= f(\varphi(x * y)) \\ &= f(\varphi(x) \bullet \varphi(y)) \\ &\leq f(\varphi(x)) \\ &= (f \circ \varphi)(x),\end{aligned}$$

proving that $f \circ \varphi : (X, *) \to P$ is also left-shrinking. □

Proposition 5.6. *Let $f : (X, *) \to P$ be left-shrinking. If $\varphi : P \to Q$ is order-preserving, then $\varphi \circ f : (X, *) \to Q$ is left-shrinking.*

Proof. If $f : (X, *) \to P$ is left-shrinking, then $f(x * y) \leq f(x)$ for all $x, y \in X$. It follows from φ is order-preserving that $\varphi(f(x * y)) \leq \varphi(f(x))$ for all $x, y \in X$, proving the proposition. □

Proposition 5.7. *Let $\varphi : (X, *) \to (Y, \bullet)$ be a homomorphism of groupoids and let $f, g : (Y, \bullet) \to P$ be mappings. If f is left-parallel to g, then $f \circ \varphi$ is left-parallel to $g \circ \varphi$.*

Proof. Let $a, b \in X$ such that $(f \circ \varphi)(a) \leq (f \circ \varphi)(b)$. Then $f(\varphi(a)) \leq f(\varphi(b))$. Since $f \,||\, g$, we have $g(\varphi(a)) \leq g(\varphi(b))$, proving the proposition. □

Given a groupoid $(X, *)$, a mapping $f : X \to P$ is said to be *left-expanding* (resp., *right-expanding*) if, for all $x, y \in X$, $f(x * y) \geq f(y)$ (resp., $f(x * y) \geq f(x)$). Note that f is expanding if and only if f is both left-expanding and right-expanding.

6. $(X, *)[f](X, \bullet)$.

Let $(X, *), (X, \bullet) \in Bin(X)$. Given a map $f : (X, *) \to P$, we define a relation $(X, *)[f](X, \bullet)$ if $f(x * y) \leq f(x \bullet y)$ for all $x, y \in X$.

Proposition 6.1. *If $(X, *)[f](X, \bullet)$ and $(X, \bullet)[f](X, *)$, then $f(x * y) = f(x \bullet y)$ for all $x, y \in X$.*

In fact, $f(x) \neq f(y)$ is possible.

Proposition 6.2. *Let $(X, *)[f](X, \bullet)$. If f is left-shrinking for $(X, \bullet)$, then f is also left-shrinking for $(X, *)$.*

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Proof. For any $x, y \in X$, we have

$$f(x * y) \leq f(x \bullet y) \leq f(x)$$

$\square$

Proposition 6.3. *Let $(X,*)[f](X,\bullet)$. If $f \,||\, g$, then $(X,*)[g](X,\bullet)$.*

Proof. Let $(X,*)[f](X,\bullet)$. Then $f(x*y) \leq f(x \bullet y)$ for all $x, y \in X$. Since $f \,||\, g$, we obtain $g(x*y) \leq g(x \bullet y)$, proving the proposition. $\square$

Theorem 6.4. *If $(X,\bullet)[f](X,\diamond)$, then*

$$(X,*)\square(X,\bullet)[f](X,*)\square(X,\diamond)$$

for all $(X,*) \in Bin(X)$.

Proof. Let $(X,\square_1) := (X,*)\square(X,\bullet)$, i.e., $x\square_1 y = (x*y)\bullet(y*x)$ for all $x, y \in X$, and let $(X,\square_2) := (X,*)\square(X,\diamond)$, i.e., $x\square_2 y = (x*y)\diamond(y*x)$ for all $x, y \in X$. If $(X,\bullet)[f](X,\diamond)$, then $f(x \bullet y) \leq f(x \diamond y)$ for all $x, y \in X$. It follows that

$$\begin{aligned} g(x\square_1 y) &= f((x*y)\bullet(y*x)) \\ &\leq f((x*y)\diamond(y*x)) \\ &= f(x\square_2 y), \end{aligned}$$

proving that $(X,*)\square(X,\bullet)[f](X,*)\square(X,\diamond)$. $\square$

7. Groupoids parallelism.

Let $(X,*), (X,\bullet) \in Bin(X)$ and let $(X,\leq)$ be a poset. A groupoid $(X,\bullet)$ is said to be *right parallel* to a groupoid $(X,*)$ with respect to the poset $(X,\leq)$ if $*(a,b) \leq *(a',b')$ implies $\bullet(a,b) \leq \bullet(a',b')$, i.e., $a*b \leq a'*b'$ implies $a \bullet b \leq a' \bullet b'$. We denote it by $(X,*) \,||\, (X,\bullet)$. Note that $(X,*) \,||\, (X,\bullet)$ and $(X,\bullet) \,||\, (X,\triangledown)$ implies $(X,*) \,||\, (X,\triangledown)$.

Example 7.1. Let $(X,\bullet)$ be a trivial groupoid, i.e., $x\bullet y = t$ for some $t \in X$, for all $x, y \in X$. Then $(X,*) \,||\, (X,\bullet)$ for all $(X,*) \in Bin(X)$. In fact, if $a*b \leq a'*b'$, then $a \bullet b = t \leq t = a' \bullet b'$.

A groupoid $(X,*)$ is said to be $\leq$*-commutative* if $a*b \leq a'*b'$ then $b*a \leq b'*a'$. Clearly, if $(X,*)$ is commutative, i.e., $x*y = y*x$ for all $x, y \in X$, then it is $\leq$-commutative. A groupoid $(X,*)$ is said to be *strictly* $\leq$*-commutative* if $a*b = a'*b'$ then $b*a = b'*a'$.

Example 7.2. Let $\leq$ be the diagonal relation, i.e., $x \leq y$ if and only if $x = y$, for all $x, y \in X$. If a groupoid $(X,*)$ is strictly $\leq$-commutative, then it is $\leq$-commutative.

A groupoid $(X, *)$ is said to be $\leq$-*ordering* if $x \leq x'$ and $y \leq y'$ implies $x * y \leq x' * y'$ (and $y * x \leq y' * x'$ also). Let $\leq$ be the diagonal relation on X. Then $x \leq x', y \leq y'$ means $x = x', y = y'$ and thus for any groupoid $(X, \bullet)$ whatsoever we have $x \bullet y = x' \bullet y'$ and $x \bullet y \leq x' \bullet y'$, whence $(X, \bullet)$ is $\leq$-ordering.

Example 7.3. Let $(X, *, 0)$ be a p-semisimple BCI-algebra (or a medial groupoid). Then $(X, *)$ is $\leq$-ordering. In fact, if $x \leq x'$ and $y \leq y'$, then $x * x' = 0 = y * y'$. It follows that $(x * y) * (x' * y') = (x * x') * (y * y') = 0$, proving that $x * y \leq x' * y'$.

Proposition 7.4. *Let $(X, *)$ be a $\leq$-ordering groupoid. If $(X, *) \,||\, (X, \bullet)$, then $(X, \bullet)$ is also $\leq$-ordering.*

Proof. Let $x \leq x'$ and $y \leq y'$. Since $(X, *)$ is $\leq$-ordering, we have $x * y \leq x' * y'$. It follows that $x \bullet y \leq x' \bullet y'$, since $(X, *) \,||\, (X, \bullet)$. $\square$

Proposition 7.5. *Let $(X, *), (X, \bullet)$ be $\leq$-ordering groupoids. If $(X, \Box) = (X, *)\Box(X, \bullet)$, then $(X, \Box)$ is also $\leq$-ordering.*

Proof. Let $x \leq x'$ and $y \leq y'$. Then $x * y \leq x' * y'$, $y * x \leq y' * x'$, since $(X, *)$ is $\leq$-ordering. Since $(X, \bullet)$ is $\leq$-ordering, we obtain $(x * y) \bullet (y * x) \leq (x' * y') \bullet (y' * x')$, i.e., $x \Box y \leq y \Box x$. This proves that $(X, \Box)$ is also $\leq$-ordering. $\square$

Theorem 7.6. *Let $(X, *)$ be a $\leq$-commutative groupoid and let $(X, \bullet)$ be a $\leq$-ordering groupoid. If $(X, \Box) = (X, *)\Box(X, \bullet)$, then $(X, *) \,||\, (X, \Box)$.*

Proof. If $a * b \leq a' * b'$, then $b * a \leq b' * a'$, since $(X, *)$ is $\leq$-commutative. Since $(X, \bullet)$ is a $\leq$-ordering groupoid, we obtain $(a * b) \bullet (b * a) \leq (a' * b') \bullet (b' * a')$, i.e., $a \Box b \leq a' \Box b'$, proving the theorem. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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Existence results for nonlinear generalized three-point boundary value problems for fractional differential equations and inclusions

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Abstract

This paper studies boundary value problems of nonlinear fractional differential equations and inclusions, of order $q \in (1, 2]$ with generalized three-point boundary conditions. Some existence and uniqueness results are obtained by using a variety of fixed point theorems. Some illustrative examples are also discussed.

Key words and phrases: Fractional differential equations; Fractional differential inclusions; three-point generalized boundary conditions; existence; contraction principle; Krasnoselskii's fixed point theorem; Leray-Schauder degree.
AMS (MOS) Subject Classifications: 26A33, 34A12, 34B15, 34A60.

1 Introduction

Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes, see [27]. These characteristics of the fractional derivatives make the fractional-order models more realistic and practical than the classical integer-order models. As a matter of fact, fractional differential equations arise in many engineering and scientific disciplines such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc. [22, 27, 28, 29]. In recent years, there are many papers dealing with the existence of solutions to various fractional differential equations. For some recent development on the topic, see [1]-[13] and the references therein.

Recently, the existence of positive solutions was studied for generalized second order three-point boundary value problems for equations or systems, see [14], [20], [21], [26], and the references cited therein.

Here, in the first part of this paper, we discuss the existence and uniqueness of solutions for a boundary value problem of nonlinear fractional differential equations of order $q \in (1, 2]$ with generalized three-point boundary conditions given by

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), \quad 0 < t < 1, \quad 1 < q \leq 2, \\ x(0) = \beta x(\eta), \quad x(1) = \alpha x(\eta), \end{cases} \tag{1.1}$$

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where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is continuous, and α, β, η are constants with $0 < \eta < 1$ and $1 - \beta + (\beta - \alpha)\eta \neq 0$. Here, $\mathcal{C} = C([0,1],\mathbb{R})$ denotes the Banach space of all continuous functions from $[0,1] \to \mathbb{R}$ endowed with a topology of uniform convergence with the norm denoted by $\|\cdot\|$.

Some new existence and uniqueness results are proved for the boundary value problem (1.1), by using a variety of fixed point theorems. Thus, in Theorem 3.1 we prove an existence and uniqueness result by using Banach's contraction principle, in Theorem 3.3 we prove the existence of a solution by using Krasnoselskii's fixed point theorem, while in Theorem 3.6 we prove the existence of a solution via Leray-Schauder nonlinear alternative. In Theorem 3.9 we prove an existence and uniqueness result by using a fixed point theorem of Boyd and Wong [15] for nonlinear contractions. Some illustrative examples are also discussed.

In the second part of this paper, we study the following generalized three-point boundary value problem for fractional differential inclusions

$$\begin{cases} {}^cD^q x(t) \in F(t, x(t)), \quad 0 < t < 1, \quad 1 < q \le 2, \\ x(0) = \beta x(\eta), \quad x(1) = \alpha x(\eta), \end{cases} \tag{1.2}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all non-empty subsets of $\mathbb{R}$, and α, β, η are as in problem (1.1).

For the problem (1.2), the aim here is to establish existence results when the right hand side is convex as well as nonconvex valued. In the first result, Theorem 4.8, we prove the existence of solutions for the problem (1.2), when the right hand side has convex values, via Leray-Schauder nonlinear alternative for Kakutani maps and F satisfying a Carathéodory condition. In the second result, Tleorem 4.16, we shall combine the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values. Finally, in the third result, Theorem 4.20, we shall use the fixed point theorem for contraction multivalued maps due to Covitz and Nadler.

It is worth mentioning that, the methods used are standard, however their exposition in the framework of problems (1.1) and (1.2) is new.

2 Preliminaries

Let us recall some basic definitions of fractional calculus [22, 29].

Definition 2.1 *For a continuous function $g : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as*

$${}^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s)ds, \quad n-1 < q < n, n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2 *The Riemann-Liouville fractional integral of order q is defined as*

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \quad q > 0,$$

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provided the integral exists.

Definition 2.3 *The Riemann-Liouville fractional derivative of order q for a continuous function $g(t)$ is defined by*

$$D^q g(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{g(s)}{(t-s)^{q-n+1}} ds, \quad n = [q]+1,$$

provided the right hand side is pointwise defined on $(0,\infty)$.

Lemma 2.4 *For a given $g \in C([0,1],\mathbb{R})$ the unique solution of the boundary value problem*

$$\begin{cases} {}^cD^q x(t) = g(t), \quad 0 < t < 1, \quad 1 < q \le 2, \\ x(0) = \beta x(\eta), \quad x(1) = \alpha x(\eta), \end{cases}$$

is given by

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s) ds + \frac{(\beta-1)t - \beta\eta}{\Delta\Gamma(q)} \int_0^1 (1-s)^{q-1} g(s) ds \\ &+ \frac{\beta + (\alpha-\beta)t}{\Delta\Gamma(q)} \int_0^\eta (\eta-s)^{q-1} g(s) ds, \quad 0 \le t \le 1, \end{aligned} \tag{2.1}$$

where $\Delta = 1 - \beta + (\beta - \alpha)\eta \neq 0$.

Proof. For some constants $c_0, c_1 \in \mathbb{R}$, we have

$$x(t) = I^q g(t) - c_0 - c_1 t = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} g(s) ds - c_0 - c_1 t. \tag{2.2}$$

We have $x(0) = -c_0$, $x(\eta) = \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} g(s) ds - c_0 - c_1\eta$ and thus from the first boundary condition we have

$$(\beta-1)c_0 + \beta\eta c_1 = \beta \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} g(s) ds. \tag{2.3}$$

Also from the second boundary condition we get

$$(\alpha-1)c_0 + (\alpha\eta-1)c_1 = \alpha \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} g(s) ds - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} g(s) ds. \tag{2.4}$$

From (2.3), (2.4) we find c_0, c_1 and substituting in (2.2) we obtain the solution (2.1). □

3 Existence results-Differential Equations

In view of Lemma 2.4, we define an operator $F : \mathcal{C} \to \mathcal{C}, \mathcal{C} = C([0,1],\mathbb{R})$ by

$$\begin{aligned} (Fx)(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(t, x(s)) ds \\ &+ \frac{(\beta-1)t - \beta\eta}{\Delta\Gamma(q)} \int_0^1 (1-s)^{q-1} f(t, x(s)) ds \\ &+ \frac{\beta + (\alpha-\beta)t}{\Delta\Gamma(q)} \int_0^\eta (\eta-s)^{q-1} f(t, x(s)) ds, \quad 0 \le t \le 1, \end{aligned} \tag{3.1}$$

For convenience, let us set

$$\lambda_1 = \frac{1}{|\Delta|}\sup_{t\in[0,1]}|(\beta-1)t-\beta\eta|, \quad \lambda_2 = \frac{1}{|\Delta|}\sup_{t\in[0,1]}|\beta+(\alpha-\beta)t|$$

and

$$\Lambda = \frac{1}{\Gamma(q+1)}(1+\lambda_1+\lambda_2\eta^q). \tag{3.2}$$

3.1 Existence result via Banach's fixed point theorem

Theorem 3.1 *Assume that $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function and satisfies the assumption*

(A_1) $|f(t,x)-f(t,y)|\le L|x-y|, \forall t\in[0,1],\ L>0,\ x,y\in\mathbb{R}$

with $L<1/\Lambda$, where Λ is given by (3.2). Then the boundary value problem (1.1) has a unique solution.

Proof. Setting $\sup_{t\in[0,1]}|f(t,0)|=M$ and choosing $\rho\ge\dfrac{\Lambda M}{1-L\Lambda}$, we show that $FB_\rho\subset B_\rho$, where $B_\rho=\{x\in\mathcal{C}:\|x\|\le\rho\}$ and F defined in (3.1). For $x\in B_\rho$, we have

$$\begin{aligned}
\|(Fx)(t)\| &\le \sup_{t\in[0,1]}\left\{\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}|f(s,x(s))|ds+\frac{\lambda_1}{\Gamma(q)}\int_0^1(1-s)^{q-1}f(s,x(s))|ds\right.\\
&\quad\left.+\frac{\lambda_2}{\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}|f(s,x(s))|ds\right\}\\
&\le \sup_{t\in[0,1]}\left\{\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}(|f(s,x(s))-f(s,0)|+|f(s,0)|)\ \|ds\right.\\
&\quad+\frac{\lambda_1}{\Gamma(q)}\int_0^1(1-s)^{q-1}(|f(s,x(s))-f(s,0)|+|f(s,0)|)ds\\
&\quad\left.+\frac{\lambda_2}{\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}(|f(s,x(s))-f(s,0)|+|f(s,0)|)ds\right\}\\
&\le (L\rho+M)\sup_{t\in[0,1]}\left\{\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}ds+\frac{\lambda_1}{\Gamma(q)}\int_0^1(1-s)^{q-1}ds\right.\\
&\quad\left.+\frac{\lambda_2}{\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}ds\right\}\\
&\le \frac{(L\rho+M)}{\Gamma(q+1)}(1+\lambda_1+\lambda_2\eta^q)=(L\rho+M)\,\Lambda\le\rho.
\end{aligned}$$

Now, for $x,y\in\mathcal{C}$ and for each $t\in[0,1]$, we obtain

$$\begin{aligned}
\|(Fx)(t)-(Fy)(t)\| &\le \sup_{t\in[0,1]}\left\{\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}|f(t,x(s))-f(s,y(s))|ds\right.\\
&\quad+\frac{\lambda_1}{\Gamma(q)}\int_0^1(1-s)^{q-1}|f(t,x(s))-f(s,y(s))|ds
\end{aligned}$$

$$
\begin{aligned}
& +\frac{\lambda_2}{\Gamma(q)}\int_0^\eta (\eta-s)^{q-1}|f(t,x(s))-f(s,y(s))|ds\Big\} \\
\leq\ & L\|x-y\| \sup_{t\in[0,1]}\left\{\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds+\frac{\lambda_1}{\Gamma(q)}\int_0^1 (1-s)^{q-1}ds\right. \\
& \left.+\frac{\lambda_2}{\Gamma(q)}\int_0^\eta (\eta-s)^{q-1}ds\right\} \\
\leq\ & \frac{L}{\Gamma(q+1)}\left(1+\lambda_1+\lambda_1\eta^q\right)\|x-y\| = L\Lambda\|x-y\|,
\end{aligned}
$$

where Λ is given by (3.2). Observe that Λ depends only on the parameters involved in the problem. As $L<1/\Lambda$, therefore F is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). □

Example 3.2 *Consider the following generalized three-point fractional boundary value problem*

$$
\begin{cases}
{}^cD^{1/2}x(t)=\dfrac{1}{(t+9)^2}\dfrac{|x|}{1+|x|}, \quad t\in[0,1], \\
x(0)=\dfrac{1}{2}x\left(\dfrac{1}{4}\right), \ x(1)=2x\left(\dfrac{1}{4}\right).
\end{cases}
\tag{3.3}
$$

Here, $q=3/2,\ \beta=1/2,\ \alpha=2,\ \eta=1/4$, and $f(t,x)=\dfrac{1}{(t+9)^2}\dfrac{|x|}{1+|x|}$. We find $\Delta=\dfrac{1}{8},\lambda_1=5,\lambda_2=16$ and $\Lambda=\dfrac{32}{3\sqrt{\pi}}$. As $|f(t,x)-f(t,y)|\leq\dfrac{1}{81}|x-y|$, therefore, (A_1) is satisfied with $L=\dfrac{1}{81}$. Further, $L\Lambda=\dfrac{32}{243\sqrt{\pi}}<1$. Thus, by the conclusion of Theorem 3.1, the boundary value problem (3.3) has a unique solution on $[0,1]$.

3.2 Existence result via Krasnoselskii's fixed point theorem

Theorem 3.3 *(Krasnoselskii's fixed point theorem)[24]. Let M be a closed, bounded, convex and nonempty subset of a Banach space X. Let A,B be the operators such that: (i) $Ax+By\in M$ whenever $x,y\in M$; (ii) A is compact and continuous; (iii) B is a contraction mapping. Then there exists $z\in M$ such that $z=Az+Bz$.*

Theorem 3.4 *Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ be a continuous function and the assumption (A_1) holds. In addition we assume that*

(A_2) $|f(t,x)|\leq\mu(t),\quad \forall(t,x)\in[0,1]\times\mathbb{R},$ *and* $\mu\in C([0,1],\mathbb{R}^+)$.

If

$$
\frac{L}{\Gamma(q+1)}(\lambda_1+\lambda_2\eta^q)<1,
\tag{3.4}
$$

then the boundary value problem (1.1) has at least one solution on $[0,1]$.

Proof. Letting $\sup_{t\in[0,1]}|\mu(t)| = \|\mu\|$, we fix

$$\overline{r} \geq \frac{\|\mu\|}{\Gamma(q+1)}(1+\lambda_1+\lambda_2\eta^q),$$

and consider $B_{\overline{r}} = \{x \in \mathcal{C} : \|x\| \leq \overline{r}\}$. We define the operators $\mathcal{P}$ and $\mathcal{Q}$ on $B_{\overline{r}}$ as

$$\begin{aligned}(\mathcal{P}x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds, \quad 0 \leq t \leq 1,\\ (\mathcal{Q}x)(t) &= \frac{(\beta-1)t-\beta\eta}{\Delta\Gamma(q)}\int_0^1 (1-s)^{q-1} f(t,x(s))ds\\ &+\frac{\beta+(\alpha-\beta)t}{\Delta\Gamma(q)}\int_0^\eta (\eta-s)^{q-1} f(t,x(s))ds, \quad 0 \leq t \leq 1.\end{aligned}$$

For $x, y \in B_{\overline{r}}$, we find that $\|\mathcal{P}x + \mathcal{Q}y\| \leq \dfrac{\|\mu\|}{\Gamma(q+1)}(1+\lambda_1+\lambda_2\eta^q) \leq \overline{r}$.

Thus, $\mathcal{P}x + \mathcal{Q}y \in B_{\overline{r}}$. It follows from the assumption (A_1) together with (3.4) that $\mathcal{Q}$ is a contraction mapping. Continuity of f implies that the operator $\mathcal{P}$ is continuous. Also, $\mathcal{P}$ is uniformly bounded on $B_{\overline{r}}$ as $\|\mathcal{P}x\| \leq \dfrac{\|\mu\|}{\Gamma(q+1)}$. Now we prove the compactness of the operator $\mathcal{P}$.

In view of (A_1), we define $\sup_{(t,x)\in[0,1]\times B_{\overline{r}}}|f(t,x)| = \overline{f}$, and consequently we have

$$\begin{aligned}\|(\mathcal{P}x)(t_1) - (\mathcal{P}x)(t_2)\| &= \left\|\frac{1}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1} - (t_1-s)^{q-1}]f(t,x(s))ds\right.\\ &\left.+\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}f(t,x(s))ds\right\|\\ &\leq \frac{\overline{f}}{\Gamma(q+1)}|2(t_2-t_1)^q + t_1^q - t_2^q|,\end{aligned}$$

which is independent of x. Thus, $\mathcal{P}$ is equicontinuous. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{P}$ is compact on $B_{\overline{r}}$. Thus all the assumptions of Theorem 3.3 are satisfied. So the conclusion of Theorem 3.3 implies that the boundary value problem (1.1) has at least one solution on $[0,1]$. □

3.3 Existence result via Leray-Schauder Alternative

Theorem 3.5 *(Nonlinear alternative for single valued maps)[19]. Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either (i) F has a fixed point in $\overline{U}$, or (ii)]there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda F(u)$.*

Theorem 3.6 *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that:*

(A_3) *There exist a function $p \in C([0,1], \mathbb{R}^+)$, and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ nondecreasing such that* $|f(t,x)| \leq p(t)\psi(\|x\|), \quad \forall (t,x) \in [0,1] \times \mathbb{R}$;

(A_4) *There exists a constant* $M > 0$ *such that*

$$\frac{M}{\dfrac{\|p\|\psi(M)}{\Gamma(q+1)}\left\{1+\lambda_1+\lambda_2\eta^q\right\}} > 1.$$

Then the boundary value problem (1.1) has at least one solution on $[0,1]$.

Proof. Consider the operator $F:\mathcal{C}\to\mathcal{C}$ defined by (3.1).

We show that F maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$. For a positive number ρ, let $B_\rho = \{x\in C([0,1],\mathbb{R}) : \|x\|\le\rho\}$ be a bounded set in $C([0,1],\mathbb{R})$. Then,

$$\begin{aligned}
|(Fx)(t)| &\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(t,x(s))|ds + \frac{\lambda_1}{\Gamma(q)}\int_0^1 (1-s)^{q-1}|f(t,x(s))|ds \\
&\quad + \frac{\lambda_2}{\Gamma(q)}\int_0^\eta (\eta-s)^{q-1}|f(t,x(s))|ds \\
&\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}p(s)\psi(\|x\|)\,ds + \frac{\lambda_1}{\Gamma(q)}\int_0^1 (1-s)^{q-1}p(s)\psi(\|x\|)\,ds \\
&\quad + \frac{\lambda_2}{\Gamma(q)}\int_0^\eta (\eta-s)^{q-1}p(s)\psi(\|x\|)\,ds \\
&\le \frac{\|p\|\psi(\|x\|)}{\Gamma(q+1)}\left\{1+\lambda_1+\lambda_2\eta^q\right\}.
\end{aligned}$$

Hence

$$\|Fx\| \le \frac{\|p\|\psi(\rho)}{\Gamma(q+1)}\left\{1+\lambda_1+\lambda_2\eta^q\right\}.$$

Next we show that F maps bounded sets into equicontinuous sets of $C([0,1],\mathbb{R})$. Let $t', t''\in[0,1]$ with $t'<t''$ and $x\in B_\rho$, where B_ρ is a bounded set of $C([0,1],\mathbb{R})$. Then we have

$$\begin{aligned}
|(Fx)(t'')-(Fx)(t')| &= \left|\frac{1}{\Gamma(q)}\int_0^{t''}(t''-s)^{q-1}f(t,x(s))ds - \frac{1}{\Gamma(q)}\int_0^{t'}(t'-s)^{q-1}f(t,x(s))ds\right| \\
&\quad + \frac{|(\beta-1)||t''-t'|}{\Delta\Gamma(q)}\int_0^1(1-s)^{q-1}f(t,x(s))ds \\
&\quad + \frac{|\alpha-\beta||t''-t'|}{\Delta\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}f(t,x(s))ds \\
&\le \frac{\|p\|\psi(\rho)}{\Gamma(q)}\int_0^{t'}|(t''-s)^{q-1}-(t'-s)^{q-1}|ds + \frac{\|p\|\psi(\rho)}{\Gamma(q)}\int_{t'}^{t''}(t''-s)^{q-1}ds \\
&\quad + \frac{\|p\|\psi(\rho)|(\beta-1)||t''-t'|}{\Delta\Gamma(q+1)} + \frac{\|p\|\psi(\rho)|\alpha-\beta||t''-t'|}{\Delta\Gamma(q+1)}.
\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x\in B_\rho$ as $t''-t'\to 0$. As F satisfies the above assumptions, therefore it follows by the Arzelá-Ascoli theorem that $F: C([0,1],\mathbb{R})\to C([0,1],\mathbb{R})$ is completely continuous.

Let x be a solution. Then for $t \in [0,1]$, and using the computations in proving that F is bounded, we have

$$\begin{aligned} |x(t)| &= |\lambda(Fx)(t)| \\ &\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(t,x(s))|ds + \frac{\lambda_1}{\Gamma(q)}\int_0^1 (1-s)^{q-1}|f(t,x(s))|ds \\ &\quad + \frac{\lambda_2}{\Gamma(q)}\int_0^\eta (\eta-s)^{q-1}|f(t,x(s))|ds \\ &\leq \frac{\|p\|\psi(\|x\|)}{\Gamma(q+1)}\{1+\lambda_1+\lambda_2\eta^q\} \end{aligned}$$

and consequently

$$\frac{\|x\|}{\dfrac{\|p\|\psi(\|x\|)}{\Gamma(q+1)}\{1+\lambda_1+\lambda_2\eta^q\}} \leq 1.$$

In view of (A_4), there exists M such that $\|x\| \neq M$. Let us set

$$U = \{x \in C([0,1],\mathbb{R}) : \|x\| < M+1\}.$$

Note that the operator $F : \overline{U} \to C([0,1],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \lambda Fx$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Theorem 3.5), we deduce that F has a fixed point $x \in \overline{U}$ which is a solution of the problem (1.1). This completes the proof. □

3.4 Existence result via nonlinear contractions

Definition 3.7 *Let E be a Banach space and let $F : E \to E$ be a mapping. F is said to be a nonlinear contraction if there exists a continuous nondecrasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Psi(0) = 0$ and $\Psi(\xi) < \xi$ for all $\xi > 0$ with the property: $\|Fx - Fy\| \leq \Psi(\|x-y\|), \quad \forall x, y \in E$.*

Lemma 3.8 *(Boyd and Wong)[15]. Let E be a Banach space and let $F : E \to E$ be a nonlinear contraction. Then F has a unique fixed point in E.*

Theorem 3.9 *Assume that:*

(A_5) $|f(t,x) - f(t,y)| \leq h(t)\dfrac{|x-y|}{H^* + |x-y|}$, $\quad t \in [0,1],\ x, y \geq 0$, *where* $h : [0,1] \to \mathbb{R}^+$ *is continuous and*

$$H^* = \frac{1}{\Gamma(q)}\int_0^1 (1-s)^{q-1}h(s)ds + \frac{\lambda_1}{\Gamma(q)}\int_0^1 (1-s)^{q-1}h(s)ds + \frac{\lambda_2}{\Gamma(q)}\int_0^\eta (\eta-s)^{q-1}h(s)ds.$$

Then the boundary value problem (1.1) has a unique solution.

Proof. Consider the operator $F : \mathcal{C} \to \mathcal{C}$ given by (3.1). Let the continuous nondecreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\Psi(0) = 0$ and $\Psi(\xi) < \xi$ for all $\xi > 0$ defined by $\Psi(\xi) =$

$\dfrac{H^*\xi}{H^*+\xi}$, $\forall\xi \geq 0$. Let $x, y \in C([0,1],\mathbb{R})$. Then $|f(s,x(s)) - f(s,y(s))| \leq \dfrac{h(s)}{H^*}\Psi(\|x-y\|)$ so that

$$\begin{aligned}
|Fx(t) - Fy(t)| \quad &\leq \quad \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)\frac{|x(s)-y(s)|}{H^*+|x(s)-y(s)|}ds \\
&+\frac{\lambda_1}{\Gamma(q)}\int_0^1 (1-s)^{q-1}h(s)\frac{|x(s)-y(s)|}{H^*+|x(s)-y(s)|}ds \\
&+\frac{\lambda_2}{\Gamma(q)}\int_0^\eta (\eta-s)^{q-1}h(s)\frac{|x(s)-y(s)|}{H^*+|x(s)-y(s)|}ds,
\end{aligned}$$

for $t \in [0,1]$. Then $\|Fx - Fy\| \leq \Psi(\|x-y\|)$ and F is a nonlinear contraction and it has a unique fixed poitn in $C([0,1],\mathbb{R})$, by Lemma 3.8. □

Example 3.10 *Let us consider the boundary value problem*

$$\begin{cases} {}^cD^{3/2}x(t) = \dfrac{t|x|}{1+|x|}, \quad 0 < t < 1, \\ x(0) = \dfrac{1}{2}x\left(\dfrac{1}{4}\right), \ x(1) = 2x\left(\dfrac{1}{4}\right). \end{cases} \tag{3.5}$$

Here, $q = 3/2, \beta = 1/2, \alpha = 2, \eta = 1/4$ and $f(t,x) = \dfrac{t|x|}{1+|x|}$. We choose $h(t) = 1+t$ and find that $H^* = 7.97$. Clearly $\Big|f(t,x) - f(t,y)\Big| = \Big|\dfrac{t(|x|-|y|)}{1+|x|+|y|+|x||y|}\Big| \leq \dfrac{(1+t)|x-y|}{7.97+|x-y|}$. Thus, the conclusion of Theorem 3.9 applies and problem (3.5) has a unique solution.

4 Existence results-Differential Inclusions

Definition 4.1 *A function $x \in C^2([0,1],\mathbb{R})$ is a solution of the problem (1.2) if $x(0) = \beta x(\eta)$, $x(1) = \alpha x(\eta)$, and exists a function $f \in L^1([0,1],\mathbb{R})$ such that $f(t) \in F(t,x(t))$ a.e. on $[0,1]$ and*

$$\begin{aligned}
x(t) \quad &= \quad \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(s)ds + \frac{(\beta-1)t-\beta\eta}{\Delta\Gamma(q)}\int_0^1 (1-s)^{q-1}f(s)ds \\
&+\frac{\beta+(\alpha-\beta)t}{\Delta\Gamma(q)}\int_0^\eta (\eta-s)^{q-1}f(s)ds.
\end{aligned}$$

4.1 The Carathéodory case

In this subsection, we are concerned with the existence of solutions for the problem (1.2) when the right hand side has convex values. We first recall some preliminary facts.

For a normed space $(X, \|\cdot\|)$, let $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$.

Definition 4.2 *A multi-valued map $G : X \to \mathcal{P}(X)$:*

(i) is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$;

(ii) is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}}G(x)$ is bounded in X for all $\mathbb{B} \in P_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$);

(iii) is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$;

(iv) is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in P_b(X)$;

(v) has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$.

Remark 4.3 *It is known that, if the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*,\ y_n \to y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$.*

Definition 4.4 *A multivalued map $G : [0; 1] \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function*

$$t \longmapsto d(y, G(t)) = \inf\{\|y - z\| : z \in G(t)\}$$

is measurable.

Definition 4.5 *A multivalued map $F : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if*

(i) *$t \longmapsto F(t, x)$ is measurable for each $x \in X$;*

(ii) *$x \longmapsto F(t, x)$ is upper semicontinuous for almost all $t \in [0, 1]$;*

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) *for each $\alpha > 0$, there exists $\varphi_\alpha \in L^1([0, 1], \mathbb{R}^+)$ such that*

$$\|F(t, x)\| = \sup\{|v| : v \in F(t, x)\} \leq \varphi_\alpha(t)$$

for all $\|x\| \leq \alpha$ and for a. e. $t \in [0, 1]$.

For each $y \in C([0, 1], \mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([0, 1], \mathbb{R}) : v(t) \in F(t, y(t)) \text{ for a.e. } t \in [0, 1]\}.$$

The consideration of this subsection is based on the following fixed point theorem ([19]).

Theorem 4.6 *(Nonlinear alternative for Kakutani maps).[19]. Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to \mathcal{P}_{cp,c}(C)$ is a upper semicontinuous compact map. Then either (i) F has a fixed point in $\overline{U}$, or (ii) there is a $u \in \partial U$ and $\lambda \in (0, 1)$ with $u \in \lambda F(u)$.*

The following lemma will be used in the sequel.

Lemma 4.7 *([25]) Let X be a Banach space. Let $F : [0,T] \times X \to \mathcal{P}_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0,1],X)$ to $C([0,1],X)$. Then the operator*

$$\Theta \circ S_F : C([0,1],X) \to P_{cp,c}(C([0,1],X)), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C([0,1],X) \times C([0,1],X)$.

Theorem 4.8 *Assume that (A_4) holds. In addition we suppose that the following conditions*

(H_1) *$F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is Carathéodory and has nonempty compact convex values;*

(H_2) *there exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in C([0,1],\mathbb{R}^+)$ such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(\|x\|) \quad \textit{for each } (t,x) \in [0,1] \times \mathbb{R},$$

are satisfied. Then the boundary value problem (1.2) has at least one solution on $[0,1]$.

Proof. In order to transform boundary value problem (1.2) into a fixed point problem, consider the multivalued operator $\Omega : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ defined by

$$\Omega(x) = \left\{ \begin{array}{l} h \in C([0,1],\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} f(s)ds \\ + \dfrac{(\beta-1)t - \beta\eta}{\Delta\Gamma(q)} \displaystyle\int_0^1 (1-s)^{q-1} f(s)ds \\ + \dfrac{\beta + (\alpha-\beta)t}{\Delta\Gamma(q)} \displaystyle\int_0^\eta (\eta-s)^{q-1} f(s)ds, \quad 0 \le t \le 1, \end{array} \right. \end{array} \right\}$$

for $f \in S_{F,x}$. Clearly, according to Lemma 2.4, the fixed points of Ω are solutions to boundary value problem (1.2). We will show that Ω satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof consists of several steps.

As a first step, *we show that Ω is convex for each $x \in C([0,1],\mathbb{R})$*. This step is obvious since $S_{F,x}$ is convex (F has convex values), and therefore we omit the proof.

Next, we show that Ω *maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$*. For a positive number ρ, let $B_\rho = \{x \in C([0,1],\mathbb{R}) : \|x\| \le \rho\}$ be a bounded set in $C([0,1],\mathbb{R})$. Then, for each $h \in \Omega(x), x \in B_\rho$, there exists $f \in S_{F,x}$ such that

$$\begin{aligned} h(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds + \frac{(\beta-1)t - \beta\eta}{\Delta\Gamma(q)} \int_0^1 (1-s)^{q-1} f(s)ds \\ &\quad + \frac{\beta + (\alpha-\beta)t}{\Delta\Gamma(q)} \int_0^\eta (\eta-s)^{q-1} f(s)ds, \end{aligned}$$

Then, as in Theorem 3.6, we have

$$|h(t)| \le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s)|ds + \frac{\lambda_1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} |f(s)|ds$$

$$
\begin{aligned}
& +\frac{\lambda_2}{\Gamma(q)}\int_0^\eta (\eta-s)^{q-1}|f(s)|ds \\
\leq\ & \frac{\|p\|\psi(\|x\|)}{\Gamma(q+1)}\left\{1+\lambda_1+\lambda_2\eta^q\right\}.
\end{aligned}
$$

Thus,

$$
\|h\| \leq \frac{\|p\|\psi(\rho)}{\Gamma(q+1)}\left\{1+\lambda_1+\lambda_2\eta^q\right\}.
$$

Now we show that Ω *maps bounded sets into equicontinuous sets of* $C([0,1],\mathbb{R})$. Let $t',t''\in[0,1]$ with $t'<t''$ and $x\in B_\rho$, where B_ρ is a bounded set of $C([0,1],\mathbb{R})$. For each $h\in\Omega(x)$, we obtain, as in Theorem 3.6,

$$
\begin{aligned}
|h(t'')-h(t')| \leq\ & \left|\frac{1}{\Gamma(q)}\int_0^{t'}[(t''-s)^{q-1}-(t'-s)^{q-1}]f(s)ds+\frac{1}{\Gamma(q)}\int_{t'}^{t''}(t''-s)^{q-1}f(s)ds\right| \\
& +\frac{\|p\|\psi(\rho)|(\beta-1)||t''-t'|}{\Delta\Gamma(q+1)}+\frac{\|p\|\psi(\rho)|\alpha-\beta||t''-t'|}{\Delta\Gamma(q+1)}.
\end{aligned}
$$

Obviously the right hand side of the above inequality tends to zero independently of $x\in B_{r'}$ as $t''-t'\to 0$. As Ω satisfies the above three assumptions, therefore it follows by the Arzelá-Ascoli theorem that $\Omega: C([0,1],\mathbb{R})\to\mathcal{P}(C([0,1],\mathbb{R}))$ is completely continuous. In our next step, we show that Ω *has a closed graph.* Let $x_n\to x_*, h_n\in\Omega(x_n)$ and $h_n\to h_*$. Then we need to show that $h_*\in\Omega(x_*)$. Associated with $h_n\in\Omega(x_n)$, there exists $f_n\in S_{F,x_n}$ such that for each $t\in[0,1]$,

$$
\begin{aligned}
h_n(t) =\ & \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f_n(s)ds+\frac{(\beta-1)t-\beta\eta}{\Delta\Gamma(q)}\int_0^1(1-s)^{q-1}f_n(s)ds \\
& +\frac{\beta+(\alpha-\beta)t}{\Delta\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}f_n(s)ds, \quad t\in[0,1].
\end{aligned}
$$

Thus we have to show that there exists $f_*\in S_{F,x_*}$ such that for each $t\in[0,1]$,

$$
\begin{aligned}
h_*(t) =\ & \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f_*(s)ds+\frac{(\beta-1)t-\beta\eta}{\Delta\Gamma(q)}\int_0^1(1-s)^{q-1}f_*(s)ds \\
& +\frac{\beta+(\alpha-\beta)t}{\Delta\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}f_*(s)ds.
\end{aligned}
$$

Let us consider the continuous linear operator $\Theta: L^1([0,1],\mathbb{R})\to C([0,1],\mathbb{R})$ given by

$$
\begin{aligned}
f\mapsto\Theta(f) =\ & \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(s)ds+\frac{(\beta-1)t-\beta\eta}{\Delta\Gamma(q)}\int_0^1(1-s)^{q-1}f(s)ds \\
& +\frac{\beta+(\alpha-\beta)t}{\Delta\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}f(s)ds.
\end{aligned}
$$

Observe that

$$
\|h_n(t)-h_*(t)\| = \left\|\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}(f_n(s)-f_*(s))ds\right.
$$

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$$+\frac{(\beta-1)t-\beta\eta}{\Delta\Gamma(q)}\int_0^1(1-s)^{q-1}(f_n(s)-f_*(s))ds$$
$$+\frac{\beta+(\alpha-\beta)t}{\Delta\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}(f_n(s)-f_*(s))ds\Bigg\| \to 0,$$

as $n \to \infty$. Thus, it follows by Lemma 4.7 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$\begin{aligned} h_*(t) &= \frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f_*(s)ds + \frac{(\beta-1)t-\beta\eta}{\Delta\Gamma(q)}\int_0^1(1-s)^{q-1}f_*(s)ds \\ &\quad +\frac{\beta+(\alpha-\beta)t}{\Delta\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}f_*(s)ds, \end{aligned}$$

for some $f_* \in S_{F,x_*}$.

Finally, *we discuss a priori bounds on solutions.* Let x be a solution of (1.2). Then there exists $f \in L^1([0,1],\mathbb{R})$ with $f \in S_{F,x}$ such that, for $t \in [0,1]$, we have

$$\begin{aligned} h(t) &= \frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f(s)ds + \frac{(\beta-1)t-\beta\eta}{\Delta\Gamma(q)}\int_0^1(1-s)^{q-1}f(s)ds \\ &\quad +\frac{\beta+(\alpha-\beta)t}{\Delta\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}f(s)ds. \end{aligned}$$

In view of (H_2), and using the computations in second step above, for each $t \in [0,1]$, we obtain

$$\begin{aligned} |h(t)| &\le \frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}|f(s)|ds + \frac{\lambda_1}{\Gamma(q)}\int_0^1(1-s)^{q-1}|f(s)|ds \\ &\quad +\frac{\lambda_2}{\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}|f(s)|ds \\ &\le \frac{\|p\|\psi(\|x\|)}{\Gamma(q+1)}\left\{1+\lambda_1+\lambda_2\eta^q\right\}. \end{aligned}$$

Consequently, we have

$$\frac{\|x\|}{\frac{\|p\|\psi(\|x\|)}{\Gamma(q+1)}\left\{1+\lambda_1+\lambda_2\eta^q\right\}} \le 1.$$

In view of (A_4), there exists M such that $\|x\| \neq M$. Let us set

$$U = \{x \in C([0,1],\mathbb{R}) : \|x\| < M+1\}.$$

Note that the operator $\Omega : \overline{U} \to \mathcal{P}(C([0,1],\mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x \in \mu\Omega(x)$ for some $\mu \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Theorem 4.6), we deduce that Ω has a fixed point $x \in \overline{U}$ which is a solution of the problem (1.2). This completes the proof. □

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Example 4.9 *Consider the following fractional boundary value problem*

$$\begin{cases} {}^cD^{3/2}x(t) \in F(t, x(t)), \quad 0 < t < 1, \\ x(0) = \dfrac{1}{2}x\left(\dfrac{1}{4}\right), \ x(1) = 2x\left(\dfrac{1}{4}\right). \end{cases} \tag{4.1}$$

Here, $q = 3/2, \beta = 1/2, \ \alpha = 2, \ \eta = 1/4$ and $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map given by $x \to F(t,x) = \left[\dfrac{|x|^3}{|x|^3+3} + 2t^3 + 1, \dfrac{|x|}{|x|+1} + t + 1\right]$. For $f \in F$, we have $|f| \leq \max\left(\dfrac{|x|^3}{|x|^3+3} + 2t^3 + 1, \dfrac{|x|}{|x|+1} + t + 1\right) \leq 4, \quad x \in \mathbb{R}$. Thus, $\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \leq 4 = p(t)\psi(\|x\|), \quad x \in \mathbb{R}$, with $p(t) = 1, \ \psi(\|x\|) = 4$. Further, using the condition (A_4) we find that $M > 21.092278$. Clearly, all the conditions of Theorem 4.8 are satisfied. So there exists at least one solution of the problem (4.1) on $[0,1]$.

4.2 The lower semi-continuous case

As a next result, we study the case when F is not necessarily convex valued. Our strategy to deal with this problems is based on the nonlinear alternative of Leray Schauder type together with the selection theorem of Bressan and Colombo [16] for lower semi-continuous maps with decomposable values.

Definition 4.10 *Let X be a nonempty closed subset of a Banach space E and $G : X \to \mathcal{P}(E)$ be a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set B in E.*

Definition 4.11 *Let A be a subset of $[0,1] \times \mathbb{R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the $\sigma-$algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[0,1]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$.*

Definition 4.12 *A subset $\mathcal{A}$ of $L^1([0,1],\mathbb{R})$ is decomposable if for all $x, y \in \mathcal{A}$ and measurable $\mathcal{J} \subset [0,1] = J$, the function $x\chi_{\mathcal{J}} + y\chi_{J-\mathcal{J}} \in \mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.*

Definition 4.13 *Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ be a multivalued operator. We say N has a property (BC) if N is lower semi-continuous (l.s.c.) and has nonempty closed and decomposable values.*

Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator $\mathcal{F} : C([0,1] \times \mathbb{R}) \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ associated with F as

$$\mathcal{F}(x) = \{w \in L^1([0,1],\mathbb{R}) : w(t) \in F(t, x(t)) \text{ for a.e. } t \in [0,1]\},$$

which is called the Nemytskii operator associated with F.

Definition 4.14 *Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Nemytskii operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.*

Lemma 4.15 *([16]) Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ be a multivalued operator satisfying the property (BC). Then N has a continuous selection, that is, there exists a continuous function (single-valued) $g : Y \to L^1([0,1],\mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

Theorem 4.16 *Assume that $(A_4), (H_2)$ and the following condition holds:*

(H_3) *$F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a nonempty compact-valued multivalued map such that*

- **(a)** *$(t,x) \longmapsto F(t,x)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable,*
- **(b)** *$x \longmapsto F(t,x)$ is lower semicontinuous for each $t \in [0,1]$.*

Then the boundary value problem (1.2) has at least one solution on $[0,1]$.

Proof. It follows from (H_3) and (H_2) that F is of l.s.c. type. Then from Lemma 4.15, there exists a continuous function $f : C^2([0,1],\mathbb{R}) \to L^1([0,1],\mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in C([0,1],\mathbb{R})$.

Consider the problem

$$\begin{cases} {}^cD^q x(t) = f(x(t)), \quad 0 < t < 1, \quad 1 < q \le 2, \\ x(0) = \beta x(\eta), \quad x(1) = \alpha x(\eta) \end{cases} \tag{4.2}$$

in the space $C^2([0,1],\mathbb{R})$. It is clear that if $x \in C^2([0,1],\mathbb{R})$ is a solution of the problem (4.2), then x is a solution to the problem (1.2). In order to transform the problem (4.2) into a fixed point problem, we define the operator $\overline{\Omega}$ as

$$\begin{aligned} \overline{\Omega}x(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(x(s))ds + \frac{(\beta-1)t - \beta\eta}{\Delta\Gamma(q)}\int_0^1 (1-s)^{q-1} f(x(s))ds \\ &\quad + \frac{\beta + (\alpha-\beta)t}{\Delta\Gamma(q)}\int_0^\eta (\eta-s)^{q-1} f(x(s))ds, \quad 0 \le t \le 1. \end{aligned}$$

It can easily be shown that $\overline{\Omega}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 4.8. So we omit it. This completes the proof. □

4.3 The Lipschitz case

Now we prove the existence of solutions for the problem (1.2) with a nonconvex valued right hand side by applying a fixed point theorem for multivalued map due to Covitz and Nadler [18].

Let (X,d) be a metric space induced from the normed space $(X; \|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by $H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\}$, where $d(A,b) = \inf_{a\in A} d(a;b)$ and $d(a,B) = \inf_{b\in B} d(a;b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space (see [23]).

Definition 4.17 *A multivalued operator $N : X \to P_{cl}(X)$ is called:*

(**a**) *$\gamma-$Lipschitz if and only if there exists $\gamma > 0$ such that*

$$H_d(N(x), N(y)) \leq \gamma d(x, y) \text{ for each } x, y \in X;$$

(**b**) *a contraction if and only if it is $\gamma-$Lipschitz with $\gamma < 1$.*

Lemma 4.18 *(Covitz-Nadler) [18]. Let (X, d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

Definition 4.19 *A measurable multi-valued function $F : [0, 1] \to \mathcal{P}(X)$ is said to be integrably bounded if there exists a function $h \in L^1([0, 1], X)$ such that for all $v \in F(t)$, $\|v\| \leq h(t)$ for a.e. $t \in [0, 1]$.*

Theorem 4.20 *Assume that the following conditions hold:*

(H_4) *$F : [0, 1] \times \mathbb{R} \to P_{cp}(\mathbb{R})$ is such that $F(\cdot, x) : [0, 1] \to P_{cp}(\mathbb{R})$ is measurable for each $x \in \mathbb{R}$;*

(H_5) *$H_d(F(t, x), F(t, \bar{x})) \leq m(t)|x - \bar{x}|$ for almost all $t \in [0, 1]$ and $x, \bar{x} \in \mathbb{R}$ with $m \in C([0, 1], \mathbb{R}^+)$ and $d(0, F(t, 0)) \leq m(t)$ for almost all $t \in [0, 1]$.*

Then the boundary value problem (1.2) has at least one solution on $[0, 1]$ if

$$\frac{\|m\|}{\Gamma(q+1)} \{1 + \lambda_1 + \lambda_2 \eta^q\} < 1.$$

Proof. We transform the problem (1.2) into a fixed point problem. Consider the set-valued map $\Omega : C([0, 1], \mathbb{R}) \to \mathcal{P}(C([0, 1], \mathbb{R}))$ defined at the begining of the proof of Theorem 4.8. It is clear that the fixed point of Ω are solutions of the problem (1.2).

Note that, by the assumption (H_4), since the set-valued map $F(\cdot, x)$ is measurable, it admits a measurable selection $f : [0, 1] \to \mathbb{R}$ (see Theorem III.6 [17]). Moreover, from assumption (H_5) $|f(t)| \leq m(t) + m(t)|x(t)|$, i.e. $f(\cdot) \in L^1([0, 1], X)$. Therefore the set $S_{F,x}$ is nonempty. Also note that since $S_{F,x} \neq \emptyset$, $\Omega(x) \neq \emptyset$ for any $x \in C([0, 1], \mathbb{R})$.

Now we show that the operator Ω satisfies the assumptions of Lemma 4.18. To show that $\Omega(x) \in \mathcal{P}_{cl}((C[0, 1], \mathbb{R}))$ for each $x \in C([0, 1], \mathbb{R})$, let $\{u_n\}_{n \geq 0} \in \Omega(x)$ be such that $u_n \to u$ $(n \to \infty)$ in $C([0, 1], \mathbb{R})$. Then $u \in C([0, 1], \mathbb{R})$ and there exists $v_n \in S_{F,x}$ such that, for each $t \in [0, 1]$,

$$\begin{aligned} u_n(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v_n(s) ds + \frac{(\beta - 1)t - \beta\eta}{\Delta\Gamma(q)} \int_0^1 (1-s)^{q-1} v_n(s) ds \\ &\quad + \frac{\beta + (\alpha - \beta)t}{\Delta\Gamma(q)} \int_0^\eta (\eta - s)^{q-1} v_n(s) ds. \end{aligned}$$

As F has compact values, we may pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1([0, 1], \mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in [0, 1]$,

$$u_n(t) \to u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v(s) ds + \frac{(\beta - 1)t - \beta\eta}{\Delta\Gamma(q)} \int_0^1 (1-s)^{q-1} v(s) ds$$

$$+\frac{\beta+(\alpha-\beta)t}{\Delta\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}v(s)ds.$$

Hence, $u\in\Omega(x)$ and $\Omega(x)$ is closed.

Next we show that Ω is a contraction on $C([0,1],\mathbb{R})$, i.e. there exists $\gamma<1$ such that

$$H_d(\Omega(x),\Omega(\bar{x}))\leq\gamma\|x-\bar{x}\| \text{ for each } x,\bar{x}\in C([0,1],\mathbb{R}).$$

Let $x,\bar{x}\in C([0,1],\mathbb{R})$ and $h_1\in\Omega(x)$. Then there exists $v_1(t)\in F(t,x(t))$ such that, for each $t\in[0,1]$,

$$\begin{aligned} h_1(t) &= \frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}v_1(s)ds+\frac{(\beta-1)t-\beta\eta}{\Delta\Gamma(q)}\int_0^1(1-s)^{q-1}v_1(s)ds \\ &\quad+\frac{\beta+(\alpha-\beta)t}{\Delta\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}v_1(s)ds. \end{aligned}$$

By (H_6), we have $H_d(F(t,x),F(t,\bar{x}))\leq m(t)|x(t)-\bar{x}(t)|$. So, there exists $w\in F(t,\bar{x}(t))$ such that $|v_1(t)-w|\leq m(t)|x(t)-\bar{x}(t)|$, $t\in[0,1]$.

Define $U:[0,1]\to\mathcal{P}(\mathbb{R})$ by $U(t)=\{w\in\mathbb{R}:|v_1(t)-w|\leq m(t)|x(t)-\bar{x}(t)|\}$. Since the multivalued operator $U(t)\cap F(t,\bar{x}(t))$ is measurable (Proposition III.4 [17]), there exists a function $v_2(t)$ which is a measurable selection for U. So $v_2(t)\in F(t,\bar{x}(t))$ and for each $t\in[0,1]$, we have $|v_1(t)-v_2(t)|\leq m(t)|x(t)-\bar{x}(t)|$.

For each $t\in[0,1]$, let us define

$$\begin{aligned} h_2(t) &= \frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}v_2(s)ds+\frac{(\beta-1)t-\beta\eta}{\Delta\Gamma(q)}\int_0^1(1-s)^{q-1}v_2(s)ds \\ &\quad+\frac{\beta+(\alpha-\beta)t}{\Delta\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}v_2(s)ds. \end{aligned}$$

Thus,

$$\begin{aligned} |h_1(t)-h_2(t)| &\leq \frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}|v_1(s)-v_2(s)|ds+\frac{\lambda_1}{\Gamma(q)}\int_0^1(1-s)^{q-1}|v_1(s)-v_2(s)|ds \\ &\quad+\frac{\lambda_2}{\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}|v_1(s)-v_2(s)|ds \\ &\leq \frac{\|m\|\|x-\overline{x}\|}{\Gamma(q+1)}\left\{1+\lambda_1+\lambda_2\eta^q\right\}. \end{aligned}$$

Hence,

$$\|h_1-h_2\|\leq\frac{\|m\|\|x-\overline{x}\|}{\Gamma(q+1)}\left\{1+\lambda_1+\lambda_2\eta^q\right\}.$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$H_d(\Omega(x),\Omega(\bar{x}))\leq\gamma\|x-\bar{x}\|\leq\frac{\|m\|\|x-\overline{x}\|}{\Gamma(q+1)}\left\{1+\lambda_1+\lambda_2\eta^q\right\}.$$

Since Ω is a contraction, it follows by Lemma 4.18 that Ω has a fixed point x which is a solution of (1.2). This completes the proof. □

Remark 4.21 *The results of this paper can easily to be generalized to boundary value problems for fractional differential equations and inclusions with deviating arguments and generalized three point boundary conditions. Thus we can study, by similar methods and obvious modifications, the following boundary value problem for fractional differential equations*

$$\begin{cases} {}^cD^q x(t) = f(t, x(\sigma(t))), \quad 0 < t < 1, \quad 1 < q \le 2, \\ x(t) = \beta x(\eta), \quad -r \le t \le 0 \\ x(1) = \alpha x(\eta), \end{cases} \tag{4.3}$$

where ${}^cD^q$ *denotes the Caputo fractional derivative of order* q, $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ *is continuous,* $-r = \min_{t\in[0,1]} \sigma(t)$, $\sigma : [0,1] \to [-r, 1]$ *is continuous with* $\sigma(t) \le t, \forall t \in [0,1]$ *and* α, β, η *are constants with* $0 < \eta < 1$ *and* $1 - \beta + (\beta - \alpha)\eta \ne 0$, *or the corresponding boundary value problem for fractional differential inclusions*

$$\begin{cases} {}^cD^q x(t) \in F(t, x(\sigma(t))), \quad 0 < t < 1, \quad 1 < q \le 2, \\ x(t) = \beta x(\eta), \quad -r \le t \le 0 \\ x(1) = \alpha x(\eta), \end{cases} \tag{4.4}$$

where ${}^cD^q$ *denotes the Caputo fractional derivative of order* q, *and* $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is a multivalued map,* $\mathcal{P}(\mathbb{R})$ *is the family of all subsets of* $\mathbb{R}$.

Remark 4.22 *It is obvious that the methods used in this paper can be applied to other types of nonlocal boundary value problems. For example for the following four point boundary value problem*

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), \quad 0 < t < 1, \quad 1 < q \le 2, \\ x(t) = \alpha x(\xi), \quad x(1) = \beta x(\eta), \end{cases} \tag{4.5}$$

where α, β, ξ, η *are constants with* $0 < \xi, \eta < 1$ *and* $\Delta := \alpha(\beta\eta - 1) - (\beta - 1)(\alpha\xi - 1) \ne 0$. *The solution of the problem (4.5) is given by*

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s))\, ds + \frac{\alpha[(\beta - 1)t - \beta\eta + 1]}{\Delta} \int_0^\xi \frac{(\xi - s)^{q-1}}{\Gamma(q)} f(s, x(s))\, ds \\ &+ \frac{\beta[\alpha\xi - 1 - \alpha t]}{\Delta} \int_0^\eta \frac{(\eta - s)^{q-1}}{\Gamma(q)} f(s, x(s))\, ds \\ &+ \frac{\alpha t - \alpha\xi + 1}{\Delta} \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s, x(s))\, ds, \quad 0 \le t \le 1. \end{aligned}$$

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QUADRATIC ρ-FUNCTIONAL INEQUALITIES IN FUZZY BANACH SPACES

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Abstract. In this paper, we solve the following quadratic ρ-functional inequalities

$$N\left(f(x+y)+f(x-y)-2f(x)-2f(y)-\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right),t\right) \geq \frac{t}{t+\varphi(x,y)}, \tag{0.1}$$

where ρ is a fixed real number with $\rho \neq 2$, and

$$N\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)-\rho\left(f(x+y)+f(x-y)-2f(x)-2f(y)\right),t\right) \geq \frac{t}{t+\varphi(x,y)}, \tag{0.2}$$

where ρ is a fixed real number with $\rho \neq \frac{1}{2}$.

Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic ρ-functional inequalities (0.1) and (0.2) in fuzzy Banach spaces.

1. Introduction and preliminaries

Katsaras [21] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [13, 24, 52]. In particular, Bag and Samanta [2], following Cheng and Mordeson [8], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [23]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [3].

We use the definition of fuzzy normed spaces given in [2, 28, 29] to investigate the Hyers-Ulam stability of quadratic ρ-functional inequalities in fuzzy Banach spaces.

Definition 1.1. [2, 28, 29, 30] Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N_1) $N(x,t) = 0$ for $t \leq 0$;
(N_2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
(N_3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N_4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;
(N_5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.
(N_6) for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a *fuzzy normed vector space*.

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [27, 28].

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Definition 1.2. [2, 28, 29, 30] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by N-$\lim_{n\to\infty} x_n = x$.

Definition 1.3. [2, 28, 29, 30] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [3]).

The stability problem of functional equations originated from a question of Ulam [51] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [17] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [40] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [14] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

The functional equation $f(x+y)+f(x-y) = 2f(x)+2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [50] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [9] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. Czerwik [10] proved the Hyers-Ulam stability of the quadratic functional equation. The functional equation $f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$ is called a *Jensen type quadratic equation*. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [4, 18, 20, 25, 36, 37, 38, 41, 42, 44, 45, 46, 47, 48, 49]).

Gilányi [15] showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(x-y)\| \leq \|f(x+y)\| \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(x+y) + f(x-y).$$

See also [43]. Fechner [12] and Gilányi [16] proved the Hyers-Ulam stability of the functional inequality (1.1). Park, Cho and Han [35] investigated the Cauchy additive functional inequality

$$\|f(x) + f(y) + f(z)\| \leq \|f(x+y+z)\| \tag{1.2}$$

and the Cauchy-Jensen additive functional inequality

$$\|f(x) + f(y) + 2f(z)\| \leq \left\|2f\left(\frac{x+y}{2} + z\right)\right\| \tag{1.3}$$

and proved the Hyers-Ulam stability of the functional inequalities (1.2) and (1.3) in Banach spaces.

Park [33, 34] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces.

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.4. [5, 11] Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;
(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

In 1996, G. Isac and Th.M. Rassias [19] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [6, 7, 22, 27, 31, 32, 38, 39]).

In Section 2, we solve the quadratic ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.1) in fuzzy Banach spaces by using the fixed point method.

In Section 3, we solve the quadratic ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.2) in fuzzy Banach spaces by using the fixed point method.

Throughout this paper, assume that X is a real vector space and (Y, N) is a fuzzy Banach space.

2. Quadratic ρ-functional inequality (0.1)

In this section, we prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.1) in fuzzy Banach spaces. Let ρ be a real number with $\rho \neq 2$. We need the following lemma to prove the main results.

Lemma 2.1. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$f(x+y) + f(x-y) - 2f(x) - 2f(y) = \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right) \tag{2.1}$$

for all $x, y \in X$. Then $f : X \to Y$ is quadratic.

Proof. Replacing y by x in (2.1), we get $f(2x) - 4f(x) = 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$\begin{aligned} f(x+y) + f(x-y) - 2f(x) - 2f(y) &= \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right) \\ &= \frac{\rho}{2} (f(x+y) + f(x-y) - 2f(x) - 2f(y)) \end{aligned}$$

and so $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in X$, as desired. □

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{4} \varphi(2x, 2y)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$N(f(x+y) + f(x-y) - 2f(x) - 2f(y) \\ -\rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right), t) \geq \frac{t}{t+\varphi(x,y)} \tag{2.2}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x) - Q(x), t\right) \geq \frac{(4-4L)t}{(4-4L)t + L\varphi(x,x)} \tag{2.3}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $y = x$ in (2.2), we get

$$N\left(f\left(2x\right) - 4f(x), t\right) \geq \frac{t}{t+\varphi(x,x)} \tag{2.4}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t+\varphi(x,x)},\ \forall x \in X, \forall t > 0\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [26, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t+\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right), L\varepsilon t\right) = N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{4}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)} \geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \frac{L}{4}\varphi(x,x)} = \frac{t}{t+\varphi(x,x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.4) that $N\left(f(x) - 4f\left(\frac{x}{2}\right), \frac{L}{4}t\right) \geq \frac{t}{t+\varphi(x,x)}$ for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{4}$.

By Theorem 1.4, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(\frac{x}{2}\right) = \frac{1}{4}Q(x) \tag{2.5}$$

for all $x \in X$. Since $f : X \to Y$ is even, $Q : X \to Y$ is a even mapping. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (2.5) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - Q(x), \mu t) \geq \frac{t}{t + \varphi(x, x)}$$

for all $x \in X$;

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f, Q) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, Q) \leq \frac{L}{4 - 4L}.$$

This implies that the inequality (2.3) holds.

By (2.2),

$$\begin{aligned} N\Bigg(4^n\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right) \\ - \rho\left(4^n\left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right), 4^n t\Bigg) \geq \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\begin{aligned} N\Bigg(4^n\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right) \\ - \rho\left(4^n\left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right), t\Bigg) \geq \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n}\varphi(x, y)} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y) = \rho\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right)$$

for all $x, y \in X$. By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic, as desired. □

Corollary 2.3. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} N\left(f(x+y) + f(x-y) - 2f(x) - 2f(y) - \rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right), t\right) \\ \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \end{aligned} \tag{2.6}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{(2^p - 4)t}{(2^p - 4)t + 2\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{2-p}$, and we get the desired result. □

Theorem 2.4. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.2). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \ge \frac{(4-4L)t}{(4-4L)t + \varphi(x,x)} \tag{2.7}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4} g(2x)$$

for all $x \in X$.

It follows from (2.4) that

$$N\left(f(x) - \frac{1}{4}f(2x), \frac{1}{4}t\right) \ge \frac{t}{t + \varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \le \frac{1}{4}$. Hence $d(f,Q) \le \frac{1}{4-4L}$, which implies that the inequality (2.7) holds.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $\theta \ge 0$ and let p be a real number with $0 < p < 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be amapping satisfying $f(0) = 0$ and (2.6). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N(f(x) - Q(x), t) \ge \frac{(4-2^p)t}{(4-2^p)t + 2\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{p-2}$, and we get the desired result. □

3. Quadratic ρ-functional inequality (0.2)

In this section, we prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.2) in fuzzy Banach spaces. Let ρ be a real number with $\rho \ne \frac{1}{2}$. We need the following lemma to prove the main results.

Lemma 3.1. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) = \rho\left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right) \tag{3.1}$$

for all $x, y \in X$. Then $f : X \to Y$ is quadratic.

Proof. Letting $y=0$ in (3.1), we get $4f\left(\frac{x}{2}\right)-f(x)=0$ and so $f(2x)=4f(x)$ for all $x\in X$. Thus

$$\begin{aligned}\frac{1}{2}f(x+y)-\frac{1}{2}f(x-y)-f(x)-f(y) &= 2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\\ &= \rho(f(x+y)+f(x-y)-2f(x)-2f(y))\end{aligned}$$

and so $f(x+y)+f(x-y)=2f(x)+2f(y)$ for all $x,y\in X$, as desired. □

Theorem 3.2. *Let $\varphi : X^2\to[0,\infty)$ be a function such that there exists an $L<1$ with*

$$\varphi(x,y)\le\frac{L}{4}\varphi(2x,2y)$$

for all $x,y\in X$. Let $f:X\to Y$ be a mapping satisfying $f(0)=0$ and

$$\begin{aligned}&N\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)-\rho\left(f(x+y)+f(x-y)-2f(x)-2f(y)\right),t\right)\\ &\qquad\ge\frac{t}{t+\varphi(x,y)}\end{aligned}\tag{3.2}$$

for all $x,y\in X$ and all $t>0$. Then $Q(x):=N\text{-}\lim_{n\to\infty}4^nf\left(\frac{x}{2^n}\right)$ exists for each $x\in X$ and defines a quadratic mapping $Q:X\to Y$ such that

$$N\left(f(x)-Q(x),t\right)\ge\frac{(1-L)t}{(1-L)t+\varphi(x,0)}\tag{3.3}$$

for all $x\in X$ and all $t>0$.

Proof. Letting $y=0$ in (3.2), we get

$$N\left(4f\left(\frac{x}{2}\right)-f(x),t\right)\ge\frac{t}{t+\varphi(x,0)}\tag{3.4}$$

for all $x\in X$.

Consider the set

$$S:=\{g:X\to Y\}$$

and introduce the generalized metric on S:

$$d(g,h)=\inf\left\{\mu\in\mathbb{R}_+ : N(g(x)-h(x),\mu t)\ge\frac{t}{t+\varphi(x,0)},\ \forall x\in X,\forall t>0\right\},$$

where, as usual, $\inf\phi=+\infty$. It is easy to show that (S,d) is complete (see [26, Lemma 2.1]).

Now we consider the linear mapping $J:S\to S$ such that

$$Jg(x):=4g\left(\frac{x}{2}\right)$$

for all $x\in X$.

Let $g,h\in S$ be given such that $d(g,h)=\varepsilon$. Then

$$N(g(x)-h(x),\varepsilon t)\ge\frac{t}{t+\varphi(x,0)}$$

for all $x\in X$ and all $t>0$. Hence

$$\begin{aligned}N(Jg(x)-Jh(x),L\varepsilon t) &= N\left(4g\left(\frac{x}{2}\right)-4h\left(\frac{x}{2}\right),L\varepsilon t\right)=N\left(g\left(\frac{x}{2}\right)-h\left(\frac{x}{2}\right),\frac{L}{4}\varepsilon t\right)\\ &\ge\frac{\frac{Lt}{4}}{\frac{Lt}{4}+\varphi\left(\frac{x}{2},0\right)}\ge\frac{\frac{Lt}{4}}{\frac{Lt}{4}+\frac{L}{4}\varphi(x,0)}=\frac{t}{t+\varphi(x,0)}\end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le Ld(g,h)$$

for all $g, h \in S$.

It follows from (3.4) that

$$N\left(f(x) - 4f\left(\frac{x}{2}\right), t\right) \ge \frac{t}{t + \varphi(x,0)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \le 1$.

By Theorem 1.4, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(\frac{x}{2}\right) = \frac{1}{4}Q(x) \tag{3.5}$$

for all $x \in X$. Since $f : X \to Y$ is even, $Q : X \to Y$ is an even mapping. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (3.5) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - Q(x), \mu t) \ge \frac{t}{t + \varphi(x,0)}$$

for all $x \in X$;

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f,Q) \le \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f,Q) \le \frac{1}{1-L}.$$

This implies that the inequality (3.4) holds.

By (3.2),

$$\begin{aligned} N\Bigg(& 4^n \left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right) \\ & -\rho\left(4^n\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right)\right), 4^n t\Bigg) \ge \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\begin{aligned} N\Bigg(& 4^n \left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right) \\ & -\rho\left(4^n\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right)\right), t\Bigg) \ge \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n}\varphi(x,y)} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$2Q\left(\frac{x+y}{2}\right) + 2\left(\frac{x-y}{2}\right) - Q(x) - Q(y) = \rho\left(Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y)\right)$$

for all $x, y \in X$=. By Lemma 3.1, the mapping $Q : X \to Y$ is quadratic, as desired. $\square$

Corollary 3.3. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$N(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \tag{3.6}$$
$$-\rho\left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right), t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x) - Q(x), t\right) \geq \frac{(2^p - 4)t}{(2^p - 4)t + 2^p\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{2-p}$, and we get the desired result. $\square$

Theorem 3.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.2). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f\left(2^n x\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x) - Q(x), t\right) \geq \frac{(1-L)t}{(1-L)t + \varphi(x, 0)} \tag{3.7}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 3.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4}g\left(2x\right)$$

for all $x \in X$.

It follows from (3.4) that

$$N\left(f(x) - \frac{1}{4}f(2x), Lt\right) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq L$. Hence

$$d(f, Q) \leq \frac{1}{1-L},$$

which implies that the inequality (3.7) holds.

The rest of the proof is similar to the proof of Theorem 3.2. $\square$

Corollary 3.5. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.6). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x) - Q(x), t\right) \geq \frac{(4 - 2^p)t}{(4 - 2^p)t + 2^p\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{p-2}$, and we get the desired result. $\square$

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Remarks on common fixed point results for cyclic contractions in ordered b-metric spaces

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Abstract. The purpose of this paper is to prove that some common fixed point theorems for cyclic contractions are equivalent to the counterpart of noncyclic contractions in the same setting. Our results improve and complement several results for cyclic contractions established in [Fixed Point Theory Appl., 2013: 256]. Furthermore, an application to the existence and uniqueness of solution for a class of integral equations is given to illustrate the superiority of the obtained assertions.

Keywords: (A, B)-weakly increasing, common fixed point, altering distance function, regular
MSC: 47H10, 54H25.

1. INTRODUCTION AND PRELIMINARIES

Since Banach fixed point theorem (see [1]) appeared in the world, there have been overwhelming trend in mathematical activities. This theorem presents numerous applications. For instance, it gives the conditions under which maps (single or multivalued) have solutions. Fixed point theory itself is a beautiful mixture of analysis (pure and applied), topology, and geometry. It has been revealed as a very powerful and important tool in the study of nonlinear phenomena. Over the last several decades, scholars have generalized this theorem greatly from several directions. Whereas, one of most influential generalizations is from spaces. Wherein, the fact from usual metric spaces to b-metric spaces is very popular. b-metric spaces, also called metric type spaces, were introduced in [2] and [3]. Afterwards, a large number of fixed point theorems have been presented in such spaces (see [4-15]). Recently, scholars cultivate some interests in fixed point theorems for cyclic contractions (see [15-19]). However, the authors of this paper find that many fixed point results for cyclic contractions are actually equivalent to those of noncyclic contractions in the same spaces. Throughout this paper, we obtain some equivalences between cyclic contractions and noncyclic contractions in the setting of b-metric spaces. Moreover, we obtain some common fixed point theorems without considering cyclic contractions. Further, as an applications, we cope with the existence and uniqueness of solutions of integral equations.

For the sake of the reader, we recall some well-known concepts and results as follows.

Definition 1.1([9]) Let X be a (nonempty) set and $s \geq 1$ a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric on X if, for all $x, y, z \in X$, the following conditions hold:

(b1) $d(x, y) = 0$ if and only if $x = y$;

(b2) $d(x, y) = d(y, x)$;

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(b3) $d(x,z) \leq s[d(x,y)+d(y,z)]$.
In this case, the pair (X,d) is called a b-metric space or metric type space. If $(X,\preceq)$ is still a partially ordered set, then $(X,\preceq,d)$ is called an ordered b-metric space.

Otherwise, for some other definitions in b-metric spaces such as convergence, Cauchy sequence, completeness, see [8-15] and the references therein.

Definition 1.2([22]) A function $\varphi:[0,\infty)\to[0,\infty)$ is called an altering distance function if the following properties hold:

(1) φ is continuous and nondecreasing;

(2) $\varphi(t)=0$ if and only if $t=0$.

Definition 1.3([21]) Let $(X,\preceq)$ be a partially ordered set, and let A and B be closed subsets of X with $A\cup B=X$. Let $f,g:X\to X$ be two mappings. The pair (f,g) is said to be (A,B)-weakly increasing if $fx\preceq gfx$ for all $x\in A$ and $gy\preceq fgy$ for all $y\in B$. In particular, (f,g) is said to be weakly increasing if $fx\preceq gfx$ and $gx\preceq fgx$ for all $x\in X$.

Definition 1.4([13]) An ordered b-metric space $(X,\preceq,d)$ is called regular if for any nondecreasing sequence $\{x_n\}$ in X such that $x_n\to x$ $(n\to\infty)$, one has $x_n\preceq x$ for all $n\in\mathbb{N}$.

Definition 1.5([16]) Let A and B be nonempty subsets of a metric space (X,d) and $T:A\cup B\to A\cup B$. Then T is called a cyclic map if $T(A)\subseteq B$ and $T(B)\subseteq A$.

Shatanawi and Postolache proved the following common fixed point results for cyclic contractions in the framework of ordered metric spaces.

Theorem 1.6([19]) Let $(X,\preceq,d)$ be a complete ordered metric space, and let A,B be closed nonempty subsets of X with $X=A\cup B$. Let $f,g:X\to X$ be (A,B)-weakly increasing mappings with respect to $\preceq$. Suppose that

(a) $X=A\cup B$ is a cyclic representation of X with respect to the pair (f,g), *i.e.*, $f(A)\subseteq B$ and $g(B)\subseteq A$;

(b) there exist $0<\delta<1$ and an altering distance function ψ such that for any comparable elements $x,y\in X$ with $x\in A$ and $y\in B$, we have

$$\psi(d(fx,gy))\leq\delta\psi\left(\max\left\{d(x,y),d(x,fx),d(y,gy),\frac{1}{2}(d(x,gy)+d(y,fx))\right\}\right);$$

(c) f or g is continuous, or

(c') $(X,\preceq,d)$ is regular.

Then f and g have a common fixed point.

It should be noted that cyclic contractions (unlike Banach-type contractions) need not to be continuous. This concept is an interesting increase in nonlinear analysis. In addition, Hussain *et al.* [15] introduced the notion of ordered cyclic weakly (ψ,φ,L,A,B)-contraction and proved the following fixed point results.

Definition 1.7 Let $(X,\preceq,d)$ be an ordered b-metric space, let $f,g:X\to X$ be two mappings, and let A and B be nonempty closed subsets of X. The pair (f,g) is called an ordered cyclic weakly (ψ,φ,L,A,B)-contraction if

(1) $X=A\cup B$ is a cyclic representation of X with respect to the pair (f,g);

(2) there exist two altering distance functions ψ,φ and a constant $L\geq 0$, such that for arbitrary comparable elements $x,y\in X$ with $x\in A$ and $y\in B$, we have

$$\psi\left(s^2d(fx,gy)\right)\leq\psi(M_s(x,y))-\varphi(M_s(x,y))+L\psi(N(x,y)),$$

where

$$M_s(x,y)=\max\left\{d(x,y),d(x,fx),d(y,gy),\frac{1}{2s}(d(x,gy)+d(y,fx))\right\} \tag{1.1}$$

and

$$N(x,y) = \min\{d(y,gy), d(x,gy), d(y,fx)\}. \tag{1.2}$$

Theorem 1.8 Let $(X, \preceq, d)$ be a complete ordered b-metric space, and let A and B be closed subsets of X. Let $f, g : X \to X$ be (A, B)-weakly increasing mappings with respect to $\preceq$. Suppose that

(a) the pair (f, g) is an ordered cyclic weakly (ψ, φ, L, A, B)-contraction;

(b) f or g is continuous.

Then f and g have a common fixed point $u \in A \cap B$.

Theorem 1.9 Let the hypothesis of Theorem 1.8 be satisfied, except that condition (b) is replaced by the following assumption:

(b') $(X, \preceq, d)$ is regular.

Then f and g have a common fixed point $u \in A \cap B$.

The following lemmas will be utilized in the proof of our main results.

Lemma 1.10([20]) If some ordinary fixed point theorem in the setting of complete metric spaces has a true cyclic-type extension, then these both theorems are equivalent.

Lemma 1.11([5]) Let $\{y_n\}$ be a sequence in a b-metric space (X, d) with $s \geq 1$ such that

$$d(y_n, y_{n+1}) \leq \lambda d(y_{n-1}, y_n)$$

for some $\lambda \in [0, \frac{1}{s})$, and each $n = 1, 2, \ldots$. Then $\{y_n\}$ is a Cauchy sequence in (X, d).

2. MAIN RESULTS

In this section, following the trend mentioned above, we extend such considerations to the simpler equivalent results so that we can enlarge, in a unified manner, the class of problems that can be investigated.

Theorem 2.1 Let $(X, \preceq, d)$ be a complete ordered metric space, and let $f, g : X \to X$ be the weakly increasing mappings. Suppose that

(a) there exist $0 < \delta < 1$ and an altering distance function ψ such that for any comparable elements $x, y \in X$, we have that

$$\psi(d(fx, gy)) \leq \delta\psi(\max\{d(x,y), d(x,fx), d(y,gy), \frac{1}{2}(d(x,gy) + d(y,fx))\});$$

(b) f or g is continuous, or

(c) $(X, \preceq, d)$ is regular.

Then f and g have a common fixed point.

The proof of Theorem 2.1 is trivial because we have the following:

Theorem 2.2 Theorem 1.6 is equivalent with Theorem 2.1.

Proof Putting $A = B = X$ in Theorem 1.6, we obtain Theorem 2.1. In other words, Theorem 1.6 implies Theorem 2.1. The proof for the converse is same as in [20-21]. Namely, we depend on Lemma 1.10. □

In the sequel, we announce the following noncyclic case result.

Theorem 2.3 Let $(X, \preceq, d)$ be a complete ordered b-metric space, and let $f, g : X \to X$ be the weakly increasing mappings. Suppose that there exist altering distance function ψ and φ, and the constants $\varepsilon > 1$, $L \geq 0$ such that

$$\psi(s^{\varepsilon} d(fx, gy)) \leq \psi(M_s(x,y)) - \varphi(M_s(x,y)) + L\psi(N(x,y)) \tag{2.1}$$

for all comparable $x, y \in X$, where $M_s(x,y)$ and $N(x,y)$ are given by (1.1) and (1.2), respectively. If either f or g is continuous, or the space $(X, \preceq, d)$ is regular, then f and g have a common fixed point.

Proof Choose $x_0 \in X$ and construct a sequence $\{x_n\}$ as follows:

$$x_{2n+1} = fx_{2n}, \qquad x_{2n+2} = gx_{2n+1}.$$

Since (f, g) is weakly increasing, then

$$x_1 \preceq x_2 \preceq x_3 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots .$$

If $x_{2n} = x_{2n+1}$ or $x_{2n+1} = x_{2n+2}$ for some n, then the proof is trivial and hence we omit it. Now we assume that $x_n \neq x_{n+1}$ for all n. We shall only prove that

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n), \tag{2.2}$$

for all $n = 1, 2, ...$, where $\lambda \in [0, \frac{1}{s})$. Indeed, by (2.1), it establishes that

$$\begin{aligned}\psi\left(s^{\varepsilon} d\left(x_{2n+1}, x_{2n+2}\right)\right) &= \psi\left(s^{\varepsilon} d\left(fx_{2n}, gx_{2n+1}\right)\right) \\ &\leq \psi\left(M_s\left(x_{2n}, x_{2n+1}\right)\right) + L\psi\left(N\left(x_{2n}, x_{2n+1}\right)\right),\end{aligned}$$

where $M_s(x_{2n}, x_{2n+1}) = \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}$ and $N(x_{2n}, x_{2n+1}) = 0$. Hence, it is not hard to verify that

$$s^{\varepsilon} d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1}). \tag{2.3}$$

Similarly, we obtain that

$$s^{\varepsilon} d(x_{2n}, x_{2n+1}) \leq d(x_{2n-1}, x_{2n}). \tag{2.4}$$

Uniting (2.3) and (2.4), ones have (2.2).

Now by Lemma 1.11, we demonstrate that $\{x_n\}$ is a Cauchy sequence and therefore there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$. Thus

$$\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} fx_{2n} = x. \tag{2.5}$$

In view of $x_{2n} \to x$, without loss of generality, assume that f is continuous. Then

$$\lim_{n\to\infty} fx_{2n} = fx. \tag{2.6}$$

It follows immediately from (2.5) and (2.6) that $x = fx$.

Further, by using $x \preceq x$ we can prove that the condition (2.1) implies the existence of common fixed point of f and g. Indeed, put $x = y$ in (2.1) it follows that

$$\psi\left(s^{\varepsilon} d\left(fx, gx\right)\right) \leq \psi\left(M_s\left(x, x\right)\right) - \varphi\left(M_s\left(x, x\right)\right) + L\psi\left(N\left(x, x\right)\right).$$

Now that $M_s(x, x) = d(x, gx)$ and $N(x, y) = 0$, one has

$$\psi\left(s^{\varepsilon} d\left(fx, gx\right)\right) \leq \psi\left(d\left(x, gx\right)\right) - \varphi\left(d\left(x, gx\right)\right) + L \cdot 0 \leq \psi\left(d\left(x, gx\right)\right),$$

which means that

$$s^{\varepsilon} d\left(fx, gx\right) = s^{\varepsilon} d\left(x, gx\right) \leq d\left(x, gx\right),$$

Consequently, $x = gx$ (because $\varepsilon > 1$).

The assumption of continuity of one of the mappings f or g can be replaced by the condition that b-metric space $(X, \preceq, d)$ is regular.

In fact, let $(X, \preceq, d)$ be regular. Via the mentioned above, we can construct an increasing sequence $\{x_n\}$ in X such that $x_n \to x$ $(n \to \infty)$ for some $x \in X$. Then $x_n \preceq x$ for all $n \in \mathbb{N}$. We shall have to show that $fx = gx = x$.

First, we have

$$\frac{1}{s} d(x, gx) \leq d(x, x_{2n+1}) + d(fx_{2n}, gx). \tag{2.7}$$

By (2.1) we get

$$\psi(s^{\varepsilon} d(fx_{2n}, gx)) \leq \psi(M_s(x_{2n}, x)) + L\psi(N(x_{2n}, x)),$$

where

$$M_s(x_{2n}, x) = \max\left\{d(x_{2n}, x), d(x_{2n}, x_{2n+1}), d(x, gx), \frac{d(x_{2n}, gx) + d(x, x_{2n+1})}{2s}\right\} \tag{2.8}$$

and

$$N(x_{2n}, x) = \min\{d(x, gx), d(x_{2n}, gx), d(x, x_{2n+1})\}. \tag{2.9}$$

Letting $n \to \infty$ in (2.8) and (2.9) and using

$$\frac{d(x_{2n}, gx) + d(x, x_{2n+1})}{2s} \leq \frac{d(x_{2n}, x) + d(x, gx)}{2} + \frac{d(x, x_{2n+1})}{2s},$$

we obtain $\lim_{n\to\infty} M_s(x_{2n}, x) = d(x, gx)$ and $\lim_{n\to\infty} N(x_{2n}, x) = 0$. Further, we deduce that

$$\overline{\lim_{n\to\infty}} \psi(s^{\varepsilon} d(fx_{2n}, gx)) \leq \psi\left(\overline{\lim_{n\to\infty}} M_s(x_{2n}, x)\right) + L \cdot \psi(0) = \psi(d(x, gx)).$$

Since ψ is nondecreasing, we arrive at

$$\overline{\lim_{n\to\infty}} s^{\varepsilon} d(fx_{2n}, gx) \leq d(x, gx). \tag{2.10}$$

Now (2.7) and (2.10) imply that $gx = x$. Similarly, we claim that $fx = x$. □

Remark 2.4 Theorem 2.3 improves and generalizes the main results of [15] (also see Theorem 1.8 and Theorem 1.9) in several directions. For one thing, the constant $\varepsilon > 1$ is arbitrary and is not only limited to $\varepsilon = 2$ stated by Theorem 1.8 and Theorem 1.9. This probably brings us more convenience in applications. For another thing, Theorem 2.3 dismisses the cyclic representation. In addition, the proof Theorem 2.3 is much simpler than the one of Theorem 1.8 and Theorem 1.9.

Finally we announce the main result of this paper:

Theorem 2.5 Theorem 1.8 together with Theorem 1.9 is equivalent to Theorem 2.3 in case of $\varepsilon = 2$.

Proof For all details and explanations see [20], [21] and the proof of Theorem 2.1.

3. APPLICATION

By using Theorem 2.3, we shall consider the existence of solutions for the following integral equation with an unknown function u:

$$u(t) = \int_0^T G(t, s) f(s, u(s)) ds, \ t \in [0, T], \tag{3.1}$$

where $T > 0$ is a constant, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$, $G : [0, T] \times [0, T] \to [0, \infty)$ are given continuous functions.

Denote $X = C[0,T]$ be the set of real continuous functions on $[0,T]$ and let $d : X \times X \to \mathbb{R}^+$ be given by

$$d(u,v) = \max_{0 \le t \le T} |u(t) - v(t)|^2, \quad \forall u, v \in X.$$

It is easy to check that (X,d) is a complete b-metric space with parameter $s = 2$. We endow X with the partial order given by

$$x \preceq y \Leftrightarrow x(t) \le y(t) \text{ for all } t \in [0,T].$$

Validly, $(X, \preceq, d)$ is regular.

Define a mapping $T : X \to X$ by

$$Tu(t) := \int_0^T G(t,z) f(z, u(z))\, dz, \quad t \in [0,T],$$

then u is a solution of the given equation (3.1) if and only if it is a fixed point of T. We shall prove that T has a fixed point under the following assumptions.

(i) For all $z \in [0,T]$, $f(z, .)$ is a decreasing function, that is, $x, y \in \mathbb{R}, x \ge y$ implies $f(z,x) \le f(z,y)$;

(ii) There exists a constant $\gamma > 0$ such that

$$\max_{0 \le t \le T} \int_0^T G(t,z)\, dz \le \frac{10}{21\sqrt{\gamma}};$$

(iii) For all $z \in [0,T]$ and for all comparable $x, y \in X$,

$$\begin{aligned} 0 &\le |f(z,x(z)) - f(z,y(z))| \\ &\le \Bigg(\gamma \max\Bigg\{ |x(z) - y(z)|^2, |x(z) - Tx(z)|^2, |y(z) - Ty(z)|^2, \\ &\quad \frac{|x(z) - Ty(z)|^2 + |y(z) - Tx(z)|^2}{4} \Bigg\}\Bigg)^{\frac{1}{2}}. \end{aligned} \tag{3.2}$$

(iv) There exists a constant $\varepsilon \in \left(1, \frac{2\ln 2.1}{\ln 2}\right)$.

Theorem 3.1 Under the conditions (i)-(iv), the equation (3.1) has a solution $x^* \in X$.

Proof First of all, if $x \preceq y$, then by (i), we have

$$Ty(t) - Tx(t) = \int_0^T G(t,z) \left[f(z, y(z)) - f(z, x(z)) \right] dz \ge 0, \quad t \in [0,T].$$

That is, $Tx \preceq Ty$. This means that T is increasing.

By virtue of (3.2), we have that

$$\begin{aligned} &[f(z,x) - f(z,y)]^2 \\ &\le \gamma \max\Bigg\{ d(x,y), d(x,Tx), d(y,Ty), \\ &\quad \frac{d(x,Ty) + d(y,Tx)}{4} \Bigg\}. \end{aligned} \tag{3.3}$$

Then for all $t \in [0,T]$ and all comparable $x, y \in X$, by (ii) and (3.3), we speculate that

$$
\begin{aligned}
d(Tx,Ty) &= \max_{t\in[0,T]} |Tx(t) - Ty(t)|^2 \\
&= \max_{t\in[0,T]} \left(\int_0^T G(t,z)\left[f(z,x(z)) - f(z,y(z))\right] dz \right)^2 \\
&\leq \frac{100}{441} \max\left\{ d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{4} \right\}.
\end{aligned}
$$

By (iv), it follows that $\frac{100}{441} < \frac{1}{2^\varepsilon} = \frac{1}{s^\varepsilon}$, thus all the conditions of Theorem 2.3 are satisfied where ψ, φ are identity mappings and $T = f = g, L = 0$. So T has a fixed point $u(t) \in X$, that is, the integral equation (3.1) has a solution $u(t) \in X = C[0,T]$. □

Remark 3.2 In the above application we use ordinary fixed point theorem, while Corollary 2 of [15] uses cyclical-type fixed point result. Actually, these both results are equivalent, then our approach has an advantage because we use the conditions (i)-(iv), while in Corollary 2 of [15] authors utilize the conditions (4.2)-(4.7) as well as two subsets A_1 and A_2. Also, our application shows that their main result is not applicable.

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A FIXED POINT METHOD TO THE STABILITY OF A JENSEN FUNCTIONAL EQUATION IN INTUITIONISTIC FUZZY 2-BANACH SPACES

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Abstract. In this paper, we recall the notion of intuitionistic fuzzy 2-normed space introduced in [1] and using the fixed point method, we investigate the Hyers-Ulam stability of the following functional equation

$$2f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)+f\left(\frac{y-x}{2}\right)=f(x)+f(y) \tag{1}$$

in intuitionistic fuzzy 2-Banach spaces.

1. Introduction

The concept of the stability for functional equations was introduced for the first time by Ulam in 1940 [2]. He proposed the famous Ulam stability problem for a metric group homomorphism. In 1941, Hyers [3] solved this stability problem for additive mappings subject to the Hyers condition on approximately additive mappings in Banach spaces. In 1951, Bourgin [4] treated the Ulam stability problem for additive mappings. Subsequently the result of Hyers was generalized by Rassias [5] for linear mapping by considering an unbounded Cauchy difference.

The functional equation $f(x+y)+f(x-y)=2f(x)+2f(y)$ is called a quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. The Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [6] for mappings $f:\mathcal{X}\to\mathcal{Y}$, where $\mathcal{X}$ is a normed space and $\mathcal{Y}$ is a Banach space.

In 1984, Katsaras [7] defined a fuzzy norm on a linear space to construct a fuzzy vector topological structure on the space. Later, some mathematicians have defined fuzzy norms on a linear space from various points of view [8, 9]. In particular, in 2003, Bag and Samanta [10], following Cheng and Mordeson [11], gave an idea of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [12]. They also established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces. Recently, considerable attention has been increasing to the problem of fuzzy stability of functional equations. Several various stability results concerning Cauchy, Jensen, simple quadratic, and cubic functional equations have been investigated [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24].

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Quite recently, the stability results in the setting of intuitionistic fuzzy normed space have been studied in [25, 26, 27, 28]; respectively, while the idea of intuitionistic fuzzy normed space was introduced in [29].

2. Preliminaries

Definition 2.1. *Let $\mathcal{X}$ be a real linear space of dimension greater than one and let $\|\cdot,\cdot\|$ be a real-valued function on $\mathcal{X}\times\mathcal{X}$ satisfying the following condition:*
(1) $\|x,y\|=\|y,x\|$ *for all* $x,y\in\mathcal{X}$*;*
(2) $\|x,y\|=0$ *if and only if* x,y *are linearly dependent;*
(3) $\|\alpha x,y\|=|\alpha|\|x,y\|$ *for all* $x,y\in\mathcal{X}$ *and* $\alpha\in\mathbb{R}$*;*
(4) $\|x,y+z\|\le\|x,y\|+\|x,z\|$ *for all* $x,y,z\in\mathcal{X}$*.*
Then the function $\|\cdot,\cdot\|$ is called a 2-norm on X and the pair $(X,\|\cdot,\cdot\|)$ is called a 2-normed linear space.

Definition 2.2. *A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is a continuous t-norm if $*$ satisfies the following conditions:*
(1) $*$ *is commutative and associative;*
(2) $*$ *is continuous;*
(3) $a*1=a$ *for all* $a\in[0,1]$*;*
(4) $a*b\le c*d$*, whenever* $a\le c$ *and* $b\le d$ *for all* $a,b,c,d\in[0,1]$.

Example 2.1. *An example of continuous t-norm is*

$$a*b=\min\{a,b\}.$$

Definition 2.3. *A binary operation $\diamond:[0,1]\times[0,1]\to[0,1]$ is a continuous t-conorm if $\diamond$ satisfies the following conditions:*
(1) $\diamond$ *is commutative and associative;*
(2) $\diamond$ *is continuous;*
(3) $a\diamond 0=a$ *for all* $a\in[0,1]$*;*
(4) $a\diamond b\le c\diamond d$*, whenever* $a\le c$ *and* $b\le d$ *for all* $a,b,c,d\in[0,1]$.

Example 2.2. *An example of continuous t-conorm is*

$$a\diamond b=\max\{a,b\}.$$

Definition 2.4. *Let $\mathcal{X}$ be a real linear space. A fuzzy subset μ of $\mathcal{X}\times\mathcal{X}\times\mathbb{R}$ is called a fuzzy 2-norm on $\mathcal{X}$ if and only if for all $x,y,z\in\mathcal{X}$, and $t,s,c\in\mathbb{R}$,*
(1) $\mu(x,y,t)=0$ *for all* $t\le 0$.
(2) $\mu(x,y,t)=1$ *if and only if* x,y *are linearly dependent for all* $t>0$.
(3) $\mu(x,y,t)$ *is invariant under any permutation of* x,y.
(4) $\mu(x,cy,t)=\mu(x,y,\frac{t}{|c|})$ *for all* $t>0$ *and* $c\neq 0$.
(5) $\mu(x+z,y,t+s)\ge\mu(x,y,t)*\mu(z,y,s)$ *for all* $t,s>0$.
(6) $\mu(x,y,.)$ *is a non-decreasing function on* $\mathbb{R}$ *and*

$$\lim_{t\to\infty}\mu(x,y,t)=1$$

Then μ is said to be a fuzzy 2-norm on a linear space $\mathcal{X}$, and the pair $(\mathcal{X},\mu)$ is called a fuzzy 2-normed linear space.

Example 2.3. *Let $(\mathcal{X}, \|\cdot,\cdot\|)$ be a 2-normed linear space. Define*

$$\mu(x, y, t) = \begin{cases} \frac{t}{t+\|x,y\|} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0, \end{cases}$$

where $x, y \in \mathcal{X}$ and $t \in \mathbb{R}$. Then $(\mathcal{X}, \mu)$ is a fuzzy 2-normed linear space.

Definition 2.5. *Let $(\mathcal{X}, \mu)$ be a fuzzy 2-normed linear space. Let $\{x_n\}$ be a sequence in $\mathcal{X}$. Then $\{x_n\}$ is said to be convergent if there exists $x \in \mathcal{X}$ such that*

$$\lim_{n\to\infty} \mu(x_n - x, y, t) = 1$$

for all $t > 0$ and all $y \in \mathcal{X}$.

Definition 2.6. *Let $(\mathcal{X}, \mu)$ be a fuzzy 2-normed linear space. Let $\{x_n\}$ be a sequence in $\mathcal{X}$. Then $\{x_n\}$ is said to be a Cauchy sequence if*

$$\lim_{n\to\infty} \mu(x_{n+p} - x_n, y, t) = 1$$

for all $t > 0$, all $y \in \mathcal{X}$ and $p = 1, 2, 3, \cdots$.

Let $(\mathcal{X}, \mu)$ be a fuzzy 2-normed linear space and $\{x_n\}$ be a Cauchy sequence in $\mathcal{X}$. If $\{x_n\}$ is convergent in $\mathcal{X}$ then $(\mathcal{X}, \mu)$ is said to be a fuzzy 2-Banach space.

Definition 2.7. *Let $\mathcal{X}$ be a real linear space. A fuzzy subset ν of $\mathcal{X} \times \mathcal{X} \times \mathbb{R}$ such that for all $x, y, z \in \mathcal{X}$, and $t, s, c \in \mathbb{R}$,*

(1) *$\nu(x, y, t) = 1$ for all $t \leq 0$.*
(2) *$\nu(x, y, t) = 0$ if and only if x, y are linearly dependent for all $t > 0$.*
(3) *$\nu(x, y, t)$ is invariant under any permutation of x, y.*
(4) *$\nu(x, cy, t) = \nu(x, y, \frac{t}{|c|})$ for all $t > 0$, $c \neq 0$.*
(5) *$\nu(x, y + z, t + s) \leq \nu(x, y, t) \diamond \nu(x, z, s)$ for all $s, t > 0$*
(6) *$\nu(x, y, .)$ is a nonincreasing function and*

$$\lim_{t\to\infty} \nu(x, y, t) = 0$$

Then ν is said to be an anti fuzzy 2-norm on a linear space $\mathcal{X}$ and the pair $(\mathcal{X}, \nu)$ is called an anti fuzzy 2-normed linear space.

Definition 2.8. *Let $(\mathcal{X}, \nu)$ be an anti fuzzy 2-normed linear space and $\{x_n\}$ be a sequence in $\mathcal{X}$. Then $\{x_n\}$ is said to be convergent if there exists $x \in \mathcal{X}$ such that*

$$\lim_{n\to\infty} \nu(x_n - x, y, t) = 0$$

for all $t > 0$ and all $y \in \mathcal{X}$.

Definition 2.9. *Let $(\mathcal{X}, \nu)$ be an anti fuzzy 2-normed linear space and $\{x_n\}$ be a sequence in $\mathcal{X}$. Then $\{x_n\}$ is said to be a Cauchy sequence if*

$$\lim_{n\to\infty} \nu(x_{n+p} - x_n, y, t) = 0$$

for all $t > 0$, all $y \in \mathcal{X}$ and $p = 1, 2, 3, \cdots$.

Let $(\mathcal{X},\nu)$ be an anti fuzzy 2-normed linear space and $\{x_n\}$ be a Cauchy sequence in $\mathcal{X}$. If $\{x_n\}$ is convergent in $\mathcal{X}$ then $(\mathcal{X},\nu)$ is said to be an anti fuzzy 2-Banach space.

The following lemma is easy to prove and we will omit it.

Lemma 2.1. *Consider the set L^* and operation $\leq_{L^*}$ defined by*

$$L^* = \{(x_1,x_2) : (x_1,x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1\}$$

$$(x_1,x_2) \leq_{L^*} (y_1,y_2) \Longleftrightarrow x_1 \leq y_1\ ,\ x_2 \geq y_2$$

for all $(x_1,x_2),(y_1,y_2) \in L^*$. Then $(L^*,\leq_{L^*})$ is a complete lattice.

Definition 2.10. *A continuous t-norm τ on $L = [0,1]^2$ is said to be continuous t-representable if there exist a continuous t-norm $*$ and a continuous t-conorm $\diamond$ on $[0,1]$ such that, for all $x = (x_1,x_2), y = (y_1,y_2) \in L$,*

$$\tau(x,y) = (x_1 * y_1, x_2 \diamond y_2).$$

Definition 2.11. *Let $\mathcal{X}$ be a set. A function $d : \mathcal{X}\times\mathcal{X} \to [0,\infty]$ is called a generalized metric on $\mathcal{X}$ if and only if d satisfies:*

(M_1) $d(x,y) = 0 \Leftrightarrow x = y\ \forall x,y \in \mathcal{X}$
(M_2) $d(x,y) = d(y,x)\ \forall x,y \in \mathcal{X}$
(M_3) $d(x,z) \leq d(x,y) + d(y,z)\ \forall x,y,z \in \mathcal{X}$

Theorem 2.1. ([30]) *Let $(\mathcal{X},d)$ be a complete generalized metric space and $\mathcal{J} : \mathcal{X} \to \mathcal{X}$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in \mathcal{X}$, either*

$$d(\mathcal{J}^n x, \mathcal{J}^{n+1}x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(a) *$d(\mathcal{J}^n x, \mathcal{J}^{n+1}x) < \infty$ for all $n \geq n_0$;*
(b) *the sequence $\{\mathcal{J}^n x\}$ converges to a fixed point y^* of $\mathcal{J}$;*
(c) *y^* is the unique fixed point of $\mathcal{J}$ in the set $\mathcal{Y} = \{y \in \mathcal{X} : d(\mathcal{J}^{n_0}x, y) < \infty\}$;*
(d) *$d(y,y^*) \leq \frac{1}{1-L} d(y,\mathcal{J}y)$ for all $y \in \mathcal{Y}$.*

This theorem was used by Cădariu and Radu (see [31, 32, 33, 34]) and then others to obtain the applications of fixed point theory in stability problems (cf. [24, 35, 36, 37, 38, 39, 40, 41, 42, 43]).

Definition 2.12. *A 3-tuple $(\mathcal{X},\rho_{\mu,\nu},\tau)$ is said to be an intuitionistic fuzzy 2-normed space(for short, IF2NS) if $\mathcal{X}$ is a real linear space, and μ and ν are a fuzzy 2-norm and an anti fuzzy 2-norm, respectively, such that $\nu(x,y,t)+\mu(x,y,t) \leq 1$, τ is continuous t-representable, and*

$$\rho_{\mu,\nu} : \mathcal{X}\times\mathcal{X}\times\mathbb{R} \to L^*$$

$$\rho_{\mu,\nu}(x,y,t) = (\mu(x,y,t),\nu(x,y,t))$$

is a function satisfying the following conditions, for all $x,y,z \in \mathcal{X}$, and $t,s,\alpha \in \mathbb{R}$,

(1) *$\rho_{\mu,\nu}(x,y,t) = (0,1) = 0_{L^*}$ for all $t \leq 0$.*
(2) *$\rho_{\mu,\nu}(x,y,t) = (1,0) = 1_{L^*}$ if and only if x,y are linearly dependent, for all $t > 0$.*
(3) *$\rho_{\mu,\nu}(\alpha x,y,t) = \rho_{\mu,\nu}(x,y,\frac{t}{|\alpha|})$ for all $t > 0$ and $\alpha \neq 0$*
(4) *$\rho_{\mu,\nu}(x,y,t)$ is invariant under any permutation of x,y.*
(5) *$\rho_{\mu,\nu}(x+z,y,t+s) \geq_{L^*} \tau(\rho_{\mu,\nu}(x,y,t),\rho_{\mu,\nu}(z,y,s))$ for all $t,s > 0$.*

(6) $\rho_{\mu,\nu}(x, y, .)$ *is continuous and*

$$\lim_{t\to 0} \rho_{\mu,\nu}(x, y, t) = 0_{L^*} \text{ and } \lim_{t\to\infty} \rho_{\mu,\nu}(x, y, t) = 1_{L^*}$$

Then $\rho_{\mu,\nu}$ is said to be an intuitionstic fuzzy 2-norm on a real linear space $\mathcal{X}$.

Example 2.4. *Let $(\mathcal{X}, \|\cdot,\cdot\|)$ be a 2-normed space,*

$$\tau(a, b) = (a_1b_1, \min(a_2 + b_2, 1))$$

be continuous t-representable for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ and μ, ν be a fuzzy and an anti fuzzy 2-norm, respectively. We define

$$\rho_{\mu,\nu}(x, y, t) = \left(\frac{t}{t + m\|x, y\|}, \frac{\|x, y\|}{t + m\|x, y\|}\right)$$

for all $t \in \mathbb{R}^+$ and $m > 1$. Then $(\mathcal{X}, \rho_{\mu,\nu}, \tau)$ is an IF2NS.

Definition 2.13. *A sequence $\{x_n\}$ in an IF2NS $(\mathcal{X}, \rho_{\mu,\nu}, \tau)$ is said to be convergent to a point $x \in \mathcal{X}$ if*

$$\lim_{n\to\infty} \rho_{\mu,\nu}(x_n - x, y, t) = 1_{L^*}$$

for all $t > 0$ and all $y \in \mathcal{X}$.

Definition 2.14. *A sequence $\{x_n\}$ in an IF2NS $(\mathcal{X}, \rho_{\mu,\nu}, \tau)$ is said to be a Cauchy sequence if for any $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathcal{N}$ such that*

$$\rho_{\mu,\nu}(x_n - x_m, y, t) \geq_{L^*} (1 - \epsilon, \epsilon)$$

for all $n, m \geq n_0$ and all $y \in \mathcal{X}$.

Definition 2.15. *An IF2NS space $(\mathcal{X}, \rho_{\mu,\nu}, \tau)$ is said to be complete if every Cauchy sequence in $(\mathcal{X}, \rho_{\mu,\nu}, \tau)$ is convergent. A complete intuitionistic fuzzy 2-normed space is called an intuitionistic fuzzy 2-Banach space.*

3. Hyers-Ulam stability of the functional equation (1) in IF2NS: an odd mapping case

Using the fixed point alternative approach, we prove the Hyers-Ulam stability of the functional equation (1) in intuitionistic fuzzy 2-Banach spaces for an odd mapping case.

Let $\mathcal{X}, \mathcal{Y}$ be real linear spaces. For a given mapping $f : \mathcal{X} \to \mathcal{Y}$, we define

$$Df(x, y) := 2f\left(\frac{x + y}{2}\right) + f\left(\frac{x - y}{2}\right) + f\left(\frac{y - x}{2}\right) - f(x) - f(y).$$

Lemma 3.1. *Let $\mathcal{X}, \mathcal{Y}$ be real linear spaces. An odd mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies*

$$2f\left(\frac{x + y}{2}\right) + f\left(\frac{x - y}{2}\right) + f\left(\frac{y - x}{2}\right) = f(x) + f(y) \tag{2}$$

if and only if it is Jensen additive.

Proof. Assume that $f : \mathcal{X} \to \mathcal{Y}$ satisfies (2). Since f is odd, we have $f(-x) = -f(x)$ for all $x, y \in \mathcal{X}$. It follows from (2) that $2f\left(\frac{x+y}{2}\right) = f(x) + f(y)$ for all $x, y \in \mathcal{X}$.

Conversely, assume that $f : \mathcal{X} \to \mathcal{Y}$ is Jensen additive. Then it is easy to show that f satisfies (2). □

Theorem 3.1. *Let $\mathcal{X}$ be a real linear space, $(\mathcal{Z}, \rho'_{\mu,\nu}, \tau')$ an intuitionistic fuzzy 2-normed space and let $\phi : \mathcal{X} \times \mathcal{X} \to \mathcal{Z}, \varphi : \mathcal{X} \times \mathcal{X} \to \mathcal{Z}$ be mappings such that for some $0 < \alpha^2 < 2$*

$$\rho'_{\mu,\nu}\left(\phi(2x,2y), \varphi(2x,2y), t\right) \geq_{L^*} \rho'_{\mu,\nu}\left(\alpha\phi(x,y), \varphi(x,y), t\right) \tag{3}$$

for all $x, y \in \mathcal{X}$ and $t \in \mathbb{R}^+$. Let $(\mathcal{Y}, \rho_{\mu,\nu}, \tau)$ be a complete intuitionistic fuzzy 2-normed space. If $\xi : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ is a mapping such that $\xi(2x, 2y) = \frac{1}{\alpha}\xi(x, y)$ for all $x, y \in \mathcal{X}$ and $f : \mathcal{X} \to \mathcal{Y}$ is an odd mapping such that

$$\rho_{\mu,\nu}(Df(x,y), \xi(x,y), t) \geq_{L^*} \rho'_{\mu,\nu}(\phi(x,y), \varphi(x,y), t) \tag{4}$$

for all $x, y \in \mathcal{X}, t > 0$, then there is a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\rho_{\mu,\nu}(f(x) - A(x), \xi(x,0), t) \geq_{L^*} \rho'_{\mu,\nu}\left(\phi(x,0), \varphi(x,0), \frac{2-\alpha^2}{\alpha^2}t\right) \tag{5}$$

Proof. Putting $y = 0$ in (4), we have

$$\rho_{\mu,\nu}\left(2f\left(\frac{x}{2}\right) - f(x), \xi(x,0), t\right) \geq_{L^*} \rho'_{\mu,\nu}\left(\phi(x,0), \varphi(x,0), t\right). \tag{6}$$

Replacing x by $2x$ in (6), we have

$$\rho_{\mu,\nu}\left(2f(x) - f(2x), \xi(2x,0), t\right) \geq_{L^*} \rho'_{\mu,\nu}\left(\phi(2x,0), \varphi(2x,0), t\right). \tag{7}$$

It follows from (3), (7) and the property of ξ that

$$\begin{aligned}\rho_{\mu,\nu}\left(f(x) - \frac{f(2x)}{2}, \xi(x,0), t\right) &\geq_{L^*} \rho'_{\mu,\nu}\left(\phi(2x,0), \varphi(2x,0), \frac{2}{\alpha}t\right)\\ &\geq_{L^*} \rho'_{\mu,\nu}\left(\frac{\alpha^2}{2}\phi(x,0), \varphi(x,0), t\right)\end{aligned}$$

for all $x \in \mathcal{X}$ and $t > 0$.

Consider the set $\Omega = \{g : \mathcal{X} \to \mathcal{Y}\}$ and define a generalized metric d on Ω by

$$d(g,h) = \inf\left\{c \in \mathbb{R}^+ : \rho_{\mu,\nu}(g(x) - h(x), \xi(x,0), t) \geq_{L^*} \rho'_{\mu,\nu}(c\phi(x,0), \varphi(x,0), t)\right\}$$

for all $x \in \mathcal{X}$ and $t > 0$ with $\inf \emptyset = \infty$. It is easy to show that (Ω, d) is complete (see [44]).

Define $J : \mathcal{X} \to \mathcal{X}$ by $Jg(x) = \dfrac{g(2x)}{2}$ for all $x \in \mathcal{X}$. Now, we prove that J is strictly contractive mapping of Ω with the Lipschitz constant $\frac{\alpha^2}{2}$.
Let $g, h \in E$ be given such that $d(g, h) < \epsilon$. Then

$$\rho_{\mu,\nu}\left(g(x) - h(x), \xi(x,0), t\right) \geq_{L^*} \rho'_{\mu,\nu}(\epsilon\phi(x,0), \varphi(x,0), t)$$

for all $x \in \mathcal{X}$ and $t > 0$. So

$$\begin{aligned}&\rho_{\mu,\nu}\left(Jg(x) - Jh(x), \xi(x,0), t\right) = \rho_{\mu,\nu}\left(g(2x) - h(2x), \xi(2x,0), \frac{2}{\alpha}t\right)\\ &\geq_{L^*} \rho'_{\mu,\nu}\left(\epsilon\phi(2x,0), \varphi(2x,0), \frac{2}{\alpha}t\right) =_{L^*} \rho'_{\mu,\nu}\left(\frac{\alpha^2}{2}\epsilon\phi(x,0), \varphi(x,0), t\right).\end{aligned}$$

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Then $d(Jg, Jh) \leq \frac{\alpha^2}{2}\ d(g, h)$ for all $g, h \in \Omega$. It follows from (7) that

$$d(f, Jf) \leq \frac{\alpha^2}{2} < \infty$$

It follows from Theorem 2.1 that there exists a mapping $A : \mathcal{X} \to \mathcal{Y}$ satisfying the following:

(1) A is a fixed point of J, that is,

$$A(2x) = 2A(x) \tag{8}$$

(2) The mapping A is a unique fixed point of J in the set

$$\Delta = \{h \in \Omega : d(g, h) < \infty\}$$

This implies that A is a unique mapping satisfying (8).

(3) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies that

$$\lim_{n\to\infty} \frac{f(2^n x)}{2^n} = A(x)$$

for all $x \in X$.

(4) $d(f, A) \leq \frac{1}{1-L} d(f, Jf)$ with $f \in \Delta$, which implies the inequality $d(f, A) \leq \frac{\alpha^2}{2-\alpha^2}$. So

$$\rho_{\mu,\nu}\left(f(x) - A(x), \xi(x, 0), t\right) \geq_{L^*} \rho'_{\mu,\nu}\left(\phi(x, 0), \varphi(x, 0), \frac{2-\alpha^2}{\alpha^2} t\right).$$

This implies that the inequality (5) holds.

It remains to show that A is an additive mapping. Replacing x and y by $2^n x$ and $2^n y$ in (4), respectively, we get

$$\rho_{\mu,\nu}\left(\frac{1}{2^n} Df(2^n x, 2^n y), \xi(2^n x, 2^n y), \frac{t}{2^n}\right) \geq_{L^*} \rho'_{\mu,\nu}\left(\phi(2^n x, 2^n y), \varphi(2^n x, 2^n y), t\right).$$

By the property of $\xi(x, y)$, we have

$$\rho_{\mu,\nu}\left(\frac{1}{2^n} Df(2^n x, 2^n y), \frac{1}{\alpha^n}\xi(x, y), \frac{t}{2^n}\right) \geq_{L^*} \rho'_{\mu,\nu}\left(\phi(2^n x, 2^n y), \varphi(2^n x, 2^n y), t\right).$$

Thus

$$\begin{aligned}\rho_{\mu,\nu}\left(\frac{1}{2^n} Df(2^n x, 2^n y), \xi(x, y), t\right) &\geq_{L^*} \rho'_{\mu,\nu}\left(\phi(2^n x, 2^n y), \varphi(2^n x, 2^n y), \frac{2^n\ t}{\alpha^n}\right) \\ \geq_{L^*} \rho'_{\mu,\nu}\left(\alpha^n \phi(x, y), \varphi(x, y), \frac{2^n\ t}{\alpha^n}\right) &=_{L^*} \rho'_{\mu,\nu}\left(\frac{\alpha^{2n}}{2^n}\phi(x, y), \varphi(x, y), t\right).\end{aligned}$$

As $n \to \infty$, we have

$$\rho_{\mu,\nu}(DA(x, y), \xi(x, y), t) \geq_{L^*} 1_{L^*}.$$

Thus A is an additive mapping, as desired. $\square$

Corollary 3.1. *Let X be a linear space, $(\mathcal{Z}, \rho'_{\mu,\nu}, \tau')$ be an IF2N-space, $(\mathcal{Y}, \rho_{\mu,\nu}, \tau)$ be a complete IF2N-space, p be real number and $z_0, z_1 \in \mathcal{Z}$. If $\xi : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ is a mapping such that $\xi(2x, 2y) = \frac{1}{2^p}\xi(x, y)$ for all $x, y \in \mathcal{X}$ and $f : \mathcal{X} \to \mathcal{Y}$ is an odd mapping such that*

$$\rho_{\mu,\nu}(Df(x, y), \xi(x, y), t) \geq_{L^*} \rho'_{\mu,\nu}((\|x\|^p + \|y\|^p)z_0, z_1, t)$$

for all $x, y \in \mathcal{X}$, $t > 0$ and $0 < p < \frac{1}{2}$, then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\rho_{\mu,\nu}(f(x) - A(x), \xi(x, 0), t) \geq_{L^*} \rho'_{\mu,\nu}\left(\|x\|^p z_0, z_1, \frac{2 - 2^{2p}}{2^{2p}} t\right)$$

for all $x \in \mathcal{X}$ and $t > 0$.

Proof. Let $\phi, \varphi : \mathcal{X} \times \mathcal{X} \to \mathcal{Z}$ be defined by $\phi(x, y) = (\|x\|^p + \|y\|^p) z_0$ and $\varphi(x, y) = z_1$. Then the result follows from Theorem 3.1 by taking $\alpha = 2^p$. □

Corollary 3.2. *Let X be a linear space, $(\mathcal{Z}, \rho'_{\mu,\nu}, \tau')$ be an IF2N-space, $(Y, \rho_{\mu,\nu}, \tau)$, be a complete IF2N-space and let $z_0, z_1 \in \mathcal{Z}$. If $f : \mathcal{X} \to \mathcal{Y}$ is an odd mapping such that*

$$\rho_{\mu,\nu}(Df(x, y), \xi(x, y), t) \geq_{L^*} \rho'_{\mu,\nu}(\epsilon z_0, z_1, t)$$

for all $x, y \in \mathcal{X}, t > 0$, then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\rho_{\mu,\nu}(f(x) - A(x), \xi(x, 0), t) \geq_{L^*} \rho'_{\mu,\nu}(\epsilon z_0, z_1, t)$$

for all $x \in \mathcal{X}$ and $t > 0$.

Proof. Let $\phi, \varphi : \mathcal{X} \times \mathcal{X} \to \mathcal{Z}$ be defined by $\phi(x, y) = z_0$ and $\varphi(x, y) = z_1$. Then the result follows from Theorem 3.1 by taking $\alpha = 1$. □

4. Hyers-Ulam stability of the functional equation (1) in IF2NS: an even mapping case

Using the fixed point alternative approach, we prove the Hyers-Ulam stability of the functional equation (1) in intuitionistic fuzzy 2-Banach spaces for an even mapping case.

Lemma 4.1. *Let $\mathcal{X}, \mathcal{Y}$ be real linear spaces. An even mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies*

$$2f\left(\frac{x + y}{2}\right) + f\left(\frac{x - y}{2}\right) + f\left(\frac{y - x}{2}\right) = f(x) + f(y) \tag{9}$$

if and only if it is Jensen quadratic.

Proof. Assume that $f : \mathcal{X} \to \mathcal{Y}$ satisfies (9). Since f is even, we have $f(-x) = f(x)$ for all $x, y \in \mathcal{X}$. It follows from (9) that $2f\left(\frac{x + y}{2}\right) + 2f\left(\frac{x - y}{2}\right) = f(x) + f(y)$ for all $x, y \in \mathcal{X}$.

Conversely, assume that $f : \mathcal{X} \to \mathcal{Y}$ is Jensen quadratic. Then it is easy to show that f satisfies (9). □

Theorem 4.1. *Let $\mathcal{X}$ be a real linear space, $(\mathcal{Z}, \rho'_{\mu,\nu}, \tau')$ an intuitionistic fuzzy 2-normed space and let $\phi : \mathcal{X} \times \mathcal{X} \to \mathcal{Z}, \varphi : \mathcal{X} \times \mathcal{X} \to \mathcal{Z}$ be mappings such that for some $0 < \alpha^2 < 4$*

$$\rho'_{\mu,\nu}\left(\phi(2x, 2y), \varphi(2x, 2y), t\right) \geq_{L^*} \rho'_{\mu,\nu}\left(\alpha\phi(x, y), \varphi(x, y), t\right) \tag{10}$$

for all $x, y \in \mathcal{X}$ and $t \in \mathbb{R}^+$. Let $(\mathcal{Y}, \rho_{\mu,\nu}, \tau)$ be a complete intuitionistic fuzzy 2-normed space. If $\xi : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ is a mapping such that $\xi(2x, 2y) = \frac{1}{\alpha}\xi(x, y)$ for all $x, y \in \mathcal{X}$ and $f : \mathcal{X} \to \mathcal{Y}$

is an even mapping satisfying $f(0) = 0$ *and* (4)*, then there is a unique quadratic mapping* $Q : \mathcal{X} \to \mathcal{Y}$ *such that*

$$\rho_{\mu,\nu}(f(x) - Q(x), \xi(x, 0), t) \geq_{L^*} \rho'_{\mu,\nu}\left(\phi(x, 0), \varphi(x, 0), \frac{4-\alpha^2}{\alpha^2}t\right) \tag{11}$$

for all $x \in \mathcal{X}$ *and* $t > 0$.

Proof. Putting $y = 0$ in (4), we have

$$\rho_{\mu,\nu}\left(4f\left(\frac{x}{2}\right) - f(x), \xi(x, 0), t\right) \geq_{L^*} \rho'_{\mu,\nu}\left(\phi(x, 0), \varphi(x, 0), t\right). \tag{12}$$

Replacing x by $2x$ in (12), we have

$$\rho_{\mu,\nu}\left(4f(x) - f(2x), \xi(2x, 0), t\right) \geq_{L^*} \rho'_{\mu,\nu}\left(\phi(2x, 0), \varphi(2x, 0), t\right). \tag{13}$$

It follows from (10), (13) and the property of ξ that

$$\begin{aligned} \rho_{\mu,\nu}\left(f(x) - \frac{f(2x)}{4}, \xi(x, 0), t\right) &\geq_{L^*} \rho'_{\mu,\nu}\left(\phi(2x, 0), \varphi(2x, 0), \frac{4}{\alpha}t\right) \\ &\geq_{L^*} \rho'_{\mu,\nu}\left(\frac{\alpha^2}{4}\phi(x, 0), \varphi(x, 0), t\right) \end{aligned}$$

for all $x \in \mathcal{X}$ and $t > 0$.

Consider the set $\Omega = \{g : \mathcal{X} \to \mathcal{Y}\}$ and define a generalized metric d on Ω as in Theorem 3.1.

Define $J : \mathcal{X} \to \mathcal{X}$ by $Jg(x) = \dfrac{g(2x)}{4}$ for all $x \in \mathcal{X}$. Now, we prove that J is strictly contractive mapping of Ω with the Lipschitz constant $\frac{\alpha^2}{4}$.
Let $g, h \in E$ be given such that $d(g, h) < \epsilon$. Then

$$\rho_{\mu,\nu}\left(g(x) - h(x), \xi(x, 0), t\right) \geq_{L^*} \rho'_{\mu,\nu}(\epsilon\phi(x, 0), \varphi(x, 0), t)$$

for all $x \in \mathcal{X}$ and $t > 0$. So

$$\begin{aligned} &\rho_{\mu,\nu}\left(Jg(x) - Jh(x), \xi(x, 0), t\right) = \rho_{\mu,\nu}\left(g(2x) - h(2x), \xi(2x, 0), \frac{4}{\alpha}t\right) \\ &\geq_{L^*} \rho'_{\mu,\nu}\left(\epsilon\phi(2x, 0), \varphi(2x, 0), \frac{4}{\alpha}t\right) =_{L^*} \rho'_{\mu,\nu}\left(\frac{\alpha^2}{4}\epsilon\phi(x, 0), \varphi(x, 0), t\right). \end{aligned}$$

Then $d(Jg, Jh) \leq \dfrac{\alpha^2}{4}\, d(g, h)$ for all $g, h \in \Omega$. It follows from (13) that $d(f, Jf) \leq \dfrac{\alpha^2}{4} < \infty$.

It follows from Theorem 2.1 that there exists a mapping $Q : \mathcal{X} \to \mathcal{Y}$ satisfying the following:

(1) Q is a fixed point of J, that is,

$$Q(2x) = 4Q(x) \tag{14}$$

(2) The mapping Q is a unique fixed point of J in the set

$$\Delta = \{h \in \Omega : d(g, h) < \infty\}$$

This implies that Q is a unique mapping satisfying (14).

(3) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies that

$$\lim_{n\to\infty} \frac{f(2^n x)}{4^n} = Q(x)$$

for all $x \in X$.

(4) $d(f, Q) \leq \dfrac{1}{1-L} d(f, Jf)$ with $f \in \Delta$, which implies the inequality $d(f, Q) \leq \frac{\alpha^2}{4-\alpha^2}$. So

$$\rho_{\mu,\nu}\left(f(x) - Q(x), \xi(x,0), t\right) \geq_{L^*} \rho'_{\mu,\nu}\left(\phi(x,0), \varphi(x,0), \frac{4-\alpha^2}{\alpha^2} t\right).$$

This implies that the inequality (11) holds.

The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 4.1. *Let X be a linear space, $(\mathcal{Z}, \rho'_{\mu,\nu}, \tau')$ be an IF2N-space, $(\mathcal{Y}, \rho_{\mu,\nu}, \tau)$ be a complete IF2N-space, p be real number and $z_0, z_1 \in \mathcal{Z}$. If $\xi : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ is a mapping such that $\xi(2x, 2y) = \frac{1}{2^p}\xi(x, y)$ for all $x, y \in \mathcal{X}$ and $f : \mathcal{X} \to \mathcal{Y}$ is an even mapping satisfying $f(0) = 0$ and*

$$\rho_{\mu,\nu}(Df(x,y), \xi(x,y), t) \geq_{L^*} \rho'_{\mu,\nu}((\|x\|^p + \|y\|^p)z_0, z_1, t)$$

for all $x, y \in \mathcal{X}$, $t > 0$ and $0 < p < 1$, then there exists a unique quadratic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\rho_{\mu,\nu}(f(x) - Q(x), \xi(x,0), t) \geq_{L^*} \rho'_{\mu,\nu}\left(\|x\|^p z_0, z_1, \frac{4-4^p}{4^p} t\right)$$

for all $x \in \mathcal{X}$ and $t > 0$.

Proof. Let $\phi, \varphi : \mathcal{X} \times \mathcal{X} \to \mathcal{Z}$ be defined by $\phi(x, y) = (\|x\|^p + \|y\|^p) z_0$ and $\varphi(x, y) = z_1$. Then the result follows from Theorem 4.1 by taking $\alpha = 2^p$. □

Corollary 4.2. *Let X be a linear space, $(\mathcal{Z}, \rho'_{\mu,\nu}, \tau')$ be an IF2N-space, $(Y, \rho_{\mu,\nu}, \tau)$, be a complete IF2N-space and let $z_0, z_1 \in \mathcal{Z}$. If $f : \mathcal{X} \to \mathcal{Y}$ is an even mapping satisfying $f(0) = 0$ and*

$$\rho_{\mu,\nu}(Df(x,y), \xi(x,y), t) \geq_{L^*} \rho'_{\mu,\nu}(\epsilon z_0, z_1, t)$$

for all $x, y \in \mathcal{X}, t > 0$, then there exists a unique quadratic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\rho_{\mu,\nu}(f(x) - Q(x), \xi(x,0), t) \geq_{L^*} \rho'_{\mu,\nu}(\epsilon z_0, z_1, 3t)$$

for all $x \in \mathcal{X}$ and $t > 0$.

Proof. Let $\phi, \varphi : \mathcal{X} \times \mathcal{X} \to \mathcal{Z}$ be defined by $\phi(x, y) = z_0$ and $\varphi(x, y) = z_1$. Then the result follows from Theorem 4.1 by taking $\alpha = 1$. □

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Characterization of modular spaces

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Abstract

In this paper we study the structure of modular spaces and random normed spaces and we show that a modular could induce a random norm

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and vice versa. Also we prove the topology generated by a modular (with a certain property) coincides with the topology generated by a random norm, and so in some situations the study of modular spaces reduces to the study of random normed spaces.

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1 Introduction

Orlicz and Birnbaum generalized the Lebesgue function spaces L^p and the theory of Orlicz spaces inspired Nakano [1] to develop the theory of modular spaces. This was generalized by Musielak and Orlicz [2]. For a good introduction to the theory of Orlicz spaces we refer the reader to Krasnoselskii and Rutickii [3]. In this paper, we show that a modular could induce a random norm and vice versa and also we show that the topology generated by a modular (with a certain property) coincides with the topology generated by a random norm.

2 Modular spaces

We start with a brief introduction to modular spaces (see [4–6, 8, 9]).

Let X be a vector space over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). A functional $\rho : X \to [0, \infty]$ is called a modular, if for f, $g \in X$, we have for any $\alpha \in \mathbb{F}$:

(*i*) $\rho(f) = 0$ if and only if $f = 0$;

(*ii*) $\rho(\alpha f) = \rho(f)$ whenever $|\alpha| = 1$;

(*iii*) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ whenever $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$.

If ρ is a modular in X, then the set defined by

$$X_\rho = \{h \in X : \lim_{\lambda \to 0} \rho(\lambda h) = 0\} \tag{2.1}$$

is called *a modular space.*

Definition 2.1. Let X_ρ be a modular space. The sequence $\{f_n\}_{n\in\mathbb{N}}$ in X_ρ is said to be ρ–convergent to $f \in X_\rho$ if $\rho(f_n - f) \to 0$, as $n \to \infty$.

The following definition plays an important role in the theory of modular function spaces.

Definition 2.2. Let X_ρ be a modular space. We say that ρ has the Ω-property if $\rho(x_n) \to 0$ implies $\rho(\lambda x_n) \to 0$ for $\lambda > 0$; here x_n is a sequence in X_ρ.

For example it is easy to see that $\rho(x) = \ln(1 + \|x\|)$ and $\rho(x) = \exp(\|x\|) - 1$ have the Ω-property (see [4]).

3 Random normed spaces

Definition 3.1. A *triangular norm* (shorter *t-norm*) is a binary operation on the unit interval $[0, 1]$, i.e., a function $T : [0, 1] \times [0, 1] \to [0, 1]$ such that for all

$a, b, c \in [0, 1]$ the following four axioms are satisfied:

(T1) $T(a, b) = T(b, a)$ (: commutativity);

(T2) $T(a, (T(b, c))) = T(T(a, b), c)$ (: associativity);

(T3) $T(a, 1) = a$ (: boundary condition);

(T4) $T(a, b) \leq T(a, c)$ whenever $b \leq c$ (: monotonicity).

The commutativity of (T1), the monotonicity (T4), and the boundary condition (T3) imply that, for any t-norm T and $x \in [0, 1]$, the following boundary conditions are also satisfied:

$$T(x, 1) = T(1, x) = x,$$

$$T(x, 0) = T(0, x) = 0,$$

and so all the t-norms coincide on the boundary of the unit square $[0, 1]^2$.

The monotonicity of a t-norm T in its second component (T4) is, together with the commutativity (T1), equivalent to the (joint) monotonicity in both components, i.e., to

$$T(x_1, y_1) \leq T(x_2, y_2) \ \text{ whenever } \ x_1 \leq x_2 \ \text{ and } \ y_1 \leq y_2. \tag{3.1}$$

Basic examples are the Łukasiewicz t-norm T_L:

$$T_L(a, b) = \max(a + b - 1, 0), \ \forall a, b \in [0, 1]$$

and the t-norms T_P, T_M, T_D, where

$$T_P(a, b) := ab,$$

$$T_M(a,b) := \min\{a,b\},$$

$$T_D(a,b) := \begin{cases} \min(a,b), & \text{if max(a,b)=1;} \\ 0, & \text{otherwise.} \end{cases}$$

If, for two t-norms T_1 and T_2, the inequality $T_1(x,y) \leq T_2(x,y)$ holds for all $(x,y) \in [0,1]^2$, then we say that T_1 is *weaker* than T_2 or, equivalently, that T_2 is stronger than T_2.

As a result of (3.1), we obtain

$$T(x,y) \leq T(x,1) = x,$$

$$T(x,y) \leq T(1,y) = y$$

for each $(x,y) \in [0,1]^2$. Since trivially $T(x,y) \geq 0 = T_D(x,y)$ for all $(x,y) \in (0,1)^2$, for an arbitrary t-norm T, we get

$$T_D \leq T \leq T_M,$$

i.e., T_D is weaker and T_M is stronger than any other t-norm, and also since $T_L < T_P$ we obtain the following ordering for the four basic t-norms

$$T_D < T_L < T_P < T_M.$$

Throughout this paper, Δ^+ is the space of distribution functions that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, \infty\} \longrightarrow [0,1]$, such that F is left-continuous, non-decreasing on $\mathbb{R}$ and $F(0) = 0$.

Now D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^-F(+\infty) = 1$, where $l^-F(x)$ denotes the left limit of the function f at the point

x, that is, $l^- F(x) = \lim_{t \to x^-} F(t)$. In particular for any $a \geq 0$, ε_a is the specific distribution function defined by

$$\varepsilon_a(t) = \begin{cases} 0 & t \leq a \\ 1 & t > a. \end{cases}$$

Definition 3.2. [10] A *random normed space* (briefly, RN-space) is a triple (X, μ, T), where X is a vector space, T is a continuous t-norm, and μ is a mapping from X into D^+ such that, if μ_x denotes the value of μ at $x \in X$, the following conditions hold:

(RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;

(RN2) $\mu_{\alpha x}(t) = \mu_x\left(\frac{t}{|\alpha|}\right)$ for all $x \in X$, $t > 0$, $\alpha \neq 0$;

(RN3) $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

Definition 3.3. Let (X, μ, T) be an RN-space. A sequence $\{x_n\}$ in X is said to be *convergent* to x in X if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x}(\epsilon) > 1 - \lambda$ whenever $n \geq N$.

Definition 3.4. Let (X, μ, T) be an RN-space. We say that μ has the $\Omega^\star$-property if $\mu_{x_n}(1) \to 1$ implies $\mu_{x_n}(t) \to 1$ for $t > 0$; here x_n is a sequence in X.

Theorem 3.5. [11] If (X, μ, T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n \to \infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.

Example 3.6. [12] Let $(X, \|.\|)$ be a normed linear space. Define

$$\mu_x(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ \frac{t}{t+\|x\|}, & \text{if } t > 0. \end{cases}$$

Then (X, μ, T_P) is a random normed space.

Example 3.7. [12] Let $(X, \|.\|)$ be a normed linear space. Define

$$\mu_x(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ e^{-\left(\frac{\|x\|}{t}\right)}, & \text{if } t > 0. \end{cases}$$

Then (X, μ, T_P) is a random normed space.

Example 3.8. [13] Let $(X, \|\cdot\|)$ be a normed linear space. Define

$$\mu_x(t) = \begin{cases} \max\{1 - \frac{\|x\|}{t}, 0\}, & \text{if } \ t > 0; \\ 0, & \text{if } \ t \leq 0. \end{cases}$$

Then $(X, \mu, \ T_L)$ is a RN-space (this was essentially proved by Musthari in [14]; see also [15]).

Definition 3.9. Let (X, μ, T) be an RN-space. We say that μ has the Ω^1-property if $\mu_x(1) = 1$ implies $x = 0$.

It is easy to see that that the RN-spaces in Examples 3.6, 3.7, 3.8 have the Ω^1-property (and also the $\Omega^\star$-property).

For more results on RN-spaces and similar spaces refer [16]– [21].

4 Main results

Theorem 4.1. *Let (X, μ, T) be a RN-space with the Ω^1-property. Define a function*

$$\varphi : [0,1] \longrightarrow [0,+\infty]$$

such that

(1) *φ is continuous and $\varphi(0) = +\infty$ and $\varphi(1) = 0$;*

(2) *φ is strictly decreasing on $[0,1]$;*

(3) *$\varphi(T(a,b)) \leq \varphi(a) + \varphi(b)$ for all $a, b \in [0,1]$.*

Let $\rho(x) = \phi(\mu_x(1))$ for $x \in X$. Then, X_ρ is a modular space.

Proof. Let (X, μ, T) be an RN-space with the Ω^1-property and let φ be a function satisfying (1)–(3).

(i) Let $x \in X$. The Ω^1-property of μ together with (RN1) imply

$$0 = \rho(x) = \varphi(\mu_x(1)) \Longleftrightarrow \mu_x(1) = 1 \Longleftrightarrow x = 0.$$

(ii) is clear. (iii) Let $x, y \in X$ and $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$. Then (note (RN3), (2) and then (RN2),(3))

$$\begin{aligned} \rho(\alpha x + \beta y) &= \varphi(\mu_{\alpha x + \beta y}(1)) \\ &\leq \varphi[T(\mu_{\alpha x}(\alpha), \mu_{\beta y}(\beta))] \\ &\leq \varphi(\mu_x(1)) + \varphi(\mu_y(1)) \\ &= \rho(x) + \rho(y). \end{aligned}$$

Now, for $x \in X$ (note (RN2)),

$$\lim_{t\to 0}\rho(tx) = \lim_{t\to 0}\varphi(\mu_{tx}(1)) = \lim_{t\to 0}\varphi\left(\mu_x\left(\frac{1}{|t|}\right)\right) = \varphi(1) = 0,$$

so, X_ρ is a modular space. $\square$

Example 4.2. Let X be a normed linear space and let (X, μ, T_P) be the random normed space in Example 3.6. Let

$$\varphi(u) = \begin{cases} +\infty, & \text{if } u = 0; \\ \ln\frac{1}{u}, & \text{if } 0 < u \leq 1. \end{cases}$$

The function φ satisfies conditions (1)–(3) in Theorem 4.1. Now Theorem 4.1 guarantees that $\phi(\mu_x(1)) = \ln(1 + \|x\|)$ is a modular (note it is also easy to check this directly).

Theorem 4.3. *Let X_ρ be a modular space. Let T be a continuous t-norm. Define a function*

$$\psi : [0, +\infty] \longrightarrow [0, 1]$$

such that

(1) ψ *is continuous and* $\psi(0) = 1$ *and* $\psi(+\infty) = 0$*;*

(2) ψ *is strictly decreasing on* $[0, +\infty]$*;*

(3) $\psi(a + b) \geq T(\psi(a), \psi(b))$ *for all* $a, b \in [0, +\infty)$.

Let

$$\mu_x(t) = \begin{cases} 0, & \textit{if} \quad t \leq 0; \\ \psi\left(\rho\left(\frac{x}{t}\right)\right), & \textit{if} \quad t > 0. \end{cases}$$

Then, (X, μ, T) *is a RN-space.*

Proof. (RN1). For $t > 0$ and $x \in X$ we have $\mu_x(t) = 1$ iff $\psi(\rho\left(\frac{x}{t}\right)) = 1$ iff $\rho\left(\frac{x}{t}\right) = 0$ iff $x = 0$.

(RN2). For $t > 0$ and $x \in X$ we have for $\alpha \neq 0$ (note (ii))

$$\mu_{\alpha x}(t) = \psi\left(\rho\left(\frac{\alpha x}{t}\right)\right) = \psi\left(\rho\left(\frac{x}{t/|\alpha|}\right)\right) = \mu_x\left(\frac{t}{|\alpha|}\right).$$

(RN3). For $t, s > 0$ and $x, y \in X$ we have (note (iii) and (3))

$$\begin{aligned}
\mu_{x+y}(t+s) &= \psi\left(\rho\left(\frac{x+y}{t+s}\right)\right) \\
&= \psi\left(\rho\left(\frac{1}{1+\frac{s}{t}}\left(\frac{x}{t}\right) + \frac{1}{1+\frac{t}{s}}\left(\frac{y}{s}\right)\right)\right) \\
&\geq \psi\left[\rho\left(\frac{x}{t}\right) + \rho\left(\frac{y}{s}\right)\right] \\
&\geq T\left(\psi\left(\rho\left(\frac{x}{t}\right)\right), \psi\left(\rho\left(\frac{y}{s}\right)\right)\right) \\
&= T(\mu_x(t), \mu_y(s)).
\end{aligned}$$

□

Example 4.4. Let X be a normed linear space. Consider the modular

$$\rho(x) = \ln(1 + \|x\|),$$

for $x \in X$. Let $\psi(t) = \exp(-t)$ for $t \in (-\infty, +\infty)$. Then the function satisfies conditions (1)–(3) in Theorem 4.3. Consider the t-norm T_P and

$$\mu_x(\lambda) = \begin{cases} 0, & \text{if } \lambda \leq 0; \\ \frac{\lambda}{\lambda + \|x\|} = \psi\left(\rho\left(\frac{x}{\lambda}\right)\right), & \text{if } 0 < \lambda. \end{cases}$$

Now Theorem 4.3 guarantees that (X, μ, T_P) is an RN-space.

Now, we consider the topology induced by a modular.

Theorem 4.5. (1). *Let (X, μ, T_P) be a RN-space with the $\Omega^\star$-property and the Ω^1-property. Let τ_μ be the topology induced by the random norm μ. Then, there exists a modular which induces a topology which coincides with τ_μ on X.*

(2). *Let (X_ρ, ρ) be a modular space with the Ω-property and let τ_ρ be the topology induced by the modular ρ. Then there exists a random norm μ which induces a topology which coincides with τ_ρ on X.*

Proof. (1). Let (X, μ, T_P) be a RN-space. Let φ be as in Example 4.2 and let $\rho(x) = \phi(\mu_x(1))$ for $x \in X$. Then, from Theorem 4.1, ρ is a modular. Now, let $\{x_n\}$ be a sequence in (X, μ, T_p) converging to x in X, i.e., $\mu_{x_n - x}(t)$ tends to 1 for $t > 0$ (so in particular $\mu_{x_n - x}(1)$ tends to 1). Then, $\rho(x_n - x) = \varphi(\mu_{x_n - x}(1))$ tends to 0, i.e., $\{x_n\}$ converges to x in the sense of Definition 2.1.

Next let $\{x_n\}$ be a sequence converging to x in X in the sense of Definition 2.1 with modular ρ (here φ is as in Example 4.2 and $\rho(x) = \phi(\mu_x(1))$ for $x \in X$) i.e., $\varphi(\mu_{x_n - x}(1))$ tends to 0. Then $\mu_{x_n - x}(1)$ tends to 1. Now since μ has the $\Omega^\star$-property, then for $t > 0$ we have that $\mu_{x_n - x}(t)$ tends to 1 i.e., $\{x_n\}$ converges to x in the sense of Definition 3.3.

Now, let A be an open set in (X, μ, T_P). Put $B = A^c$. We show B is a closed set in (X_ρ, ρ). Let x be an element in the closure of B in (X_ρ, ρ). Then there exists a sequence $\{x_n\}$ in B with x_n converging to x in the sense of Definition 2.1 with modular ρ. Now from the above x_n converges to x in the sense of Definition

3.3. Now since B is a closed set in (X, μ, T_P) then $x \in B$. Thus B is a closed set in (X_ρ, ρ) so A is an open set in (X_ρ, ρ). A similar argument show that if C is an open set in (X_ρ, ρ) then C is an open set in (X, μ, T_P).

(2). Let (X_ρ, ρ) be a modular space with the Ω-property. Let ψ be as in Example 4.4. Then Theorem 4.3 guarantees that (X, μ, T_P) is a RN-space (here μ is as in Theorem 4.3). Now, let $\{x_n\}$ be a sequence in (X_ρ, ρ) converging to x in X, i.e., $\rho(x_n - x)$ tends to 0. Now since ρ has the Ω-property, then for $t > 0$ we have that $\mu_{x_n - x}(t) = \psi\left(\rho\left(\frac{x_n - x}{t}\right)\right)$ tends to 1, i.e., $\{x_n\}$ converges to x in the sense of Definition 3.3.

Next let $\{x_n\}$ be a sequence converging to x in X in the sense of Definition 3.3 i.e., $\mu_{x_n - x}(t) = \psi\left(\rho\left(\frac{x_n - x}{t}\right)\right)$ tends to 1 for $t > 0$ (here ψ is as in Example 4.4 and μ is as in Theorem 4.3). Then $\rho\left(\frac{x_n - x}{t}\right)$ tends to 0 for $t > 0$ so in particular $\rho(x_n - x)$ tends to 0 i.e., $\{x_n\}$ converges to x in the sense of Definition 2.1.

Now, let A be an open set in (X_ρ, ρ). Put $B = A^c$. We show B is a closed set in (X, μ, T_P). Let x be an element in the closure of B in (X, μ, T_P). Then there exists a sequence $\{x_n\}$ in B with x_n converging to x in the sense of Definition 3.3 with random norm μ. Now from the above x_n converges to x in the sense of Definition 2.1. Now since B is a closed set in (X_ρ, ρ) then $x \in B$. Thus B is a closed set in (X, μ, T_P) so A is an open set in (X, μ, T_P). A similar argument show that if C is an open set in (X, μ, T_P) then C is an open set in (X_ρ, ρ). $\square$

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Ulam-Hyers stability of 3-Jordan homomorphisms in C^*-ternary algebras

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Abstract. In this paper, we investigate the Ulam-Hyers stability of C^*-ternary 3-Jordan homomorphisms for the functional equation

$$f(x_1+x_2, y_1+y_2, z_1+z_2) = \sum_{1\le i,j,k\le 2} f(x_i, y_j, z_k)$$

in C^*-ternary algebras.

1. Introduction

Ternary algebraic operations were considered in the 19th century by several mathematicians and physicists such as Cayley [8] who introduced the notion of cubic matrix which in turn was generalized by Kapranov, Gelfand and Zelevinskii [14]. As an application in physics, the quark model inspired a particular brand of ternary algebraic systems. The so-called Nambu mechanics which has been proposed by Nambu [11] in 1973, is based on such structures. There are also some applications, although still hypothetical, in the fractional quantum Hall effect, the non-standard statistics (the anyons), supersymmetric theories, Yang-Baxter equation, etc, (cf. [15, 16, 26]).

The comments on physical applications of ternary structures can be found in [1, 6, 14].

A C^*-ternary algebra is a complex Banach space, equipped with a ternary product $(x, y, z) \to [x, y, z]$ of A^3 into A, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $\Big[x, y, [z, u, v]\Big] = \Big[x, [y, z, u], v\Big] = \Big[[x, y, z], u, v\Big]$, and satisfies $\|[x, y, z]\| \le \|x\|\cdot\|y\|\cdot\|z\|$, $\|[x, x, x]\| = \|x\|^3$ (see [3, 28]).

Every left Hilbert C^*-module is a C^*-ternary algebra via the ternary product $[x, y, z] := \langle x, y\rangle z$.

Let A and B be two Banach ternary algebras. An additive mapping $H : (A, [\ ]_A) \to (B, [\ ]_B)$ is called a ternary ring homomorphism if

$$H([x, y, z]_A) = [H(x), H(y), H(z)]_B$$

for all $x, y, z \in A$. An additive mapping $H : (A, [\ ]_A) \to (B, [\ ]_B)$ is called a Jordan homomorphism if

$$H([x, x, x]_A) = [H(x), H(x), H(x)]_B$$

for all $x \in A$.

Definition 1.1. Let A and B be C^*-ternary algebras. A 3-linear mapping $H : A \times A \times A \to B$ over $\mathbb{C}$ is called a C^*-ternary 3-homomorphism if it satisfies

$$H([x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]) \quad = [H(x_1, x_2, x_3), H(y_1, y_2, y_3), H(z_1, z_2, z_3)]$$

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for all $x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3 \in A$. A 3-linear mapping $H : A \times A \times A \to B$ over C is called a C^*-ternary algebra 3-Jordan homomorphism if it satisfies

$$H([x, x, x], [y, y, y], [z, z, z]) = [H(x, x, x), H(y, y, y), H(z, z, z)]$$

for all $x, y, z \in A$

The study of stability problems originated from a famous talk given by Ulam [27] in 1940: "Under what condition does there exist a homomorphism near an approximate homomorphism?" In the next year 1941, Hyers [13] answered affirmatively the question of Ulam for additive mappings between Banach spaces. Then, Aoki [4] considered the stability problem with unbounded Cauchy differences. A generalized version of the theorem of Hyers for approximately additive maps was given by Rassias [20] in 1978. Let X and Y be real or complex vector spaces. For a mapping $f : X \times X \times X \to Y$, consider the functional equation:

$$f(x_1 + x_2, y_1 + y_2, z_1 + z_2) = \sum_{1 \le i,j,k \le 2} f(x_i, y_j, z_k) \tag{1.1}$$

In 2006, Park and Bae [19] showed that a mapping $f : X \times X \times X \to Y$ satisfies the equation (1.1) if and only if the mapping f is 3-additive. We investigate the Ulam-Hyers stability in C^*-ternary algebras for the 3-additive mappings satisfying (1.1). The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2, 7, 9, 10, 17, 18, 21, 22, 23, 24, 25, 29, 30]).

2. Ulam-Hyers stability of C^*-ternary 3-Jordan homomorphisms

The following lemma was proved in [5].

Lemma 2.1. *Let X and Y be real or complex vector spaces. Let $f : X \times X \times X \to Y$ be a 3-additive mapping such that $f(\lambda x, \mu y, \nu z) = \lambda\mu\nu f(x, y, z)$for all $\lambda, \mu, \nu \in T_1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $x, y, z \in X$. Then f is 3-linear over $\mathbb{C}$.*

Using the above lemma, one can obtain the following result.

The following lemma was proved in [5].

Lemma 2.2. *Let X and Y be complex vector spaces and let $f : X \times X \times X \to Y$ be a mapping such that*

$$f(\lambda x_1 + \lambda x_2, \mu y_1 + \mu y_2, \nu z_1 + \nu z_2) = \lambda\mu\nu \sum_{1 \le i,j,k \le 2} f(x_i, y_j, z_k) \tag{2.1}$$

for all $\lambda, \mu, \nu \in T_1$ and all $x_1, x_2, y_1, y_2, z_1, z_2 \in X$. Then f is 3-linear over $\mathbb{C}$.

Lemma 2.3. *Let A and B be two Banach ternary algebras. Let $f : A \to B$ be an additive mapping. Then the following assertions are equivalent*

$$H([x, x, x], [y, y, y], [z, z, z]) = [H(x, x, x), H(y, y, y), H(z, z, z)] \tag{2.2}$$

for all $x, y, z \in A$.

$$\begin{aligned} \|f\Big(&([x_1, x_2, x_3] + [x_2, x_3, x_1] + [x_3, x_1, x_2]), ([y_1, y_2, y_3] + [y_2, y_3, y_1] + [y_3, y_1, y_2]), \\ &([z_1, z_2, z_3] + [z_2, z_3, z_1] + [z_3, z_1, z_2])\Big) \\ = \Big(&f([x_1, x_2, x_3] + [x_2, x_3, x_1] + [x_3, x_1, x_2]), f([y_1, y_2, y_3] + [y_2, y_3, y_1] + [y_3, y_1, y_2]), \\ &f([z_1, z_2, z_3] + [z_2, z_3, z_1] + [z_3, z_1, z_2])\Big)\|_B, \end{aligned} \tag{2.3}$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$.

Proof. The proof is similar to the proof of [12, Lemma 2.1]. If we replace x, y, z by $x_1 + x_2 + x_3, y_1 + y_2 + y_3, z_1 + z_2 + z_3$ in (2.2), respectively, then we can easily obtain (2.3).

For the converse, if we replace x_1, x_2, x_3 by x, y_1, y_2, y_3 by y and z_1, z_2, z_3 by z in (2.3), we can easily obtain (2.2). □

From now on, assume that A is a C^*-ternary algebra with norm $\|.\|_A$ and that B is a C^*-ternary algebra with norm $\|.\|_B$. For a given mapping $f : A \times A \times A \to B$, we define

$$D_{\lambda,\mu,\nu}f(x_1, x_2, y_1, y_2, z_1, z_2) := f(\lambda x_1 + \lambda x_2, \mu y_1 + \mu y_2, \nu z_1 + \nu z_2) - \lambda\mu\nu \sum_{1\le i,j,k\le 2} f(x_i, y_j, z_k). \tag{2.4}$$

Theorem 2.4. *Let $p, q, r \in (0,\infty)$ with $p + q + r < 3$ and $\theta \in (0,\infty)$, and let $f : A \times A \times A \to B$ be a mapping such that*

$$\|D_{\lambda,\mu,\nu}f(x_1, x_2, y_1, y_2, z_1, z_2)\|_B \le \theta \cdot \max\{\|x_1\|_A, \|x_2\|_A\}^p \cdot \max\{\|y_1\|_A, \|y_2\|_A\}^q \cdot \max\{\|z_1\|_A, \|z_2\|_A\}^r, \tag{2.5}$$

$$\begin{aligned}&\|f\Big(([x_1, x_2, x_3] + [x_2, x_3, x_1] + [x_3, x_1, x_2]), ([y_1, y_2, y_3] + [y_2, y_3, y_1] + [y_3, y_1, y_2]), ([z_1, z_2, z_3] + [z_2, z_3, z_1] + [z_3, z_1, z_2])\Big)\\ &-\Big(f([x_1, x_2, x_3] + [x_2, x_3, x_1] + [x_3, x_1, x_2]), f([y_1, y_2, y_3] + [y_2, y_3, y_1] + [y_3, y_1, y_2]),\\ &\qquad f([z_1, z_2, z_3] + [z_2, z_3, z_1] + [z_3, z_1, z_2])\Big)\|_B \le \theta \sum_{i=1}^{3}(\|x_i\|_A^p \cdot \|y_i\|_A^q \cdot \|z_i\|_A^r)\end{aligned} \tag{2.6}$$

for all $\lambda, \mu, \nu \in T_1$ and all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. Then there exists a unique C^*-ternary 3-Jordan homomorphism $H : A \times A \times A \to B$

$$\|f(x, y, z) - H(x, y, z)\|_B \le \frac{\theta}{8 - 2^{p+q+r}}\|x\|_A^p \cdot \|y\|_A^q \cdot \|z\|_A^r \tag{2.7}$$

for all $x, y, z \in A$.

Proof. By the same reasoning as in the proof of [5, Theorem 2.3], there exists a unique 3-additive mapping $H : A\times A\times A \to B$ satisfying (2.7). By Lemma 2.1, the 3-linear mapping $H : A \times A \times A \to B$ is given by

$$H(\lambda x, \mu y, \nu z) := \lim_{n\to\infty} \frac{1}{8^n} f(2^n\lambda x, 2^n\mu y, 2^n\nu z) = \lim_{n\to\infty} \lambda\mu\nu\frac{1}{8^n} f(2^n x, 2^n y, 2^n z) = \lambda\mu\nu H(x, y, z)$$

for all $\lambda, \mu, \nu \in T_1$ and all $x, y, z \in A$.
It follows from (2.6) that

$$\begin{aligned}&\|H\Big(([x_1, x_2, x_3] + [x_2, x_3, x_1] + [x_3, x_1, x_2]), ([y_1, y_2, y_3] + [y_2, y_3, y_1] + [y_3, y_1, y_2]), ([z_1, z_2, z_3] + [z_2, z_3, z_1] + [z_3, z_1, z_2])\Big)\\ &- \Big(H([x_1, x_2, x_3] + [x_2, x_3, x_1] + [x_3, x_1, x_2]), H([y_1, y_2, y_3] + [y_2, y_3, y_1] + [y_3, y_1, y_2]), H([z_1, z_2, z_3] + [z_2, z_3, z_1] + [z_3, z_1, z_2])\Big)\|_B\\ &= \lim_{n\to\infty}\frac{1}{8^n}\|f\Big(([2^n x_1, 2^n x_2, 2^n x_3] + [2^n x_2, 2^n x_3, 2^n x_1] + [2^n x_3, 2^n x_1, 2^n x_2]),\\ &([2^n y_1, 2^n y_2, 2^n y_3] + [2^n y_2, 2^n y_3, 2^n y_1] + [2^n y_3, 2^n y_1, 2^n y_2]), ([2^n z_1, 2^n z_2, 2^n z_3] + [2^n z_2, 2^n z_3, 2^n z_1] + [2^n z_3, 2^n z_1, 2^n z_2])\Big)\\ &- \Big(f([2^n x_1, 2^n x_2, 2^n x_3] + [2^n x_2, 2^n x_3, 2^n x_1] + [2^n x_3, 2^n x_1, 2^n x_2]),\\ &f([2^n y_1, 2^n y_2, 2^n y_3] + [2^n y_2, 2^n y_3, 2^n y_1] + [2^n y_3, 2^n y_1, 2^n y_2]), f([2^n z_1, 2^n z_2, 2^n z_3] + [2^n z_2, 2^n z_3, 2^n z_1] + [2^n z_3, 2^n z_1, 2^n z_2])\Big)\|_B\\ &\le \lim_{n\to\infty}\frac{\theta}{8^n}\sum_{i=1}^{3}\|x_i\|_A^p \cdot \|y_i\|_A^q \cdot \|z_i\|_A^r = 0\end{aligned}$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. So

$$H\Big(([x_1,x_2,x_3]+[x_2,x_3,x_1]+[x_3,x_1,x_2]),([y_1,y_2,y_3]+[y_2,y_3,y_1]+[y_3,y_1,y_2]),([z_1,z_2,z_3]+[z_2,z_3,z_1]+[z_3,z_1,z_2])\Big)$$
$$=\Big(H([x_1,x_2,x_3]+[x_2,x_3,x_1]+[x_3,x_1,x_2]),H([y_1,y_2,y_3]+[y_2,y_3,y_1]+[y_3,y_1,y_2]),H([z_1,z_2,z_3]+[z_2,z_3,z_1]+[z_3,z_1,z_2)\Big)\|_B$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$.

Now, let $T : A \times A \times A \to B$ be another 3-additive mapping satisfying (2.7). Then we have

$$\begin{aligned}
&\|H(x,y,z)-T(x,y,z)\|_B = \frac{1}{8^n}\|H(2^nx,2^ny,2^nz)-T(2^nx,2^ny,2^nz)\|_B \\
&\le \frac{1}{8^n}\|H(2^nx,2^ny,2^nz)-f(2^nx,2^ny,2^nz)\|_B+\frac{1}{8^n}\|f(2^nx,2^ny,2^nz)-T(2^nx,2^ny,2^nz)\|_B \\
&\le \frac{2^{(p+q+r-3)n+1}\theta}{8-2^{p+q+r}}\|x\|_A^p\cdot\|y\|_A^q\cdot\|z\|_A^r,
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x, y, z \in A$. So we can conclude that $H(x, y, z) = T(x, y, z)$ for all $x, y, z \in A$. This proves the uniqueness of H.

Thus the mapping $H : A \to B$ is a unique C^*-ternary 3-Jordan homomorphism satisfying (2.7). □

Putting $p = q = r = 0$ and $\theta = \varepsilon$ in Theorem 2.3, we obtain the Ulam stability for the 3-additive functional equation (1.1).

Corollary 2.5. *Let $\varepsilon \in (0,\infty)$ and let $f : A \times A \times A \to B$ be a mapping satisfying*

$$\|D_{\lambda,\mu,\nu}f(x_1,x_2,y_1,y_2,z_1,z_2)\|_B \le \varepsilon,$$
$$\begin{aligned}
&\|f\Big(([x_1,x_2,x_3]+[x_2,x_3,x_1]+[x_3,x_1,x_2]),([y_1,y_2,y_3]+[y_2,y_3,y_1]+[y_3,y_1,y_2]),([z_1,z_2,z_3]+[z_2,z_3,z_1]+[z_3,z_1,z_2])\Big) \\
&-\Big(f([x_1,x_2,x_3]+[x_2,x_3,x_1]+[x_3,x_1,x_2]),f([y_1,y_2,y_3]+[y_2,y_3,y_1]+[y_3,y_1,y_2]), \\
&\qquad f([z_1,z_2,z_3]+[z_2,z_3,z_1]+[z_3,z_1,z_2)\Big)\|_B \le 3\varepsilon
\end{aligned}$$

for all $\lambda, \mu, \nu \in T_1$ and all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. Then there exists a unique C^*-ternary 3-Jordan homomorphism $H : A \times A \times A \to B$

$$\|f(x,y,z)-H(x,y,z)\|_B \le \frac{\varepsilon}{7}$$

for all $x, y, z \in A$.

Theorem 2.6. *Let $p \in (0,3)$ and $\theta \in (0,8)$, and let $f : A \times A \times A \to B$ be a mapping such that*

$$\|D_{\lambda,\mu,\nu}f(x_1,x_2,y_1,y_2,z_1,z_2)\|_B \le \theta\sum_{i=1}^{2}(\|x_i\|_A^p+\|y_i\|_A^q+\|z_i\|_A^r), \tag{2.8}$$

$$\begin{aligned}
&\|f\Big(([x_1,x_2,x_3]+[x_2,x_3,x_1]+[x_3,x_1,x_2]),([y_1,y_2,y_3]+[y_2,y_3,y_1]+[y_3,y_1,y_2]),([z_1,z_2,z_3]+[z_2,z_3,z_1]+[z_3,z_1,z_2])\Big) \\
&-\Big(f([x_1,x_2,x_3]+[x_2,x_3,x_1]+[x_3,x_1,x_2]),f([y_1,y_2,y_3]+[y_2,y_3,y_1]+[y_3,y_1,y_2]),f([z_1,z_2,z_3]+[z_2,z_3,z_1]+[z_3,z_1,z_2)\Big)\|_B \\
&\qquad\le \theta\sum_{i=1}^{3}(\|x_i\|_A^p+\|y_i\|_A^q+\|z_i\|_A^r)
\end{aligned} \tag{2.9}$$

for all $\lambda, \mu, \nu \in T_1$ and all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. Then there exists a unique C^*-ternary 3-Jordan homomorphism $H : A \times A \times A \to B$

$$\|f(x,y,z)-H(x,y,z)\|_B \le \frac{2\theta}{8-2^p}(\|x\|_A^p+\|y\|_A^q+\|z\|_A^r)$$

for all $x, y, z \in A$.

Proof. The proof is similar to the proof of Theorem 2.4. □

Theorem 2.7. *Let* $p, q, r \in (0, \infty)$ *with* $p+q+r < 3$, $s \in (0,3)$ *and* $\theta, \eta \in (0, \infty)$, *and let* $f : A \times A \times A \to B$ *be a mapping such that*

$$\begin{aligned}\|D_{\lambda,\mu,\nu} f(x_1, x_2, y_1, y_2, z_1, z_2)\|_B \leq \theta \cdot \max\{\|x_1\|_A, \|x_2\|_A\}^p \cdot \max\{\|y_1\|_A, \|y_2\|_A\}^q \cdot \max\{\|z_1\|_A, \|z_2\|_A\}^r \\ + \eta \sum_{i=1}^{2} (\|x_i\|_A^s + \|y_i\|_A^s + \|z_i\|_A^s),\end{aligned} \tag{2.10}$$

$$\begin{aligned}&\|f\Big(([x_1, x_2, x_3] + [x_2, x_3, x_1] + [x_3, x_1, x_2]), ([y_1, y_2, y_3] + [y_2, y_3, y_1] + [y_3, y_1, y_2]), ([z_1, z_2, z_3] + [z_2, z_3, z_1] + [z_3, z_1, z_2])\Big) \\ &-\Big(f([x_1, x_2, x_3] + [x_2, x_3, x_1] + [x_3, x_1, x_2]), f([y_1, y_2, y_3] + [y_2, y_3, y_1] + [y_3, y_1, y_2]), \\ &f([z_1, z_2, z_3] + [z_2, z_3, z_1] + [z_3, z_1, z_2])\Big)\|_B \leq \theta \sum_{i=1}^{3} (\|x_i\|_A^p \cdot \|y_i\|_A^q \cdot \|z_i\|_A^r) + \eta \sum_{i=1}^{2} (\|x_i\|_A^s + \|y_i\|_A^s + \|z_i\|_A^s)\end{aligned} \tag{2.11}$$

for all $\lambda, \mu, \nu \in T_1$ *and all* $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. *Then there exists a unique* C^**-ternary 3-Jordan homomorphism* $H : A \times A \times A \to B$

$$\|f(x, y, z) - H(x, y, z)\|_B \leq \frac{\theta}{8 - 2^{p+q+r}} \|x\|_A^p \cdot \|y\|_A^q \cdot \|z\|_A^r + \frac{2\eta}{8 - 2^s} (\|x_i\|_A^s + \|y_i\|_A^s + \|z_i\|_A^s)$$

for all $x, y, z \in A$.

Proof. The proof is similar to the proof of Theorem 2.4. □

Theorem 2.8. *Let* $p \in (0, 3)$ *and* $\theta \in (0, 8)$, *and let* $f : A \times A \times A \to B$ *be a mapping satisfying (2.8), (2.9) and* $f(0, 0, 0) = 0$. *Then there exists a unique* C^**-ternary 3-Jordan homomorphism* $H : A \times A \times A \to B$ *such that*

$$\|f(x, y, z) - H(x, y, z)\|_B \leq \frac{2\theta}{2^p - 8} (\|x\|_A^p + \|y\|_A^q + \|z\|_A^r)$$

for all $x, y, z \in A$.

Theorem 2.9. *Letp,* $q, r \in (0, \infty)$ *with* $p + q + r > 3$, $s \in (0, 3)$ *and* $\theta, \eta \in (0, \infty)$, *and let* $f : A \times A \times A \to B$ *be a mapping satisfying (2.10), (2.11) and* $f(0, 0, 0) = 0$. *Then there exists a unique* C^**-ternary 3-Jordan homomorphism* $H : A \times A \times A \to B$ *such that*

$$\|f(x, y, z) - H(x, y, z)\|_B \leq \frac{\theta}{2^{p+q+r-8}} \|x\|_A^p \cdot \|y\|_A^q \cdot \|z\|_A^r + \frac{2\eta}{2^s - 8} (\|x_i\|_A^s + \|y_i\|_A^s + \|z_i\|_A^s)$$

for all $x, y, z \in A$.

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3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

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SOME NEW RESULTS ON PRODUCTS OF THE APOSTOL-GENOCCHI POLYNOMIALS

YUAN HE

ABSTRACT. We perform a further investigation for the Apostol-Genocchi polynomials numbers. By making use of the generating function methods and summation transform techniques, we establish some new formulae for products of any arbitrary number of the Apostol-Genocchi polynomials and numbers. The results presented here are the corresponding generalizations of some known formulae on the classical Genocchi polynomials and numbers.

1. INTRODUCTION

The classical Bernoulli polynomials $B_n(x)$ and the classical Genocchi polynomials $G_n(x)$ are usually defined by means of the following generating functions:

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\quad (|t|<2\pi) \tag{1.1}$$

and

$$\frac{2te^{xt}}{e^t+1}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}\quad (|t|<\pi). \tag{1.2}$$

The rational numbers B_n and G_n given by

$$B_n=B_n(0)\quad\text{and}\quad G_n=G_n(0) \tag{1.3}$$

are called the classical Bernoulli numbers and the classial Genocchi numbers, respectively. These polynomials and numbers play important roles in different areas of mathematics such as number theory, combinatorics, special functions and analysis. Numerous interesting properties for them can be found in many books and papers; see for example, [7, 14, 17, 18, 19, 26, 27, 29, 30, 31].

We now turn to some widely-investigated analogues of the classical Bernoulli polynomials $B_n(x)$ and the classical Genocchi polynomials $G_n(x)$, i.e., the Apostol-Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$ and the Apostol-Genocchi polynomials $\mathcal{G}_n(x;\lambda)$. They are usually defined by means of the following generating functions (see, e.g., [20, 21, 24]):

$$\frac{te^{xt}}{\lambda e^t-1}=\sum_{n=0}^{\infty}\mathcal{B}_n(x;\lambda)\frac{t^n}{n!} \tag{1.4}$$

$$(|t|<2\pi \text{ when } \lambda=1;\ |t|<|\log\lambda| \text{ when } \lambda\neq 1)$$

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and

$$\frac{2te^{xt}}{\lambda e^t+1}=\sum_{n=0}^{\infty}\mathcal{G}_n(x;\lambda)\frac{t^n}{n!} \tag{1.5}$$

$$(|t|<\pi \text{ when } \lambda=1;\ |t|<|\log(-\lambda)| \text{ when } \lambda\neq 1)$$

In particular, $\mathcal{B}_n(\lambda)$ and $\mathcal{G}_n(\lambda)$ given by

$$\mathcal{B}_n(\lambda)=\mathcal{B}_n(0;\lambda) \quad \text{and} \quad \mathcal{G}_n(\lambda)=\mathcal{G}_n(0;\lambda) \tag{1.6}$$

are called the Apostol-Bernoulli numbers and the Apostol-Genocchi numbers, respectively. Obviously, $\mathcal{B}_n(x;\lambda)$ and $\mathcal{G}_n(x;\lambda)$, respectively, reduces to $B_n(x)$ and $G_n(x)$ when $\lambda=1$. It is worth mentioning that the Apostol-Bernoulli polynomials were firstly introduced by Apostol [3] (see also Srivastava [28] for a systematic study) in order to evaluate the value of the Hurwitz-Lerch zeta function. For some related results on the Apostol type polynomials and numbers, one can consult to [6, 8, 11, 16, 22, 24, 33].

The idea of the present paper stems from the work of Agoh [1, 2]. We establish some new formulae of products of any arbitrary number of the Apostol-Genocchi polynomials and numbers by making use of the generating function methods and summation transform techniques. It turns out that some results presented here are the corresponding generalizations of several known formulae including the recent ones discovered by Agoh [2] on the classical Genocchi polynomials and numbers.

2. The statement of results

Let n be a positive integer and let $m_1,\ldots,m_n$ be non-negative integers. In the following we denote by $[t_1^{m_1}\cdots t_n^{m_n}]f(t_1,\ldots,t_n)$ the coefficients of $t_1^{m_1}\cdots t_n^{m_n}$ in $f(t_1,\ldots,t_n)$. We first recall the elementary and beautiful idea contributed to Euler, namely (see, e.g., [4, 5])

$$(1+x_1)(1+x_2)(1+x_3)\cdots=(1+x_1)+x_2(1+x_1)+x_3(1+x_1)(1+x_2)+\cdots. \tag{2.1}$$

Obviously, the finite form of (2.1) can be expressed as

$$\begin{aligned}(1+x_1)(1+x_2)\cdots(1+x_n)&=(1+x_1)+x_2(1+x_1)+\cdots\\&\quad+x_n(1+x_1)(1+x_2)\cdots(1+x_{n-1}).\end{aligned} \tag{2.2}$$

We shall make use of (2.2) to establish some new formulae for products of any arbitrary number of the Apostol-Genocchi polynomials and numbers. It is easily seen that for $1\leq r\leq n$, substituting x_r-1 for x_r in (2.2) gives

$$x_1\cdots x_n-1=\sum_{r=1}^{n}(x_r-1)x_1\cdots x_{r-1}, \tag{2.3}$$

where the product $x_1\cdots x_{r-1}$ is considered to be equal to 1 when $r=1$. If we take $x_r=-\lambda_r e^{t_r}$ for $1\leq r\leq n$ in (2.3) then we have

$$(-1)^n\lambda_1\cdots\lambda_n e^{t_1+\cdots+t_n}-1=\sum_{r=1}^{n}(-1)^r(\lambda_r e^{t_r}+1)\prod_{k=1}^{r-1}\lambda_k e^{t_k}. \tag{2.4}$$

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It follows from (2.4) that

$$\prod_{i=1}^{n}\frac{2t_ie^{x_it_i}}{\lambda_ie^{t_i}+1}=\frac{1}{(-1)^n\lambda_1\cdots\lambda_ne^{t_1+\cdots+t_n}-1}\times\sum_{r=1}^{n}(-1)^r(\lambda_re^{t_r}+1)\prod_{k=1}^{r-1}\lambda_ke^{t_k}\prod_{i=1}^{n}\frac{2t_ie^{x_it_i}}{\lambda_ie^{t_i}+1}. \tag{2.5}$$

Observe that

$$(\lambda_re^{t_r}+1)\prod_{k=1}^{r-1}\lambda_ke^{t_k}\prod_{i=1}^{n}\frac{2t_ie^{x_it_i}}{\lambda_ie^{t_i}+1}=2t_re^{x_r(t_1+\cdots+t_n)}\prod_{i=1}^{r-1}\lambda_i\frac{2t_ie^{(x_i-x_r+1)t_i}}{\lambda_ie^{t_i}+1}\prod_{i=r+1}^{n}\frac{2t_ie^{(x_i-x_r)t_i}}{\lambda_ie^{t_i}+1}. \tag{2.6}$$

Hence, by applying (2.6) to (2.5), we get

$$\prod_{i=1}^{n}\frac{2t_ie^{x_it_i}}{\lambda_ie^{t_i}+1}=\sum_{r=1}^{n}(-1)^r\frac{2t_re^{x_r(t_1+\cdots+t_n)}}{(-1)^n\lambda_1\cdots\lambda_ne^{t_1+\cdots+t_n}-1}\times\prod_{i=1}^{r-1}\lambda_i\frac{2t_ie^{(x_i-x_r+1)t_i}}{\lambda_ie^{t_i}+1}\prod_{i=r+1}^{n}\frac{2t_ie^{(x_i-x_r)t_i}}{\lambda_ie^{t_i}+1}, \tag{2.7}$$

which means

$$\begin{aligned}&\left[\frac{t_1^{m_1}}{m_1!}\cdots\frac{t_n^{m_n}}{m_n!}\right]\left(\prod_{i=1}^{n}\frac{2t_ie^{x_it_i}}{\lambda_ie^{t_i}+1}\right)\\&=m_1!\cdots m_n!\sum_{r=1}^{n}(-1)^r[t_1^{m_1}\cdots t_{r-1}^{m_{r-1}}t_r^{m_r-1}t_{r+1}^{m_{r+1}}\cdots t_n^{m_n}]\\&\quad\times\left(\frac{2e^{x_r(t_1+\cdots+t_n)}}{(-1)^n\lambda_1\cdots\lambda_ne^{t_1+\cdots+t_n}-1}\prod_{i=1}^{r-1}\lambda_i\frac{2t_ie^{(x_i-x_r+1)t_i}}{\lambda_ie^{t_i}+1}\right.\\&\quad\left.\times\prod_{i=r+1}^{n}\frac{2t_ie^{(x_i-x_r)t_i}}{\lambda_ie^{t_i}+1}\right).\end{aligned} \tag{2.8}$$

It is trivial to get

$$\left[\frac{t_1^{m_1}}{m_1!}\cdots\frac{t_n^{m_n}}{m_n!}\right]\left(\prod_{i=1}^{n}\frac{2t_ie^{x_it_i}}{\lambda_ie^{t_i}+1}\right)=\prod_{i=1}^{n}\mathcal{G}_{m_i}(x_i;\lambda_i). \tag{2.9}$$

We next consider the right hand side of (2.8). Since $\mathcal{B}_0(x;\lambda)=0$ when $\lambda\neq1$ and $\mathcal{G}_0(x;\lambda)=0$ (see, e.g., [20, 23]) , then by (1.3) we have

$$\frac{e^{xt}}{\lambda e^t-1}=\sum_{n=0}^{\infty}\frac{\mathcal{B}_{n+1}(x;\lambda)}{n+1}\frac{t^n}{n!}\quad(\lambda\neq1), \tag{2.10}$$

and

$$\frac{2e^{xt}}{\lambda e^t+1}=\sum_{n=0}^{\infty}\frac{\mathcal{G}_{n+1}(x;\lambda)}{n+1}\frac{t^n}{n!}. \tag{2.11}$$

Notice that for non-negative integer N (see, e.g., [32]),

$$(t_1+\cdots+t_n)^N=\sum_{\substack{k_1+\cdots+k_n=N\\k_1,\ldots,k_n\geq 0}}\binom{N}{k_1,\ldots,k_n}t_1^{k_1}\cdots t_n^{k_n}, \tag{2.12}$$

where $\binom{n}{r_1,\ldots,r_k}$ denotes by the multinomials coefficient given by

$$\binom{n}{r_1,\ldots,r_k}=\frac{n!}{r_1!\cdots r_k!}\quad (n,r_1,\ldots,r_k\geq 0). \tag{2.13}$$

So from (2.10) and (2.11), we obtain that for even integer n and $\lambda_1\cdots\lambda_n\neq 1$,

$$\begin{aligned}&\frac{e^{x_r(t_1+\cdots+t_n)}}{(-1)^n\lambda_1\cdots\lambda_n e^{t_1+\cdots+t_n}-1}\\&\qquad=\sum_{N=0}^{\infty}\frac{\mathcal{B}_{N+1}(x_r;\lambda_1\cdots\lambda_n)}{N+1}\sum_{\substack{k_1+\cdots+k_n=N\\k_1,\ldots,k_n\geq 0}}\frac{t_1^{k_1}}{k_1!}\cdots\frac{t_n^{k_n}}{k_n!},\end{aligned} \tag{2.14}$$

and for odd integer n,

$$\begin{aligned}&\frac{2e^{x_r(t_1+\cdots+t_n)}}{(-1)^n\lambda_1\cdots\lambda_n e^{t_1+\cdots+t_n}-1}\\&\qquad=-\sum_{N=0}^{\infty}\frac{\mathcal{G}_{N+1}(x_r;\lambda_1\cdots\lambda_n)}{N+1}\sum_{\substack{k_1+\cdots+k_n=N\\k_1,\ldots,k_n\geq 0}}\frac{t_1^{k_1}}{k_1!}\cdots\frac{t_n^{k_n}}{k_n!}.\end{aligned} \tag{2.15}$$

It follows from (1.3), (1.4), (2.8), (2.9), (2.14) and (2.15) that if n is an even integer, then for positive integers $m_1,\ldots,m_n$ and $\lambda_1\cdots\lambda_n\neq 1$,

$$\begin{aligned}&\mathcal{G}_{m_1}(x_1;\lambda_1)\mathcal{G}_{m_2}(x_2;\lambda_2)\cdots\mathcal{G}_{m_n}(x_n;\lambda_n)\\&\quad=2\sum_{r=1}^{n}(-1)^r\sum_{k_1,\cdots,k_{r-1},k_{r+1},\cdots,k_n\geq 0}\frac{m_1!\cdots m_n!}{k_1!\cdots k_{r-1}!\cdot(m_r-1)!\cdot k_{r+1}!\cdots k_n!}\\&\qquad\times\frac{\mathcal{B}_{k_1+\cdots+k_{r-1}+(m_r-1)+k_{r+1}+\cdots+k_n+1}(x_r;\lambda_1\cdots\lambda_n)}{k_1+\cdots+k_{r-1}+(m_r-1)+k_{r+1}+\cdots+k_n+1}\\&\qquad\times\prod_{i=1}^{r-1}\lambda_i\frac{\mathcal{G}_{m_i-k_i}(x_i-x_r+1;\lambda_i)}{(m_i-k_i)!}\prod_{i=r+1}^{n}\frac{\mathcal{G}_{m_i-k_i}(x_i-x_r;\lambda_i)}{(m_i-k_i)!}.\end{aligned} \tag{2.16}$$

and if n is an odd integer, then for positive integers $m_1,\ldots,m_n$,

$$\begin{aligned}&\mathcal{G}_{m_1}(x_1;\lambda_1)\mathcal{G}_{m_2}(x_2;\lambda_2)\cdots\mathcal{G}_{m_n}(x_n;\lambda_n)\\&\quad=\sum_{r=1}^{n}(-1)^{r-1}\sum_{k_1,\cdots,k_{r-1},k_{r+1},\cdots,k_n\geq 0}\frac{m_1!\cdots m_n!}{k_1!\cdots k_{r-1}!\cdot(m_r-1)!\cdot k_{r+1}!\cdots k_n!}\\&\qquad\times\frac{\mathcal{G}_{k_1+\cdots+k_{r-1}+(m_r-1)+k_{r+1}+\cdots+k_n+1}(x_r;\lambda_1\cdots\lambda_n)}{k_1+\cdots+k_{r-1}+(m_r-1)+k_{r+1}+\cdots+k_n+1}\\&\qquad\times\prod_{i=1}^{r-1}\lambda_i\frac{\mathcal{G}_{m_i-k_i}(x_i-x_r+1;\lambda_i)}{(m_i-k_i)!}\prod_{i=r+1}^{n}\frac{\mathcal{G}_{m_i-k_i}(x_i-x_r;\lambda_i)}{(m_i-k_i)!}.\end{aligned} \tag{2.17}$$

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Thus, by replacing k_i by $m_i - k_i$ for $i \neq r$ in (2.16) and (2.17), we obtain the following formulae for products of an arbitrary number of the Apostol-Genocchi polynomials.

Theorem 2.1. *Let $m_1, \cdots, m_n$ be n positive integers. If n is an even integer, then*

$$\begin{aligned}
&\mathcal{G}_{m_1}(x_1;\lambda_1)\mathcal{G}_{m_2}(x_2;\lambda_2)\cdots\mathcal{G}_{m_n}(x_n;\lambda_n) \\
&\quad = 2\sum_{\substack{k_1+\cdots+k_n=m_1+\cdots+m_n \\ k_1,\ldots,k_n\geq 0}} \sum_{r=1}^{n}(-1)^r \frac{m_r}{k_r}\mathcal{B}_{k_r}(x_r;\lambda_1\cdots\lambda_n) \\
&\qquad\qquad \times \prod_{i=1}^{r-1}\binom{m_i}{k_i}\lambda_i\mathcal{G}_{k_i}(x_i - x_r + 1;\lambda_i) \\
&\qquad\qquad\quad \times \prod_{i=r+1}^{n}\binom{m_i}{k_i}\mathcal{G}_{k_i}(x_i - x_r;\lambda_i) \quad (\lambda_1\cdots\lambda_n \neq 1). \qquad (2.18)
\end{aligned}$$

If n is an odd integer, then

$$\begin{aligned}
&\mathcal{G}_{m_1}(x_1;\lambda_1)\mathcal{G}_{m_2}(x_2;\lambda_2)\cdots\mathcal{G}_{m_n}(x_n;\lambda_n) \\
&\quad = \sum_{\substack{k_1+\cdots+k_n=m_1+\cdots+m_n \\ k_1,\ldots,k_n\geq 0}} \sum_{r=1}^{n}(-1)^{r-1} \frac{m_r}{k_r}\mathcal{G}_{k_r}(x_r;\lambda_1\cdots\lambda_n) \\
&\qquad \times \prod_{i=1}^{r-1}\binom{m_i}{k_i}\lambda_i\mathcal{G}_{k_i}(x_i - x_r + 1;\lambda_i)\prod_{i=r+1}^{n}\binom{m_i}{k_i}\mathcal{G}_{k_i}(x_i - x_r;\lambda_i). \qquad (2.19)
\end{aligned}$$

It follows that we show some special cases of Theorem 2.1. Since the Apostol-Genocchi polynomials satisfy the following difference equation (see, e.g., [23]):

$$\lambda\mathcal{G}_n(x+1;\lambda) + \mathcal{G}_n(x;\lambda) = 2nx^{n-1} \quad (n \geq 0), \qquad (2.20)$$

by taking $n = 2$ in Theorem 2.1, we get that for positive integers m, n and $\lambda\mu \neq 1$,

$$\begin{aligned}
\mathcal{G}_m(x;\lambda)\mathcal{G}_n(y;\mu) &= 2n\sum_{k=0}^{m}\binom{m}{k}\{2k(x-y)^{k-1} - \mathcal{G}_k(x-y;\lambda)\}\frac{\mathcal{B}_{m+n-k}(y;\lambda\mu)}{m+n-k} \\
&\quad - 2m\sum_{k=0}^{n}\binom{n}{k}\mathcal{G}_k(y-x;\mu)\frac{\mathcal{B}_{m+n-k}(x;\lambda\mu)}{m+n-k}. \qquad (2.21)
\end{aligned}$$

The identity (2.21) can be also found in [12] where it was further considered the case $\lambda\mu = 1$. We also refer to [9, 10, 35] for some similar formulae to (2.21). If we take $n = 3$ in Theorem 2.1, in light of the symmetric relation for the Apostol-Genocchi polynomials (see, e.g., [23]):

$$\lambda\mathcal{G}_n(1-x;\lambda) = (-1)^{n+1}\mathcal{G}_n\left(x;\frac{1}{\lambda}\right) \quad (n \geq 0), \qquad (2.22)$$

we obtain that for positive integers m_1, m_2, m_3,

$$\begin{aligned}
&\mathcal{G}_{m_1}(x_1;\lambda_1)\mathcal{G}_{m_2}(x_2;\lambda_2)\mathcal{G}_{m_3}(x_3;\lambda_3)\\
&=\sum_{\substack{k_1+k_2+k_3=m_1+m_2+m_3\\ k_1,k_2,k_3\geq 0}}\left\{\frac{m_1}{k_1}\binom{m_2}{k_2}\binom{m_3}{k_3}\mathcal{G}_{k_1}(x_1;\mu)\mathcal{G}_{k_2}(x_2-x_1;\lambda_2)\mathcal{G}_{k_3}(x_3-x_1;\lambda_3)\right.\\
&\quad+\frac{m_2}{k_2}\binom{m_1}{k_1}\binom{m_3}{k_3}(-1)^{k_1}\mathcal{G}_{k_2}(x_2;\mu)\mathcal{G}_{k_1}(x_2-x_1;1/\lambda_1)\mathcal{G}_{k_3}(x_3-x_2;\lambda_3)\\
&\quad\quad+\frac{m_3}{k_3}\binom{m_1}{k_1}\binom{m_2}{k_2}(-1)^{k_1+k_2}\mathcal{G}_{k_3}(x_3;\mu)\mathcal{G}_{k_1}(x_3-x_1;1/\lambda_1)\\
&\quad\quad\quad\left.\times\mathcal{G}_{k_2}(x_3-x_2;1/\lambda_2)\right\}\quad(\mu=\lambda_1\lambda_2\lambda_3).
\end{aligned}\tag{2.23}$$

Remark 2.2. *Note that (2.19) does not require the condition $\lambda_1\cdots\lambda_n\neq 1$. However, we were unable to get the formula analogous (2.18) in the case $\lambda_1\cdots\lambda_n=1$.*

We next give some higher-order convolution formulae for the Apostol-Genocchi polynomials, which are the corresponding generalization of Agoh's convolution formula on the classical Genocchi polynomials presented in [1, 12]. Clearly, by substituting k for n and u_it for t_i with $u_1+u_2+\cdots+u_k=1$ in (2.7), we discover that for positive integer k,n,

$$\begin{aligned}
\left[\frac{t^n}{n!}\right]\left(\prod_{i=1}^k\frac{2u_ite^{x_iu_it}}{\lambda_ie^{u_it}+1}\right)=&\sum_{r=1}^k(-1)^r\left[\frac{t^n}{n!}\right]\left(\frac{2u_rte^{x_rt}}{(-1)^k\lambda_1\cdots\lambda_ke^t-1}\right.\\
&\left.\times\prod_{i=1}^{r-1}\lambda_i\frac{2u_ite^{(x_i-x_r+1)u_it}}{\lambda_ie^{u_it}+1}\prod_{i=r+1}^k\frac{2u_ite^{(x_i-x_r)u_it}}{\lambda_ie^{u_it}+1}\right).
\end{aligned}\tag{2.24}$$

It is easy to see from (1.4) that the left hand side of (2.24) can be rewritten as

$$\begin{aligned}
\left[\frac{t^n}{n!}\right]\left(\prod_{i=1}^k\frac{2u_ite^{x_iu_it}}{\lambda_ie^{u_it}+1}\right)=&\sum_{\substack{j_1+j_2\cdots+j_k=n\\ j_1,j_2,\ldots,j_k\geq 0}}\frac{n!\cdot u_1^{j_1}u_2^{j_2}\cdots u_k^{j_k}}{j_1!\cdot j_2!\cdots j_k!}\\
&\times\mathcal{G}_{j_1}(x_1;\lambda_1)\mathcal{G}_{j_2}(x_2;\lambda_2)\cdots\mathcal{G}_{j_k}(x_k;\lambda_k),
\end{aligned}\tag{2.25}$$

and the right hand side of (2.24) can be rewritten in the following ways: if k is an even integer then

$$\begin{aligned}
&\left[\frac{t^n}{n!}\right]\left(\prod_{i=1}^k\frac{u_ite^{x_iu_it}}{\lambda_ie^{u_it}-1}\right)\\
&=-2\sum_{r=1}^k\sum_{\substack{j_1+j_2\cdots+j_k=n\\ j_1,j_2,\ldots,j_k\geq 0}}\frac{n!\cdot u_1^{j_1}u_2^{j_2}\cdots u_{r-1}^{j_{r-1}}u_ru_{r+1}^{j_{r+1}}\cdots u_k^{j_k}}{j_1!\cdot j_2!\cdots j_k!}\mathcal{B}_{j_r}(x_r;\lambda_1\lambda_2\cdots\lambda_k)\\
&\quad\times\prod_{i=1}^{r-1}\{-\lambda_i\mathcal{G}_{j_i}(x_i-x_r+1;\lambda_i)\}\prod_{i=r+1}^k\mathcal{G}_{j_i}(x_i-x_r;\lambda_i),
\end{aligned}\tag{2.26}$$

and if k is an odd integer then

$$\begin{aligned}&\left[\frac{t^n}{n!}\right]\left(\prod_{i=1}^{k}\frac{u_ite^{x_iu_it}}{\lambda_ie^{u_it}-1}\right)\\&\quad=\sum_{r=1}^{k}\sum_{\substack{j_1+j_2\cdots+j_k=n\\j_1,j_2,\ldots,j_k\geq0}}\frac{n!\cdot u_1^{j_1}u_2^{j_2}\cdots u_{r-1}^{j_{r-1}}u_ru_{r+1}^{j_{r+1}}\cdots u_k^{j_k}}{j_1!\cdot j_2!\cdots j_k!}\mathcal{G}_{j_r}(x_r;\lambda_1\lambda_2\cdots\lambda_k)\\&\qquad\times\prod_{i=1}^{r-1}\{-\lambda_i\mathcal{G}_{j_i}(x_i-x_r+1;\lambda_i)\}\prod_{i=r+1}^{k}\mathcal{G}_{j_i}(x_i-x_r;\lambda_i).\end{aligned}\tag{2.27}$$

Since for positive integer $k\geq2$ and complex numbers $\alpha_1,\alpha_2,\ldots,\alpha_k$ with $\mathrm{Re}(\alpha_j)>-1$ for $j=1,2,\ldots,k$, (see, e.g., [2, 34])

$$\begin{aligned}&\int_0^1\int_0^{1-u_1}\cdots\int_0^{1-u_1-\cdots-u_{k-2}}u_1^{\alpha_1}u_2^{\alpha_2}\cdots u_k^{\alpha_k}du_1d_2\cdots du_{k-1}\\&\quad=\frac{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)\cdots\Gamma(\alpha_k+1)}{\Gamma(\alpha_1+\alpha_2+\cdots+\alpha_k+k)}\quad(u_1+u_2+\cdots+u_k=1).\end{aligned}\tag{2.28}$$

by equating (2.25), (2.26) and (2.27) and making the above integral operation, with the help of (2.28), we get that if k is an even integer then

$$\begin{aligned}&(n+k)\sum_{\substack{j_1+j_2\cdots+j_k=n\\j_1,j_2,\ldots,j_k\geq0}}\mathcal{G}_{j_1}(x_1;\lambda_1)\mathcal{G}_{j_2}(x_2;\lambda_2)\cdots\mathcal{G}_{j_k}(x_k;\lambda_k)\\&=-2\sum_{r=1}^{k}\sum_{\substack{j_1+j_2\cdots+j_k=n\\j_1,j_2,\ldots,j_k\geq0}}\binom{n+k}{j_r}\mathcal{B}_{j_r}(x_r;\lambda_1\lambda_2\cdots\lambda_k)\prod_{i=1}^{r-1}\{-\lambda_i\mathcal{G}_{j_i}(x_i-x_r+1;\lambda_i)\}\\&\qquad\times\prod_{i=r+1}^{k}\mathcal{G}_{j_i}(x_i-x_r;\lambda_i),\end{aligned}\tag{2.29}$$

and if k is an odd integer then

$$\begin{aligned}&(n+k)\sum_{\substack{j_1+j_2\cdots+j_k=n\\j_1,j_2,\ldots,j_k\geq0}}\mathcal{G}_{j_1}(x_1;\lambda_1)\mathcal{G}_{j_2}(x_2;\lambda_2)\cdots\mathcal{G}_{j_k}(x_k;\lambda_k)\\&=\sum_{r=1}^{k}\sum_{\substack{j_1+j_2\cdots+j_k=n\\j_1,j_2,\ldots,j_k\geq0}}\binom{n+k}{j_r}\mathcal{G}_{j_r}(x_r;\lambda_1\lambda_2\cdots\lambda_k)\prod_{i=1}^{r-1}\{-\lambda_i\mathcal{G}_{j_i}(x_i-x_r+1;\lambda_i)\}\\&\qquad\times\prod_{i=r+1}^{k}\mathcal{G}_{j_i}(x_i-x_r;\lambda_i).\end{aligned}\tag{2.30}$$

Notice that from (2.20) we have

$$\begin{aligned}&\prod_{i=1}^{r-1}\{-\lambda_i\mathcal{G}_{j_i}(x_i-x_r+1;\lambda_i)\}\\&\quad=\sum_{T\subseteq\{1,\ldots,r-1\}}\prod_{i\in T}\mathcal{G}_{j_i}(x_i-x_r;\lambda_i)\times\prod_{i\in\overline{T}}\{-2j_i(x_i-x_r)^{j_i-1}\}.\end{aligned}\tag{2.31}$$

Thus, by applying (2.31) to the right hand sides of (2.29) and (2.30) and then taking $x_1 = x_2 = \cdots = x_k = x$, we immediately obtain the following result.

Theorem 2.3. *Let k, n be positive integers. If k is an even integer, then*

$$\begin{aligned}(n+k) &\sum_{\substack{j_1+j_2\cdots+j_k=n\\ j_1,j_2,\ldots,j_k\geq 0}} \mathcal{G}_{j_1}(x;\lambda_1)\mathcal{G}_{j_2}(x;\lambda_2)\cdots\mathcal{G}_{j_k}(x;\lambda_k)\\ &= \sum_{r=1}^{k}\binom{k}{r-1}(-2)^{k-r+1}\sum_{\substack{j_1+j_2\cdots+j_r=n-k+r\\ j_1,j_2,\ldots,j_r\geq 0}}\binom{n+k}{j_r}\mathcal{B}_{j_r}(x;\lambda_1\lambda_2\cdots\lambda_k)\\ &\qquad\times\mathcal{G}_{j_1}(\lambda_1)\mathcal{G}_{j_2}(\lambda_2)\cdots\mathcal{G}_{j_{r-1}}(\lambda_{r-1}).\end{aligned}\tag{2.32}$$

If k is an odd integer, then

$$\begin{aligned}(n+k) &\sum_{\substack{j_1+j_2\cdots+j_k=n\\ j_1,j_2,\ldots,j_k\geq 0}} \mathcal{G}_{j_1}(x;\lambda_1)\mathcal{G}_{j_2}(x;\lambda_2)\cdots\mathcal{G}_{j_k}(x;\lambda_k)\\ &= \sum_{r=1}^{k}\binom{k}{r-1}(-2)^{k-r}\sum_{\substack{j_1+j_2\cdots+j_r=n-k+r\\ j_1,j_2,\ldots,j_r\geq 0}}\binom{n+k}{j_r}\mathcal{G}_{j_r}(x;\lambda_1\lambda_2\cdots\lambda_k)\\ &\qquad\times\mathcal{G}_{j_1}(\lambda_1)\mathcal{G}_{j_2}(\lambda_2)\cdots\mathcal{G}_{j_{r-1}}(\lambda_{r-1}).\end{aligned}\tag{2.33}$$

It becomes obvious that setting $k = 2$ in Theorem 2.3 gives that for positive integer n,

$$\begin{aligned}\sum_{k=0}^{n}\mathcal{G}_k(x;\lambda)\mathcal{G}_{n-k}(x;\mu) + \frac{4}{n+2}\sum_{k=0}^{n}\binom{n+2}{k}\mathcal{B}_k(x;\lambda\mu)\mathcal{G}_{n-k}(\lambda)&\\ = \frac{2n(n+1)}{3}\mathcal{B}_{n-1}(x;\lambda\mu).&\end{aligned}\tag{2.34}$$

Since the classical Genocchi polynomials can be expressed in terms of the classical Bernoulli polynomials, as follows,

$$G_n(x) = 2B_n(x) - 2^{n+1}B_n\left(\frac{x}{2}\right)\quad (n\geq 0),\tag{2.35}$$

by $B_1 = -1/2$, we have $G_0 = 0$ and $G_1 = 1$. Hence, the case $\lambda = \mu = 1$ in (2.34) gives the convolution identity on the classical Genocchi polynomials due to Agoh [1, 12], namely

$$\sum_{k=1}^{n-1}G_k(x)G_{n-k}(x) + \frac{4}{n+2}\sum_{k=0}^{n-2}\binom{n+2}{k}B_k(x)G_{n-k} = 0\quad (n\geq 2).\tag{2.36}$$

It is worth noticing that $x = 0$ in (2.36) can give the result (see, e.g., [1]):

$$\sum_{k=2}^{n-2}G_kG_{n-k} + 4\sum_{k=2}^{n-2}\binom{n+1}{k-1}\frac{B_kG_{n-k}}{k} = -\frac{4}{n+2}G_n\quad (n\geq 4),\tag{2.37}$$

which is very analogous to the convolution identity on the classical Bernoulli numbers due to Matiyasevich [25], in an equivalent form, as follows,

$$\sum_{k=2}^{n-2}B_kB_{n-k} - 2\sum_{k=2}^{n-2}\binom{n+1}{k-1}\frac{B_kB_{n-k}}{k} = \frac{n(n+1)}{n+2}B_n\quad (n\geq 4).\tag{2.38}$$

For some similar convolution formulae to (2.37) and (2.38), one is referred to [12, 13, 15]. If we take $k = 3$ in Theorem 2.3, we obtain that for positive integer $n \geq 2$,

$$\begin{aligned}\sum_{j_1+j_2+j_3=n} &\mathcal{G}_{j_1}(x;\lambda_1)\mathcal{G}_{j_2}(x;\lambda_2)\mathcal{G}_{j_3}(x;\lambda_3)\\ &-\frac{3}{n+3}\sum_{j_1+j_2+j_3=n}\binom{n+3}{j_3}\mathcal{G}_{j_1}(\lambda_1)\mathcal{G}_{j_2}(\lambda_2)\mathcal{G}_{j_3}(x;\mu)\\ &+\frac{6}{n+3}\sum_{k=0}^{n-1}\binom{n+3}{k}\mathcal{G}_k(x;\mu)\mathcal{G}_{n-1-k}(\lambda_1)\\ &=\frac{4}{n+3}\binom{n+3}{5}\mathcal{G}_{n-2}(x;\mu)\quad(\mu=\lambda_1\lambda_2\lambda_3).\end{aligned}\tag{2.39}$$

The case $\lambda_1 = \lambda_2 = \lambda_3 = 1$ in (2.39) gives the corresponding formula of products of the classical Genocchi polynomials, as follows,

$$\begin{aligned}\sum_{j_1+j_2+j_3=n} &G_{j_1}(x)G_{j_2}(x)G_{j_3}(x)-\frac{3}{n+3}\sum_{j_1+j_2+j_3=n}\binom{n+3}{j_3}G_{j_1}G_{j_2}G_{j_3}(x)\\ &+\frac{6}{n+3}\sum_{k=0}^{n-1}\binom{n+3}{k}G_k(x)G_{n-1-k}=\frac{4}{n+3}\binom{n+3}{5}G_{n-2}(x),\end{aligned}\tag{2.40}$$

which is very analogous to the convolution identity on the classical Euler polynomials presented in [2, Corollary 3].

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FUNCTIONAL INEQUALITIES IN FUZZY NORMED SPACES

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ABSTRACT. In this paper, we solve the following additive functional inequality

$$N(f(x+y)-f(x)-f(y),t) \quad \geq \quad N\left(f\left(\frac{x+y}{2}\right)-\frac{1}{2}f(x)-\frac{1}{2}f(y),t\right) \tag{0.1}$$

and the following quadratic functional inequality

$$\begin{aligned} &N(f(x+y)+f(x-y)-2f(x)-2f(y),t) \\ &\geq N\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y),t\right) \end{aligned} \tag{0.2}$$

in fuzzy normed spaces.

Using the fixed point method, we prove the Hyers-Ulam stability of the additive functional inequality (0.1) and the quadratic functional inequality (0.2) in fuzzy Banach spaces.

1. Introduction and preliminaries

Katsaras [21] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [13, 24, 52]. In particular, Bag and Samanta [2], following Cheng and Mordeson [8], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [23]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [3].

We use the definition of fuzzy normed spaces given in [2, 28, 29] to investigate the Hyers-Ulam stability of a quadratic functional inequality in fuzzy Banach spaces.

Definition 1.1. [2, 28, 29, 30] Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N_1) $N(x,t) = 0$ for $t \leq 0$;
(N_2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
(N_3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N_4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;
(N_5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.
(N_6) for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a *fuzzy normed vector space.*

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [27, 28].

Definition 1.2. [2, 28, 29, 30] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by N-$\lim_{n\to\infty} x_n = x$.

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Definition 1.3. [2, 28, 29, 30] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [3]).

The stability problem of functional equations originated from a question of Ulam [51] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [17] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [40] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [14] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach. The functional equation $f\left(\frac{x+y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$ is called the *Jensen equation*.

The functional equation $f(x+y)+f(x-y) = 2f(x)+2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [50] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [9] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. Czerwik [10] proved the Hyers-Ulam stability of the quadratic functional equation. The functional equation $f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$ is called a *Jensen type quadratic equation*. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [4, 18, 20, 25, 36, 37, 38, 41, 42, 44, 45, 46, 47, 48, 49]).

Gilányi [15] showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(x-y)\| \leq \|f(x+y)\| \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(x+y) + f(x-y).$$

See also [43]. Fechner [12] and Gilányi [16] proved the Hyers-Ulam stability of the functional inequality (1.1). Park, Cho and Han [35] investigated the Cauchy additive functional inequality

$$\|f(x) + f(y) + f(z)\| \leq \|f(x+y+z)\| \tag{1.2}$$

and the Cauchy-Jensen additive functional inequality

$$\|f(x) + f(y) + 2f(z)\| \leq \left\|2f\left(\frac{x+y}{2} + z\right)\right\| \tag{1.3}$$

and proved the Hyers-Ulam stability of the functional inequalities (1.2) and (1.3) in Banach spaces.

Park [33, 34] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces.

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.4. [5, 11] Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;
(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

In 1996, G. Isac and Th.M. Rassias [19] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [6, 7, 22, 27, 31, 32, 38, 39]).

In Section 2, we solve the additive functional inequality (0.1) and prove the Hyers-Ulam stability of the additive functional inequality (0.1) in fuzzy Banach spaces by using the fixed point method.

In Section 3, we solve the quadratic functional inequality (0.2) and prove the Hyers-Ulam stability of the quadratic functional inequality (0.2) in fuzzy Banach spaces by using the fixed point method.

Throughout this paper, assume that X is a real vector space and (Y, N) is a fuzzy Banach space.

2. Additive functional inequality (0.1)

In this section, we prove the Hyers-Ulam stability of the additive functional inequality (0.1) in fuzzy Banach spaces. We need the following lemma to prove the main results.

Lemma 2.1. *Let $f : X \to Y$ be a mapping such that*

$$N(f(x+y) - f(x) - f(y), t) \quad \geq \quad N\left(f\left(\frac{x+y}{2}\right) - \frac{1}{2}f(x) - \frac{1}{2}f(y), t\right) \tag{2.1}$$

for all $x, y \in X$ and all $t > 0$. Then f is Cauchy additive, i.e., $f(x+y) = f(x) + f(y)$ for all $x, y \in X$.

Proof. Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $N(f(0), t) = N(0, t) = 1$. So $f(0) = 0$.

Letting $y = x$ in (2.1), we get $N(f(2x) - 2f(x), t) \geq N(0, t) = 1$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2}f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} N(f(x+y)-f(x)-f(y),t) &\geq N\left(f\left(\frac{x+y}{2}\right)-\frac{1}{2}f(x)-\frac{1}{2}f(y),t\right) \\ &= N\left(\frac{1}{2}(f(x+y)-f(x)-f(y)),t\right) \\ &= N\left(f(x+y)-f(x)-f(y),2t\right) \end{aligned}$$

for all $t>0$. By (N_5) and (N_6), $N(f(x+y)-f(x)-f(y),t)=1$ for all $t>0$. It follows from (N_2) that

$$f(x+y)=f(x)+f(y)$$

for all $x,y\in X$. □

Theorem 2.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L<1$ with*

$$\varphi(x,y)\leq \frac{L}{2}\varphi\left(2x,2y\right)$$

for all $x,y\in X$. Let $f:X\to Y$ be an odd mapping satisfying

$$\begin{aligned} &N\left(f(x+y)-f(x)-f(y),t\right) \\ &\quad\geq \min\left\{N\left(f\left(\frac{x+y}{2}\right)-\frac{1}{2}f(x)-\frac{1}{2}f(y),t\right), \frac{t}{t+\varphi(x,y)}\right\} \end{aligned} \tag{2.3}$$

for all $x,y\in X$ and all $t>0$. Then $A(x):=N\text{-}\lim_{n\to\infty}2^n f\left(\frac{x}{2^n}\right)$ exists for each $x\in X$ and defines an additive mapping $A:X\to Y$ such that

$$N\left(f(x)-A(x),t\right)\geq \frac{(2-2L)t}{(2-2L)t+L\varphi(x,x)} \tag{2.4}$$

for all $x\in X$ and all $t>0$.

Proof. Letting $y=x$ in (2.3), we get

$$N\left(f\left(2x\right)-2f(x),t\right)\geq \frac{t}{t+\varphi(x,x)} \tag{2.5}$$

for all $x\in X$.

Consider the set

$$S:=\{g:X\to Y\}$$

and introduce the generalized metric on S:

$$d(g,h)=\inf\left\{\mu\in\mathbb{R}_+ : N(g(x)-h(x),\mu t)\geq \frac{t}{t+\varphi(x,x)},\ \forall x\in X,\forall t>0\right\},$$

where, as usual, $\inf\phi=+\infty$. It is easy to show that (S,d) is complete (see [26, Lemma 2.1]).

Now we consider the linear mapping $J:S\to S$ such that

$$Jg(x):=2g\left(\frac{x}{2}\right)$$

for all $x\in X$.

Let $g,h\in S$ be given such that $d(g,h)=\varepsilon$. Then

$$N(g(x)-h(x),\varepsilon t)\geq \frac{t}{t+\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right), L\varepsilon t\right) = N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{2}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)} \geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \frac{L}{2}\varphi(x,x)} = \frac{t}{t + \varphi(x,x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (2.5) that

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), \frac{L}{2}t\right) \geq \frac{t}{t + \varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{2}$.

By Theorem 1.4, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \tag{2.6}$$

for all $x \in X$. Since $f : X \to Y$ is odd, $A : X \to Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - A(x), \mu t) \geq \frac{t}{t + \varphi(x,x)}$$

for all $x \in X$;

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \leq \frac{1}{1-L}d(f, Jf)$, which implies the inequality

$$d(f, A) \leq \frac{L}{2 - 2L}.$$

This implies that the inequality (2.4) holds.

By (2.3),

$$\begin{aligned} &N\left(2^n\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right), 2^n t\right) \\ &\geq \min\left\{N\left(2^n f\left(\frac{x+y}{2^{n+1}}\right) - 2^{n-1} f\left(\frac{x}{2^n}\right) - 2^{n-1} f\left(\frac{y}{2^n}\right), 2^n t\right), \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}\right\} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\begin{aligned}&N\left(2^n\left(f\left(\frac{x+y}{2^n}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right),t\right)\\&\geq \min\left\{N\left(2^n f\left(\frac{x+y}{2^{n+1}}\right)-2^{n-1}f\left(\frac{x}{2^n}\right)-2^{n-1}f\left(\frac{y}{2^n}\right),t\right),\frac{\frac{t}{2^n}}{\frac{t}{2^n}+\frac{L^n}{2^n}\varphi(x,y)}\right\}\end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty}\frac{\frac{t}{2^n}}{\frac{t}{2^n}+\frac{L^n}{2^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$N\left(A(x+y)-A(x)-A(y),t\right)\geq N\left(A\left(\frac{x+y}{2}\right)-\frac{1}{2}A(x)-\frac{1}{2}A(y),t\right)$$

for all $x, y \in X$ and all $t > 0$. By Lemma 2.1, the mapping $A : X \to Y$ is Cauchy additive. □

Corollary 2.3. *Let $\theta \geq 0$ and let p be a real number with $p > 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying*

$$\begin{aligned}&N\left(f(x+y)-f(x)-f(y),t\right)\\&\quad\geq \min\left\{N\left(f\left(\frac{x+y}{2}\right)-\frac{1}{2}f(x)-\frac{1}{2}f(y),t\right),\frac{t}{t+\theta(\|x\|^p+\|y\|^p)}\right\}\end{aligned}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x)-A(x),t\right)\geq\frac{(2^p-2)t}{(2^p-2)t+2\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{1-p}$, and we get the desired result. □

Theorem 2.4. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y)\leq 2L\varphi\left(\frac{x}{2},\frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying (2.3). Then $A(x) := N\text{-}\lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x)-A(x),t\right)\geq\frac{(2-2L)t}{(2-2L)t+\varphi(x,x)} \tag{2.7}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

It follows from (2.5) that

$$N\left(f(x)-\frac{1}{2}f(2x),\frac{1}{2}t\right)\geq\frac{t}{t+\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2}g(2x)$$

for all $x \in X$. So $d(f, Jf) \leq \frac{1}{2}$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\begin{aligned} &N\left(f(x+y) - f(x) - f(y), t\right) \\ &\quad \geq \min\left\{N\left(f\left(\frac{x+y}{2}\right) - \frac{1}{2}f(x) - \frac{1}{2}f(y), t\right), \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}\right\} \end{aligned}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{(2-2^p)t}{(2-2^p)t + 2\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{p-1}$, and we get the desired result. □

3. Quadratic functional inequality (0.2)

In this section, we prove the Hyers-Ulam stability of the quadratic functional inequality (0.2) in fuzzy Banach spaces. We need the following lemma to prove the main results.

Lemma 3.1. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} &N(f(x+y) + f(x-y) - 2f(x) - 2f(y), t) \\ &\quad \geq N\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y), t\right) \end{aligned} \tag{3.1}$$

for all $x, y \in X$ and all $t > 0$. Then f is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $y = x$ in (3.1), we get $N(f(2x) - 4f(x), t) \geq N(0, t) = 1$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{3.2}$$

for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} &N(f(x+y) + f(x-y) - 2f(x) - 2f(y), t) \\ &\quad \geq N\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y), t\right) \\ &\quad = N\left(\frac{1}{2}\left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right), t\right) \\ &\quad = N\left(f(x+y) + f(x-y) - 2f(x) - 2f(y), 2t\right) \end{aligned}$$

for all $t > 0$. By (N_5) and (N_6), $N(f(x+y) + f(x-y) - 2f(x) - 2f(y), t) = 1$ for all $t > 0$. It follows from (N_2) that $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in X$. □

Theorem 3.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{4}\varphi(2x, 2y)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\begin{aligned} &N(f(x+y)+f(x-y)-2f(x)-2f(y),t) \\ &\geq \min\left\{N\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y),t\right), \frac{t}{t+\varphi(x,y)}\right\} \end{aligned} \tag{3.3}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x) - Q(x), t\right) \geq \frac{(4-4L)t}{(4-4L)t + L\varphi(x,x)} \tag{3.4}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $y = x$ in (3.3), we get

$$N\left(f\left(2x\right) - 4f(x), t\right) \geq \frac{t}{t+\varphi(x,x)} \tag{3.5}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : N(g(x)-h(x),\mu t) \geq \frac{t}{t+\varphi(x,x)},\ \forall x \in X, \forall t > 0\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [26, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x)-h(x),\varepsilon t) \geq \frac{t}{t+\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x)-Jh(x),L\varepsilon t) &= N\left(4g\left(\frac{x}{2}\right)-4h\left(\frac{x}{2}\right),L\varepsilon t\right) = N\left(g\left(\frac{x}{2}\right)-h\left(\frac{x}{2}\right),\frac{L}{4}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{4}}{\frac{Lt}{4}+\varphi\left(\frac{x}{2},\frac{x}{2}\right)} \geq \frac{\frac{Lt}{4}}{\frac{Lt}{4}+\frac{L}{4}\varphi(x,x)} = \frac{t}{t+\varphi(x,x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (3.5) that $N\left(f(x) - 4f\left(\frac{x}{2}\right), \frac{L}{4}t\right) \geq \frac{t}{t+\varphi(x,x)}$ for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{4}$.

By Theorem 1.4, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(\frac{x}{2}\right) = \frac{1}{4}Q(x) \tag{3.6}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (3.6) such that there exists a $\mu \in (0,\infty)$ satisfying

$$N(f(x) - Q(x), \mu t) \geq \frac{t}{t + \varphi(x,x)}$$

for all $x \in X$;

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f,Q) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f,Q) \leq \frac{L}{4-4L}.$$

This implies that the inequality (3.4) holds.

By (3.3),

$$\begin{aligned} & N\left(4^n\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right), 4^n t\right) \\ & \geq \min\left\{N\left(4^n\left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right), 4^n t\right), \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}\right\} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\begin{aligned} & N\left(4^n\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right), t\right) \\ & \geq \min\left\{N\left(4^n\left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right), t\right), \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n}\varphi(x,y)}\right\} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$\begin{aligned} & N(Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y), t) \\ & \geq N\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y), t\right) \end{aligned}$$

for all $x, y \in X$ and all $t > 0$. By Lemma 3.1, the mapping $Q : X \to Y$ is quadratic, as desired. ☐

Corollary 3.3. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} & N(f(x+y) + f(x-y) - 2f(x) - 2f(y), t) \\ & \geq \min\left\{N\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y), t\right), \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}\right\} \end{aligned}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{(2^p - 4)t}{(2^p - 4)t + 2\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{2-p}$, and we get the desired result. □

Theorem 3.4. *Let* $\varphi : X^2 \to [0,\infty)$ *be a function such that there exists an* $L < 1$ *with*

$$\varphi(x,y) \leq 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. *Let* $f : X \to Y$ *be a mapping satisfying* $f(0) = 0$ *and* (3.3). *Then* $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ *exists for each* $x \in X$ *and defines a quadratic mapping* $Q : X \to Y$ *such that*

$$N(f(x) - Q(x), t) \geq \frac{(4-4L)t}{(4-4L)t + \varphi(x,x)} \tag{3.7}$$

for all $x \in X$ *and all* $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 3.2.

It follows from (3.5) that

$$N\left(f(x) - \frac{1}{4}f(2x), \frac{1}{4}t\right) \geq \frac{t}{t + \varphi(x,x)}$$

for all $x \in X$ and all $t > 0$.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4}g(2x)$$

for all $x \in X$. So $d(f, Jf) \leq \frac{1}{4}$.

The rest of the proof is similar to the proof of Theorem 3.2. □

Corollary 3.5. *Let* $\theta \geq 0$ *and let* p *be a real number with* $0 < p < 2$. *Let* X *be a normed vector space with norm* $\|\cdot\|$. *Let* $f : X \to Y$ *be a mapping satisfying* $f(0) = 0$ *and*

$$\begin{aligned} &N(f(x+y) + f(x-y) - 2f(x) - 2f(y), t) \\ &\geq \min\left\{N\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y), t\right), \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}\right\} \end{aligned}$$

for all $x, y \in X$ *and all* $t > 0$. *Then* $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ *exists for each* $x \in X$ *and defines a quadratic mapping* $Q : X \to Y$ *such that*

$$N(f(x) - Q(x), t) \geq \frac{(4-2^p)t}{(4-2^p)t + 2\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{p-2}$, and we get the desired result. □

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A generalization of Simpson type inequality via differentiable functions using extended $(s, m)_\phi$-preinvex functions

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Abstract

Some new class of preinvex functions which called (s, m)-preinvex, extended (s, m)-preinvex, $(s, m)_\phi$-preinvex and extended $(s, m)_\phi$-preinvex function are introduced respectively in this paper. An integral identity is established, and then we prove some Simpson type integral inequalities, dealing with the existing similar type integral inequalities in a relatively uniform frame. In particular, we also show some results obtained by these inequalities for extended $(s, m)_\phi$-preinvex under some suitable conditions, which improve the previously known results.

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1 Introduction

The following notations are used throughout this paper. I is an interval on the real line $\mathbb{R}$, $\mathbb{R}_0 = [0, \infty)$. $\mathbb{R}^n$ is used to denote a generic n-dimensional vector space, $\mathbb{R}_0^n$ denotes an n-dimensional nonegative vector space, and $\mathbb{R}_+^n$ denotes an n-dimensional positive vector space. For any subset $K \subseteq \mathbb{R}^n$, $L_1[a, b]$ is the set of integrable functions over the interval $[a, b]$. Let us firstly recall some definitions of various convex type functions.

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Definition 1.1 *([6]) A function $f : I \subseteq \mathbb{R} \to \mathbb{R}_0$ is said to be a Godunova-Levin function if f is nonnegative and for all $x, y \in I$, $\lambda \in (0,1)$ we have that*

$$f\big(\lambda x + (1-\lambda)y\big) \leq \frac{f(x)}{\lambda} + \frac{f(y)}{1-\lambda}.$$

Definition 1.2 *([5]) For some $(s,m) \in (0,1]^2$, a function $f : [0,b] \to \mathbb{R}$ is said to be (s,m)-convex in the second sense if for every $x, y \in [0,b]$ and $\lambda \in (0,1]$ we have that*

$$f\big(\lambda x + m(1-\lambda)y\big) \leq \lambda^s f(x) + m(1-\lambda)^s f(y).$$

Definition 1.3 *([36]) For some $s \in [-1,1]$ and $m \in (0,1]$, a function $f : [0,b] \to \mathbb{R}_0$ is said to be extended (s,m)-convex if for all $x, y \in [0,b]$ and $\lambda \in (0,1)$ we have that*

$$f\big(\lambda x + m(1-\lambda)y\big) \leq \lambda^s f(x) + m(1-\lambda)^s f(y).$$

Definition 1.4 *([1]) A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the map $\eta : K \times K \to \mathbb{R}^n$, if $x + t\eta(y,x) \in K$ for every $x, y \in K$ and $t \in [0,1]$.*

Notice that every convex set is invex with respect to the map $\eta(y,x) = y - x$, but the converse is not necessarily true. For more details please refer to [1, 37] and the references therein.

Definition 1.5 *([1]) Let $K \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : K \times K \to \mathbb{R}^n$, for every $x, y \in K$, the η-path $P_{x\nu}$ joining the points x and $\nu = x + \eta(y,x)$ is defined by*

$$P_{x\nu} = \big\{z | z = x + t\eta(y,x),\ t \in [0,1]\big\}.$$

Definition 1.6 *([27]) The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect to η if for every $x, y \in K$ and $t \in [0,1]$ we have that*

$$f\big(x + t\eta(y,x)\big) \leq (1-t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the map $\eta(y,x) = y - x$, but the converse is not true.

Definition 1.7 *([13]) The function f defined on the invex set $K \subseteq [0,b^*]$ with $b^* > 0$ is said to be m-preinvex with respect to η if for all $x, y \in K$, $t \in [0,1]$ and for some fixed $m \in (0,1]$, we have that*

$$f\big(x + t\eta(y,x)\big) \leq (1-t)f(x) + mtf\Big(\frac{y}{m}\Big).$$

Remark 1.1 *Notice that if $y \in [0,b^*]$, then for any $0 < m < 1$, $\dfrac{y}{m}$ could be greater than b^*, which is not in the domain of f. Thus, the right hand side of the inequality in this definition could be meaningless. To fix this flaw, we suggest to replace $[0,b^*]$ by the half real line $\mathbb{R}_0$.*

Definition 1.8 *([14]) Let $K \subseteq \mathbb{R}_0$ be an invex set with respect to η. A function $f : K \to \mathbb{R}$ is said to be s-preinvex with respect to η, if for all $x, y \in K$, $t \in [0,1]$ and some fixed $s \in (0,1]$ we have that*

$$f\big(x + t\eta(y,x)\big) \leq (1-t)^s f(x) + t^s f(y).$$

Definition 1.9 *([21]) The set $K_{\phi\eta} \subseteq \mathbb{R}^n$ is said to be ϕ-invex at u with respect to $\phi(.)$, if there exists a bifunction $\eta(.,.) : K_{\phi\eta} \times K_{\phi\eta} \to \mathbb{R}^n$, such that*

$$u + te^{i\phi}\eta(v,u) \in K_{\phi\eta}, \quad \forall u, v \in K_{\phi\eta}, t \in [0,1].$$

The ϕ-invex set $K_{\phi\eta}$ is also called $\phi\eta$-connected set. Note that the convex set with $\phi = 0$ and $\eta(v,u) = v - u$ is a ϕ-invex set, but the converse is not true (see [21]).

Definition 1.10 *([22]) For some fixed $s \in (0,1]$, a function f on the set $K_{\phi\eta}$ is said to be s_φ-preinvex function with respect to ϕ and η, if*

$$f\big(u + te^{i\phi}\eta(v,u)\big) \leq (1-t)^s f(u) + t^s f(v), \quad \forall u, v \in K_{\phi\eta}, t \in [0,1].$$

Definition 1.11 *([22]) A function f on the set C_ϕ is said to be ϕ-convex function with respect to ϕ, if and only if*

$$f\big(u + te^{i\phi}(v-u)\big) \leq (1-t)f(u) + tf(v), \quad \forall u, v \in C_\phi, t \in [0,1].$$

The following inequality is very remarkable and well known in the literature as Simpson type inequality, which plays an important role in analysis. Particularly, it is well applied in numerical integration.

Theorem 1.1 *([4]) Let $f : [a,b] \to \mathbb{R}$ be a four times continuously differentiable mapping on (a,b) and $||f^{(4)}||_\infty = \sup_{x\in(a,b)} |f^{(4)}(x)| < \infty$. Then the following inequality holds:*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f(\frac{a+b}{2})\right] - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leq \frac{1}{2880}||f^{(4)}||_\infty (b-a)^4. \quad (1.1)$$

In recent decades, a lot of inequalities of Simpson type and Hadamard type for various kinds of convex functions have been established and developed by many scholars, some of them may be reformulated as follows.

Theorem 1.2 *([30]) Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L_1[a,b]$, where $a, b \in I^\circ$ with $a < b$. If $|f'|$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$, then*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a) + f(b) + 4f\Big(\frac{a+b}{2}\Big)\right] - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\ &\leq \frac{(s-4)6^{s+1} + 2\times 5^{s+2} - 2\times 3^{s+2} + 2}{6^{s+2}(s+1)(s+2)}(b-a)\Big[|f'(a)| + |f'(b)|\Big]. \quad (1.2)\end{aligned}$$

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Theorem 1.3 *([4]) Let $f : [a, b] \to \mathbb{R}$ is a differentiable mapping whose derivative is continuous on (a, b) and $||f'||_1 = \int_a^b |f'(x)|\mathrm{d}x < \infty$. Then we have the inequality:*

$$\Big| \int_a^b f(x)\mathrm{d}x - \frac{b-a}{3}\Big[\frac{f(a)+f(b)}{2} + 2f\Big(\frac{a+b}{2}\Big)\Big]\Big| \leq \frac{1}{3}||f'||_1 (b-a)^2. \quad (1.3)$$

Theorem 1.4 *([35]) Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I$ with $a < b$, $f' \in L_1[a, b]$ and $0 \leq \lambda, \mu \leq 1$. If $|f'(x)|^q$ for $q \geq 1$ is an extended s-convex on $[a, b]$ for some $s \in [-1, 1]$, specially, when $q = 1$ and $s = -1$, the following inequality holds:*

$$\Big|f\Big(\frac{a+b}{2}\Big) - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\Big| \leq (b-a)\ln 2\Big(|f'(a)| + |f'(b)|\Big). \quad (1.4)$$

Theorem 1.5 *([2, 29]) Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If $|f'|$ is preinvex on K, then, for every $a, b \in K$ with $\eta(b, a) \neq 0$ we have that:*

$$\begin{aligned} &\Big|\frac{f(a) + f\big(a + \eta(b,a)\big)}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x\Big| \\ &\leq \frac{|\eta(b,a)|}{8}\Big(|f'(a)| + |f'(b)|\Big) \end{aligned} \quad (1.5)$$

and

$$\Big|f\Big(\frac{2a+\eta(b,a)}{2}\Big) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x\Big| \leq \frac{|\eta(b,a)|}{8}\Big(|f'(a)| + |f'(b)|\Big). (1.6)$$

Theorem 1.6 *([33]) Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function. If $q > 1$, $q \geq r$, $s \geq 0$ and $|f'|$ is preinvex on A, then for every $a, b \in A$ with $\eta(a, b) \neq 0$, we have that*

$$\begin{aligned} &\Big|f\Big(\frac{2b+\eta(a,b)}{2}\Big) - \frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\mathrm{d}x\Big| \\ &\leq \frac{|\eta(a,b)|}{4}\Bigg\{\Big(\frac{1}{r+1}\Big)^{\frac{1}{q}}\Big(\frac{q-1}{2q-r-1}\Big)^{1-\frac{1}{q}}\Big[\frac{(r+1)|f'(a)|^q + (r+3)|f'(b)|^q}{2(r+2)}\Big]^{\frac{1}{q}} \\ &\quad + \Big(\frac{1}{s+1}\Big)^{\frac{1}{q}}\Big(\frac{q-1}{2q-s-1}\Big)^{1-\frac{1}{q}}\Big[\frac{(s+3)|f'(a)|^q + (s+1)|f'(b)|^q}{2(s+2)}\Big]^{\frac{1}{q}}\Bigg\}. \end{aligned}$$

Corollary 1.1 *([33]) Under the conditions of Theorem 1.6, when $r = s = 0$,*

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the following inequality holds:

$$\begin{aligned}&\left|f\left(\frac{2b+\eta(a,b)}{2}\right)-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\mathrm{d}x\right|\\&\le\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\frac{|\eta(a,b)|}{4}\left[\left(\frac{1}{4}|f'(a)|^q+\frac{3}{4}|f'(b)|^q\right)^{\frac{1}{q}}\right.\\&\quad\left.+\left(\frac{3}{4}|f'(a)|^q+\frac{1}{4}|f'(b)|^q\right)^{\frac{1}{q}}\right].\end{aligned}\tag{1.7}$$

Theorem 1.7 *([22]) Let $I\subseteq\mathbb{R}$ be an open ϕ-invex set with respect to $\eta: I\times I\to\mathbb{R}$. Suppose that $f: I\to\mathbb{R}$ is a differentiable function such that $f'\in L_1[a,a+e^{i\phi}\eta(b,a)]$. If $|f'|$ is ϕ-preinvex on I , then, for $\eta(b,a)>0$,*

$$\begin{aligned}&\left|\frac{f(a)+f\big(a+e^{i\phi}\eta(b,a)\big)}{2}-\frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)}f(x)\mathrm{d}x\right|\\&\le\frac{|e^{i\phi}\eta(b,a)|}{8}\Big(|f'(a)|+|f'(b)|\Big).\end{aligned}\tag{1.8}$$

Currently, the Simpson type inequalities concerning different kinds of preinvex and ϕ-convex functions are still interesting research topics to many researchers in the field of convex analysis. For more information please refer to [7–12, 15, 17–20, 23–26, 31, 32, 34] and references cited therein.

Motivated by the inspiring idea in [3, 13, 21, 28] and based on our previous works [16, 38], in this paper we are mainly going to introduce the $(s,m)_\phi$-preinvex function and the extended $(s,m)_\phi$-preinvex function, and then we will establish some Simpson type integral inequalities for extended $(s,m)_\phi$-preinvex functions. In Section 2, we will introduce new definitions and an integral identity. Section 3 will be devoted of presenting the main results.

2 New definitions and an integral identity

We now mainly introduce some new concepts about preinvex function. The class of $(s,m)_\phi$-preinvex function is quite a general and unifying one. This is one of the main motivation of this paper.

Definition 2.1 *Let $K\subseteq\mathbb{R}_0^n$ be an open invex set with respect to $\eta: K\times K\to\mathbb{R}_+^n$. For $f: K\to\mathbb{R}$ and some fixed $(s,m)\in(0,1]\times(0,1]$, if*

$$f\Big(x+\lambda\eta(y,x)\Big)\le(1-\lambda)^sf(x)+m\lambda^sf\Big(\frac{y}{m}\Big)\tag{2.1}$$

is valid for all $x,y\in K$, $\lambda\in[0,1]$, then we say that $f(x)$ is an (s,m)-preinvex function with respect to η.

Definition 2.2 *Let $K \subseteq \mathbb{R}_0^n$ be an open invex set with respect to $\eta : K \times K \to \mathbb{R}_+^n$. For $f : K \to \mathbb{R}_0$ and some fixed $(s, m) \in [-1, 1] \times (0, 1]$, if*

$$f\Big(x + \lambda\eta(y, x)\Big) \le (1 - \lambda)^s f(x) + m\lambda^s f\Big(\frac{y}{m}\Big) \tag{2.2}$$

is valid for all $x, y \in K$, $\lambda \in [0, 1]$, then we say that $f(x)$ is an extended (s, m)-preinvex function with respect to η.

Remark 2.1 *In Definition 2.1, if $s = 1$ then one obtains the usual definition of m-preinvex function. If $m = 1$ then one obtains the usual definition of s-preinvex function. It is also worthwhile to note that every (s, m)-preinvex function is (s, m)-convex and every extended (s, m)-preinvex functions is extended (s, m)-convex with respect to $\eta(y, x) = y - x$ respectively.*

Definition 2.3 *A function f on the set $K_{\phi\eta} \subseteq \mathbb{R}_0^n$ is said to be $(s, m)_\varphi$-preinvex function with respect to $\phi(.)$ and $\eta(.,.)$. For $f : K_{\phi\eta} \to \mathbb{R}$ and some fixed $(s, m) \in (0, 1] \times (0, 1]$, if*

$$f\Big(x+\lambda e^{i\phi}\eta(y, x)\Big) \le (1-\lambda)^s f(x)+m\lambda^s f\Big(\frac{y}{m}\Big), \quad \forall x, y \in K_{\phi\eta},\ \lambda \in [0, 1]. \tag{2.3}$$

Remark 2.2 *In Definition 2.3, if $\phi = 0$ then it reduces to the definition for (s, m)-preinvex function. If $m = 1$ then it reduces to the definition for s_φ-preinvex function. Also, it is obvious that Definition 2.3 is the ϕ-convex function when $\eta(y, x) = y - x$ and $s = m = 1$.*

Definition 2.4 *A function f on the set $K_{\phi\eta} \subseteq \mathbb{R}_0^n$ is said to be extended $(s, m)_\varphi$-preinvex function with respect to $\phi(.)$ and $\eta(.,.)$. For $f : K_{\phi\eta} \to \mathbb{R}_0$ and some fixed $(s, m) \in [-1, 1] \times (0, 1]$, if*

$$f\Big(x+\lambda e^{i\phi}\eta(y, x)\Big) \le (1-\lambda)^s f(x)+m\lambda^s f\Big(\frac{y}{m}\Big), \quad \forall x, y \in K_{\phi\eta},\ \lambda \in [0, 1]. \tag{2.4}$$

In order to establish some new Simpson type integral inequalities, we need the following key integral identity, which will be used in the sequel.

Lemma 2.1 *Let $K_{\phi\eta} \subseteq \mathbb{R}$ be a ϕ-invex subset with respect to $\phi(.)$ and $\eta : K_{\phi\eta} \times K_{\phi\eta} \subseteq \mathbb{R}$, $a, b \in K_{\phi\eta}$ with $a < a + \eta(b, a)$. If $k, t \in \mathbb{R}$, $f : K_{\phi\eta} \to \mathbb{R}$ is a differentiable function and $f' \in L[a, a + e^{i\phi}\eta(b, a)]$ we have that*

$$\begin{aligned} & tf(a) + (1 - k)f\Big(a + e^{i\phi}\eta(b, a)\Big) + (k - t)f\Big(a + \frac{e^{i\phi}\eta(b, a)}{2}\Big) \\ & -\frac{1}{e^{i\phi}\eta(b, a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x \\ & = e^{i\phi}\eta(b, a)\Big[\int_0^{\frac{1}{2}} (\lambda - t)f'\Big(a + \lambda e^{i\phi}\eta(b, a)\Big)\mathrm{d}\lambda \\ & + \int_{\frac{1}{2}}^1 (\lambda - k)f'\Big(a + \lambda e^{i\phi}\eta(b, a)\Big)\mathrm{d}\lambda\Big]. \end{aligned} \tag{2.5}$$

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Proof. Set

$$J = e^{i\phi}\eta(b,a)\Big[\textstyle\int_0^{\frac{1}{2}}(\lambda - t)f'\Big(a+\lambda e^{i\phi}\eta(b,a)\Big)\mathrm{d}\lambda + \textstyle\int_{\frac{1}{2}}^{1}(\lambda - k)f'\Big(a+\lambda e^{i\phi}\eta(b,a)\Big)\mathrm{d}\lambda\Big].$$

Since $a, b \in K_{\phi\eta}$ and $K_{\phi\eta}$ is ϕ-invex subset with respect to ϕ and η, for every $t \in [0,1]$, we have $a + \lambda e^{i\phi}\eta(b,a) \in K_{\phi\eta}$. Integrating by part, it yields that

$$\begin{aligned}
J = e^{i\phi}\eta(b,a)\Bigg\{&\frac{1}{e^{i\phi}\eta(b,a)}\Bigg[(\lambda - t)f\Big(a+\lambda e^{i\phi}\eta(b,a)\Big)\Big|_0^{\frac{1}{2}} \\
&- \int_0^{\frac{1}{2}} f\Big(a+\lambda e^{i\phi}\eta(b,a)\Big)\mathrm{d}\lambda\Bigg] \\
&+ \frac{1}{e^{i\phi}\eta(b,a)}\Bigg[(\lambda - k)f\Big(a+\lambda e^{i\phi}\eta(b,a)\Big)\Big|_{\frac{1}{2}}^{1} - \int_{\frac{1}{2}}^{1} f\Big(a+\lambda e^{i\phi}\eta(b,a)\Big)\mathrm{d}\lambda\Bigg]\Bigg\} \\
= {}&\Big(\frac{1}{2} - t\Big)f\left(a + \frac{e^{i\phi}\eta(b,a)}{2}\right) + tf(a) - \int_0^{\frac{1}{2}} f\Big(a+\lambda e^{i\phi}\eta(b,a)\Big)\mathrm{d}\lambda \\
&+ (1-k)f\Big(a + e^{i\phi}\eta(b,a)\Big) - \Big(\frac{1}{2} - k\Big)f\left(a + \frac{e^{i\phi}\eta(b,a)}{2}\right) \\
&- \int_{\frac{1}{2}}^{1} f\Big(a+\lambda e^{i\phi}\eta(b,a)\Big)\mathrm{d}\lambda \\
= {}&tf(a) + (1-k)f\Big(a + e^{i\phi}\eta(b,a)\Big) + (k-t)f\left(a + \frac{e^{i\phi}\eta(b,a)}{2}\right) \\
&- \int_0^1 f\Big(a+\lambda e^{i\phi}\eta(b,a)\Big)\mathrm{d}\lambda.
\end{aligned}$$

Let $x = a + \lambda e^{i\phi}\eta(b,a)$, then $\mathrm{d}x = e^{i\phi}\eta(b,a)\mathrm{d}\lambda$ and we have

$$\begin{aligned}
J = {}&tf(a) + (1-k)f\Big(a + e^{i\phi}\eta(b,a)\Big) + (k-t)f\left(a + \frac{e^{i\phi}\eta(b,a)}{2}\right) \\
&- \frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x,
\end{aligned}$$

which is required.

Remark 2.1 *Clearly, applying Lemma 2.1 for $\phi = 0$, $\eta(b,a) = b - a$, $t = \dfrac{1}{6}$, and $k = \dfrac{5}{6}$, then we obtain the Lemma 2.1 in ([28], 2013).*

3 Some Simpson type integral inequalities

In what follows, we establish another refinement of the Simpson's inequality for extended $(s,m)_\phi$-preinvex functions in the second sense.

Theorem 3.1 *Let $A_{\phi\eta} \subseteq \mathbb{R}_0$ be an open ϕ-invex subset with respect to $\phi(.)$ and $\eta : A_{\phi\eta} \times A_{\phi\eta} \to \mathbb{R}_0$, $a, b \in A_{\phi\eta}$ with $a < a + e^{i\phi}\eta(b,a)$, $a < b$. Let $k, t \in \mathbb{R}$. Suppose that $f : A_{\phi\eta} \to \mathbb{R}_0$ is a differentiable function and f' is integrable on $[a, a + e^{i\phi}\eta(b,a)]$. If $|f'|$ is extended $(s,m)_\phi$-preinvex on $A_{\phi\eta}$ for some fixed $(s,m) \in [-1,1] \times (0,1]$ then the following inequality holds:*
1. when $s \in (-1,1]$, we have

$$\begin{aligned}
&\bigg| tf(a) + (1-k)f\Big(a + e^{i\phi}\eta(b,a)\Big) + (k-t)f\left(a + \frac{e^{i\phi}\eta(b,a)}{2}\right) \\
&\quad - \frac{1}{e^{i\phi}\eta(b,a)} \int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x \bigg| \\
&\leq \left|e^{i\phi}\eta(b,a)\right| \left[\nu_1|f'(a)| + m\nu_2\left|f'\Big(\frac{b}{m}\Big)\right|\right],
\end{aligned} \tag{3.1}$$

where

$$v_1 = \frac{2(1-t)^{s+2} + 2(1-k)^{s+2} + \big[2(k+t)(s+2) - 2(s+3)\big]\frac{1}{2^{s+2}} + (ts + 2t - 1)}{(s+1)(s+2)}$$

and

$$v_2 = \frac{2t^{s+2} + 2k^{s+2} + \big[2(s+1) - 2(s+2)(k+t)\big]\frac{1}{2^{s+2}} + (s + 1 - ks - 2k)}{(s+1)(s+2)};$$

2. when $s = -1$, $t = 0$ and $k = 1$, we have

$$\begin{aligned}
&\left| f\left(a + \frac{e^{i\phi}\eta(b,a)}{2}\right) - \frac{1}{e^{i\phi}\eta(b,a)} \int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x \right| \\
&\leq \left|e^{i\phi}\eta(b,a)\right| \ln 2\left[|f'(a)| + m\left|f'\Big(\frac{b}{m}\Big)\right|\right].
\end{aligned} \tag{3.2}$$

Proof. 1. When $-1 < s \leq 1$, by Lemma 2.1 and using the extended $(s,m)_{\phi\eta}$-preinvexity of $|f'|$ on $A_{\phi\eta}$, we have

$$\begin{aligned}
&\bigg| tf(a) + (1-k)f\Big(a + e^{i\phi}\eta(b,a)\Big) + (k-t)f\left(a + \frac{e^{i\phi}\eta(b,a)}{2}\right) \\
&\quad - \frac{1}{e^{i\phi}\eta(b,a)} \int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x \bigg| \\
&\leq \left|e^{i\phi}\eta(b,a)\right| \bigg[\int_0^{\frac{1}{2}} |\lambda - t| \left|f'\Big(a + e^{i\phi}\lambda\eta(b,a)\Big)\right| \mathrm{d}\lambda \\
&\quad + \int_{\frac{1}{2}}^1 |\lambda - k| \left|f'\Big(a + \lambda e^{i\phi}\eta(b,a)\Big)\right| \mathrm{d}\lambda \bigg]
\end{aligned}$$

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$$\begin{aligned}
&\leq \left|e^{i\phi}\eta(b,a)\right|\bigg\{\int_0^{\frac{1}{2}}|\lambda-t|(1-\lambda)^s|f'(a)|\mathrm{d}\lambda+m\int_0^{\frac{1}{2}}|\lambda-t|\lambda^s\Big|f'\Big(\frac{b}{m}\Big)\Big|\mathrm{d}\lambda\\
&\quad+\int_{\frac{1}{2}}^1|\lambda-k|(1-\lambda)^s|f'(a)|\mathrm{d}\lambda+m\int_{\frac{1}{2}}^1|\lambda-k|\lambda^s\Big|f'\Big(\frac{b}{m}\Big)\Big|\mathrm{d}\lambda\bigg\}\\
&=\left|e^{i\phi}\eta(b,a)\right|\bigg\{\bigg[\int_0^{\frac{1}{2}}|\lambda-t|(1-\lambda)^s\mathrm{d}\lambda+\int_{\frac{1}{2}}^1|\lambda-k|(1-\lambda)^s\mathrm{d}\lambda\bigg]\left|f'(a)\right|\\
&\quad+\bigg[m\int_0^{\frac{1}{2}}|\lambda-t|\lambda^s\mathrm{d}\lambda+m\int_{\frac{1}{2}}^1|\lambda-k|\lambda^s\mathrm{d}\lambda\bigg]\Big|f'\Big(\frac{b}{m}\Big)\Big|\bigg\}.
\end{aligned}$$

Using the fact that

$$\begin{aligned}
&\int_0^{\frac{1}{2}}|\lambda-t|(1-\lambda)^s\mathrm{d}\lambda+\int_{\frac{1}{2}}^1|\lambda-k|(1-\lambda)^s\mathrm{d}\lambda\\
&=\frac{2(1-t)^{s+2}+2(1-k)^{s+2}+\big[2(k+t)(s+2)-2(s+3)\big]\frac{1}{2^{s+2}}+(ts+2t-1)}{(s+1)(s+2)}
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{\frac{1}{2}}|\lambda-t|\lambda^s\mathrm{d}\lambda+\int_{\frac{1}{2}}^1|\lambda-k|\lambda^s\mathrm{d}\lambda\\
&=\frac{2t^{s+2}+2k^{s+2}+\big[2(s+1)-2(s+2)(k+t)\big]\frac{1}{2^{s+2}}+(s+1-ks-2k)}{(s+1)(s+2)},
\end{aligned}$$

the desired inequality (3.1) is established.
2. When $s=-1$, $t=0$, and $k=1$, utilizing Lemma 2.1 again and the extended $(-1,m)_{\phi\eta}$-preinvexity of $|f'|$ on $A_{\phi\eta}$, we have that

$$\begin{aligned}
&\left|f\Big(a+\frac{e^{i\phi}\eta(b,a)}{2}\Big)-\frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)}f(x)\mathrm{d}x\right|\\
&\leq\left|e^{i\phi}\eta(b,a)\right|\bigg[\int_0^{\frac{1}{2}}|\lambda|\Big|f'\Big(a+\lambda e^{i\phi}\eta(b,a)\Big)\Big|\mathrm{d}\lambda\\
&\quad+\int_{\frac{1}{2}}^1|\lambda-1|\Big|f'\Big(a+\lambda e^{i\phi}\eta(b,a)\Big)\Big|\mathrm{d}\lambda\bigg]\\
&\leq\left|e^{i\phi}\eta(b,a)\right|\bigg\{\int_0^{\frac{1}{2}}\bigg[\frac{\lambda}{1-\lambda}|f'(a)|+m\frac{\lambda}{\lambda}\Big|f'\Big(\frac{b}{m}\Big)\Big|\bigg]\mathrm{d}\lambda
\end{aligned}$$

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$$
\begin{aligned}
&+\int_{\frac{1}{2}}^{1}\left[\frac{1-\lambda}{1-\lambda}|f'(a)|+m\frac{1-\lambda}{\lambda}\Big|f'\Big(\frac{b}{m}\Big)\Big|\right]\mathrm{d}\lambda\Big\}\\
&=\big|e^{i\phi}\eta(b,a)\big|\ln 2\Big[|f'(a)|+m\Big|f'\Big(\frac{b}{m}\Big)\Big|\Big].
\end{aligned}
$$

This proves as required.
Direct computation yields the following corollaries.

Corollary 3.1 *Under the conditions of Theorem 3.1 and* $s\in(-1,1]$,
1. if $t=\dfrac{1}{6}$ *and* $k=\dfrac{5}{6}$, *we have*

$$
\begin{aligned}
&\left|\frac{1}{6}\Big[f(a)+f\Big(a+e^{i\phi}\eta(b,a)\Big)+4f\Big(a+\frac{e^{i\phi}\eta(b,a)}{2}\Big)\Big]\right.\\
&\quad\left.-\frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)}f(x)\mathrm{d}x\right|\\
&\le\frac{\big[(s-4)6^{s+1}+2\times 5^{s+2}-2\times 3^{s+2}+2\big]}{6^{s+2}(s+1)(s+2)}\\
&\quad\times\big|e^{i\phi}\eta(b,a)\big|\Big[|f'(a)|+m\Big|f'\Big(\frac{b}{m}\Big)\Big|\Big];
\end{aligned}
\tag{3.3}
$$

2. if $\phi=0$, $\eta(b,a)=b-a$, *and* $m=1$ *in inequality (3.3), we have*

$$
\begin{aligned}
&\left|\frac{1}{6}\Big[f(a)+f(b)+4f\Big(\frac{a+b}{2}\Big)\Big]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\\
&\le\frac{(s-4)6^{s+1}+2\times 5^{s+2}-2\times 3^{s+2}+2}{6^{s+2}(s+1)(s+2)}(b-a)\Big[|f'(a)|+|f'(b)|\Big];
\end{aligned}
\tag{3.4}
$$

3. if $t=k=\dfrac{1}{2}$ *and* $s=m=1$ *in inequality (3.1), we have*

$$
\begin{aligned}
&\left|\frac{f(a)+f\big(a+e^{i\phi}\eta(b,a)\big)}{2}-\frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)}f(x)\mathrm{d}x\right|\\
&\le\frac{|e^{i\phi}\eta(b,a)|}{8}\Big(|f'(a)|+|f'(b)|\Big);
\end{aligned}
\tag{3.5}
$$

4. if $\phi=0$ *in inequality (3.5), we have*

$$
\begin{aligned}
&\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}x\right|\\
&\le\frac{|\eta(b,a)|}{8}\Big(|f'(a)|+|f'(b)|\Big).
\end{aligned}
\tag{3.6}
$$

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Remark 3.1 *Inequality (3.4) is the same as inequality of (1.2) presented by Sarikaya et al. in ([30], 2010). Inequality (3.5) is the same as inequality of (1.8) presented by Noor et al. in ([22], 2015). Inequality (3.6) is the same as inequality of (1.5) established by Barani et al. in ([2], 2012). Thus, inequality (3.1) is a generalization of these inequalities.*

Corollary 3.2 *The upper bound of the midpoint inequality for the first derivative is developed as follows:*
1. By putting $f(a) = f\Big(a + e^{i\phi}\eta(b,a)\Big) = f\Big(a + \frac{e^{i\phi}\eta(b,a)}{2}\Big)$ *in inequality (3.1), we have:*

$$\left|f\Big(a + \frac{e^{i\phi}\eta(b,a)}{2}\Big) - \frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x\right| \leq \left|e^{i\phi}\eta(b,a)\right|\Big[\nu_1|f'(a)| + m\nu_2\Big|f'\Big(\frac{b}{m}\Big)\Big|\Big], \tag{3.7}$$

where v_1 and v_2 are defined in Theorem 3.1.
2. Putting $\phi = 0$, $s = 1$, $m = 1$, $t = \dfrac{1}{6}$, and $k = \dfrac{5}{6}$ in the above inequality (3.7), it yields that

$$\left|f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x\right| \leq \frac{5|\eta(b,a)|}{72}\Big(|f'(a)| + |f'(b)|\Big). \tag{3.8}$$

Remark 3.2 *It is noted that the above midpoint inequality (3.8) is better than the inequality (1.6) presented by Sarikaya et al. in ([29], 2012).*

Corollary 3.3 *Under the conditions of Theorem 3.1 and $s = -1$, if $\phi = 0$, $\eta(b,a) = b - a$, we have*

$$\left|f\Big(\frac{a+b}{2}\Big) - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leq (b-a)\ln 2\Big[|f'(a)| + m\Big|f'\Big(\frac{b}{m}\Big)\Big|\Big]. \tag{3.9}$$

Remark 3.3 *When applying $m = 1$ to inequality (3.9), then we get inequality (1.4). Thus, Theorem 3.1 and its consequences generalize the main result in ([35], 2015).*

Theorem 3.2 *Let f be defined as in Theorem 3.1 with $\frac{1}{p} + \frac{1}{q} = 1$. If $|f'|^q$ for $q > 1$ is extended $(s,m)_{\phi\eta}$-preinvex on $A_{\phi\eta}$ for some fixed $(s,m) \in (-1,1] \times (0,1]$*

then the following inequality holds:

$$
\begin{aligned}
&\left|tf(a)+(1-k)f\Big(a+e^{i\phi}\eta(b,a)\Big)+(k-t)f\left(a+\frac{e^{i\phi}\eta(b,a)}{2}\right)\right.\\
&\left.\quad-\frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)}f(x)\mathrm{d}x\right|\\
&\le\frac{|e^{i\phi}\eta(b,a)|}{(p+1)^{\frac{1}{p}}(s+1)^{\frac{1}{q}}}\left\{\left[t^{p+1}+\Big(\frac{1}{2}-t\Big)^{p+1}\right]^{\frac{1}{p}}\right.\\
&\quad\times\left[\Big(1-\Big(\frac{1}{2}\Big)^{s+1}\Big)\Big|f'(a)\Big|^q+m\Big(\frac{1}{2}\Big)^{s+1}\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\right]^{\frac{1}{q}}\\
&\quad+\left[\Big(k-\frac{1}{2}\Big)^{p+1}+(1-k)^{p+1}\right]^{\frac{1}{p}}\\
&\quad\left.\times\left[\Big(\frac{1}{2}\Big)^{s+1}\Big|f'(a)\Big|^q+m\Big(1-\Big(\frac{1}{2}\Big)^{s+1}\Big)\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}\tag{3.10}
$$

Proof. By Lemma 2.1 and using the famous Hölder's inequality, we have

$$
\begin{aligned}
&\left|tf(a)+(1-k)f\Big(a+e^{i\phi}\eta(b,a)\Big)+(k-t)f\left(a+\frac{e^{i\phi}\eta(b,a)}{2}\right)\right.\\
&\left.\quad-\frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)}f(x)\mathrm{d}x\right|\\
&\le\Big|e^{i\phi}\eta(b,a)\Big|\left[\int_0^{\frac{1}{2}}|\lambda-t|\Big|f'\Big(a+\lambda e^{i\phi}\eta(b,a)\Big)\Big|\mathrm{d}\lambda\right.\\
&\quad\left.+\int_{\frac{1}{2}}^1|\lambda-k|\Big|f'\Big(a+\lambda e^{i\phi}\eta(b,a)\Big)\Big|\mathrm{d}\lambda\right]\\
&\le\Big|e^{i\phi}\eta(b,a)\Big|\left\{\left(\int_0^{\frac{1}{2}}|\lambda-t|^p\mathrm{d}\lambda\right)^{\frac{1}{p}}\left[\int_0^{\frac{1}{2}}\Big|f'\Big(a+\lambda e^{i\phi}\eta(b,a)\Big)\Big|^q\mathrm{d}\lambda\right]^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_{\frac{1}{2}}^1|\lambda-k|^p\mathrm{d}\lambda\right)^{\frac{1}{p}}\left[\int_{\frac{1}{2}}^1\Big|f'\Big(a+\lambda e^{i\phi}\eta(b,a)\Big)\Big|^q\mathrm{d}\lambda\right]^{\frac{1}{q}}\right\}.
\end{aligned}
$$

Also, making use of the extended $(s,m)_{\phi\eta}$-convexity of $|f'|^q$, it follows that

$$
\begin{aligned}
&\left|tf(a)+(1-k)f\Big(a+e^{i\phi}\eta(b,a)\Big)+(k-t)f\left(a+\frac{e^{i\phi}\eta(b,a)}{2}\right)\right.\\
&\left.\quad-\frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)}f(x)\mathrm{d}x\right|
\end{aligned}
$$

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$$
\begin{aligned}
&\le \left|e^{i\phi}\eta(b,a)\right|\left\{\left(\int_0^{\frac{1}{2}}|\lambda-t|^p\mathrm{d}\lambda\right)^{\frac{1}{p}}\right.\\
&\quad\times\left[\int_0^{\frac{1}{2}}\left((1-\lambda)^s\left|f'(a)\right|^q+m\lambda^s\left|f'\left(\frac{b}{m}\right)\right|^q\right)\mathrm{d}\lambda\right]^{\frac{1}{q}}\\
&\quad\left.+\left(\int_{\frac{1}{2}}^1|\lambda-k|^p\mathrm{d}\lambda\right)^{\frac{1}{p}}\left[\int_{\frac{1}{2}}^1\left((1-\lambda)^s\left|f'(a)\right|^q+m\lambda^s\left|f'\left(\frac{b}{m}\right)\right|^q\right)\mathrm{d}\lambda\right]^{\frac{1}{q}}\right\}.
\end{aligned}
$$

Direct calculation yields that

$$
\int_0^{\frac{1}{2}}|\lambda-t|^p\mathrm{d}\lambda=\frac{t^{p+1}+\left(\frac{1}{2}-t\right)^{p+1}}{p+1},\quad \int_{\frac{1}{2}}^1|\lambda-k|^p\mathrm{d}\lambda=\frac{\left(k-\frac{1}{2}\right)^{p+1}+(1-k)^{p+1}}{p+1},
$$

similarly, we have

$$
\int_0^{\frac{1}{2}}(1-\lambda)^s\mathrm{d}\lambda=\int_{\frac{1}{2}}^1\lambda^s\mathrm{d}\lambda=\frac{1-\left(\frac{1}{2}\right)^{s+1}}{s+1},\quad \int_0^{\frac{1}{2}}\lambda^s\mathrm{d}\lambda=\int_{\frac{1}{2}}^1(1-\lambda)^s\mathrm{d}\lambda=\frac{\left(\frac{1}{2}\right)^{s+1}}{s+1}.
$$

Therefore, combining the above four equalities can lead to the desired result. The statement in Theorem 3.2 is proved.

Corollary 3.4 *Under the condition of Theorem 3.2,*
1. when $s=1$, we have

$$
\begin{aligned}
&\left|tf(a)+(1-k)f\left(a+e^{i\phi}\eta(b,a)\right)+(k-t)f\left(a+\frac{e^{i\phi}\eta(b,a)}{2}\right)\right.\\
&\left.-\frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)}f(x)\mathrm{d}x\right|\\
&\le\frac{|e^{i\phi}\eta(b,a)|}{2^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\left\{\left[t^{p+1}+\left(\frac{1}{2}-t\right)^{p+1}\right]^{\frac{1}{p}}\left[\frac{3|f'(a)|^q}{4}+\frac{m|f'(\frac{b}{m})|^q}{4}\right]^{\frac{1}{q}}\right.\\
&\quad\left.+\left[\left(k-\frac{1}{2}\right)^{p+1}+(1-k)^{p+1}\right]^{\frac{1}{p}}\left[\frac{|f'(a)|^q}{4}+\frac{3m|f'(\frac{b}{m})|^q}{4}\right]^{\frac{1}{q}}\right\};
\end{aligned}\tag{3.11}
$$

2. when $\phi=0$, $k=1$, $t=0$, and $m=1$ in inequality (3.11), we can get

$$
\begin{aligned}
&\left|f\left(a+\frac{\eta(b,a)}{2}\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}x\right|\\
&\le\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{|\eta(b,a)|}{4}\left[\left(\frac{3}{4}|f'(a)|^q+\frac{1}{4}|f'(b)|^q\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\frac{1}{4}|f'(a)|^q+\frac{3}{4}|f'(b)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}\tag{3.12}
$$

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Remark 3.4 *As* $p = \dfrac{q}{q-1}$, *exchange* a *and* b *in inequality (3.12), then we can deduce the inequality (1.7).*

In the following corollary, we have the midpoint inequality for powers in terms of the first derivative.

Corollary 3.5 *By substituting* $f(a) = f\Big(a + e^{i\phi}\eta(b,a)\Big) = f\Big(a + \frac{e^{i\phi}\eta(b,a)}{2}\Big)$, $t = \dfrac{1}{6}$, *and* $k = \dfrac{5}{6}$ *in Theorem 3.2, we have*

$$\begin{aligned}
&\left|\frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x - f\left(a + \frac{e^{i\phi}\eta(b,a)}{2}\right)\right| \\
&\le \frac{|e^{i\phi}\eta(b,a)|}{2^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\left[\left(\frac{1}{6}\right)^{p+1} + \left(\frac{1}{3}\right)^{p+1}\right]^{\frac{1}{p}} \\
&\times\left\{\left[\frac{3|f'(a)|^q}{4} + \frac{m|f'(\frac{b}{m})|^q}{4}\right]^{\frac{1}{q}} + \left[\frac{|f'(a)|^q}{4} + \frac{3m|f'(\frac{b}{m})|^q}{4}\right]^{\frac{1}{q}}\right\}. \quad (3.13)
\end{aligned}$$

In the following theorem, we obtain another form of Simpson type inequality for powers in term of the first derivative.

Theorem 3.3 *Let* f *be defined as in Theorem 3.1. If the mapping* $|f'|^q$ *for* $q \ge 1$ *is extended* $(s,m)_{\phi\eta}$*-preinvex on* $A_{\phi\eta}$ *for some fixed* $(s,m) \in (-1,1] \times (0,1]$ *then*

$$\begin{aligned}
&\left|tf(a) + (1-k)f\Big(a + e^{i\phi}\eta(b,a)\Big) + (k-t)f\left(a + \frac{e^{i\phi}\eta(b,a)}{2}\right)\right. \\
&\left.\quad - \frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x\right| \\
&\le \left|e^{i\phi}\eta(b,a)\right|\left\{\left(t^2 - \frac{1}{2}t + \frac{1}{8}\right)^{1-\frac{1}{q}}\left[\xi_1|f'(a)|^q + m\xi_2\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}\right. \\
&\quad \left. + \left(k^2 - \frac{3}{2}k + \frac{5}{8}\right)^{1-\frac{1}{q}}\left[\xi_3|f'(a)|^q + m\xi_4\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}\right\}, \qquad (3.14)
\end{aligned}$$

where

$$\xi_1 = \frac{t(s+2) - 1 + 2(1-t)^{s+2} + (2ts + 4t - s - 3)\frac{1}{2^{s+2}}}{(s+1)(s+2)},$$

$$\xi_2 = \frac{2t^{s+2} + (s + 1 - 2ts - 4t)\frac{1}{2^{s+2}}}{(s+1)(s+2)},$$

$$\xi_3 = \frac{2(1-k)^{s+2} + (2ks + 4k - s - 3)\frac{1}{2^{s+2}}}{(s+1)(s+2)},$$

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and

$$\xi_4 = \frac{2k^{s+2} + (s+1-2ks-4k)\dfrac{1}{2^{s+2}} + (s+1-ks-2k)}{(s+1)(s+2)}.$$

Proof. By Lemma 2.1 and power-mean inequality, it follows that

$$\begin{aligned}
&\Bigg| tf(a) + (1-k)f\Big(a + e^{i\phi}\eta(b,a)\Big) + (k-t)f\Big(a + e^{i\phi}\frac{\eta(b,a)}{2}\Big) \\
&\quad - \frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x \Bigg| \\
&\le \big|e^{i\phi}\eta(b,a)\big| \Bigg[\int_0^{\frac{1}{2}} |\lambda - t| \Big| f'\Big(a + \lambda e^{i\phi}\eta(b,a)\Big)\Big| \mathrm{d}\lambda \\
&\quad + \int_{\frac{1}{2}}^1 |\lambda - k| \Big| f'\Big(a + \lambda e^{i\phi}\eta(b,a)\Big)\Big| \mathrm{d}\lambda\Bigg] \\
&\le \big|e^{i\phi}\eta(b,a)\big| \Bigg\{ \Bigg(\int_0^{\frac{1}{2}} |\lambda - t| \mathrm{d}\lambda\Bigg)^{1-\frac{1}{q}} \Bigg[\int_0^{\frac{1}{2}} |\lambda - t| \Big| f'\Big(a + \lambda e^{i\phi}\eta(b,a)\Big)\Big|^q \mathrm{d}\lambda\Bigg]^{\frac{1}{q}} \\
&\quad + \Bigg(\int_{\frac{1}{2}}^1 |\lambda - k| \mathrm{d}\lambda\Bigg)^{1-\frac{1}{q}} \Bigg[\int_{\frac{1}{2}}^1 |\lambda - k| \Big| f'\Big(a + \lambda e^{i\phi}\eta(b,a)\Big)\Big|^q \mathrm{d}\lambda\Bigg]^{\frac{1}{q}} \Bigg\}.
\end{aligned}$$

Using the extended $(s,m)_{\phi\eta}$-convexity of $|f'|^q$, we have that

$$\begin{aligned}
&\Bigg| tf(a) + (1-k)f\Big(a + e^{i\phi}\eta(b,a)\Big) + (k-t)f\Big(a + e^{i\phi}\frac{\eta(b,a)}{2}\Big) \\
&\quad - \frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x \Bigg| \\
&\le \big|e^{i\phi}\eta(b,a)\big| \Bigg\{ \Bigg(\int_0^{\frac{1}{2}} |\lambda - t| \mathrm{d}\lambda\Bigg)^{1-\frac{1}{q}} \\
&\quad \times \Bigg[\int_0^{\frac{1}{2}} |\lambda - t| \Bigg((1-\lambda)^s \Big|f'(a)\Big|^q + m\lambda^s \Big|f'\Big(\frac{b}{m}\Big)\Big|^q\Bigg)\mathrm{d}\lambda\Bigg]^{\frac{1}{q}} + \Bigg(\int_{\frac{1}{2}}^1 |\lambda - k| \mathrm{d}\lambda\Bigg)^{1-\frac{1}{q}} \\
&\quad \times \Bigg[\int_{\frac{1}{2}}^1 |\lambda - k| \Bigg((1-\lambda)^s \Big|f'(a)\Big|^q + m\lambda^s \Big|f'\Big(\frac{b}{m}\Big)\Big|^q\Bigg)\mathrm{d}\lambda\Bigg]^{\frac{1}{q}} \Bigg\}.
\end{aligned}$$

By simple calculations, we can get

$$\int_0^{\frac{1}{2}} |\lambda - t| \mathrm{d}\lambda = t^2 - \frac{1}{2}t + \frac{1}{8}, \quad \int_{\frac{1}{2}}^1 |\lambda - k| \mathrm{d}\lambda = k^2 - \frac{3}{2}k + \frac{5}{8}, \tag{3.15}$$

$$\int_0^{\frac{1}{2}} |\lambda - t|(1-\lambda)^s \mathrm{d}\lambda = \frac{t(s+2) - 1 + 2(1-t)^{s+2} + (2ts + 4t - s - 3)\dfrac{1}{2^{s+2}}}{(s+1)(s+2)}, \tag{3.16}$$

$$\int_0^{\frac{1}{2}} |\lambda - t|\lambda^s \mathrm{d}\lambda = \frac{2t^{s+2} + (s+1-2ts-4t)\dfrac{1}{2^{s+2}}}{(s+1)(s+2)}, \tag{3.17}$$

$$\int_{\frac{1}{2}}^1 |\lambda - k|(1-\lambda)^s \mathrm{d}\lambda = \frac{2(1-k)^{s+2} + (2ks + 4k - s - 3)\dfrac{1}{2^{s+2}}}{(s+1)(s+2)}, \tag{3.18}$$

and

$$\int_{\frac{1}{2}}^1 |\lambda - k|\lambda^s \mathrm{d}\lambda = \frac{2k^{s+2} + (s+1-2ks-4k)\dfrac{1}{2^{s+2}} + (s+1-ks-2k)}{(s+1)(s+2)}. \tag{3.19}$$

Thus, our desired result can be obtained by combining equalities (3.15)-(3.19), the proof is completed.

Corollary 3.6 *Let f be defined as in Theorem 3.3, if $s=1$, $t=\frac{1}{6}$, and $k=\frac{5}{6}$, the inequality holds for m-convex functions:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a) + f\Big(a + e^{i\phi}\eta(b,a)\Big) + 4f\left(a + \frac{e^{i\phi}\eta(b,a)}{2}\right)\right]\right. \\
&\quad \left. - \frac{1}{e^{i\phi}\eta(b,a)} \int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x\right| \\
&\leq \left|e^{i\phi}\eta(b,a)\right|\Big(\frac{5}{72}\Big)^{1-\frac{1}{q}}\left[\Big(\frac{61}{1296}|f'(a)|^q + \frac{29m}{1296}\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\Big)^{\frac{1}{q}}\right. \\
&\quad \left. + \Big(\frac{29}{1296}|f'(a)|^q + \frac{61m}{1296}\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\Big)^{\frac{1}{q}}\right].
\end{aligned} \tag{3.20}$$

In particular, if $m=1$, $\phi=0$, and $\eta(b,a)=b-a$ in inequality (3.20), the inequality holds for convex function. If $|f'(x)| \leq Q$, $\forall x \in I$, then we have

$$\left|\frac{1}{6}\left[f(a) + 4f\Big(\frac{b+a}{2}\Big) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leq \frac{5(b-a)}{36}Q. \tag{3.21}$$

Remark 3.5 *It is observed that the inequality (3.21) is an improvement compared with inequality (1.3). Thus, Theorem 3.3 and its consequences generalize the main results in ([4], 2000).*

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ISOMETRIC EQUIVALENCE OF LINEAR OPERATORS ON SOME SPACES OF ANALYTIC FUNCTIONS

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ABSTRACT. In this paper, we are interested in the isometric equivalence problem for the weighted composition operator $W_{u,\varphi}$ and the composition operator C_φ on the Hardy and the Dirichlet space. We show what properties the operators must satisfy to insure that they are isometric equivalent.

1. INTRODUCTION

Let $\mathbb{D}$ be the unit disk in the complex plane, and $S(\mathbb{D})$ be the set of analytic self-maps of $\mathbb{D}$. The algebra of all holomorphic functions with domain $\mathbb{D}$ will be denoted by $H(\mathbb{D})$.

Let φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$, the multiplication operator M_u is defined by $(M_u f)(z) = u(z)f(z)$, and the weighted composition operator $W_{u,\varphi}$ induced by u and φ is defined by $(W_{u,\varphi}f)(z) = u(z)f(\varphi(z))$ for $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$. If let $u \equiv 1$, then $W_{u,\varphi} = C_\varphi$, which is often called composition operator. As is well known, the weighted composition operator can be regarded as a generalization of a multiplication operator and a composition operator.

Let X and Y be two Banach spaces and T_1 and T_2 are bounded linear operators on X and Y respectively . We say that T_1 and T_2 are isometrically equivalent if there exists surjective isometries U_X and U_Y on X and Y respectively such that $U_X T_1 = T_2 U_Y$. For $X = Y$, two operators T_1 and T_2 are said to be similar if there is a bounded invertible operator S such that $ST_2 = T_1 S$. If S could be chosen to be an isometry as well, then T_1 and T_2 are said to be isometrically isomorphic. If X is a Hilbert space as well as a Banach space, then isometric isomorphism on X is referred to as unitary equivalence.

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The "unitary equivalence" problem for operators on a Hilbert space has received a lot of attention over the yeas. The analogous problem for operators on Banach spaces, due to the lack of inner product structure, requires different techniques directly related to the specific settings under consideration. The "isometric equivalence problem" arises as to what properties the operators must satisfy to insure that they are isometrically equivalent. There has been some recent work on the isometric equivalence problem in spaces of analytic functions. In [1], Wright investigated the isometric equivalence of composition operators for $X = Y = H^p(\mathbb{D})$ for $1 \leq p < \infty$ and $p \neq 2$, he obtained that if two composition operators C_{φ_1} and C_{φ_2} are isometrically equivalent on Hardy space H^p, then $\varphi_1(z) = e^{i\theta}\varphi_2(e^{-i\theta}z)$. In [2, 3], Hornor and Jamison studied isometric equivalence of composition operators on several important Banach spaces of analytic function spaces on the unit disk $\mathbb{D}$. In [4], Jamison studied isometric equivalence of composition operators for $X = Y = \mathcal{B}$, where $\mathcal{B}$ is a Bloch space. He obtained that if two composition operators C_{φ_1} and C_{φ_2} are isometrically equivalent on $\mathcal{B}$, then there is an automorphism φ such that $\varphi_1(\varphi(z)) = \varphi(\varphi_2(z))$; he also investigated the isometric equivalence problem of certain operators on some specific types of Banach spaces. In [5], Nadia studied isometric equivalence of differentiated composition operators on some analytic function spaces. He obtained that two operators DC_{φ_1} and $DC_{\varphi_2} : H^p \to H^q(1 < p, q < \infty$, and $p, q \neq 2)$, then $DC_{\varphi_1}W_p = W_q DC_{\varphi_2}$ if and only if $\varphi_1(z) = e^{-i\theta_p}\varphi_2(e^{i\theta_q}z)$, here $DC_\varphi : H^p \to H^q$ is defined to be $DC_\varphi f = (f \circ \varphi)'$, W_p and W_q are surjective isometries on H^p and H^q.

Building on those foundation, the present paper continues this line of research. More precisely, we first investigated the case of weighted composition operators on the Hardy spaces.

2. Isometric equivalence of weighted composition operators on Hardy spaces

Let $H^\infty(\mathbb{D})$ denote the space of bounded holomorphic functions f on the unit disk with the supremum

$$\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|.$$

The surjective linear isometries of $H^\infty(\mathbb{D})$ were determined in [6]. It was proven that, a surjective linear isometry T of $H^\infty(\mathbb{D})$ is of the form:

$$Tf = \alpha f(\tau) \tag{1}$$

for every $f \in H^\infty(\mathbb{D})$. Where τ is a conformal map of $\mathbb{D}$ and α is a unimodular complex number.

The Hardy space $H^p(\mathbb{D})$ for $1 \leq p < \infty$ is defined to be the Banach space of analytic functions in $\mathbb{D}$ such that

$$\|f\|_{H^p} = \sup_{0<r<1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p d\theta \right\}^{1/p} < \infty.$$

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In [7], it was proved that if $p \neq 2$, a surjective isometry T of H^p is of the form:

$$Tf = \alpha(\tau')^{1/p} f(\tau) \tag{2}$$

where $|\alpha| = 1$ and τ is a conformal map of the disk.

In the following theorem, we are interested in the isometric equivalence for the weighted composition operators on the space $H^\infty(\mathbb{D})$:

Lemma 1. *If φ_1 and φ_2 are analytic functions of the disk into itself, then C_{φ_1} and C_{φ_2} are isometrically equivalent on $H^\infty(\mathbb{D})$ if and only if $\varphi_1(\tau) = \tau(\varphi_2)$; M_{u_1} and M_{u_2} are isometrically equivalent on $H^\infty(\mathbb{D})$ if and only if $u_1(\tau) = u_2$, where $u_1, u_2 \in H(\mathbb{D})$ and τ is a conformal map of the disk.*

Proof. Suppose C_{φ_1} and C_{φ_2} are isometrically equivalent on $H^\infty(\mathbb{D})$, then there is a surjective isometry T on $H^\infty(\mathbb{D})$ such that $TC_{\varphi_1} = C_{\varphi_2}T$. From (1),we have

$$\alpha f(\varphi_1(\tau)) = \alpha f(\tau(\varphi_2))$$

for any $f \in H^\infty(\mathbb{D})$, so $\varphi_1(\tau) = \tau(\varphi_2)$. For the converse, if $\varphi_1(\tau) = \tau(\varphi_2)$, then

$$TC_{\varphi_1} f = C_{\varphi_2} T f$$

for any $f \in H^\infty(\mathbb{D})$.

Similarly, suppose M_{u_1} and M_{u_2} are isometrically equivalent on $H^\infty(\mathbb{D})$, then $TM_{u_1} f = M_{u_2} T f$ for any $f \in H^\infty(\mathbb{D})$. It's easy to get that

$$\alpha u_1(\tau) f(\tau) = \alpha u_2 f(\tau),$$

so $u_1(\tau) = u_2$. For the converse, if $u_1(\tau) = u_2$, then

$$\alpha u_1(\tau) f(\tau) = \alpha u_2 f(\tau)$$

for any $f \in H^\infty(\mathbb{D})$. This implies that M_{u_1} and M_{u_2} are isometrically equivalent on $H^\infty(\mathbb{D})$. The converse is obviously. □

Theorem 2. *Let $u_1, u_2 \in H(\mathbb{D})$ and $\varphi_1, \varphi_2 \in S(\mathbb{D})$. Two weighted composition operators W_{u_1,φ_1} and W_{u_2,φ_2} are isometrically equivalent on $H^\infty(\mathbb{D})$ if and only if C_{φ_1} and C_{φ_2}, M_{u_1} and M_{u_2} are isometrically equivalent on $H^\infty(\mathbb{D})$ respectively .*

Proof. For the sufficiently, suppose C_{φ_1} and C_{φ_2}, M_{u_1} and M_{u_2} are isometrically equivalent, then there exists surjective isometric T on $H^\infty(\mathbb{D})$ such that

$$TC_{\varphi_2} = C_{\varphi_1} T, \quad \text{and} \quad TM_{u_2} = M_{u_1} T.$$

It follows from (1) that

$$TW_{u_2,\varphi_2} f = TM_{u_2} C_{\varphi_2} f = M_{u_1} T T^{-1} C_{\varphi_1} T f = M_{u_1} C_{\varphi_1} T f = W_{u_1,\varphi_1} T f.$$

For the necessity, now suppose $TW_{u_1,\varphi_1} f = W_{u_2,\varphi_2} T f$, where T is a surjective isometry on $H^\infty(\mathbb{D})$. It follows from (1) that

$$\alpha u_1(\tau) f(\varphi_1(\tau)) = \alpha u_2 f(\tau(\varphi_2)) \tag{3}$$

for any $f \in H^\infty(\mathbb{D})$.

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Let $f = z$ in (3), then $\varphi_1(\tau) = \tau(\varphi_2)$. By Lemma 1, we know that C_{φ_1} and C_{φ_2} are isometrically equivalent on $H^\infty(\mathbb{D})$.

Let $f = 1$in (3), then $u_1(\tau) = u_2$. Consequently, from Lemma 1, this implies that M_{u_1} and M_{u_2} are isometrically equivalent on $H^\infty(\mathbb{D})$. The proof of this theorem is completed. □

Theorem 3. *Suppose M_{u_1} and M_{u_2} are two multiplication operators on $H^p(\mathbb{D})$,$1 \le p < \infty$ and $p \ne 2$, then M_{u_1} and M_{u_2} are isometrically equivalent if and only if there exists a conformal map τ of the unit disk onto itself such that $u_1 = u_2(\tau)$.*

Proof. Suppose that M_{u_1} and M_{u_2} are isometrically equivalent, and T is an isometry of $H^p(\mathbb{D})$ onto itself. Then

$$M_{u_1}Tf = TM_{u_2}f$$

for any $f \in H^p(\mathbb{D})$. It follows from (2) that

$$bu_1 f(\tau)(\tau')^{1/p} = bu_2(\tau)f(\tau)(\tau')^{1/p}.$$

So $u_1 = u_2(\tau)$. The converse is obvious. □

Theorem 4. *Let C_{φ_1} is a composition operator on $H^p(\mathbb{D})$,$1 \le p < \infty$ and $p \ne 2$, C_{φ_2} is a composition operator on H^∞ then C_{φ_1} and C_{φ_2} are isometrically equivalent if and only if φ_1 and φ_2 are constants or $\varphi_1(z) = e^{i\theta}\varphi_2(\tau(z))$, here τ is a conformal map of the disk.*

Proof. Suppose C_{φ_1} and C_{φ_2} are isometrically equivalent, then there exists an isometry T_1 on H^p and an isometry T_2 on H^∞ such that $C_{\varphi_1}T_1 f = T_2 C_{\varphi_2} f$. By (1) and (2), from the expression of T_1 and T_2, we have

$$\alpha(\tau_1'(\varphi_1))^{1/p} f(\tau(\varphi_1)) = \beta f(\varphi_2(\tau_2)), \tag{4}$$

where α and β are unimodular complex numbers; $\tau_i, i = 1, 2$ is conformal map of D onto itself.

Let $f = 1$ in (4), then $\alpha(\tau_1'(\varphi_1))^{1/p} = \beta$.

Let $f = z$ in (4), then $\alpha(\tau_1'(\varphi_1))^{1/p}\tau_1(\varphi_1) = \beta\varphi_2(\tau_2)$. Here $\tau_i(z) = \lambda_i \frac{w_i - z}{1 - \bar{w}_i z}$, where $|\lambda_i| = 1$, $|w_i| < 1$.

If $w_i \ne 0, i = 1, 2$, we can get φ_1 and φ_2 are constant functions. If $w_i = 0, i = 1, 2$, we can get $\lambda_1\varphi_1 = \varphi_2(\lambda_2 z)$, that is: $\varphi_1(z) = e^{i\theta_1}\varphi_2(e^{i\theta_2}z)$. If φ_1 and φ_2 are not constant functions, from $\alpha(\tau_1'(\varphi_1))^{1/p} = \beta$, we can get $w_1 = 0$ and $\alpha(\lambda_1)^{1/p} = \beta$, then $\lambda_1\varphi_1(z) = \varphi_2(\tau_2(z))$, that is $\varphi_1(z) = e^{i\theta}\varphi_2(\tau_2(z))$

The converse is obvious. □

3. Isometric equivalence of composition operators on Dirichlet space

For $1 \le p < \infty$, let L_a^p denote the Bergman space of the unit disk $\mathbb{D}$ and $\|\cdot\|_p$ denote the usual norm. $\mathcal{D}^p$ will denote the space of analytic functions on $\mathbb{D}$ for which $f' \in L_a^p$. The norm on $\mathcal{D}^p$ is defined as the following:

$$\|f\|_{\mathcal{D}^p} = (|f(0)|^p + \|f'\|_p)^{1/p}.$$

In [8], the linear isometry T of $\mathcal{D}^p$ $(p \neq 2)$ onto itself is given by:

$$Tf(z) = \lambda[f(0) + \mu \int_0^z [\phi'(\xi)]^{2/p} f'(\phi(\xi))d\xi] \tag{5}$$

where $|\lambda| = |\mu| = 1$ and ϕ is a conformal map of the disk.

We have the following theorem for isometric equivalence of composition operators on $\mathcal{D}^p$:

Theorem 5. *Let φ_1 and φ_2 be analytic maps of the disk.The composition operators C_{φ_1} and C_{φ_2} are isometrically equivalent, as operators on $\mathcal{D}^p$ ($1 < p < \infty$ and $p \neq 2$) if and only if $\varphi_1(z) = \varphi_2(z) = z$.*

Proof. First assume that C_{φ_1} and C_{φ_2} are isometrically equivalent then for any $f \in \mathcal{D}^p$ and T a surjective linear isometry on $\mathcal{D}^p$,we have

$$TC_{\varphi_1}f = C_{\varphi_2}Tf$$

By (5):

$$TC_{\varphi_1}f(z) = \lambda[f(\varphi_1(0)) + \mu \int_0^z [\phi'(\xi)]^{2/p} f'(\varphi_1(\phi(\xi)))\varphi_1'(\phi(\xi))d\xi]$$

and

$$C_{\varphi_2}Tf(z) = \lambda[f(0) + \mu \int_0^{\varphi_2(z)} [\phi'(\xi)]^{2/p} f'(\phi(\xi))d\xi]$$

From $TC_{\varphi_1}f = C_{\varphi_2}Tf$, we can get:

$$\begin{aligned} &\lambda[f(\varphi_1(0)) + \mu \int_0^z [\phi'(\xi)]^{2/p} f'(\varphi_1(\phi(\xi)))\varphi_1'(\phi(\xi))d\xi] \\ &= \lambda[f(0) + \mu \int_0^{\varphi_2(z)} [\phi'(\xi)]^{2/p} f'(\phi(\xi))d\xi] \end{aligned}$$

for any $f \in \mathcal{D}^p$. Derivative with respect to z on both sides of above equation, we can get:

$$[\phi'(z)]^{2/p} f'(\varphi_1(\phi(z)))\varphi_1'(\phi(z)) = [\phi'(\varphi_2(z))]^{2/p} f'(\phi(\varphi_2(z)))\varphi_2'(z).$$

Let $f = z$,we get

$$[\phi'(z)]^{2/p}\varphi_1'(\phi(z)) = [\phi'(\varphi_2(z))]^{2/p}\varphi_2'(z); \tag{6}$$

Let$f = z^2$,we get

$$[\phi'(z)]^{2/p}\varphi_1(\phi(z))\varphi_1'(\phi(z)) = [\phi'(\varphi_2(z))]^{2/p}\phi(\varphi_2(z))\varphi_2'(z);$$

Let$f = z^n$,we can get

$$[\phi'(z)]^{2/p}(\varphi_1(\phi(z)))^{n-1}\varphi_1'(\phi(z)) = [\phi'(\varphi_z(z))]^{2/p}(\phi(\varphi_2(z)))^{n-1}\varphi_2'(z).$$

For $p \neq 1$, we can get:

$$\varphi_1(\phi(z)) = \phi(\varphi_2(z)). \tag{7}$$

Derivative with respect to z, we can get:

$$\varphi_1'(\phi(z))\phi'(z) = \phi'(\varphi_2(z))\varphi_2'(z) \tag{8}$$

The following equations can be obtained from (6),(7) and (8)

$$\phi'(z) = \phi'(\varphi_2(z)) \quad and \quad \varphi_2'(z) = \varphi_1'(\phi(z)). \tag{9}$$

The conformal map ϕ of the unit disk can be written as the form $\phi(z) = \lambda\frac{z-a}{1-\bar{a}z}$, where $|\lambda| = 1, |a| < 1$, so

$$\phi'(z) = \lambda\frac{1-|a|^2}{(1-\bar{a}z)^2}. \tag{10}$$

Then $\varphi_1(z) = \varphi_2(z) = z$ can be got from (7),(9)and (10).

The sufficient condition is obvious. □

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S-fuzzy subalgebras and their S-products in BE-algebras

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Abstract. Using the s-norm S, the notions of an S-fuzzy subalgebra of BE-algebras are introduced, and some properties are investigated. The S-product of S-fuzzy subalgebras is discussed.

1. Introduction

In [5], H. S. Kim and Y. H. Kim introduced the notion of a BE-algebra. S. S. Ahn and K. S. So [3,4] introduced the notion of ideals in BE-algebras. S. S. Ahn et al. [1] fuzzified the concept of BE-algebras, investigated some of their properties. Y. B. Jun and S. S. Ahn ([6]) provided several degrees in defining a fuzzy filter and a fuzzy implicative filter. It is a generalization of a fuzzy filter in BE-algebras.

In this paper, we introduce the notion of an S-fuzzy subalgebra of BE-algebras over an s-norm S, and we investigate some related properties. We also discuss the S-product of S-fuzzy subalgebras of BE-algebras.

2. Preliminaries

An algebra $(X; *, 1)$ of type (2, 0) is called a BE-*algebra* ([5]) if

(BE1) $x * x = 1$ for all $x \in X$;
(BE2) $x * 1 = 1$ for all $x \in X$;
(BE3) $1 * x = x$ for all $x \in X$;
(BE4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$ (*exchange*)

We introduce a relation "$\leq$" on a BE-algebra X by $x \leq y$ if and only if $x * y = 1$. A non-empty subset S of a BE-algebra X is said to be a *subalgebra* of X if it is closed under the operation "$*$". Noticing that $x * x = 1$ for all $x \in X$, it is clear that $1 \in S$. A BE-algebra $(X; *, 1)$ is said to be *self distributive* if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$. A mapping

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$f : X \to Y$ of BE-algebras is called a *homomorphism* if $f(x * y) = f(x) * f(y)$ for any $x, y \in X$. A homomorphism f of BE-algebras is called an *epimorphism* if f is onto. Note that if f is a homomorphism of BE-algebras, then $f(1) = 1$.

Proposition 2.1([5])**.** *Let $(X; *, 1)$ be a self distributive BE-algebra. Then the following hold: for any $x, y, z \in X$,*

(i) *if $x \le y$, then $z * x \le z * y$ and $y * z \le x * z$;*
(ii) *$y * z \le (z * x) * (y * z)$;*
(iii) *$y * z \le (x * y) * (x * z)$.*

A BE-algebra $(X; *, 1)$ is said to be *transitive* if it satisfies Proposition 2.1(iii).

We now review some fuzzy logic concepts. Let X be a non-empty set. A fuzzy set μ in X is a function $\mu : X \to [0, 1]$. Given a fuzzy set μ in X and $\alpha \in [0, 1]$, the set

$$U(\mu; \alpha) := \{x \in X | \mu(x) \ge \alpha\} (\text{resp. } L(\mu; \alpha) := \{x \in X | \mu(x) \le \alpha\})$$

is called an *upper* (resp. *lower*) *level subset* of μ.

Definition 2.2([1])**.** Let μ be a fuzzy set in a BE-algebra X. Then μ is called a *fuzzy BE-algebra* of X if $\mu(x * y) \ge \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

Definition 2.3([6]). A binary operation S on $[0, 1]$ is called a *t-conorm* if

(S1) boundary condition: $S(x, 0) = x$;
(S2) commutativity: $S(x, y) = S(y, x)$;
(S3) associativity: $S(x, S(y, z)) = S(S(x, y), z)$;
(S4) monotonicity: $S(x, y) \le S(x, z)$ whenever $y \le z$, for all $x, y, z \in [0, 1]$.

We call such a t-conorm an *s-norm* in this paper.

Note that $\max\{x, y\} \le S(x, y)$ for all $x, y \in [0, 1]$. Moreover, $([0, 1]; S)$ is a commutative semigroup with 0 as the neutral element. In particular,

$$S(S(x, y), S(z, t)) = S(S(x, z), S(y, t))$$

holds for all $x, y, z, t \in [0, 1]$.

The set of all idempotents with respect to S, i.e., the set

$$E_S := \{x \in [0, 1] | S(x, x) = x\}$$

is a subsemigroup of $([0, 1]; S)$. If $Im(\mu) \subseteq E_S$, then the fuzzy set μ is said to be *idempotent*.

3. S-fuzzy subalgebras

In what follows, let S and X denote an s-norm and a BE-algebra respectively, unless otherwise specified.

Definition 3.1. A fuzzy set μ in X is called a *fuzzy subalgebra* of X over S (briefly, an S-*fuzzy subalgebra* of X) if it satisfies
(SF_0) $\mu(x * y) \leq S(\mu(x), \mu(y))$
for all $x, y \in X$. An S-fuzzy subalgebra μ of X is said to be *idempotent* if $Im(\mu) \subseteq E_S$, where $E_S = \{x \in [0,1] | S(x,x) = x\}$.

Example 3.2. Let $X := \{1, a, b, c, d\}$ be a BE-algebra([5]) with the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Let $S_m : [0,1] \times [0,1] \to [0,1]$ be a function defined by $S_m(x,y) := \min(x+y, 1)$ for all $x, y \in [0,1]$. Then it is easy to see that S_m is an s-norm. Define a fuzzy set μ in X by $\mu(1) = 0, \mu(a) = \mu(b) = 0.5$ and $\mu(c) = \mu(d) = 1$. Then μ is an S_m-fuzzy subalgebra of X, which is not idempotent, since $0.5 \in Im(\mu)$ and $0.5 \notin E_{S_m}$.

Let $S_M : [0,1] \times [0,1] \to [0,1]$ be a function defined by $S_M(x,y) := \max(x,y)$ for all $x, y \in [0,1]$. Then S_M is also an s-norm. It follows that μ is an idempotent S_M-fuzzy subalgebra of X.

Proposition 3.3. *Let S_m be the s-norm defined in Example 3.2 and let S be a subalgebra of X. Then the fuzzy set μ in X defined by*

$$\mu(x) := \begin{cases} 0 & \text{if } x \in S \\ 1 & \text{otherwise,} \end{cases}$$

is an idempotent S_m-fuzzy subalgebra of X.

Proof. Let $x, y \in X$. If $x, y \in S$, then $x * y \in S$ and so $\mu(x * y) = 0 \leq S_m(\mu(x), \mu(y))$. If $x \notin S$ and $y \notin S$, then $\mu(x) = 1 = \mu(y)$. Hence $S_m(\mu(x), \mu(y)) = \min\{1+1, 1\} = 1 \geq \mu(x * y)$. If exactly one of x and y belongs to S, then exactly one of $\mu(x)$ and $\mu(y)$ is equal to 0. It follows that $S_m(\mu(x), \mu(y)) = min\{1+0, 1\} = 1 \geq \mu(x * y)$. Therefore μ is an S_m-fuzzy subalgebra of X. Obviously, $Im(\mu) \subseteq E_{S_m}$, completing the proof. □

Proposition 3.4. *If μ is an idempotent S-fuzzy subalgebra of X, then $\mu(1) \leq \mu(x)$ and $\mu(x*y) \leq \max\{\mu(x), \mu(y)\}$ for all $x, y \in X$.*

Proof. For any $x, y \in X$, we have $\mu(1) = \mu(x*x) \leq S(\mu(x), \mu(x)) = \mu(x)$ and $\max\{\mu(x), \mu(y)\} = S(\max\{\mu(x), \mu(y)\}, \max\{\mu(x), \mu(y)\}) \geq S(\mu(x), \mu(y)) \geq \mu(x*y)$. This competes the proof. ☐

Proposition 3.5. *Let μ be a fuzzy set of X. If every non-empty lower level subset $L(\mu;\alpha)$ of μ is a subalgebra of X, then μ is an S-fuzzy subalgebra of X.*

Proof. Assume that there exist $a, b \in X$ such that $\mu(a*b) > S(\mu(a), \mu(b))$. If we take $m_0 := \frac{1}{2}(\mu(a*b) + S(\mu(a), \mu(b)))$, then $\mu(a*b) > m_0 > S(\mu(a), \mu(b)) \geq \max(\mu(a), \mu(b))$. Hence $a, b \in L(\mu; m_0)$, but $a*b \notin L(\mu; m_0)$. This is a contradiction and so μ satisfies the inequality $\mu(x*y) \leq S(\mu(x), \mu(y))$ for all $x, y \in X$. This completes the proof. ☐

The converse of Proposition 3.5 may not be true as seen in the following example.

Example 3.6. In Example 3.2, define a fuzzy set ν in X by $\nu(1) = 0, \nu(b) = \nu(d) = 0.5$ and $\nu(a) = \nu(c) = 1$. Let S_m be the s-norm in Example 3.2. Then it is easy to see that ν is an S_m-fuzzy subalgebra of X, but the lower level subset $L(\nu; 0.5) = \{1, b, d\}$ is not a subalgebra of X, since $b*d = c \notin L(\nu; 0.5)$.

Proposition 3.7. *Let μ be an idempotent S-fuzzy subalgebra of X. Then the non-empty lower level subset $L(\mu;\alpha)$ of μ is a subalgebra of X.*

Proof. Let $x, y \in L(\mu;\alpha)$, where $\alpha \in [0,1]$. Then $\mu(x) \leq \alpha$ and $\mu(y) \leq \alpha$. Hence $\mu(x*y) \leq S(\mu(x), \mu(y)) \leq S(\alpha, \alpha) = \alpha$ and so $x*y \in L(\mu;\alpha)$. Thus $L(\mu;\alpha)$ is a subalgebra of X. ☐

Proposition 3.8. *Let μ be an S-fuzzy subalgebra of X. If there is a sequence $\{x_n\}$ in X such that $lim_{n\to\infty} S(\mu(x_n), \mu(x_n)) = 0$, then $\mu(1) = 0$.*

Proof. For any $x \in X$, we have $\mu(1) = \mu(x*x) \leq S(\mu(x), \mu(x))$. Therefore $\mu(1) \leq S(\mu(x_n), \mu(x_n))$ for each $n \in \mathbb{N}$ and so $0 \leq \mu(1) \leq lim_{n\to\infty} S(\mu(x_n), \mu(x_n)) = 0$. It follows that $\mu(1) = 0$. ☐

Let f be a mapping defined on X and let μ be a fuzzy set in $f(X)$. The fuzzy set $f^{-1}(\mu)$ in X defined by $[f^{-1}(\mu)](x) := \mu(f(x))$ for all $x \in X$ is called the *preimage* of μ under f.

Theorem 3.9. *Let $f : X \to Y$ be an epimorphism of BE-algebras and let μ be an S-fuzzy subalgebra of Y. Then the preimage $f^{-1}(\mu)$ of μ under f is also an S-fuzzy subalgebra of X.*

Proof. Assume that μ is an S-fuzzy subalgebra of Y. Let $x, y \in X$. Then

$$\begin{aligned}[f^{-1}(\mu)](x*y)) =& \mu(f(x*y)) = \mu(f(x)*f(y)) \\ \leq& S(\mu(f(x)), \mu(f(y))) = S([f^{-1}(\mu)](x), [f^{-1}(\mu)](y)).\end{aligned}$$

Hence $f^{-1}(\mu)$ is an S-fuzzy subalgebra of X. ☐

Let μ be a fuzzy set in X and let f be a mapping defined on X. The fuzzy set μ^f in $f(X)$ defined by $\mu^f(y) := \inf_{x\in f^{-1}(y)} \mu(x)$ for all $y \in f(X)$ is called the *anti-image* of μ under f.

Definition 3.10. An s-norm S on $[0,1]$ is said to be *continuous* if S is a continuous function from $[0,1]\times[0,1]$ to $[0,1]$ with respect to the usual topology.

Theorem 3.11 *Let S be a continuous s-norm and let $f : X \to Y$ be an epimorphism of BE-algebras. If μ is an S-fuzzy subalgebra of X, then anti-image μ^f is also an S-fuzzy subalgebra of Y.*

Proof. Let $A_1 := f^{-1}(y_1), A_2 := f^{-1}(y_2)$ and $A_{12} := f^{-1}(y_1 * y_2)$, where $y_1, y_2 \in Y$. Consider the set $A_1 * A_2 := \{x \in X | x = a_1 * a_2 \text{ for some } a_1 \in A_1, a_2 \in A_2\}$. If $x \in A_1 * A_2$, then $x = x_1 * x_2$ for some $x_1 \in A_1, x_2 \in A_2$ and so $f(x) = f(x_1 * x_2) = f(x_1) * f(x_2) = y_1 * y_2$, i.e., $x \in f^{-1}(y_1 * y_2) = A_{12}$. Hence $A_1 * A_2 \subseteq A_{12}$. It follows that

$$\begin{aligned}\mu^f(y_1 * y_2) &= \inf_{x\in f^{-1}(y_1*y_2)} \mu(x) = \inf_{x\in A_{12}} \mu(x)\\ &\le \inf_{x\in A_1*A_2} \mu(x) = \inf_{x\in A_1, x_2\in A_2} \mu(x_1 * x_2)\\ &\le \inf_{x\in A_1, x_2\in A_2} S(\mu(x_1), \mu(x_2)).\end{aligned}$$

Since S is continuous, if ϵ is any positive number, then there exists a number $\delta > 0$ such that $S(x_1^*, x_2^*) \subseteq S(\inf_{x_1\in A_1} \mu(x_1), \inf_{x_2\in A_2} \mu(x_2)) + \epsilon$, whenever $x_1^* \le \inf_{x_1\in A_1} \mu(x_1) + \delta$ and $x_2^* \le \inf_{x_2\in A_2} \mu(x_2) + \delta$. Choose $a_1 \in A_1, a_2 \in A_2$ such that $\mu(a_1) \le \inf_{x_1\in A_1} \mu(x_1) + \delta$ and $\mu(a_2) \le \inf_{x_2\in A_2} \mu(x_2) + \delta$. Then $S(\mu(a_1), \mu(a_2)) \le S(\inf_{x_1\in A_1} \mu(x_1), \inf_{x_2\in A_2} \mu(x_2)) + \epsilon$. Hence we have

$$\begin{aligned}\mu^f(y_1 * y_2) &\le \inf_{x_1\in A_1, x_2\in A_2} S(\mu(x_1), \mu(x_2))\\ &\le S(\inf_{x_1\in A_1} \mu(x_1), \inf_{x_2\in A_2} \mu(x_2))\\ &= S(\mu^f(y_1), \mu^f(y_2)).\end{aligned}$$

Thus μ^f is an S-fuzzy subalgebra of Y. □

Theorem 3.12. *Let μ be an idempotent S-fuzzy subalgebra of X. Then the set*

$$X_\mu := \{x \in X | \mu(x) = \mu(1)\}$$

is a subalgebra of X.

Proof. Noticing that $\mu(1) \le \mu(x)$ for all $x \in X$, we have $L(\mu; \mu(1)) = \{x \in X | \mu(x) \le \mu(1)\} = \{x \in X | \mu(x) = \mu(1)\} = X_\mu$. By Proposition 3.7, X_μ is a subalgebra of X. □

Proposition 3.13. *Let μ, ν be idempotent S-fuzzy subalgebras of X. If $\mu \subset \nu$ and $\mu(1) = \nu(1)$, then $X_\mu \subset X_\nu$.*

Proof. Assume that $\mu \subset \nu$ and $\mu(1) = \nu(1)$. Let $x \in X_\mu$. Then $\nu(x) > \mu(x) = \mu(1) = \nu(1)$. Noticing $\nu(x) \le \nu(1)$ for all $x \in X$, we have $\nu(x) = \nu(1)$, i.e., $x \in X_\nu$. This completes the proof. □

Theorem 3.14. *Let S_M be an s-norm defined in Example 3.2. Let μ be an S_M-fuzzy subalgebra of X and let $f : [\mu(1), 1] \to [0, 1]$ be an increasing function. Define a fuzzy set $\mu_f : X \to [0, 1]$ by*

$$\mu_f(x) := f(\mu(x))$$

for all $x \in X$. Then μ_f is an S_M-fuzzy subalgebra of X. Furthermore, if $f(\alpha) \geq \alpha$ for all $\alpha \in [\mu(1), 1]$, then $\mu \subseteq \mu_f$.

Proof. Let $x, y \in X$. Then

$$\begin{aligned}\mu_f(x * y) =&f(\mu(x * y)) \leq f(S_M(\mu(x), \mu(y))) \\ \leq& S_M(f(\mu(x)), f(\mu(y))) = S_M(\mu_f(x), \mu_f(y)).\end{aligned}$$

Hence μ_f is an S_M-fuzzy subalgebra of X. Assume that $f(\alpha) \geq \alpha$ for all $\alpha \in [\mu(1), 1]$. Then $\mu_f(x) = f(\mu(x)) \geq \mu(x)$ for all $x \in X$, which proves that $\mu \subset \mu_f$. □

4. **Direct products and s-normed products**

Definition 4.1. Let μ and ν be fuzzy sets of X and let S be an s-norm of X. Then the S-*product* of μ and ν is defined by

$$[\mu \cdot \nu]_S(x) := S(\mu(x), \nu(x))$$

for all $x \in X$ and we denote it by $[\mu \cdot \nu]_S$.

Theorem 4.2. *Let μ, ν be two S-fuzzy subalgebras of X and let S^* be an s-norm which dominates S, i.e.,*

$$S^*(S(a, b), S(c, d)) \leq S(S^*(a, c), S^*(b, d))$$

for all a, b, c and $d \in [0, 1]$. Then the S^*-product $[\mu \cdot \nu]_{S^*}$ of μ and ν is an S-fuzzy subalgebra of X.

Proof. For any $x, y \in X$, we have

$$\begin{aligned}[\mu \cdot \nu]_{S^*}(x * y) &= S^*\{\mu(x * y), \nu(x * y)\} \\ &\leq S^*\{S\{\mu(x), \mu(y)\}, S\{\nu(x), \nu(y)\}\} \\ &\leq S\{S^*\{\mu(x), \nu(x)\}, S^*\{\mu(y), \nu(y)\}\} \\ &= S\{[\mu \cdot \nu]_{S^*}(x), [\mu \cdot \nu]_{S^*}(y)\}.\end{aligned}$$

Hence $[\mu \cdot \nu]_{S^*}$ is an S-fuzzy subalgebra of X. □

Let $f : X \to Y$ be an epimorphism of BE-algebras. If μ and ν are S-fuzzy subalgebras of Y, then the S^*-product $[\mu \cdot \nu]_{S^*}$ of μ and ν is also an S-fuzzy subalgebra of Y whenever S^* dominates S. Since every epimorphic preimage of an S-fuzzy subalgebra is also an S-fuzzy subalgebra, the

preimages $f^{-1}(\mu), f^{-1}(\nu)$ and $f^{-1}([\mu\cdot\nu]_{S^*})$ are S-fuzzy subalgebras. The next theorem provides the relation between $f^{-1}([\mu\cdot\nu]_{S^*})$ and the S^*-product $[f^{-1}(\mu)\cdot f^{-1}(\nu)]_{S*}$ of $f^{-1}(\mu)$ and $f^{-1}(\nu)$.

Proposition 4.3. *Assume that $f : X \to Y$ is an epimorphism of BE-algebras and S, S^* are s-norms such that S^* dorminates S. For any S-fuzzy subalgebras μ and ν of Y, we have*

$$f^{-1}([\mu\cdot\nu]_{S^*}) = [f^{-1}(\mu)\cdot f^{-1}(\nu)]_{S^*}.$$

Proof. For any $x \in X$, we obtain

$$\begin{aligned}\{f^{-1}([\mu\cdot\nu]_{S*})\}(x) =&[\mu\cdot\nu]_{S^*}(f(x))\\ =&S^*\{\mu(f(x)),\nu(f(x))\}\\ =&S^*([f^{-1}(\mu)](x),[f^{-1}(\nu)](x))\\ =&[f^{-1}(\mu)\cdot f^{-1}(\nu)]_{S^*}(x),\end{aligned}$$

completing the proof. □

Let $(X_1, *_1, 1_1)$ and $(X_2, *_2, 1_2)$ be BE-algebras. Define a binary operation "$*$" on $X_1 \times X_2$ by

$$(x_1, x_2) * (y_1, y_2) := (x_1 *_1 x_2, y_1 *_2 y_2)$$

for all $(x_1, x_2), (y_1, y_2) \in X$. Then $(X, *, 1)$ is a BE-algebra, where $1 = (1_1, 1_2)$.

Theorem 4.4. *Let $X = X_1 \times X_2$ be the direct product of BE-algebras X_1 and X_2. If μ_1 (resp., μ_2) is an S-fuzzy subalgebra of X_1(resp., X_2), then $\mu := \mu_1 \times \mu_2$ is an S-fuzzy subalgebra of X defined by*

$$\mu(x_1, x_2) = (\mu_1 \times \mu_2)(x_1, x_2) = S(\mu_1(x_1), \mu_2(x_2))$$

for all $(x_1, x_2) \in X_1 \times X_2$.

Proof. Let $x = (x_1, x_2), y = (y_1, y_2) \in X$. Then we have

$$\begin{aligned}\mu(x*y) =&\mu((x_1,x_2)*(y_1,y_2))\\ =&\mu(x_1*y_1,x_2*y_2)\\ =&S(\mu_1(x_1*y_1),\mu_2(x_2*y_2))\\ \le&S(S(\mu_1(x_1),\mu_1(y_1)),S(\mu_2(x_2),\mu_2(y_2)))\\ =&S(S(\mu_1(x_1),\mu_2(x_2)),S(\mu_1(y_1),\mu_2(y_2)))\\ =&S(\mu(x_1,x_2),\mu(y_1,y_2))\\ =&S(\mu(x),\mu(y)).\end{aligned}$$

Hence $\mu = \mu_1 \times \mu_2$ is an S-fuzzy subalgebra of X. □

Now, we generalize the idea to the product of S-fuzzy subalgebras. We first need to generalize the domain of an s-norm to $\prod_{i=1}^{n}[0,1]$ as follows.

Definition 4.5. We define a map $S_n : \prod_{i=1}^{n}[0,1] \to [0,1]$ is defined by $S_n(\alpha_1, \alpha_2, \cdots, \alpha_n) := S(\alpha_i, S_{n-1}(\alpha_1, \cdots, \alpha_{i-1}, \alpha_{i+1}, \cdots, \alpha_n))$ for all $1 \le i \le n$, where $S_2 = S$ and $S_1 = \mathrm{id}_{[0,1]}$.

Using the induction on n, we have following two lemmas:

Lemma 4.6. *For an s-norm S and every α_i, β_i, where $1 \le i \le n$ and $n \ge 2$, we have*

$$\begin{aligned}S_n(S(\alpha_1, \beta_1),&S(\alpha_2, \beta_2), \cdots, S(\alpha_n, \beta_n)\\ &=S(S_n(\alpha_1, \alpha_2, \cdots, \alpha_n), S_n(\beta_1, \beta_2, \cdots, \beta_n)).\end{aligned}$$

Lemma 4.7. *For an s-norm S and every $\alpha_1, \alpha_2, \cdots, \alpha_n \in [0,1]$, where $n \ge 2$, we have*

$$\begin{aligned}S_n(\alpha_1, \alpha_2, \cdots, \alpha_n) =&S(\cdots, S(S(S(\alpha_1, \alpha_2), \alpha_3, \alpha_4), \cdots, \alpha_n)\\ =&S(\alpha_1, S(\alpha_2, S(\alpha_3, \cdots, S(\alpha_{n-1}, \alpha_n)\cdots))).\end{aligned}$$

Theorem 4.8. *Let $X := \prod_{i=1}^{n} X_i$ be the direct product of BE-algebras $\{X_i\}_{i=1}^{n}$. If μ_i is an S-fuzzy subalgebra of X_i, where $1 \le i \le n$, then $\mu = \prod_{i=1}^{n}\mu_i$ defined by*

$$\mu(x_1, \cdots, x_n) = (\prod_{i=1}^{n}\mu_i)(x_1, \cdots, x_n) = S(\mu(x_1), \cdots, \mu(x_n))$$

is an S-fuzzy subalgebra of X.

Proof. Let $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n)$ be any elements of X. Using Lemmas 4.6 and 4.7, we have

$$\begin{aligned}\mu(x * y) =&\mu((x_1 * y_1), (x_2 * y_2), \cdots, (x_n * y_n))\\ =&S_n(\mu_1(x_1 * y_1), \cdots, \mu_n(x_n * y_n))\\ \le&S_n(S(\mu_1(x_1), \mu_1(y_1)), \cdots, S(\mu_n(x_n), \mu_n(y_n)))\\ =&S(S_n(\mu_1(x_1), \mu_2(x_2), \cdots, \mu_n(x_n)),\\ &\ S_n(\mu_1(y_1), \mu_2(y_2), \cdots, \mu_n(y_n)))\\ =&S(\mu((x_1, \cdots, x_n) * (y_1, \cdots, y_n)))\\ =&S(\mu(x * y)).\end{aligned}$$

Hence $\mu = \prod_{i=1}^{n}\mu_i$ is an S-fuzzy subalgebra of X. □

Theorem 4.9. *Let S be a continuous s-norm and $f : X \to Y$ be an epimorphism of BE-algebras, and let μ and ν be S-fuzzy subalgebras of X. If an s-norm S^* dominates S, then*

$$([\mu \cdot \nu]_{S^*})^f \subseteq [\mu^f \cdot \nu^f]_{S^*}.$$

Proof. By Theorems 4.2 and 3.11, the S^*-product $[\mu \cdot \nu]_{S^*}$ is an S-fuzzy subalgebra of X, and the S^*-product $[\mu^f \cdot \nu^f]_{S^*}$ is an S-fuzzy subalgebra of Y. Moreover, for each $y \in Y$, we have

$$\begin{aligned} ([\mu \cdot \nu]_{S^*})^f(y) &= \inf_{x \in f^{-1}(y)} [\mu \cdot \nu]_{S^*}(x) \\ &= \inf_{x \in f^{-1}(y)} S^*(\mu(x), \nu(x)) \\ &\leq S^*(\inf_{x \in f^{-1}(y)} \mu(x), \inf_{x \in f^{-1}(y)} \nu(x)) \\ &= S^*(\mu^f(y), \nu^f(y)) \\ &= ([\mu^f \cdot \nu^f]_{S^*})(y), \end{aligned}$$

proving that $([\mu \cdot \nu]_{S^*})^f \subseteq [\mu^f \cdot \nu^f]_{S^*}$. □

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Fixed point results for generalized g-quasi-contractions of Perov-type in cone metric spaces over Banach algebras without the assumption of normality

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Abstract: In this paper, we introduce the concept of generalized g-quasi-contractions of Perov-type in the setting of cone metric spaces over Banach algebras. By omitting the assumption of normality of the cone we establish common fixed point theorems for generalized g-quasi-contractions of Perov-type with the spectral radius $r(\lambda)$ of the g-quasi-contractive constant vector λ satisfying $r(\lambda) \in [0,\ 1)$ in the setting of cone metric spaces over Banach algebras. The main results generalize, extend and unify several well-known comparable results in the literature. As a result, we extend the famous Ćirić fixed point theorem to the version in the setting of cone metric spaces over Banach algebras.

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1 Introduction

It is known that the modern metric fixed point theory was motivated from Banach contraction principle (see, e.g., [1]) which plays an important role in various fields of applied mathematical analysis. In 1922, Polish mathematician proved the following classical Banach contraction principle:

Theorem 1.1 ([1]) Let $T : X \to X$ be a contraction on a complete metric space (X, d). Then T possesses exactly one fixed point $x^* \in X$. Moreover, for any point $x \in X$, the sequence $\{T^n(x) : n = 0, 1, 2, ...\}$ converges to $x^* \in X$. That is $\lim_{n\to\infty} T^n(x) = x^*$, for each $x \in X$, where T^n denotes the n-fold composition of T.

Since 1922, many authors have obtained all kinds of versions to extend the famous Banach contraction principle. In general, people did such extensions by means of two methods. One is to extend Banach contraction to other more general mapping or mappings (for example, when two or more mappings are involved and discussed, the common fixed point(s) is(are) usually investigated). The other is to extend classical metric space to more general spaces (usually called abstract spaces). There are many generalizations of the concept metric space in the literature. In 1964, Perov [34] introduced vector valued metric space, instead of general metric space, and obtained a Banach type fixed point theorem on such a complete generalized metric space. Later on, following Perov, many authors studied fixed point results of Perov-type in more general abstract spaces, such as cone metric spaces, etc (see [35]-[39]). Among them, Cvetković and Rakočević [36] introduced the concept of f-quasi-contraction of Perov-type and obtained fixed point results for such kind mappings, which is a generalization of the famous Ćirić mappings. Let $(X,\, d)$ be a complete metric space. Recall that a mapping $T : X \to X$ is called a quasi-contraction if, for some $k \in [0,\, 1)$ and for all $x,\, y \in X$, one has

$$d(Tx,\, Ty) \leqslant k \max\{d(x,\, y),\, d(x,\, Tx),\, d(y,\, Ty),\, d(x,\, Ty),\, d(y,\, Tx)\}.$$

Ćirić [13] introduced and studied quasi-contractions as one of the most general classes of contractive-type mappings. He proved the well-known theorem that any quasi-contraction T has a unique fixed point. Recently, many authors obtained various similar results on cone b-metric spaces (some authors call such spaces cone metric type spaces) and cone metric spaces. See, for instance, [7]-[15].

Since 2010, some authors have investigated the problem of whether cone metric spaces are equivalent to metric spaces in terms of the existence of the fixed points of the mappings involved. They used to establish the equivalence between some fixed point results in metric and in (topological vector spaces valued) cone metric spaces by means of the nonlinear scalarization function ξ_e where e denotes the vector in the internal of the underlying solid cone (see [16]-[19]). Very recently, based on the concept of cone metric spaces, Liu and Xu [21] studied cone metric spaces with Banach algebras, replacing Banach spaces by Banach algebras as the underlying spaces of cone metric spaces. In [21], the authors proved some fixed point theorems of quasi-contractions in cone metric spaces over Banach algebras, but the proof relied strongly on the assumption that the underlying cone is normal. We need state that it is significant to study cone metric spaces with Banach algebras (which we would like to call in this paper cone metric spaces over Banach algebras). This is because there are examples to show that one is unable to conclude that the cone metric space (X, d) over a Banach algebra $\mathcal{A}$ discussed is equivalent to the metric space (X, d^*), where the metric d^* is defined by $d^* = \xi_e \circ d$, here the nonlinear scalarization function $\xi_e : \mathcal{A} \to \mathbb{R}$ $(e \in \text{int}P)$ is defined by

$$\xi_e(y) = \inf\{r \in \mathbb{R} : y \in re - P\}. \tag{1.1}$$

See [20] for more details.

In the present paper we introduce the concept of generalized g-quasi-contractions of Perov-type in cone metric spaces over Banach algebras and obtain common fixed point theorems for two weakly compatible self-mappings satisfying g-quasi-contractive condition in the case of g-quasi-contractive constant vector with $r(\lambda) \in [0, 1/s)$ in cone metric spaces without the assumption of normality. Our main results extend the fixed point theorem of quasi-contractions of Das-Naik in metric spaces to the case in cone metric spaces over Banach algebras. As consequences, we obtain the versions of Ćirić fixed point theorem and Banach contraction principle in the setting of cone metric spaces over Banach algebras. Our main results generalize and extend the relevant results in the literature (see, for example, [3]-[9], [13], [15], [21], [23], [25], [27]).

In addition, we give an example to show that the main results are genuine generalizations of the corresponding results in the literature.

3

2 Preliminaries

Let $\mathcal{A}$ always be a real Banach algebra. That is, $\mathcal{A}$ is a real Banach space in which an operation of multiplication is defined, subject to the following properties (for all x, y, $z \in \mathcal{A}$, $\alpha \in \mathbb{R}$):

1. $(xy)z = x(yz)$;
2. $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$;
3. $\alpha(xy) = (\alpha x)y = x(\alpha y)$;
4. $\|xy\| \leqslant \|x\| \, \|y\|$.

Throughout this paper, we shall assume that a Banach algebra $\mathcal{A}$ has a unit (i.e., a multiplicative identity) e such that $ex = xe = x$ for all $x \in \mathcal{A}$. An element $x \in \mathcal{A}$ is said to be invertible if there is an inverse element $y \in \mathcal{A}$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}. For more details, we refer to [28].

The following proposition is well known (see [28]).

Proposition 2.1 Let $\mathcal{A}$ be a Banach algebra with a unit e, and $x \in \mathcal{A}$. If the spectral radius $r(x)$ of x is less than 1, i.e.,

$$r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_{n\geqslant 1} \|x^n\|^{\frac{1}{n}} < 1,$$

then $e - x$ is invertible. Actually,

$$(e-x)^{-1} = \sum_{i=0}^{\infty} x^i.$$

Now let us recall the concepts of cone and semi-order for a Banach algebra $\mathcal{A}$. A subset P of $\mathcal{A}$ is called a cone if

1. P is non-empty closed and $\{\theta, e\} \subset P$;
2. $\alpha P + \beta P \subset P$ for all non-negative real numbers α, β;
3. $P^2 = PP \subset P$;
4. $P \cap (-P) = \{\theta\}$,

where θ denotes the null of the Banach algebra $\mathcal{A}$. For a given cone $P \subset \mathcal{A}$, we can define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. $x \prec y$ will stand for $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \mathrm{int}P$, where $\mathrm{int}P$ denotes the interior of P.

The cone P is called normal if there is a number $M > 0$ such that for all $x, y \in \mathcal{A}$,

$$\theta \preceq x \preceq y \Rightarrow \|x\| \leqslant M\|y\|.$$

The least positive number satisfying above is called the normal constant of P.

In the following we always assume that P is a cone in Banach algebra $\mathcal{A}$ with $\mathrm{int}P \neq \emptyset$ and $\preceq$ is the partial ordering with respect to P.

Definition 2.1 (See [2], [3], [20], [21]) Let X be a non-empty set. Suppose the mapping $d: X \times X \to \mathcal{A}$ satisfies

1. $0 \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, y) \preceq d(x, z) + d(z, x)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space over a Banach algebra $\mathcal{A}$.

Definition 2.2 (See [2], [3], [20], [21]) Let (X, d) be a cone metric space with a solid cone P over a Banach algebra $\mathcal{A}$, $x \in X$ and $\{x_n\}$ a sequence in X. Then

1. $\{x_n\}$ converges to x whenever for each $c \in \mathcal{A}$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geqslant N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.
2. $\{x_n\}$ is a Cauchy sequence whenever for each $c \in \mathcal{A}$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geqslant N$.
3. (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

Now, we shall appeal to the following lemmas in the sequel.

Lemma 2.1 (See [12]) If E is a real Banach space with a cone P and if $a \preceq \lambda a$ with $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.

Lemma 2.2 (See [27]) If E is a real Banach space with a solid cone P and if $\theta < u \ll c$ for each $\theta \ll c$, then $u = \theta$.

Lemma 2.3 (See [27]) If E is a real Banach space with a solid cone P and if $\|x_n\| \to 0 (n \to \infty)$, then for any $\theta \ll \epsilon$, there exists $N \in \mathbb{N}$ such that for any $n > N$, we have $x_n \ll \epsilon$.

Finally, let us recall the concept of quasi-contraction defining on the cone metric spaces over Banach algebras, which is introduced in [21].

Definition 2.3 (See [21]) Let $(X,\, d)$ be a cone metric space over a Banach algebra $\mathcal{A}$. A mapping $T : X \to X$ is called a quasi-contraction if for some $k \in P$ with $r(k) < 1$ and for all $x,\, y \in X$, one has

$$d(Tx,\, Ty) \preceq ku,$$

where

$$u \in \{d(x,\, y),\, d(x,\, Tx),\, d(y,\, Ty),\, d(x,\, Ty),\, d(y,\, Tx)\}.$$

Remark 2.1 (See [29]) If $r(k) < 1$, then $\|k^m\| \to 0 (m \to \infty)$.

Lemma 2.4 (See [4]-[6], [23], [32]) Let $\preceq$ be the partial ordering with respect to P, where P is the given solid cone P of the Banach algebra $\mathcal{A}$. The following properties are often used while dealing with cone metric spaces where the underlying cone is solid but not necessarily normal.

(1) If $u \ll v$ and $v \preceq w$, then $u \ll w$.

(2) If $\theta \preceq u \ll c$ for each $c \in \text{int}P$, then $u = \theta$.

(3) If $a \preceq b + c$ for each $c \in \text{int}P$, then $a \preceq b$.

(4) If $c \in \text{int}P$ and $a_n \to \theta$, then there exists $n_0 \in \mathbb{N}$ such that $a_n \ll c$ for all $n > n_0$.

(5) Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$, $x \in X$ and $\{x_n\}$ be a sequence in X. If $d(x_n, x) \preceq b_n$ and $b_n \to \theta$, then $x_n \to x$.

Lemma 2.5 (See [3]) The limit of a convergent sequence in cone metric space is unique.

Definition 2.6 (See [4], [11]) The mappings $f, g : X \to X$ are called weakly compatible, if for every $x \in X$ holds $fgx = gfx$ whenever $fx = gx$.

Definition 2.7 (See [4], [11], [14]) Let f and g be self-maps of a set X. If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

Lemma 2.6 (See (See [4], [11], [14]) Let f and g be weakly compatible self-maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.

Definition 2.8 (See [23] Let (X, d) be a cone metric space. A mapping $f : X \to X$ such that, for some constant $\lambda \in [0, 1)$ and for every $x, y \in X$, there exists an element

$$u \in C(g; x, y) = \{d(gx, gy), d(gx, fx), d(gy, fy), d(gx, fy), d(gy, fx)\}$$

for which $d(fx, fy) \preceq \lambda u$ is called a g-quasi-contraction, where $g : X \to X$, $f(X) \subset g(X)$.

Definition 2.9 Let $(X,\ d)$ be a cone metric space over a Banach algebra $\mathcal{A}$. A mapping $f : X \to X$ is called a generalized g-quasi-contractions of Perov-type, if there exist a mapping $g : X \to X$ with $f(X) \subset g(X)$ and some $\lambda \in P$ with $r(\lambda) \in [0,\ 1)$, for all $x,\ y \in X$, one has

$$d(fx, fy) \preceq \lambda u, \tag{2.1}$$

where

$$u \in C(g; x, y) = \{d(gx, gy), d(gx, fx), d(gy, fy), d(gx, fy), d(gy, fx)\}.$$

Definition 2.10 (See [30], [31]) Let P be a solid cone in a Banach space $\mathcal{A}$. A sequence $\{u_n\} \subset P$ is a c-sequence if for each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for $n \geq n_0$.

It is easy to show the following propositions.

Proposition 2.2 (See [30]) Let P be a solid cone in a Banach space $\mathcal{A}$ and let $\{u_n\}$ and $\{v_n\}$ be sequences in P. If $\{u_n\}$ and $\{v_n\}$ are c-sequences and $\alpha, \beta > 0$, then $\{\alpha u_n + \beta v_n\}$ is a c-sequence.

In addition to Proposition 2.2 above, the following propositions are crucial to the proof of our main results.

Proposition 2.3 (See [30]) Let P be a solid cone in a Banach algebra $\mathcal{A}$ and let $\{u_n\}$ be a sequence in P. Then the following conditions are equivalent.

(1) $\{u_n\}$ is a c-sequence.

(2) For each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $u_n \prec c$ for $n \geq n_0$.

(3) For each $c \gg \theta$ there exists $n_1 \in \mathbb{N}$ such that $u_n \preceq c$ for $n \geq n_1$.

Proposition 2.4 (See [30]) Let P be a solid cone in a Banach algebra $\mathcal{A}$ and let $\{u_n\}$ be a sequence in P. Suppose that $k \in P$ is an arbitrarily given vector and $\{u_n\}$ is a c-sequence in P. Then $\{ku_n\}$ is a c-sequence.

Proposition 2.5 Let $\mathcal{A}$ be a Banach algebra with a unit e, P be a cone in $\mathcal{A}$ and $\preceq$ be the semi-order be yielded by the cone P. Let $\lambda \in P$. If the spectral radius $r(\lambda)$ of λ is less than 1, then the following assertions hold true.

(i) We have $(e-\lambda)^{-1} \succ \theta$. In addition, we have $\theta \preceq \lambda^n \preceq (e-\lambda)^{-1}\lambda^n \preceq (e-\lambda)^{-1}\lambda$ for any integer $n \geq 1$.

(ii) For any $u \succ \theta$, we have $u \npreceq \lambda u$. Moreover, we have $u \npreceq \lambda^n u$ for any integer $n \geq 1$.

Proposition 2.6 (See [29]) Let (X, d) be a complete cone metric space over a Banach algebra $\mathcal{A}$ and let P be the underlying solid cone in Banach algebra $\mathcal{A}$. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to $x \in X$, then we have

(i) $\{d(x_n, x)\}$ is a c-sequence;

(ii) for any $p \in \mathbb{N}$, $\{d(x_n, x_{n+p})\}$ is a c-sequence.

3 Main results

In this section, without the assumption of normality of the underlying cone, we give some common fixed point theorems for generalized g-quasi-contractions of Perov-type with

the spectral radius $r(\lambda)$ of the g-quasi-contractive constant vector λ satisfying $r(\lambda) \in [0,\ 1)$ in the setting of cone metric spaces over Banach algebras.

Theorem 3.1 Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$ and the underlying solid cone P. Let the mapping $f : X \to X$ be the g-quasi-contractions of Perov-type with the spectral radius $r(\lambda)$ of the g-quasi-contractive constant vector λ satisfying $r(\lambda) \in [0,\ 1)$. If the range of g contains the range of f and $g(X)$ or $f(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

We begin the proof of Theorem 3.1 with a useful lemma. For each $x_0 \in X$, set $gx_1 = fx_0$ and $gx_{n+1} = fx_n$. We will prove that $\{gx_n\}$ is a Cauchy sequence. First, we shall show the following lemmas. Note that for these lemmas, we suppose that all the conditions in Theorem 3.1 are satisfied.

Lemma 3.1 For any $N \geq 2$ and $1 \leq m \leq N - 1$, one has that

$$d(gx_N, gx_m) \preceq \lambda(e - \lambda)^{-1} d(gx_1, gx_0). \tag{3.1}$$

Proof. We now prove Lemma 3.1 by induction. When $N = 2, m = 1$, since $f : X \to X$ is a generalized g-quasi-contractions of Perov-type satisfying (2.1), there exists

$$u_1 \in C(g; x_1, x_0) = \{d(gx_1, gx_0), d(gx_1, gx_2), d(gx_0, gx_1), d(gx_1, gx_1), d(gx_0, gx_2)\}$$

such that

$$d(gx_2, gx_1) \preceq \lambda u_1.$$

Hence, $u_1 = d(gx_1, gx_0)$ or $u_1 = d(gx_0, gx_2)$. (Note that it is obvious that $u_1 \neq d(gx_1, gx_2)$ since $d(gx_2, gx_1) \npreceq \lambda d(gx_1, gx_2)$ and $u_1 \neq d(gx_1, gx_1)$ since $d(gx_1, gx_2) \neq \theta$.)

When $u_1 = d(gx_1, gx_0)$, we have

$$d(gx_2, gx_1) \preceq \lambda d(gx_0, gx_1) \preceq \lambda(e - \lambda)^{-1} d(gx_1, gx_0).$$

When $u_1 = d(gx_2, gx_0)$, then we have

$$d(gx_2, gx_1) \preceq \lambda d(gx_2, gx_0) \preceq \lambda[d(gx_2, gx_1) + d(gx_1, gx_0)].$$

So we get

$$(e - \lambda) d(gx_2, gx_1) \preceq \lambda d(gx_1, gx_0),$$

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which implies that

$$d(gx_2, gx_1) \preceq \lambda(e-\lambda)^{-1}d(gx_1, gx_0).$$

Hence, (3.1) holds for $N = 2$ and $m = 1$.

Suppose that for some $N \geq 2$ and for any $2 \leq p \leq N$ and $1 \leq n \leq p$, one has

$$d(gx_p, gx_n) \preceq \lambda(e-\lambda)^{-1}d(gx_1, gx_0). \tag{3.2}$$

That is,

$$d(gx_p, gx_1) \preceq \lambda(e-\lambda)^{-1}d(gx_1, gx_0), \tag{3.2.1}$$

$$d(gx_p, gx_2) \preceq \lambda(e-\lambda)^{-1}d(gx_1, gx_0), \tag{3.2.2}$$

......

$$d(gx_p, gx_{p-1}) \preceq \lambda(e-\lambda)^{-1}d(gx_1, gx_0). \tag{3.2.$p-1$}$$

Then, we need to prove that for $N+1 \geq 2$ and any $1 \leq n < N+1$, one has

$$d(gx_{N+1}, gx_n) \preceq \lambda(e-\lambda)^{-1}d(gx_1, gx_0). \tag{3.3}$$

That is,

$$d(gx_{N+1}, gx_1) \preceq \lambda(e-\lambda)^{-1}d(gx_1, gx_0), \tag{3.3.1}$$

$$d(gx_{N+1}, gx_2) \preceq \lambda(e-\lambda)^{-1}d(gx_1, gx_0), \tag{3.3.2}$$

......

$$d(gx_{N+1}, gx_{N-1}) \preceq \lambda(e-\lambda)^{-1}d(gx_1, gx_0), \tag{3.3.$N-1$}$$

$$d(gx_{N+1}, gx_N) \preceq \lambda(e-\lambda)^{-1}d(gx_1, gx_0). \tag{3.3.N}$$

In fact, since $f : X \to X$ is a g-quasi-contraction, there exists

$$u_1 \in C(g; x_N, x_0) = \{d(gx_N, gx_0), d(gx_N, gx_{N+1}), d(gx_0, gx_1), d(gx_N, gx_1), d(gx_0, gx_{N+1})\}$$

such that

$$d(gx_{N+1}, gx_1) \preceq \lambda u_1.$$

If $u_1 = d(gx_N, gx_1)$, then by (3.2.1) we have

$$d(gx_{N+1}, gx_1) \preceq \lambda^2(e-\lambda)^{-1}d(gx_1, gx_0) \preceq \lambda^2(e-\lambda)^{-1}d(gx_1, gx_0) \preceq \lambda(e-\lambda)^{-1}d(gx_1, gx_0).$$

If $u_1 = d(gx_0, gx_1)$, then we have

$$d(gx_{N+1}, gx_1) \preceq \lambda d(gx_1, gx_0) \preceq \lambda d(gx_1, gx_0) \preceq \lambda(e-\lambda)^{-1}d(gx_1, gx_0).$$

If $u_1 = d(gx_N, gx_0)$, then by (3.2.1) we have

$$\begin{aligned} d(gx_{N+1}, gx_1) &\preceq \lambda d(gx_N, gx_0) \preceq \lambda(d(gx_N, gx_1) + d(gx_1, gx_0)) \\ &\preceq \lambda(\lambda(e-\lambda)^{-1} d(gx_1, gx_0) + d(gx_1, gx_0)) \\ &= \lambda(\lambda(e-\lambda)^{-1} + e) d(gx_1, gx_0) \\ &= \lambda(e-\lambda)^{-1} d(gx_1, gx_0). \end{aligned}$$

If $u_1 = d(gx_0, gx_{N+1})$, then we have

$$d(gx_{N+1}, gx_1) \preceq \lambda d(gx_0, gx_{N+1}) \preceq \lambda(d(gx_0, gx_1) + d(gx_1, gx_{N+1})).$$

Hence, we see

$$(e-\lambda) d(gx_{N+1}, gx_1) \preceq \lambda d(gx_0, gx_1),$$

which implies that

$$d(gx_{N+1}, gx_1) \preceq (e-\lambda)^{-1} \lambda d(gx_0, gx_1).$$

Without loss of generality, suppose that $u_1 = d(gx_N, gx_{N+1})$. Since $f : X \to X$ is a g-quasi-contraction, there exists $u_2 \in C(g; x_{N-1}, x_N)$ such that

$$u_1 = d(gx_N, gx_{N+1}) \preceq \lambda u_2,$$

where

$$\begin{aligned} C(g; x_{N-1}, x_N) = \{&d(gx_{N-1}, gx_N), d(gx_{N-1}, gx_N), d(gx_N, gx_{N+1}), \\ &d(gx_{N-1}, gx_{N+1}), d(gx_N, gx_N)\}. \end{aligned}$$

So, we get

$$d(gx_{N+1}, gx_1) \preceq \lambda u_1 \preceq \lambda^2 u_2.$$

Similarly, it is easy to see that $u_2 \neq d(gx_N, gx_N)$ since $u_2 \neq \theta$ and $u_2 \neq d(gx_N, gx_{N+1})$ since $d(gx_N, gx_{N+1}) \npreceq \lambda^2 d(gx_N, gx_{N+1})$.

If $u_2 = d(gx_{N-1}, gx_N)$, then by the induction assumption (3.2) we have

$$\begin{aligned} d(gx_{N+1}, gx_1) &\preceq \lambda^2 u_2 \preceq \lambda^3 (e-\lambda)^{-1} d(gx_1, gx_0) \\ &\preceq \lambda^3 (e-\lambda)^{-1} d(gx_1, gx_0) \\ &\preceq \lambda(e-\lambda)^{-1} d(gx_1, gx_0). \end{aligned}$$

Without loss of generality, suppose that $u_2 = d(gx_{N-1}, gx_{N+1})$. There exists $u_3 \in C(g; x_{N-2}, x_N)$ such that

$$u_2 = d(gx_{N-1}, gx_{N+1}) \preceq \lambda u_3,$$

where

$$\begin{aligned} C(g; x_{N-2}, x_N) = \{&d(gx_{N-2}, gx_N), d(gx_{N-2}, gx_{N-1}), d(gx_N, gx_{N+1}), \\ &d(gx_{N-2}, gx_{N+1}), d(gx_N, gx_{N-1})\}. \end{aligned}$$

In general, suppose that $u_{i-1} = d(gx_{N-i+2}, gx_{N+1})$. Since $f : X \to X$ is a g-quasi-contraction, by the similar arguments above, there exists $u_i \in C(g; x_{N-i+1}, x_N)$ such that

$$u_{i-1} = d(gx_{N-i+2}, gx_{N+1}) \preceq \lambda u_i,$$

for which we obtain

$$d(gx_{N+1}, gx_1) \preceq \lambda u_1 \preceq \lambda^2 u_2 \preceq \cdots \preceq \lambda^i u_i,$$

where

$$\begin{aligned} C(g; x_{N-i+1}, x_N) = \{&d(gx_{N-i+1}, gx_N), d(gx_{N-i+1}, gx_{N-i+2}), d(gx_N, gx_{N+1}), \\ &d(gx_{N-i+1}, gx_{N+1}), d(gx_N, gx_{N-i+2})\}. \end{aligned}$$

Similarly, it is easy to see that $u_i \neq d(gx_N, gx_{N+1})$. This is because by Proposition 2.5(iii) we have

$$u_1 = d(gx_N, gx_{N+1}) \npreceq \lambda^{i-1} d(gx_N, gx_{N+1}).$$

So we know that if $u_i = d(gx_{N-i+1}, gx_N)$ or $u_i = d(gx_{N-i+1}, gx_{N-i+2})$ or $u_i = d(gx_N, gx_{N-i+2})$ then by the induction assumption (3.2) we have $u_i \preceq \lambda(e-\lambda)^{-1} d(gx_1, gx_0)$. Hence,

$$\begin{aligned} d(gx_{N+1}, gx_1) &\preceq \lambda^i u_i \preceq \lambda^{i+1}(e-\lambda)^{-1} d(gx_1, gx_0) \\ &\preceq (\lambda)^{i+1}(e-\lambda)^{-1} d(gx_1, gx_0) \\ &\preceq \lambda(e-\lambda)^{-1} d(gx_1, gx_0), \end{aligned}$$

which means that (3.3.1) holds true. Without loss of generality, suppose that $u_i = d(gx_{N-i+1}, gx_{N+1})$. Then by the similar arguments as above we have $u_i \preceq \lambda u_{i+1}$, where $u_{i+1} \in C(g; x_{N-i}, x_N)$. Hence, there is a sequence $\{u_n\}$ such that

$$d(gx_{N+1}, gx_1) \preceq \lambda u_1 \preceq \lambda^2 u_2 \preceq \cdots \preceq \lambda^{N-1} u_{N-1} \preceq \lambda^N u_N,$$

where

$$u_{N-1} = d(gx_2, gx_{N+1}) \preceq \lambda u_N$$

and

$$u_N \in C(g; x_1, x_N) = \{d(gx_1, gx_N), d(gx_1, gx_2), d(gx_N, gx_{N+1}), d(gx_N, gx_2), d(gx_1, gx_{N+1})\}.$$

Obviously, $u_N \neq d(gx_1, gx_{N+1})$ and $u_N \neq d(gx_N, gx_{N+1})$. On the contrary, if $u_N = d(gx_1, gx_{N+1})$, then $u_N \preceq \lambda^N u_N$, a contradiction. If $u_N = d(gx_N, gx_{N+1}) = u_1$, then we have

$$u_1 = d(gx_N, gx_{N+1}) \preceq \lambda^2 u_2 \preceq \cdots \preceq \lambda^{N-1} u_{N-1} \preceq \lambda^{N-1} u_1,$$

a contradiction. Hence, it follows that $u_N = d(gx_1, gx_N)$, $u_N = d(gx_1, gx_2)$ or $u_N = d(gx_N, gx_2)$. By the induction assumption (3.2), in any case, we have

$$u_N \preceq \lambda(e-\lambda)^{-1} d(gx_1, gx_0). \tag{3.4}$$

Therefore, we get

$$\begin{aligned} d(gx_{N+1}, gx_1) &\preceq \lambda u_1 \preceq \lambda^2 u_2 \preceq \cdots \preceq \lambda^N u_N \\ &\preceq \lambda^N (e-\lambda)^{-1} \lambda d(gx_1, gx_0) \\ &\preceq (\lambda)^{N+1} (e-\lambda)^{-1} d(gx_1, gx_0) \\ &\preceq \lambda(e-\lambda)^{-1} d(gx_1, gx_0). \end{aligned} \tag{3.5}$$

That is to say, (3.3.1) is true. By (3.5), we have

$$u_1 \preceq \lambda^{N-1} \lambda (e-\lambda)^{-1} d(gx_1, gx_0).$$

Thus,

$$\begin{aligned} d(gx_N, gx_{N+1}) = u_1 &\preceq \lambda^{N-1} \lambda (e-\lambda)^{-1} d(gx_1, gx_0) \\ &\preceq (\lambda)^N (e-\lambda)^{-1} d(gx_1, gx_0) \\ &\preceq \lambda (e-\lambda)^{-1} d(gx_1, gx_0), \end{aligned}$$

which implies that (3.3.N) is true. Similarly, since

$$u_2 = d(gx_{N-1}, gx_{N+1}), \ldots, u_i = d(gx_{N-i+1}, gx_{N+1}), \ldots,$$

by (3.4) and (3.5) we get

$$u_i \preceq \lambda^{N-i} u_N \preceq \lambda^{n-i+1} (e-\lambda)^{-1} d(gx_1, gx_0). \tag{3.6}$$

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Hence, it follows from (3.6) that (3.3.2)-(3.3.$N-1$) are all true. That is, (3.3) is true. Therefore, we conclude that Lemma 3.1 holds true.

By Lemma 3.1, we immediately obtain the following result.

Lemma 3.2 For all $i, j \in \mathbb{N}_+$, one has

$$d(gx_i, gx_j) \preceq \lambda(e-\lambda)^{-1} d(gx_0, gx_1). \tag{3.7}$$

Now, we begin to prove Theorem 3.1. First, we need to show that $\{gx_n\}$ is a Cauchy sequence. For all $n > m$, there exists

$$\begin{aligned}\nu_1 \in C(g; x_{n-1}, x_{m-1}) = &\{d(gx_{n-1}, gx_{m-1}), d(gx_{n-1}, gx_n),\\ &d(gx_{m-1}, gx_m), d(gx_{n-1}, gx_m), d(gx_{m-1}, gx_n)\}\end{aligned}$$

such that

$$d(fx_{n-1}, fx_{m-1}) \preceq \lambda\nu_1.$$

Using the g-quasi-contractive condition repeatedly, we easily show by induction that there must exist

$$\nu_k \in \{d(gx_i, gx_j) : 0 \leq i < j \leq n\} \ (k = 2, 3, \ldots, m)$$

such that

$$\nu_k \preceq \lambda\nu_{k+1} \ (k = 1, 2, \ldots, m-1). \tag{3.8}$$

For convenience, we write $\nu_m = d(gx_i, gx_j)$ where $0 \leq i < j \leq n$.

Using the triangular inequality, we have

$$d(gx_i, gx_j) \preceq d(gx_i, gx_0) + d(gx_0, gx_j) \quad (0 \leq i, j \leq n),$$

and by Lemma 3.2 we obtain

$$\begin{aligned}d(gx_n, gx_m) &= d(fx_{n-1}, fx_{m-1}) \preceq \lambda\nu_1 \preceq \lambda^2\nu_2 \preceq \cdots \preceq \lambda^m\nu_m\\ &\preceq \lambda^m d(gx_i, gx_j)\\ &= \lambda^{m+1}(e-\lambda)^{-1} d(gx_1, gx_0).\end{aligned}$$

Since $r(\lambda) < 1$, by Remark 2.1 we have that $\lambda^{m+1}(e-\lambda)^{-1}d(gx_1, gx_0) \to \theta$ as $m \to \infty$, so by Proposition 2.4, it is easy to see that for any $c \in \mathrm{int}P$, there exists $n_0 \in \mathbb{N}$ such that for all $n > m > n_0$,

$$d(gx_n, gx_m) \preceq \lambda^{m+1}(e-\lambda)^{-1}d(gx_1, gx_0) \ll c.$$

So $\{gx_n\}$ is a Cauchy sequence in $g(X)$. If $g(X) \subset X$ is complete, there exist $q \in g(X)$ and $p \in X$ such that $gx_n \to q$ as $n \to \infty$ and $gp = q$.

Now, from (2.1) we get

$$d(fx_n, fp) \preceq \lambda\nu$$

where

$$\nu \in C(g; x_n, p) = \{d(gx_n, gp), d(gx_n, fx_n), d(gp, fp), d(gx_n, fp), d(fx_n, gp)\}.$$

Clearly at least one of the following five cases holds for infinitely many n.

(1) $d(fx_n, fp) \preceq \lambda d(gx_n, gp) \preceq \lambda d(gx_{n+1}, gp) + \lambda d(gx_{n+1}, gx_n)$;

(2) $d(fx_n, fp) \preceq \lambda d(gx_n, fx_n) = \lambda d(gx_n, gx_{n+1})$;

(3) $d(fx_n, fp) \preceq \lambda d(gp, fp) \preceq \lambda d(gx_{n+1}, gp) + \lambda d(gx_{n+1}, fp)$,
that is, $d(fx_n, fp) \preceq \lambda(e-\lambda)^{-1}d(gx_{n+1}, gp)$;

(4) $d(fx_n, fp) \preceq \lambda d(gx_n, fp) \preceq \lambda d(gx_{n+1}, fp) + \lambda d(gx_{n+1}, gx_n)$,
that is, $d(fx_n, fp) \preceq \lambda(e-\lambda)^{-1}d(gx_{n+1}, gx_n)$;

(5) $d(fx_n, fp) \preceq \lambda d(fx_n, gp) = \lambda d(gx_{n+1}, gp)$.

As $\lambda \preceq \lambda(e-\lambda)^{-1}$ (since $\theta \preceq \lambda$ and $r(\lambda) < 1$), we obtain that

$$d(gx_{n+1}, fp) \preceq \lambda(e-\lambda)^{-1}[d(gx_{n+1}, gx_n) + d(gx_{n+1}, q)].$$

Since $gx_n \to q$ as $n \to \infty$, we get that for any $c \in \mathrm{int}P$, there exists $n_1 \in \mathbb{N}$ such that for all $n > n_1$, one has

$$d(gx_{n+1}, fp) \ll c.$$

By Lemmas 2.4 and 2.5, we have $gx_n \to fp$ as $n \to \infty$ and $q = fp$.

Now if w is another point such that $gu = fu = w$, hence

$$d(w, q) = d(fu, fp) \preceq \lambda\nu,$$

where $r(\lambda) \in [0,\ 1)$ and

$$\nu \in C(g; u, p) = \{d(gu, gp), d(gu, fu), d(gp, fp), d(gu, fp), d(fu, gp)\}.$$

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It is obvious that $d(w,q)=\theta$, i.e., $w=q$. Therefore, q is the unique point of coincidence of f and g in X. Moreover, the mappings f and g are weakly compatible, by Lemma 2.6 we know that q is the unique common fixed point of f and g.

Similarly, if $f(X)$ is complete, the above conclusion is also established.

According to Das-Naik version of the known theorem in the setting of metric spaces from [33], we have following result similar to Theorem 3.1.

Theorem 3.2 Suppose one of the following conditions holds:

(1) As in Theorem 3.1, let $(X,\, d)$ be a complete cone metric space over a Banach algebra $\mathcal{A}$. Assume one of $f\,(X)$ or $g\,(X)$ is closed and the other conditions in Theorem 3.1 are not changeable;

(2) As in Theorem 3.1, let (X,d) be a complete cone metric space over a Banach algebra $\mathcal{A}$. Assume f,g are cone compatible and both continuous and the other conditions in Theorem 3.1 are not changeable;

(3) As in Theorem 3.1, let (X,d) be a complete cone metric space over a Banach algebra $\mathcal{A}$. Assume f commutes with g, f or g is continuous (see Theorem 3.2 in Cvetković-Rakočević [36]) and the other conditions in Theorem 3.1 are not changeable.

Then the conclusions of Theorem 3.1 are also true.

Proof. (1) The proof of this case is the same as that in Theorem 3.1.

(2) The sequence $y_n = fx_n = gx_{n+1}, y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$ converges to some $z \in X$ as $n \to \infty$. Further, since $fx_n \to z$ and $gx_n \to z$ we get that

$$d\,(fz, gz) \leq d\,(fz, fgx_n) + d\,(fgx_n, gfx_n) + d\,(gfx_n, gz) \to 0 + 0 + 0 = 0.$$

Hence $fz = gz = \omega$. Hence, f, g has (a unique) point of coincidence. Since f and g are compatible then they are weakly compatible. Therefore by standard result they have a unique common fixed point (in this case it is ω).□

(3) Let g be continuous.

Then we get $gy_n \to gz$ and $fy_n \to gz$ since f commutes with g. Indeed, $fy_n = fgx_{n+1} = gfy_{n+1} \to gz$.

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So we get

$$\begin{aligned} d(fz, gz) &\preceq d(fz, fy_n) + d(fy_n, gz) \\ &\preceq \lambda u + d(fy_n, gz), \end{aligned}$$

where

$$u \in \{d(gz, gy_n), d(gz, fz), d(gy_n, fy_n), d(gz, fy_n), d(gy_n, fz) + d(fy_n, gz)\}.$$

If $u = d(gz, gy_n)$ or $u = d(gy_n, fy_n)$ or $d(gz, fy_n)$, then we obtain that $\lambda u + d(fy_n, gz)$ is a c-sequence. This mens that $fz = gz$. If $u = d(gz, fz)$ or $u = d(gy_n, fz) \preceq d(gy_n, gz) + d(gz, fz)$ we get

$$d(fz, gz) \preceq \lambda d(gz, fz) + d(fy_n, gz)$$

or

$$d(fz, gz) \preceq \lambda d(gz, fz) + \lambda d(gy_n, gz) + d(fy_n, gz).$$

In both cases we have that

$$d(fz, gz) \preceq (e - \lambda)^{-1} c_n,$$

where c_n is a c-sequence. Hence, f, g have a unique point of coincidence. Since f commutes with g then they are weakly compatible and by known result have a unique fixed point.

Now let f be continuous.

Again, $fy_n \to fz$ and $gy_n = gfx_n = fgx_n = fy_{n-1} \to fz$.

Further we get

$$d(fz., z) \preceq d(fz, fy_n) + d(fy_n, y_n) + d(y_n, y).$$

Since $d(fz, fy_n) + d(y_n, y) = c_n$ is c-sequence it is sufficient to estimate $d(fy_n, y_n)$.

We have

$$d(fy_n, y_n) = d(fy_n, fx_n) \preceq \lambda u,$$

where

$$\begin{aligned} u &\in \{d(gy_n, gx_n), d(gy_n, fy_n), d(gx_n, fx_n), d(gy_n, fx_n), d(gx_n, fy_n)\} \\ &= \{d(fy_{n-1}, y_{n-1}), d(fy_{n-1}, fy_n), d(fy_{n-1}, y_n), d(y_{n-1}, fy_n), d(y_{n-1}, y_n)\}. \end{aligned}$$

Now we get the following cases:

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I) $u = d(fy_{n-1}, y_{n-1})$. Then

$$\begin{aligned} d(fz., z) &\preceq c_n + \lambda d(fy_{n-1}, y_{n-1}) \\ &\preceq c_n + \lambda (d(fy_{n-1}, fy) + d(fy, y) + d(y, y_{n-1})) \end{aligned}$$

or

$$\begin{aligned} d(fz., z) &\preceq (e-\lambda)^{-1} c_n + (e-\lambda)^{-1} \lambda d(fy_{n-1}, fy) + (e-\lambda)^{-1} \lambda d(y, y_{n-1}) \\ &= d_n \text{ where } d_n \text{ is a new } c\text{-sequence.} \end{aligned}$$

II) $u = d(fy_{n-1}, fy_{n-1})$
III) $u = d(fy_{n-1}, y_n)$
IV) $u = d(y_{n-1}, fy_n)$
V) $u = d(y_{n-1}, y_n)$
In all cases we obtained that $fz = z$.
For details see Theorem 3.2 in Cvetković-Rakočević [36].

Corollary 3.1 Let (X, d) be a complete cone metric space over a Banach algebra $\mathcal{A}$ and let P be the underlying cone with $k \in P$. If the mapping $T : X \to X$ is a quasi-contraction, then T has a unique fixed point in X. And for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Proof. Set $g = I_X$, the identity mapping from X to X. It is obvious to see that Theorem 3.1 yields Corollary 3.1.

Remark 3.1 Corollary 3.1 does not need to require the assumption of normality of the cone P. So Corollary 3.1 improves and generalizes Theorem 9 in [21].

Remark 3.2 From the proof of Lemma 3.1, we note that the technique of induction appearing in Theorem 3.1 is somewhat different from that in Theorem 9 from [21], and also different from that in Theorem 2.6 from [11], which is more interesting and easily to understood. In addition, the proof of Theorem 3.1 is a valuable addition to [9] since Theorem 3.1 is a generalization of Theorem 3 from [9] but some main results in the proof of Theorem 3 from [9] were not proved in general.

Remark 3.3 Taking $E = \mathbb{R}$, $P = [0, +\infty)$, $\|.\| = |.|$, $\lambda \in [0, 1)$ in Theorem 3.1, we get Das-Naik's result from [33]; if $g = I_X$ we get Ćirić's result from [13], both in the setting of

metric spaces.

The following corollary is the Jungck's result in the setting of cone metric spaces over Banach algebras.

Corollary 3.2 Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$ with the underlying solid cone P. Let the mappings $f,\ g : X \to X$ satisfy the condition that for $\lambda \in P$ with $r(\lambda) \in [0,\ 1)$ and for every $x, y \in X$ holds $d(fx, fy) \preceq \lambda d(gx, gy)$. If $g(X) \subset f(X)$ and $g(X)$ or $f(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

The next result is the Banach contraction principle in the setting of cone metric spaces over Banach algebras.

Corollary 3.3 (see [29]) Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$ the underlying solid cone P. Let the mapping $f : X \to X$ satisfy the condition that for $\lambda \in P$ with $r(\lambda) \in [0,\ 1)$ and for every $x, y \in X$ holds $d(fx, fy) \preceq \lambda d(x, y)$ (namely, f is a generalized Lipschitz contraction). If $f(X)$ is a complete subspace of X, then f has a unique point in X.

We will present an example to show that the results presented above are real generalizations of the corresponding results in the literature.

Example 3.1 Let $X = [1, \infty)$ and $\mathcal{A}$ be a set of all real valued function on $[0, 1]$ which also have continuous derivates on $[0, 1]$ with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and the usual multiplication. Let $P = \{x \in \mathcal{A} : x\,(t) \geq 0, t \in [0, 1]\}$. It is clear that P is a nonrmal cone and $\mathcal{A}$ is a Banach algebra with a unit $e = 1$. Define a mapping

$$d : X \times X \to \mathcal{A}$$

by

$$d\,(x, y)\,(t) := |x - y|\, e^t.$$

We make a conclusion that (X, d) is a complete cone metric space over Banach algebra $\mathcal{A}$. Now define the mappings $f, g : X \to X$ by $f\,(x) = 3x - 2, g\,(x) = 4x - 3$. Choose

$\lambda(t) = \frac{1}{12}t + \frac{3}{4}$. Since $f(X) \subseteq g(X)$ and $r(\lambda) = \frac{5}{6}$, thus, all the conditions of Theorem 3.1 are satisfied and consequently f and g have a unique comon fixed point $x = 1$. Indeed, for $x, y \in X$ we can putting $u(x, y) = d(g(x), g(y)) = 4|x - y|$. In this case we have

$$d(f(x), f(y)) = 3|x - y|e^t \leq \left(\frac{1}{12}t + \frac{3}{4}\right) 4|x - y|e^t \Leftrightarrow \frac{3}{4} \leq \frac{1}{12}t + \frac{3}{4},$$

which is indeed true. On the other hand, we see that

$$f(g(x)) = f(4x - 3) = 3(4x - 3) - 2 = 12x - 11 = 4(3x - 2) - 3 = g(f(x)),$$

that is, f commutes with g and other words f, g are weakly compatible.

Now let us estimate $r(\lambda) = \lim_{n\to\infty} \|\lambda^n\|^{\frac{1}{n}}$. Since

$$\lambda^n(t) = \left(\frac{1}{12}t + \frac{3}{4}\right)^n, (\lambda^n(t))' = \frac{n}{12}\left(\frac{1}{12}t + \frac{3}{4}\right)^{n-1},$$

we have $(t = 1)$

$$\|\lambda\|_\infty + \|\lambda'\|_\infty = \left(\frac{5}{6}\right)^n + \frac{n}{12}\left(\frac{5}{6}\right)^{n-1} = \frac{n}{12}\left(\frac{5}{6}\right)^{n-1}\left(\frac{12}{n} \cdot \frac{5}{6} + 1\right) = \frac{n}{12}\left(\frac{5}{6}\right)^{n-1}\left(1 + \frac{10}{n}\right).$$

Further we get

$$\|\lambda^n\|^{\frac{1}{n}} = \left(\frac{n}{12}\right)^{\frac{1}{n}}\left(\frac{5}{6}\right)^{\frac{n-1}{n}}\left(1 + \frac{10}{n}\right)^{\frac{1}{n}} \to \frac{5}{6} < 1.$$

However, both f and g are not quasi-contraction. Indeed, for $x = 2, y = 1$ and for all λ with $r(\lambda) \in [0, 1)$, we get

$$d(f2, f1)(t) = d(4, 1)(t) = 3e^t > \lambda(t)u,$$

for all

$$\begin{aligned} u &\in \left\{d(2,1)e^t, d(2, f2)e^t, d(1, f1)e^t, d(2, f1)e^t, d(1, f2)e^t\right\} \\ &= \left\{e^t, 2e^t, 0, e^t, 3e^t\right\}, \end{aligned}$$

and similarly

$$d(g2, g1)(t) = d(5, 1)(t) = 4e^t > \lambda(t)u$$

for all

$$\begin{aligned} u &\in \left\{d(2,1)e^t, d(2, g2)e^t, d(1, g1)e^t, d(2, g1)e^t, d(1, g2)e^t\right\} \\ &= \left\{e^t, 3e^t, 0, e^t, 4e^t\right\}. \end{aligned}$$

Hence, Theorem 3.1 is a genuine generalization of Theorem 9 from [21].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors contribute equally and significantly in writing this paper. All the authors read and approve the final manuscript.

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On some inequalities of the Bateman's $G-$function

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Abstract

In the paper, we prove that the Bateman's $G-$function satisfies the double inequality

$$\sum_{n=1}^{2m} \frac{(2^n-1)B_{2n}}{nx^{2n}} < G(x) - \frac{1}{x} < \sum_{n=1}^{2m-1} \frac{(2^n-1)B_{2n}}{nx^{2n}}, \qquad m \in \mathbb{N}$$

with best bounds, where $B_r's$ are the Bernoulli numbers and we study the monotonicity of some functions involving the function $G(x)$. Also, we present some estimates for the error term of a class of the alternating series, which improve and generalize some recent resutls and we prove the increasing monotonicity of a sequence arising from computation of the intersecting probability between a plane couple and a convex body.

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Key Words: Digamma function, Bateman's G-function, sharp inequality, monotonicity, alternating series, sequence.

1 Introduction.

The ordinary gamma function is defined by [3]

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \qquad x > 0$$

and the derivative of $\log\Gamma(x)$ is called the digamma function and is denoted by $\psi(x)$. We can considered to the gamma function, the digamma function and the the Riemann zeta function as the most important special functions [5]. For more details on bounding the gamma function and its logarithmic derivatives, please refer to the papers [2]-[5], [7], [8], [14]-[20] [22]-[26], [35]-[41]

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and plenty of references therein.

The Bateman's $G-$function is defined by Erdélyi [6] as

$$G(x) = \psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right), \qquad x \neq 0, -1, -2, ... \tag{1}$$

which satisfies [6]:

$$G(1+x) + G(x) = \frac{2}{x} \tag{2}$$

and

$$G(1-x) + G(x) = 2\pi \csc(\pi x). \tag{3}$$

The function $G(x)$ can be defined by the hypergeometric function as

$$G(x) = \frac{2}{x}\, {}_2F_1(1, x; 1+x; -1).$$

From the integral representation of the function $\psi(z)$ [3]

$$\psi(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}}\right) dt, \qquad x > 0$$

we obtain the following integral representation

$$G(x) = \int_0^\infty \frac{2\, e^{-xt}}{1+e^{-t}} dt, \qquad x > 0. \tag{4}$$

The function $G(x)$ is very useful in summing and estimating certain numerical and algebraic series [27] . For example:

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{s\,k+u} = \frac{1}{2s} G\left(\frac{u}{s}\right), \qquad u \neq 0, -s, -2s, ... \tag{5}$$

and its n^{th} partial sum is given by

$$\sum_{k=0}^{n} \frac{(-1)^k}{s\,k+u} = \frac{1}{2s}\left[G\left(\frac{u}{s}\right) + (-1)^n G\left(\frac{u}{s}+n+1\right)\right], \qquad u \neq 0, -s, -2s, ... \tag{6}$$

Qiu and Vuorinen [43] deduced the inequality

$$\frac{4(1.5 - \log 4)}{x^2} < G(x) - \frac{1}{x} < \frac{1}{2x^2}, \qquad x > \frac{1}{2}. \tag{7}$$

Mahmoud and Agarwal [16] presented an asymptotic formula for Bateman's G-function $G(x)$ and deduced the double inequality

$$\frac{1}{2x^2+1.5} < G(x) - \frac{1}{x} < \frac{1}{2x^2}, \qquad x > 0 \tag{8}$$

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which improve the lower bound of the inequality (7) and they posed a sharp double inequality of the function $G(x)$ as a conjecture. Mortici [21] established the inequality

$$0 < \psi(x+u) - \psi(x) \leq \psi(u) + \gamma + \frac{1}{u} - u \qquad x \geq 1;\ u \in (0,1), \tag{9}$$

where γ is the Euler constant, which also improves the result of Qiu and Vuorinen. Also, Alzer presented the double inequality [2]

$$\frac{1}{x} - A_n(u;x) - \delta_n(u;x) < \psi(x+u) - \psi(x) < \frac{1}{x} - A_n(u;x),$$

where $n \geq 0$ be an integer, $x > 0$, $u \in (0,1)$,

$$A_n(u;x) = (1-u)\left[\frac{1}{u+n+1} + \sum_{i=0}^{n-1} \frac{1}{(x+i+1)(x+i+u)}\right]$$

and

$$\delta_n(u;x) = \frac{1}{x+n+u} \log \frac{(x+n)^{(x+n)(1-u)}(x+n+1)^{(x+n+1)u}}{(x+n+u)^{x+n+u}}.$$

In this paper, we prove the conjecture posed by Mahmoud and Agarwal [16] about a sharp double inequality of the function $G(x)$. We will study the completely monotonicity property of some functions involving the Bateman's $G-$function. Our results generalize and improve some inequalities about the error term of a class of alternating series and will prove the main result of [9] about the increasing monotonicity of a certain sequence .

2 Main results.

Theorem 1. *The Bateman's $G-$function satisfies*

$$G(x) = \frac{1}{x} + \sum_{n=1}^{m} \frac{(2^{2n}-1)B_{2n}}{nx^{2n}} + \frac{(2^{2m+2}-1)B_{2m+2}}{(m+1)x^{2m+2}}\theta_1, \qquad m = 1,2,3,... \tag{10}$$

where $B_i's$ are Bernoulli numbers, θ_1 is independent of x and $0 < \theta_1 < 1$.

Proof. Using the integral representation of the function $G(x)$ and the formula [1]

$$\frac{1}{x^s} = \frac{1}{(s-1)!}\int_0^\infty t^{s-1} e^{-xt} dt, \qquad s \in \mathbb{N}$$

we get

$$G(x) - \frac{1}{x} = \int_0^\infty \tanh(t/2)\, e^{-xt} dt. \tag{11}$$

We will apply a technique which used later by Qi and Guo [34]. By the expansion [1]

$$\tanh(t/2) = \sum_{k=1}^{\infty} \frac{4t}{t^2 + \pi^2(2k-1)^2}$$

and the identity

$$\frac{4t}{t^2+\pi^2(2k-1)^2} = \sum_{n=1}^{m} \frac{4(-1)^{n-1}t^{2n-1}}{\pi^{2n}(2k-1)^{2n}} + \frac{4(-1)^m t^{2m+1}}{\pi^{2m}(2k-1)^{2m}} \frac{1}{t^2+\pi^2(2k-1)^2}, \quad m \in \mathbb{N}$$

we obtain

$$G(x) - \frac{1}{x} = \int_0^\infty \sum_{k=1}^{\infty} \left(\sum_{n=1}^{m} \frac{4(-1)^{n-1}t^{2n-1}}{\pi^{2n}(2k-1)^{2n}} + \frac{4(-1)^m t^{2m+1}}{\pi^{2m}(2k-1)^{2m}} \frac{1}{t^2+\pi^2(2k-1)^2} \right) e^{-xt}dt \quad m \in \mathbb{N}.$$

Now

$$\sum_{k=1}^{\infty}\sum_{n=1}^{m} \frac{4(-1)^{n-1}t^{2n-1}}{\pi^{2n}} \frac{1}{(2k-1)^{2n}} = \sum_{n=1}^{m} \frac{4(-1)^{n-1}t^{2n-1}}{\pi^{2n}} (1-2^{-2n})\zeta(2n),$$

where $\zeta(t)$ is the Riemann zeta function which satisfies [3]

$$\zeta(2s) = \frac{(-1)^{s-1}\pi^{2s}2^{2s-1}}{(2s)!} B_{2s}, \qquad s \in \mathbb{N}.$$

Then

$$\sum_{k=1}^{\infty}\sum_{n=1}^{m} \frac{4(-1)^{n-1}t^{2n-1}}{\pi^{2n}} \frac{1}{(2k-1)^{2n}} = \sum_{n=1}^{m} \frac{2(2^{2n}-1)B_{2n}}{(2n)!} t^{2n-1}, \qquad m \in \mathbb{N}. \tag{12}$$

Also,

$$\sum_{k=1}^{\infty} \frac{4(-1)^m t^{2m+1}}{\pi^{2m}(2k-1)^{2m}} \frac{1}{t^2+\pi^2(2k-1)^2} = \frac{4(-1)^m t^{2m+1}}{\pi^{2m+2}} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^{2m+2}} \frac{1}{\left(\frac{t}{\pi(2k-1)}\right)^2+1} \quad m \in \mathbb{N}$$

and hence

$$\sum_{k=1}^{\infty} \frac{4(-1)^m t^{2m+1}}{\pi^{2m}(2k-1)^{2m}} \frac{1}{t^2+\pi^2(2k-1)^2} = \frac{4(-1)^m t^{2m+1}}{\pi^{2m+2}} \theta(t) \sum_{k=1}^{\infty} \frac{1}{(2k-1)^{2m+2}}, \quad m \in \mathbb{N}$$

where $0 < \theta(t) < 1$. Then

$$\sum_{k=1}^{\infty} \frac{4(-1)^m t^{2m+1}}{\pi^{2m}(2k-1)^{2m}} \frac{1}{t^2+\pi^2(2k-1)^2} = \frac{2(2^{2m+2}-1)t^{2m+1}B_{2m+2}}{(2m+2)!} \theta(t), \qquad 0 < \theta(t) < 1; \ m \in \mathbb{N}. \tag{13}$$

Now

$$G(x) - \frac{1}{x} = \sum_{n=1}^{m} \frac{2(2^n-1)B_{2n}}{(2n)!} \int_0^\infty t^{2n-1}e^{-xt}dt + \frac{2(2^{2m+2}-1)B_{2m+2}}{(2m+2)!} \int_0^\infty \theta(t)t^{2m+1}e^{-xt}dt. \tag{14}$$

Using the ordinary gamma function and its functional equation $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$, we get

$$G(x) - \frac{1}{x} = \sum_{n=1}^{m} \frac{(2^{2n}-1)B_{2n}}{nx^{2n}} + \frac{(2^{2m+2}-1)B_{2m+2}}{(m+1)x^{2m+2}} \theta_1, \qquad m \in \mathbb{N}$$

where θ_1 is independent of x and $0 < \theta_1 < 1$. □

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Theorem 2 ([16], Conjecture 1). *The Bateman's $G-$function satisfies the following double inequality*

$$\sum_{n=1}^{2m} \frac{(2^{2n}-1)B_{2n}}{nx^{2n}} < G(x) - \frac{1}{x} < \sum_{n=1}^{2m-1} \frac{(2^{2n}-1)B_{2n}}{nx^{2n}} \qquad m = 1,2,3,... \tag{15}$$

with sharp bounds, where $B_i's$ are Bernoulli numbers.

Proof. The inequality (15) satisfies from the relation (10) and the following property of Bernoulli constants [12]:

$$B_{2r+2} < 0 \quad and \quad B_{2r+4} > 0 \quad for\ r = 1,3,5,... . \tag{16}$$

Now, we will prove the sharpness of the inequality (15) using Mortici's technique [25]. From the definition [11], the asymptotic expansion of a function $T(x)$ of the form

$$T(x) = K(x) + b_0 + \sum_{k=1}^{\infty} \frac{b_k}{x^k}$$

satisfies for every fixed r, that

$$\lim_{x\to\infty} x^r \left[T(x) - \left(K(x) + b_0 + \sum_{k=1}^{r} \frac{b_k}{x^k} \right) \right] = 0.$$

Using the relation (10), we have

$$\lim_{x\to\infty} x^{2m} \left[G(x) - \frac{1}{x} - \sum_{n=1}^{m-1} \frac{(2^{2n}-1)B_{2n}}{nx^{2n}} \right] = \frac{(2^{2m}-1)B_{2m}}{m}, \quad m = 1,2,3,... . \tag{17}$$

If we have other constants $h_2, h_4, h_6, ...$ satisfy

$$\sum_{i=1}^{2} \frac{h_{2i}}{x^{2i}} < G(x) - \frac{1}{x} < \sum_{i=1}^{1} \frac{h_{2i}}{x^{2i}},$$

$$\sum_{i=1}^{4} \frac{h_{2i}}{x^{2i}} < G(x) - \frac{1}{x} < \sum_{i=1}^{3} \frac{h_{2i}}{x^{2i}},$$

$$\sum_{i=1}^{6} \frac{h_{2i}}{x^{2i}} < G(x) - \frac{1}{x} < \sum_{i=1}^{5} \frac{h_{2i}}{x^{2i}},$$

etc. Then these inequalities give us that

$$\begin{aligned} &\lim\nolimits_{x\to\infty} x^2 \left[G(x) - \tfrac{1}{x} \right] = h_2, \\ &\lim\nolimits_{x\to\infty} x^4 \left[G(x) - \tfrac{1}{x} - \tfrac{h_2}{x^2} \right] = h_4, \\ &\lim\nolimits_{x\to\infty} x^6 \left[G(x) - \tfrac{1}{x} - \tfrac{h_2}{x^2} - \tfrac{h_4}{x^4} \right] = h_6, \end{aligned} \tag{18}$$

etc. Comparing the relations (17) and (18), gives us that

$$h_{2j} = \frac{(2^{2j}-1)B_{2j}}{j}, \qquad \forall\, j \in N. \tag{19}$$

This means that the constants $\frac{(2^{2j}-1)B_{2j}}{j}$ in the inequality (15) are the best. Also, the constant one in the function $G(x) - \frac{1}{x}$ can not be improved whatsoever, see [16]. $\square$

Remark 1. As a special case of the inequality (15), we get

$$\frac{1}{2x^2} - \frac{1}{4x^4} < G(x) - \frac{1}{x} < \frac{1}{2x^2} - \frac{1}{4x^4} + \frac{1}{2x^6}, \tag{20}$$

which improve the right hand side of the inequality (8) for $x > 0$ and its left hand side for $x > \sqrt{\frac{3}{2}}$.

Lemma 2.1. *For $m \in \mathbb{N}$, the functions*

$$F_m(x) = G(x) - \frac{1}{x} - \sum_{n=1}^{2m} \frac{(2^{2n}-1)B_{2n}}{nx^{2n}}$$

and

$$H_m(x) = -G(x) + \frac{1}{x} + \sum_{n=1}^{2m-1} \frac{(2^{2n}-1)B_{2n}}{nx^{2n}}$$

are strictly completely monotonic.

Proof. Using the relation (14), we have

$$G(x) - \frac{1}{x} - \sum_{n=1}^{m} \frac{(2^{2n}-1)B_{2n}}{nx^{2n}} = \frac{2(2^{2m+2}-1)B_{2m+2}}{(2m+2)!} \int_0^\infty \theta(t) t^{2m+1} e^{-xt} dt.$$

Then

$$(-1)^k \frac{d^k}{dx^k} \left(G(x) - \frac{1}{x} - \sum_{n=1}^{m} \frac{(2^{2n}-1)B_{2n}}{nx^{2n}} \right) = \frac{2(2^{2m+2}-1)B_{2m+2}}{(2m+2)!} \int_0^\infty \theta(t) t^{2m+k+1} e^{-xt} dt.$$

Using the Bernoulli number's property (16), we get

$$(-1)^k \frac{d^k}{dx^k} \left(G(x) - \frac{1}{x} - \sum_{n=1}^{2m} \frac{(2^{2n}-1)B_{2n}}{nx^{2n}} \right) > 0$$

and

$$(-1)^k \frac{d^k}{dx^k} \left(G(x) - \frac{1}{x} - \sum_{n=1}^{2m-1} \frac{(2^{2n}-1)B_{2n}}{nx^{2n}} \right) < 0.$$

$\square$

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Corollary 2.2. *For odd k, we have*

$$\sum_{n=1}^{2m-1} \frac{(2^{2n}-1)(2n)(2n+1)...(2n+k-1)B_{2n}}{nx^{2n}} < G^{(k)}(x) - \frac{k!}{x^{k+1}}$$

$$< \sum_{n=1}^{2m} \frac{(2^{2n}-1)(2n)(2n+1)...(2n+k-1)B_{2n}}{nx^{2n}}; \qquad m = 1, 2, 3, ...$$

and the inequality will reverse for even $k's$.

3 Applications

3.1 Bounds of the error of some alternating series

A series of the form

$$\sum_{r=1}^{\infty} (-1)^r a_r$$

where $a_r > 0$ for all r, is called an alternating series. By Leibnitz's Theorem [11], the alternating series converges if a_r decreases monotonically and $a_r \to 0$ as $r \to \infty$. Moreover, let S denote the sum of the series and S_n its n^{th} partial sum, then

$$|S_n - S| < a_{n+1}, \qquad n \in \mathbb{N}.$$

For further details about finding estimates for the error $|S_n - S|$, please refer to [13], [28]-[33]. The alternating series [10]

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = \ln 2 \quad and \quad \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} = \frac{\pi}{4}$$

presented early important results of the calculus. Kazarinoff [10] deduced the following error estimates

$$\frac{1}{4n+2} < \left| \sum_{r=1}^{n} \frac{(-1)^r}{2r-1} - \frac{\pi}{4} \right| < \frac{1}{4n-2}, \qquad n \in \mathbb{N} \tag{21}$$

and

$$\frac{1}{2(n+1)} < \left| \sum_{r=1}^{n} \frac{(-1)^{r+1}}{r} - \ln 2 \right| < \frac{1}{2n}, \qquad n \in \mathbb{N} \tag{22}$$

by studying the function

$$E_n = \int_0^{\pi/4} \tan^n \theta d\theta, \qquad n \in \mathbb{N}.$$

Tóth [32] improved Kazarinoff's estimates by

$$\frac{1}{4n + 2\sqrt{19} - 8} < \left| \sum_{r=1}^{n} \frac{(-1)^r}{2r-1} - \frac{\pi}{4} \right| < \frac{1}{4n}, \qquad n \in \mathbb{N} \tag{23}$$

and

$$\frac{1}{2n+2\sqrt{7}-4} < \left|\sum_{r=1}^{n} \frac{(-1)^{r+1}}{r} - \ln 2\right| < \frac{1}{2n+1}, \qquad n \in \mathbb{N}. \tag{24}$$

Also, Tóth and Bukor [33] shown that the best constants a and b such that the inequalities

$$\frac{1}{2n+a} \leq \left|\sum_{r=1}^{n} \frac{(-1)^{r+1}}{r} - \ln 2\right| < \frac{1}{2n+b}, \qquad n \geq 1 \tag{25}$$

hold are $a = \frac{2\ln 2-1}{1-\ln 2}$ and $b = 1$.

Koumandos [13] refined Kazarinoff's estimate (21) by

$$\frac{1}{4n+c} \leq \left|\sum_{r=1}^{n} \frac{(-1)^{r}}{2r-1} - \frac{\pi}{4}\right| < \frac{1}{4n+d}, \qquad n \in \mathbb{N} \tag{26}$$

where the constants $c = \frac{4}{4-\pi} - 4$ and $d = 0$ are the best possible.

In [16], Mahmoud and Agarwal presented the following generalization

$$\frac{4(l+n)^2+10(l+n)+9}{2(l+n+1)\left[4(l+n)^2+8(l+n)+7\right]} < \left|\sum_{r=n+1}^{\infty} \frac{(-1)^{r-1}}{r+l}\right| < \frac{2(l+n)+3}{4(l+n+1)^2}, \tag{27}$$

where $l > -n-1$ and $-l \notin \mathbb{N}$. The double inequality (27) improved the two inequalities (25) and (26) for $n > 1$.

Now, using (5) and (6), we have

$$\left|\sum_{r=n+1}^{\infty} \frac{(-1)^{r-1}}{r+l}\right| = \left|\frac{(-1)^n}{2} G\left(l+n+1\right)\right| = \frac{1}{2} G\left(l+n+1\right), \qquad -l \notin \mathbb{N}. \tag{28}$$

Then our double inequality (15) will give us sharp bounds of the the error $\left|\sum_{r=n+1}^{\infty} \frac{(-1)^{r-1}}{r+l}\right|$, for $-l \notin \mathbb{N}$.

Lemma 3.1.

$$\frac{2}{n} + \sum_{r=1}^{2n} \frac{2(2^{2r}-1)B_{2r}}{r(l+n+1)^{2r}} < \left|\sum_{r=n+1}^{\infty} \frac{(-1)^{r-1}}{r+l}\right| < \frac{2}{n} + \sum_{r=1}^{2n-1} \frac{2(2^{2r}-1)B_{2r}}{r(l+n+1)^{2r}} \qquad n \in \mathbb{N} \tag{29}$$

with sharp bounds, where $l > -n-1$ and $-l \notin \mathbb{N}$.

Remark 2. The inequality (29) improve the inequalities (25) and (26) for special values of the parameter l. Also, it is a generalization of the inequality (27).

3.2 New proof of the increasing monotonicity of a sequence arising from computation of the intersecting probability between a plane couple and a convex body

The increasing of the sequence

$$P_k = \frac{k-1}{2}\left(\int_0^{\pi/2} \sin^{k-1} v \, dv\right)^2, \qquad k \in \mathbb{N}$$

was a question arises from computation of the intersecting probability between a plane couple and a convex body [9]. To prove the increasing monotonicity of the sequences P_k, Guo and Qi [9] studied equivalently the increasing monotonicity of the sequence

$$Q_k = \frac{1}{k}\frac{\Gamma^2\left(\frac{k+1}{2}\right)}{\Gamma^2\left(\frac{k}{2}\right)} \qquad k \in \mathbb{N}.$$

Qi, Mortici and Guo [42] investigated an asymptotic formula for the function

$$\phi(t) = 2\left(\log\Gamma\left(\frac{t+1}{2}\right) - \log\Gamma\left(\frac{t}{2}\right)\right) - \log t \qquad t > 0$$

and proved some properties of the sequence Q_k. Also, Mahmoud [17] generalized some properties of the function $\phi(t)$ and answered about the two posed questions in [42] about the sequence Q_k.

The first derivative of the function $\phi(t)$ can be represented by

$$\phi'(t) = G(t) - \frac{1}{t}$$

and then the function $\phi'(t)$ is strictly completely monotonic, that is

$$(-1)^r \left(\phi'(t)\right)^{(r)} > 0, \qquad r = 0, 1, 2, \dots .$$

Hence the function $\phi(t)$ is increasing and also the function

$$Q(t) = \frac{1}{t}\frac{\Gamma^2\left(\frac{t+1}{2}\right)}{\Gamma^2\left(\frac{t}{2}\right)} \qquad t > 0$$

since $Q'(t) = Q(t)\phi'(t)$. Then $Q(t)$ is increasing function and hence the sequence Q_k is increasing sequence, which is the main result of [9].

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3-VARIABLE ADDITIVE ρ-FUNCTIONAL INEQUALITIES IN FUZZY NORMED SPACES

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Abstract. In this paper, we introduce and investigate the following additive ρ-functional inequalities

$$\begin{aligned} &N(f(x+y+z)-f(x)-f(y)-f(z),t) \\ &\qquad \geq N\left(\rho\left(2f\left(\frac{x+y}{2}+z\right)-f(x)-f(y)-2f(z)\right),t\right), \\ &N\left(2f\left(\frac{x+y}{2}+z\right)-f(x)-f(y)-2f(z),t\right) \\ &\qquad \geq N\left(\rho\left(2f\left(\frac{x+y+z}{2}\right)-f(x)-f(y)-f(z)\right),t\right), \\ &N(f(x+y+z)-f(x)-f(y)-f(z),t) \\ &\qquad \geq N\left(\rho\left(2f\left(\frac{x+y+z}{2}\right)-f(x)-f(y)-f(z)\right),t\right) \end{aligned}$$

in fuzzy normed spaces.

Furthermore, we prove the Hyers-Ulam stability of the above additive ρ-functional inequalities in fuzzy Banach spaces.

1. Introduction and preliminaries

Katsaras [16] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [10, 20, 44]. In particular, Bag and Samanta [2], following Cheng and Mordeson [8], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [19]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [3].

We use the definition of fuzzy normed spaces given in [2, 24, 25] to investigate the Hyers-Ulam stability of additive ρ-functional inequalities in fuzzy Banach spaces.

Definition 1.1. [2, 24, 25, 26] Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N_1) $N(x,t) = 0$ for $t \leq 0$;
(N_2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
(N_3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N_4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;
(N_5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.
(N_6) for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X,N) is called a *fuzzy normed vector space.*

We know that $N(-x,t) = N(x,t)$ for all $x \in X$ by (N_3).

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The other properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [23, 24].

Definition 1.2. [2, 24, 25, 26] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by $N\text{-}\lim_{n\to\infty} x_n = x$.

Definition 1.3. [2, 24, 25, 26] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [3]).

The stability problem of functional equations originated from a question of Ulam [43] concerning the stability of group homomorphisms. The functional equation $f(x+y) = f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [12] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [36] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [11] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach. The functional equation $f\left(\frac{x+y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$ is called the *Jensen equation*. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [7, 13, 15, 17, 18, 21, 31, 32, 33, 34, 37, 38, 39, 40, 41, 42]).

Park [29, 30] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces.

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.4. [4, 9] Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either

$$d(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geq n_0$;
(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

In 1996, G. Isac and Th.M. Rassias [14] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using

fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5, 6, 23, 27, 28, 34, 35]).

Lemma 1.5. *Let $N(x,t) \geq N(\lambda x, t)$ for all $t > 0$. Assume that λ is a fixed real with $|\lambda| < 1$. Then $x = 0$.*

Proof. Putting $\frac{t}{|\lambda|^{n-1}}$ instead of t, we get

$$N\left(x, \frac{t}{|\lambda|^{n-1}}\right) \geq N\left(|\lambda| x, \frac{t}{|\lambda|^{n-1}}\right) \geq N\left(x, \frac{t}{|\lambda|^{n}}\right)$$

So we get

$$N(x,t) \quad \geq \quad N\left(x, \frac{t}{|\lambda|^{n}}\right)$$

for all positive integers n. Passing the limit $n \to \infty$, we get $N(x,t) = 1$ by (N_5), and so $x = 0$ by (N_2). $\square$

In this paper, we introduce and investigate additive ρ-functional inequalities associated with the following additive functional equations

$$\begin{aligned} f(x+y+z) - f(x) - f(y) - f(z) &= 0 \\ 2f\left(\frac{x+y}{2} + z\right) - f(x) - f(y) - 2f(z) &= 0 \\ 2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z) &= 0 \end{aligned}$$

in fuzzy normed spaces.

Furthermore, we prove the Hyers-Ulam stability of the additive ρ-functional inequalities in fuzzy Banach spaces.

Throughout this paper, assume that X is a real fuzzy normed space with norm $N(\cdot, t)$ and that Y is a fuzzy Banach space with norm $N(\cdot, t)$.

2. Additive ρ-functional inequality I

In this section, we investigate the additive ρ-functional inequality

$$\begin{aligned} &N\left(f(x+y+z) - f(x) - f(y) - f(z), t\right) \\ &\geq N\left(\rho\left(2f\left(\frac{x+y}{2} + z\right) - f(x) - f(y) - 2f(z)\right), t\right) \end{aligned} \tag{2.1}$$

in fuzzy normed spaces. Assume that ρ is a fixed real number with $|\rho| < \frac{1}{2}$.

Lemma 2.1. *Let $f : X \to Y$ be a mapping satisfying (2.1) for all $x, y, z \in X$. Then $f : X \to Y$ is additive.*

Proof. Letting $x = y = z = 0$ in (2.1), we get $N(2f(0), t) \geq N(2\rho f(0), t)$ and so $f(0) = 0$ by Lemma 1.5.

Replacing y by x and z by $-x$ in (2.1), we get $N(f(x) + f(-x), t) \geq N(2\rho(f(x) + f(-x)), t)$ and so $f(-x) = -f(x)$ for all $x \in X$ by Lemma 1.5.

Replacing y by x and z by $-2x$ in (2.1), we get

$$N(f(2x) - 2f(x), t) \geq N(2\rho(f(2x) - 2f(x)), t)$$

and so $f(2x) = 2f(x)$ for all $x \in X$ by Lemma 1.5.

Replacing z by $-x-y$ in (2.1), we get

$$N\left(f(x+y) - f(x) - f(y), t\right) \geq N\left(\rho\left(f(x+y) - f(x) - f(y)\right), t\right)$$

and so $f(x+y) = f(x) + f(y)$ for all $x, y \in X$ by Lemma 1.5.

Hence $f : X \to Y$ is additive. $\square$

We prove the Hyers-Ulam stability of the additive ρ-functional inequality (2.1) in fuzzy Banach spaces.

Theorem 2.2. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ satisfying*

$$\varphi(x, y, z) \le \frac{L}{2}\varphi(2x, 2y, 2z), \quad \varphi(0, 0, 0) = 0$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying

$$\begin{aligned} (2.2)\qquad & N(f(x+y+z) - f(x) - f(y) - f(z), t) \\ & \ge \min\left\{ N\left(\rho\left(2f\left(\frac{x+y}{2} + z\right) - f(x) - f(y) - 2f(z)\right), t\right), \frac{t}{t + \varphi(x, y, z)} \right\} \end{aligned}$$

for all $x, y, z \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N(f(x) - A(x), t) \ge \frac{(2-2L)t}{(2-2L)t + L\varphi(x, x, 0)} \tag{2.3}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $x = y = z = 0$ in (2.2), we get $N(2f(0), t) \ge N(2\rho f(0), t)$ and so $f(0) = 0$ by Lemma 1.5.

Replacing y by x and z by 0 in (2.2), we get

$$N(f(2x) - 2f(x), t) \ge \frac{t}{t + \varphi(x, x, 0)} \tag{2.4}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf\left\{ \mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \ge \frac{t}{t + \varphi(x, x, 0)}, \ \forall x \in X, \forall t > 0 \right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [22, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \ge \frac{t}{t + \varphi(x, x, 0)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right), L\varepsilon t\right) = N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{2}\varepsilon t\right) \\ &\ge \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right)} \ge \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \frac{L}{2}\varphi(x, x, 0)} = \frac{t}{t + \varphi(x, x, 0)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.4) that $N\left(f(x) - 2f\left(\frac{x}{2}\right), \frac{L}{2}t\right) \geq \frac{t}{t+\varphi(x,x,0)}$ for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{2}$.

By Theorem 1.4, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e., $A\left(\frac{x}{2}\right) = \frac{1}{2}A(x)$ for all $x \in X$. Since $f : X \to Y$ is odd, $A : X \to Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0, \infty)$ satisfying $N(f(x) - A(x), \mu t) \geq \frac{t}{t+\varphi(x,x,0)}$ for all $x \in X$;

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \leq \frac{1}{1-L}d(f, Jf)$, which implies the inequality $d(f, A) \leq \frac{L}{2-2L}$. This implies that the inequality (2.3) holds.

By (2.2),

$$\begin{aligned}
&N\left(2^n\left(f\left(\frac{x+y+z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right), 2^n t\right) \\
&\geq \min\left\{N\left(2^n\rho\left(2f\left(\frac{x+y+2z}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - 2f\left(\frac{z}{2^n}\right)\right), 2^n t\right),\right. \\
&\qquad\left.\frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right)}\right\}
\end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$.

Replacing t by $\frac{t}{2^n}$, we get

$$\begin{aligned}
&N\left(2^n\left(f\left(\frac{x+y+z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right), t\right) \\
&\geq \min\left\{N\left(2^n\rho\left(2f\left(\frac{x+y+2z}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - 2f\left(\frac{z}{2^n}\right)\right), t\right),\right. \\
&\qquad\left.\frac{\frac{t}{2^n}}{\frac{t}{2^n} + \frac{L^n}{2^n}\varphi(x,y,z)}\right\}
\end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \frac{L^n}{2^n}\varphi(x,y,z)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$\begin{aligned}
&N\left(A(x+y+z) - A(x) - A(y) - A(z), t\right) \\
&\quad\geq N\left(\rho\left(2A\left(\frac{x+y}{2} + z\right) - A(x) - A(y) - 2A(z)\right), t\right)
\end{aligned}$$

for all $x, y \in X$ and all $t > 0$. By Lemma 2.1, the mapping $A : X \to Y$ is Cauchy additive, as desired. $\square$

Theorem 2.3. *Let $\varphi : X^3 \to [0,\infty)$ be a function such that there exists an $L < 1$ satisfying*

$$\varphi(x,y,z) \le 2L\varphi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right), \quad \varphi(0,0,0) = 0$$

for all $x,y,z \in X$. Let $f : X \to Y$ be a mapping satisfying (2.2). Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N(f(x) - A(x), t) \ge \frac{(2-2L)t}{(2-2L)t + \varphi(x,x,0)} \tag{2.5}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S,d) be the generalized metric space defined in the proof of Theorem 2.2.

It follows from (2.4) that $f(0) = 0$ and

$$N(f(2x) - 2f(x), t) \ge \frac{t}{t + \varphi(x,x,0)}$$

for all $x \in X$. Consider the linear mapping $J : S \to S$ such that $Jg(x) := \frac{1}{2}g(2x)$.

$$N(Jf(x) - f(x), t) \ge \frac{t}{t + \frac{1}{2}\varphi(x,x,0)}$$

So, we can get $d(Jf,f) \ge \frac{1}{2}$

The rest of the proof is similar to the proof of Theorem 2.2. □

Lemma 2.4. *Let $f : X \to Y$ be a mapping satisfying*

$$f(x+y+z) - f(x) - f(y) - f(z) = \rho\left(2f\left(\frac{x+y}{2}+z\right) - f(x) - f(y) - 2f(z)\right) \tag{2.6}$$

for all $x,y,z \in X$. Then $f : X \to Y$ is additive.

Proof. Letting $x = y = z = 0$ in (2.6), we get $-2f(0) = -2\rho f(0)$ and so $f(0) = 0$.

Replacing y by x and letting $z = 0$ in (2.6), we get $f(2x) - 2f(x) = 0$ and so $f(2x) = 2f(x)$ for all $x \in X$.

Letting $z = 0$ in (2.6), we get

$$f(x+y) - f(x) - f(y) = \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right) = \rho(f(x+y) - f(x) - f(y))$$

and so $f(x+y) = f(x) + f(y)$ for all $x,y \in X$. □

Now, we prove the Hyers-Ulam stability of an additive ρ-functional inequality associated with (2.6) in fuzzy Banach spaces.

Theorem 2.5. *Let $\varphi : X^3 \to [0,\infty)$ be a function such that there exists an $L < 1$ satisfying*

$$\varphi(x,y,z) \le \frac{L}{2}\varphi(2x,2y,2z), \quad \varphi(0,0,0) = 0$$

for all $x,y,z \in X$. Let $f : X \to Y$ be a mapping satisfying

$$\begin{aligned} &N((f(x+y+z) - f(x) - f(y) - f(z) \\ &\quad -\rho\left(2f\left(\frac{x+y}{2}+z\right) - f(x) - f(y) - 2f(z)\right), t\Big) \ge \frac{t}{t+\varphi(x,y,z)} \end{aligned} \tag{2.7}$$

for all $x,y,z \in X$ and all $t > 0$. Then there exists an unique additive mapping $A : X \to Y$ satisfying (2.3).

Proof. Letting $x = y = z = 0$ in (2.7), we get $N(2(1-\rho)f(0), t) = 1$. So $f(0) = 0$.

Replacing y by x and z by 0 in (2.7), we get

$$N(f(2x) - 2f(x), t) \geq \frac{t}{t + \varphi(x, x, 0)} \tag{2.8}$$

for all $x \in X$. So

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), t\right) \geq \frac{t}{t + \varphi(\frac{x}{2}, \frac{x}{2}, 0)} \geq \frac{t}{t + \frac{L}{2}\varphi(x, x, 0)}$$

for all $x \in X$. Consider the linear mapping $J : S \to S$ such that $Jg(x) = 2g\left(\frac{x}{2}\right)$.

$$N\left(f(x) - Jf(x), t\right) \geq \frac{t}{t + \frac{L}{2}\varphi(x, x, 0)}$$

and so $d(f, Jf) \leq \frac{L}{2}$

The rest of the proof is similar to the proof of Theorem 2.2. □

Theorem 2.6. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ satisfying*

$$\varphi(x, y, z) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right), \varphi(0, 0, 0) = 0$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying (2.7). *Then there exists an unique additive mapping $A : X \to Y$ satisfying* (2.5).

Proof. It follows from (2.8) that $f(0) = 0$ and

$$N\left(f\left(2x\right) - 2f(x), t\right) \geq \frac{t}{t + \varphi(x, x, 0)}$$

for all $x \in X$. Consider the linear mapping $J : S \to S$ such that $Jg(x) = \frac{1}{2}g(2x)$.

$$N\left(Jf(x) - f(x), t\right) \geq \frac{t}{t + \frac{1}{2}\varphi(x, x, 0)}$$

So, we can get $d(f, Jf) \geq \frac{1}{2}$

The rest of the proof is similar to the proof of Theorem 2.2. □

3. Additive ρ-functional inequality II

In this section, we investigate the additive ρ-functional inequality

$$\begin{aligned} N\left(2f\left(\frac{x+y}{2} + z\right) - f(x) - f(y) - 2f(z), t\right) \\ \geq N\left(\rho\left(2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z)\right), t\right) \end{aligned} \tag{3.1}$$

in fuzzy normed spaces. Assume that ρ is a fixed real with $|\rho| < 1$.

Lemma 3.1. *Let $f : X \to Y$ be a mapping satisfying* (3.1) *for all $x, y, z \in X$. Then $f : X \to Y$ is additive.*

Proof. Letting $x = y = z = 0$ in (3.1), we get $N(2f(0), t) \geq N(\rho f(0), t)$ and so $f(0) = 0$ by Lemma 1.5.

Replacing z by x and letting $y = 0$ in (3.1), we get $N\left(2f\left(\frac{3x}{2}\right) - 3f(x), t\right) = 1$ and so $f(\frac{3x}{2}) = \frac{3}{2}f(x)$ for all $x \in X$ by Lemma 1.5.

Replacing y by x and z by x in (3.1), we get $N\left(2f(2x) - 4f(x), t\right) = 1$ and so $f(2x) = 2f(x)$ for all $x \in X$ by Lemma 1.5.

Replacing y by $-x$ and z by y in (3.1), we get

$$N\left(f(x)+f(-x),t\right) \ge N\left(\rho\left(f(x)+f(-x)\right),t\right)$$

and so $f(-x)=-f(x)$ for all $x\in X$ by Lemma 1.5.

Replacing z by $-x-y$ in (3.1), we get

$$N\left(f(x+y)-f(x)-f(y),t\right) \ge N\left(\rho\left(f(x+y)-f(x)-f(y)\right),t\right)$$

and so $f(x+y)=f(x)+f(y)$ for all $x,y\in X$ by Lemma 1.5. So $f:X\to Y$ is additive. □

We prove the Hyers-Ulam stability of the additive ρ-functional inequality (3.1) in fuzzy Banach spaces.

Theorem 3.2. *Let $\varphi : X^3 \to [0,\infty)$ be a function such that there exists an $L<1$ satisfying*

$$\varphi(x,y,z) \le \frac{2}{3}L\varphi\left(\frac{3}{2}x,\frac{3}{2}y,\frac{3}{2}z\right), \quad \varphi(0,0,0)=0$$

for all $x,y,z\in X$. Let $f:X\to Y$ be a mapping satisfying

$$\begin{aligned}(3.2)\qquad & N\left(2f\left(\frac{x+y}{2}+z\right)-f(x)-f(y)-2f(z),t\right)\\ &\ge \min\left\{N\left(\rho\left(2f\left(\frac{x+y+z}{2}\right)-f(x)-f(y)-f(z)\right),t\right),\frac{t}{t+\varphi(x,y,z)}\right\}\end{aligned}$$

for all $x,y,z\in X$ and all $t>0$. Then $A(x):=N\text{-}\lim_{n\to\infty}2^n f\left(\frac{x}{2^n}\right)$ exists for each $x\in X$ and defines an additive mapping $A:X\to Y$ such that

$$N\left(f(x)-A(x),t\right) \ge \frac{(3-3L)t}{(3-3L)t+L\varphi(x,0,x)} \tag{3.3}$$

for all $x\in X$ and all $t>0$.

Proof. Letting $x=y=z=0$ in (3.2), we get $N(2f(0),t)\ge N(\rho f(0),t)$ and so $f(0)=0$ by Lemma 1.5.

Replacing y by 0 and z by x, we get

$$N\left(2f\left(\frac{3}{2}x\right)-3f(x),t\right) \ge \frac{t}{t+\varphi(x,0,x)} \tag{3.4}$$

for all $x\in X$.

Consider the set

$$S:=\{g:X\to Y\}$$

and introduce the generalized metric on S:

$$d(g,h)=\inf\left\{\mu\in\mathbb{R}_+ : N(g(x)-h(x),\mu t)\ge \frac{t}{t+\varphi(x,0,x)},\ \forall x\in X, \forall t>0\right\},$$

where, as usual, $\inf\phi=+\infty$. It is known that (S,d) is complete.

Now we consider the linear mapping $J:S\to S$ such that

$$Jg(x):=\frac{3}{2}g\left(\frac{2}{3}x\right)$$

for all $x\in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then $N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t+\varphi(x,0,x)}$ for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(\frac{3}{2}g\left(\frac{2}{3}x\right) - \frac{3}{2}h\left(\frac{2}{3}x\right), L\varepsilon t\right) = N\left(g\left(\frac{2}{3}x\right) - h\left(\frac{2}{3}x\right), \frac{2}{3}L\varepsilon t\right) \\ &\geq \frac{\frac{2Lt}{3}}{\frac{2Lt}{3} + \varphi\left(\frac{2}{3}x, 0, \frac{2}{3}x\right)} \geq \frac{\frac{2Lt}{3}}{\frac{2Lt}{3} + \frac{2L}{3}\varphi(x,0,x)} = \frac{t}{t+\varphi(x,0,x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (3.4) that

$$N\left(f(x) - \frac{3}{2}f\left(\frac{2}{3}x\right), \frac{L}{3}t\right) \geq \frac{t}{t+\varphi(x,0,x)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{3}$.

By Theorem 1.4, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e., $A\left(\frac{2}{3}x\right) = \frac{2}{3}A(x)$ for all $x \in X$. Since $f : X \to Y$ is odd, $A : X \to Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0, \infty)$ satisfying $N(f(x) - A(x), \mu t) \geq \frac{t}{t+\varphi(x,0,x)}$ for all $x \in X$;

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} \left(\frac{3}{2}\right)^n f\left(\left(\frac{2}{3}\right)^n x\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \leq \frac{1}{1-L}d(f, Jf)$, which implies the inequality $d(f, A) \leq \frac{L}{3-3L}$. This implies that the inequality (3.3) holds.

By (3.2),

$$\begin{aligned} &N\left(\left(\frac{3}{2}\right)^n \left[2f\left(\left(\frac{2}{3}\right)^n \left(\frac{x+y}{2} + z\right)\right) - f\left(\left(\frac{2}{3}\right)^n x\right) - f\left(\left(\frac{2}{3}\right)^n y\right) - 2f\left(\left(\frac{2}{3}\right)^n z\right)\right],\right. \\ &\qquad \left.\left(\frac{3}{2}\right)^n t\right) \\ &\geq \min\left\{N\left(\left(\frac{3}{2}\right)^n \rho\left[2f\left(\left(\frac{2}{3}\right)^n \frac{x+y+z}{2}\right) - f\left(\left(\frac{2}{3}\right)^n x\right)\right], \left(\frac{3}{2}\right)^n t\right),\right. \\ &\qquad \left.\frac{t}{t+\varphi\left(\left(\frac{2}{3}\right)^n x, \left(\frac{2}{3}\right)^n y, \left(\frac{2}{3}\right)^n z\right)}\right\} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$.

Replacing t by $\frac{t}{2^n}$, we get that

$$N\left(\left(\frac{3}{2}\right)^n\left[2f\left(\left(\frac{2}{3}\right)^n\left(\frac{x+y}{2}+z\right)\right)-f\left(\left(\frac{2}{3}\right)^n x\right)-f\left(\left(\frac{2}{3}\right)^n y\right)-2f\left(\left(\frac{2}{3}\right)^n z\right)\right],t\right)$$
$$\geq \min\left\{N\left(\left(\frac{3}{2}\right)^n\rho\left[2f\left(\left(\frac{2}{3}\right)^n\frac{x+y+z}{2}\right)-f\left(\left(\frac{2}{3}\right)^n x\right)\right],t\right),\frac{\left(\frac{2}{3}\right)^n t}{\left(\frac{2}{3}\right)^n t+\left(\frac{2L}{3}\right)^n\varphi(x,y,z)}\right\}$$

for all $x,y\in X$, all $t>0$ and all $n\in\mathbb{N}$. Since $\lim_{n\to\infty}\frac{\left(\frac{2}{3}\right)^n t}{\left(\frac{2}{3}\right)^n t+\left(\frac{2L}{3}\right)^n\varphi(x,y,z)}=1$ for all $x,y\in X$ and all $t>0$,

$$N\left(2A\left(\frac{x+y}{2}+z\right)-A(x)-A(y)-2A(z),t\right)$$
$$\geq N\left(\rho\left(2A\left(\frac{x+y+z}{2}\right)-A(x)-A(y)-A(z)\right),t\right)$$

for all $x,y\in X$ and all $t>0$. By Lemma 2.1, the mapping $A:X\to Y$ is Cauchy additive, as desired.

The rest of the proof is similar to the proof of Theorem 2.2. □

Theorem 3.3. *Let $\varphi:X^3\to[0,\infty)$ be a function such that there exists an $L<1$ satisfying*

$$\varphi(x,y,z)\leq\frac{3}{2}L\varphi\left(\frac{2}{3}x,\frac{2}{3}y,\frac{2}{3}z\right),\quad \varphi(0,0,0)=0$$

for all $x,y,z\in X$. Let $f:X\to Y$ be a mapping satisfying (3.2). Then $A(x):=N$-$\lim_{n\to\infty}\left(\frac{2}{3}\right)^n f\left(\left(\frac{3}{2}\right)^n x\right)$ exists for each $x\in X$ and defines an additive mapping $A:X\to Y$ such that

$$N(f(x)-A(x),t)\geq\frac{(3-3L)\,t}{(3-3L)\,t+\varphi(x,0,x)}\tag{3.5}$$

for all $x\in X$ and all $t>0$.

Proof. Let (S,d) be the generalized metric space defined in the proof of Theorem 3.2. It follows from (3.4) that $f(0)=0$ and

$$N\left(2f\left(\frac{3}{2}x\right)-3f(x),t\right)\geq\frac{t}{t+\varphi(x,0,x)}$$

for all $x\in X$. Consider the linear mapping $J:S\to S$ such that $Jg(x):=\frac{2}{3}g\left(\frac{3}{2}x\right)$.

$$N(Jf(x)-f(x),t)\geq\frac{t}{t+\frac{1}{3}\varphi(x,x,0)}$$

So, we can get $d(Jf,f)\geq\frac{1}{3}$

The rest of the proof is similar to the proof of Theorem 3.2. □

From now on, we investigate another additive ρ-functional inequality

$$N\left(2f\left(\frac{x+y+z}{2}\right)-f(x)-f(y)-f(z),t\right)\tag{3.6}$$
$$\geq N\left(\rho\left(2f\left(\frac{x+y}{2}+z\right)-f(x)-f(y)-2f(z)\right),t\right)$$

in fuzzy normed spaces. Assume that ρ is a fixed real with $|\rho|<\frac{1}{2}$.

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Lemma 3.4. *Let $f : X \to Y$ be a mapping satisfying* (3.6) *for all $x, y, z \in X$. Then $f : X \to Y$ is additive.*

Proof. Letting $x = y = z = 0$ in (3.6), we get $N(f(0), t) \geq N(2|\rho|f(0), t)$. and so $f(0) = 0$ by Lemma 1.5.

Letting $x = y = 0$ in (3.6), we get $N\left(2f\left(\frac{z}{2}\right) - f(z), t\right) = 1$ and so $f(\frac{x}{2}) = \frac{1}{2}f(x)$ for all $x \in X$.

Replacing z by $-x - y$ in (3.6), we get

$$N\left(f(-x-y) + f(x) + f(y), t\right) \geq N\left(\rho\left(f(-x-y) + f(x) + f(y)\right), t\right)$$

and so

$$f(-x-y) = -f(x) - f(y)$$

for all $x, y \in X$.

Letting $y = 0$ in (3.6), we get $f(-x) = -f(x)$ for all $x \in X$.

Thus $f(x) + f(y) = -f(-x-y) = f(x+y)$ for all $x, y \in X$. Hence $f : X \to Y$ is additive. $\square$

4. Additive ρ-functional inequality III

In this section, we investigate the additive ρ-functional inequality

$$\begin{aligned} N\left(f(x+y+z) - f(x) - f(y) - f(z), t\right) \\ \geq N\left(\rho\left(2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z)\right), t\right) \end{aligned} \tag{4.1}$$

in fuzzy normed spaces. Assume that ρ is a fixed real with $|\rho| < 1$.

Lemma 4.1. *Let $f : X \to Y$ be a mapping satisfying* (4.1) *for all $x, y, z \in X$. Then $f : X \to Y$ is additive.*

Proof. Letting $x = y = z = 0$ in (4.1), we get $N(2f(0), t) \geq N(\rho f(0), t)$. and so $f(0) = 0$ by Lemma 1.5.

Replacing z by $-x - y$ in (4.1), we get

$$N(f(x) + f(y) + f(-x-y), t) \geq N(\rho(f(x) + f(y) + f(-x-y)), t)$$

and so

$$f(x) + f(y) + f(-x-y) = 0 \tag{4.2}$$

for all $x, y \in X$.

Letting $y = -x$ in (4.2), we get $f(-x) = -f(x)$ for all $x \in X$.

Thus $f(x) + f(y) = -f(-x-y) = f(x+y)$ for all $x, y \in X$. So $f : X \to Y$ is additive. $\square$

We prove the Hyers-Ulam stability of the additive ρ-functional inequality (4.1) in fuzzy Banach spaces.

Theorem 4.2. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ satisfying*

$$\varphi(x, y, z) \leq \frac{L}{2}\varphi\left(2x, 2y, 2z\right), \varphi(0, 0, 0) = 0$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying

$$\begin{aligned} N\left(f\left(x+y+z\right) - f(x) - f(y) - f(z), t\right) \\ \geq \min\left\{N\left(\rho\left(2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z)\right), t\right), \frac{t}{t + \varphi(x, y, z)}\right\} \end{aligned} \tag{4.3}$$

for all $x, y, z \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{(2-2L)t}{(2-2L)t + L\varphi(x,x,0)} \tag{4.4}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $x = y = z = 0$ in (4.3), we get $N(2f(0), t) \geq N(\rho f(0), t)$ and so $f(0) = 0$ by Lemma 1.5.

Replacing y by x and letting $z = 0$ in (4.3), we get

$$N\left(f\left(2x\right) - 2f(x), t\right) \geq \frac{t}{t + \varphi(x,x,0)} \tag{4.5}$$

for all $x \in X$.

Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Consider the linear mapping $J : S \to S$ such that $Jg(x) = 2g\left(\frac{x}{2}\right)$. Then

$$N(f(x) - Jf(x), t) \geq \frac{t}{t + \varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right)} \geq \frac{t}{t + \frac{L}{2}\varphi(x,x,0)}$$

So $d(f, Jf) \leq \frac{L}{2}$.

The rest of the proof is similar to the proof of the Theorem 2.2. □

Theorem 4.3. *Let $\varphi : X^3 \to [0,\infty)$ be a function such that there exists an $L < 1$ satisfying*

$$\varphi(x,y,z) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right), \quad \varphi(0,0,0) = 0$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying (4.3). Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{(2-2L)t}{(2-2L)t + \varphi(x,x,x)} \tag{4.6}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

It follows from (4.5) that $f(0) = 0$ and

$$N\left(f\left(2x\right) - 2f(x), t\right) \geq \frac{t}{t + \varphi(x,x,0)}$$

for all $x \in X$.

Consider the linear mapping $J : S \to S$ such that $Jg(x) = \frac{1}{2}g(2x)$. Then

$$N(f(x) - Jf(x), t) \geq \frac{2t}{2t + \varphi(x,x,0)} \geq \frac{t}{t + \frac{1}{2}\varphi(x,x,0)}$$

So $d(f, Jf) \leq \frac{1}{2}$. The rest of the proof is similar to the proof of the Theorem 4.2. □

Lemma 4.4. *Let $f : X \to Y$ be a mapping satisfying*

$$f(x+y+z) - f(x) - f(y) - f(z) = \rho\left(2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z)\right) \tag{4.7}$$

for all $x, y, z \in X$. Then $f : X \to Y$ is additive.

Proof. Letting $x = y = z = 0$ in (4.7), we get $-2f(0) = -\rho f(0)$ and so $f(0) = 0$.

Replacing y by x and letting $z = 0$ in (4.7), we get $f(2x) - 2f(x) = 0$ and so $f(2x) = 2f(x)$ for all $x \in X$.

Letting $z = 0$ in (4.7), we get

$$f(x+y) - f(x) - f(y) = \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right) = \rho(f(x+y) - f(x) - f(y))$$

and so $f(x+y) = f(x) + f(y)$ for all $x, y \in X$. □

Now, we prove the Hyers-Ulam stability of an additive ρ-functional inequality associated with (4.7) in fuzzy Banach spaces.

Theorem 4.5. *Let $\varphi : X^3 \to [0,\infty)$ be a function such that there exists an $L < 1$ satisfying*

$$\varphi(x, y, z) \le \frac{L}{2}\varphi(2x, 2y, 2z), \quad \varphi(0,0,0) = 0$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying

$$N\left((f(x+y+z) - f(x) - f(y) - f(z) - \rho\left(2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z)\right), t\right) \ge \frac{t}{t + \varphi(x,y,z)} \tag{4.8}$$

for all $x, y, z \in X$ and all $t > 0$. Then there exists an unique additive mapping $A : X \to Y$ satisfying (4.4).

Proof. Letting $x = y = z = 0$ in (4.8), we get $N((2-\rho)f(0), t) = 1$ and so $f(0) = 0$.

Replacing y by x and letting $z = 0$ in (4.8), we get

$$N\left(f\left(2x\right) - 2f(x), t\right) \ge \frac{t}{t + \varphi(x,x,0)} \tag{4.9}$$

for all $x \in X$.

Let (S,d) be the generalized metric space defined in the proof of Theorem 2.2.

Let $Jg(x) = 2g\left(\frac{x}{2}\right)$. Then

$$N(f(x) - Jf(x), t) \ge \frac{t}{t + \varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right)} \ge \frac{t}{t + \frac{L}{2}\varphi(x,x,0)}$$

So $d(f, Jf) \le \frac{L}{2}$.

The rest of the proof is similar to the proof of Theorem 4.2. □

Theorem 4.6. *Let $\varphi : X^3 \to [0,\infty)$ be a function such that there exists an $L < 1$ satisfying*

$$\varphi(x, y, z) \le 2L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right), \quad \varphi(0,0,0) = 0$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying (4.8)*. Then there exists an unique additive mapping $A : X \to Y$ satisfying* (4.6).

Proof. It follows from (4.9) that $f(0) = 0$ and

$$N\left(f\left(2x\right) - 2f(x), t\right) \ge \frac{t}{t + \varphi(x,x,0)}$$

for all $x \in X$.

Let (S,d) be the generalized metric space defined in the proof of Theorem 2.2.

Consider the linear mapping $J : S \to S$ such that $Jg(x) = \frac{1}{2}g(2x)$. Then

$$N(f(x) - Jf(x), t) \ge \frac{2t}{2t + \varphi(x,x,0)} \ge \frac{t}{t + \frac{1}{2}\varphi(x,x,0)}$$

So $d(f, Jf) \leq \frac{1}{2}$.

The rest of the proof is similar to the proof of Theorem 4.2. □

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An Approach to Separability of Integrable Hamiltonian System

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Abstract

Directly from Benenti's theorem, which characterizes separability with one single Killing tensor, we adopt an algorithm to execute the task of separability test. The algorithm is applied to generalized quartic and quintic polynomial potentials as well as some multi-separable systems on (pseudo)-Euclidean spaces. It yields many well-known integrable systems in a unified and straight manner, in contrast to some complicated techniques employed in the literature to derive them.

Keywords: completely integrable system; separable system; Killing two-tensor; Hénon-Heiles systems

1. Introduction

Finite dimensional completely integrable system has always attracted much attention. Recently authors adopted various methods or perspectives to investigate them. Prominent of all, are the separability theory of Hamilton-Jacobi equation [1], the approach of Lax representations [2], and the bi-Hamiltonian theory [3, 4] among others (see e.g. the references above and therein).

For a given Hamilton system finding canonical separation coordinates is very non-trivial. The above approaches can partly solve this problem. Sklyanin developed a method based on a Lax pair [5]. The separable coordinates are obtained from the spectrum of the Lax operator. Another approach is based on the existence of a bi-Hamiltonian representation [6, 4]. The separable variables, called Darboux-Nijenhuis coordinates, are related with the recursion operator constructed from the Poisson pencil.

For a generic system there is no intrinsic criterion of the existence of a Lax or bi-Hamiltonian formulation. Benenti [7, 8] has developed an intrinsic characterization for a Hamiltonian system being separable (see Theorem 1). It is based on geometric properties of the Killing tensors corresponding to the first integrals of the system. We can make a comparison of these approaches to the separability of the Hamilton-Jacobi equation. While the technique based on the Lax or bi-Hamiltonian formulation may be more effective in studying particular examples (for which such formulation has been found beforehand), the Benenti approach is more rigorous from the mathematical point of view.

Though integrability does not necessarily imply separability, the separable class constitute the vital examples among all integrable systems. Directly from Benenti's theorem, we can adopt a strategy to cope with the problem of separability test. We present this method as an executable algorithm, which are especially applicable to families of Hamiltonian systems containing some numeric constants. In this paper we employ this algorithm to test several natural systems, recovering some known models obtained by other approaches such as Painlevé analysis or differential Galois theory [9].

This paper is organised as follows. In Section 2, some basic concepts in Killing tensors method of H-J separability are reviewed, then based on it we suggest an executable algorithm to make concrete separability test. The algorithm is, in Section 3, applied to test several potentials, including inhomogeneous quartic polynomial potential, homogeneous quintic potential, as well as some multi-separable systems on Euclidean and Minkowski planes. These will yield many well-known integrable systems in a unified and straight manner, in contrast to complicated techniques employed in the literature. The last section is devoted to some concluding remarks.

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2. The Geometric Method to Variables Separation and an Executable Algorithm

We briefly recall some necessary facts about the separation of variables method, considered in the framework of symplectic geometry. Let $(\mathcal{M}, \omega)$ be a symplectic manifold with symplectic form ω. Then the Poisson bivector is $P = \omega^{-1}$. Note here we view ω (and P) as transformation taking vector field to one-form (and vice-versa, respectively). It is well known that the Schouten bracket vanishes, $[P, P] = 0$.

Let $(\boldsymbol{q}, \boldsymbol{p})$, $\boldsymbol{q} = (q_1, \ldots, q_n)$, $\boldsymbol{p} = (p_1, \ldots, p_n)$ be the (local) canonical coordinates on $\mathcal{M}$, then the Poisson bivector is $P = \sum_{i=1}^{n} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}$. The *Hamiltonian vector field* corresponding to a smooth function $H = H(\boldsymbol{q}, \boldsymbol{p})$ is defined as $X_H = P\,\mathrm{d}H$. The triple (M, P, H) is called a *Hamiltonian system.*

In this paper we will focus on Hamiltonian system in the setting of Riemannian geometry. That is to say, the phase space is the cotangent bundle T^*M of some (pseudo)-Riemannian manifold (M, g). We remind that we are finding separable coordinates related to the original physics coordinates $(\boldsymbol{q}, \boldsymbol{p})$ via a point transformation. In the setting of Riemannian geometry a natural Hamiltonian $H = T + V$ takes the form as follows

$$H = \sum_{i,j=1}^{n} \frac{1}{2} G^{ij}(\boldsymbol{q})\, p_i\, p_j + V(\boldsymbol{q}) \tag{2.1}$$

where G^{ij} is the inverse of metric g and $V(\boldsymbol{q})$ the potential. The classical *Hamilton-Jacobi equation* reads

$$\sum_{i,j=1}^{n} \frac{1}{2}\, G^{ij}\, \partial_i W\, \partial_j W + V = E \tag{2.2}$$

where E is the constant of conserved energy (Hamiltonian). It is a first-order partial differential equation of the unknown W.

Definition 1. *The Hamiltonian H (2.1) is* separable *in the canonical variable $(\boldsymbol{q}, \boldsymbol{p})$, if the Hamilton-Jacobi equation (2.2) admits a complete integral of additive form*

$$W(\boldsymbol{q}, \boldsymbol{\alpha}) = \sum\nolimits_{i=1}^{n} W_i(q^i, \boldsymbol{\alpha}), \tag{2.3}$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ are integration constants, such that $\det \left[\frac{\partial^2 W}{\partial q_i\, \partial \alpha_j}\right] \neq 0$. *The variables $(\boldsymbol{q}, \boldsymbol{p})$ are called* separable variables.

It is well known that the n first integrals obtained by solving the Hamilton-Jacobi equation (2.2) are either quadratic or linear in momenta and thus correspond to Killing 2-tensors or Killing vectors, respectively.

Definition 2. *A* Killing tensor *K of valence p defined on (M, g) is a symmetric $(p, 0)$-tensor satisfying the Killing tensor equation*

$$[K, G] = 0, \tag{2.4}$$

where $[\ ,\ ]$ denotes the Schouten bracket.

All Killing p-tensors constitute a vector space $\mathcal{K}^p(M)$. For manifold of constant curvature its dimension attains the maximum, see e.g. [10].

Separability of the natural Hamiltonian $H = T + V$ depends on the Killing 2-tensor of the underlying manifold (M, g). This idea, due to Eisenhart, has been extensively exploited by many authors, see e.g. [11, 12]. The intrinsic criterion given by Benenti [7, 8] allows one to characterize separability by a single Killing 2-tensor (orthogonal case), or a Killing 2-tensor together with an abelian algebra of Killing vectors (non-orthogonal case). Here we focus on the orthogonal case since it is more common.

Theorem 1 (Benenti)**.** *A Hamiltonian $H = T + V$ is separable in some orthogonal coordinates if and only if there exists a Killing 2-tensor $\mathbf{K}$ with pointwise simple and real eigenvalues, orthogonally integrable eigenvectors and such that*

$$\mathrm{d}\,(\mathbf{K}\,\mathrm{d}V) = 0. \tag{2.5}$$

The $(0,2)$-tensor $\mathbf{K}$ is called *characteristic Killing tensor*. On a Riemannian manifold it can be viewed as $(2,0)$- or $(1,1)$-tensor by lowering or raising indices via metric g or G. Here $\mathbf{K}$ in (2.5) is seen as a $(1,1)$-tensor which takes a one-form to another. In local coordinates (2.5) entails the following one-form is closed,

$$\mathbf{K}\,\mathrm{d}V = \sum\nolimits_{i,j,l} g_{ij}\,\mathbf{K}^{jl}\,\partial_l V\,\mathrm{d}q^i\,. \tag{2.6}$$

This theorem elegantly and intrinsically characterizes the orthogonal separability of the natural Hamiltonian (2.1). Often in the literature one is faced with a general system with some parameters involved in the Hamiltonian. Many sophisticated methods have to be invented and applied to identify the rare cases which are separable, or integrable. We will revisit some of these examples in later sections.

From the Benenti's theorem 1 we come up with an approach to deal with the problem of searching for separable case of parameters, presented by an Algorithm as below:

Algorithm. Let H be a natural Hamiltonian with potential $V(\boldsymbol{q};a_i)$ defined on some pseudo-Riemannian manifold (M,g), where a_i's are some constant parameters. The special values of parameters, that guarantees the system H is H-J separable, are achieved during execution of the algorithm.

Begin.

Step 1. For the pseudo-Riemannian manifold (M,g), using the Killing tensor equation (2.4) to calculate the general Killing 2-tensor K. All of them constitute a vector space $\mathcal{K}^2(M)$ of dimension d. The expression of a general Killing 2-tensor is $\mathbf{K} = \sum_{i=1}^{d} C_i\mathbf{K}_i$, where $(\mathbf{K}_i)$ is the basis, $C_i \in \mathbb{R}$.

If $d < \dim(M)$, then by theorems due to Kalnins & Miller [11] there exists no separable potential — Stop.

Essentially, this step is a pure problem of differential geometry.

Step 2. The Killing tensor $\mathbf{K}$ obtained above is of covariant $(2,0)$-type. Using metric g, transform it to a $(1,1)$-tensor $\widehat{\mathbf{K}}$ which can be regarded as an endomorphism of the cotangent bundle T^*M.

By abuse of notation we use $\mathbf{K}$ to denote the new tensor below. Note that in matrix form $\mathbf{K}$ is always symmetric, $\widehat{\mathbf{K}}$ is not so in general.

Step 3. Insert the $(1,1)$-tensor into the core equation (2.5). The vanishing of form $\mathrm{d}(\mathbf{K}\mathrm{d}V)$ entails the vanishing of all its coefficients. Thus a system of equations involving variables q_k and constants a_i, C_j follows, which are usually (or can be transformed to) polynomials of q_k.

Simplify this system of equations, eventually we obtain algebraic equations of the parameters a_i, C_j only. Solve the system of algebraic equations. The obtained solutions are candidates of separable cases.

Step 4. Substitute the C_j's back into the general expression of Killing tensor. Calculate its eigenvalues and eigenvectors. Check whether they satisfy the additional condition in Benenti's theorem. These gives the complete set of all separable cases.

Step 5 (Optional). For a separable case, by using the eigenvalues and eigenvectors obtained in step 4, we can figure out which concrete coordinate system permits the separability of the corresponding system. (see e.g. [12])

End.

3. Applications

In this section, we first review the Killing vectors (tensors) of $\mathbb{E}^2$ (see e.g. [13]), then we use them to make a detailed analysis of several systems involved with some constants.

In the situation of $\mathbb{E}^2$ we will write the familiar (x, y) for (q^1, q^2), and (p_x, p_y) for (p_1, p_2), here (x, y) is the usual Cartesian coordinate. The space of Killing vectors has dim $\mathcal{K}^1(\mathbb{E}^2) = 3$ with a basis [13] being

$$\partial_x, \partial_y \ \ \text{(two translations)}, \qquad y\partial_x - x\partial_y \ \ \text{(rotation)} \tag{3.1}$$

where we adopt the notation $\partial_x = \frac{\partial}{\partial x}$, $\partial_y = \frac{\partial}{\partial y}$. The second space $\mathcal{K}^2(\mathbb{E}^2)$ has a basis [13]

$$\begin{aligned} &\mathbf{K}_1 = \partial_x^2, \qquad \mathbf{K}_2 = \partial_y^2, \qquad \mathbf{K}_3 = \partial_x\partial_y + \partial_y\partial_x \ \ (= G)\ , \\ &\mathbf{K}_4 = -2y\,\partial_x^2 + x\,\partial_x\partial_y + x\,\partial_y\partial_x, \quad \mathbf{K}_5 = -2x\,\partial_y^2 + y\,\partial_x\partial_y + y\,\partial_y\partial_x, \\ &\mathbf{K}_6 = y^2\,\partial_x^2 + x^2\,\partial_y^2 - xy\,\partial_x\partial_y - xy\,\partial_y\partial_x. \end{aligned} \tag{3.2}$$

So the expression for a general Killing 2-tensor is

$$\begin{aligned} \mathbf{K} = \sum\nolimits_{i=1}^{6} C_i\mathbf{K}_i = (C_6 y^2 - 2C_4\,y + C_1)\,\partial_x^2 + (C_6 x^2 - 2C_5\,x + C_2)\,\partial_y^2 \, + \\ + (-C_6\,xy + C_4\,x + C_5\,y + C_3)\,(\partial_x\partial_y + \partial_y\partial_x) \end{aligned} \tag{3.3}$$

or, in matrix form

$$(\mathbf{K}^{ij}) = \begin{pmatrix} C_6 y^2 - 2C_4\,y + C_1 & -C_6\,xy + C_4\,x + C_5\,y + C_3 \\ -C_6\,xy + C_4\,x + C_5\,y + C_3 & C_6 x^2 - 2C_5\,x + C_2 \end{pmatrix} \tag{3.4}$$

where C_i are constants.

At last, we mention a special non-separable situation, that is,

$$(\mathbf{NS}) \qquad C_1 = C_2, \quad C_3 = C_4 = C_5 = C_6 = 0.$$

In such a case the matrix $\mathbf{K} = C_1\,I_2$, where I_2 denotes the identity matrix. It is not simple as it admits two coincident eigenvalues. This means the characteristic tensor $\mathbf{K}$ does not exist, hence the system is not separable. Such a special case arises several times during our arguments later.

3.1. System with a General Quartic Potential

We shall use these general results to several specified systems defined on $\mathbb{E}^2$. In this section we consider a system with a quartic polynomial potential, whose Hamiltonian is

$$H = \frac{1}{2}(p_x^2 + p_y^2) + (\lambda\,x^2 + \mu\,y^2) + (c\,x^4 + b\,x^2y^2 + a\,y^4) \tag{3.5}$$

in which a, b, c, λ, μ are constants. Note that the system (3.5) is called Yang-Mills-type system in [16]. In general, the system with arbitrary parameters are non-integrable and display chaotic behaviour.

After applying the celebrated Painlevé analysis or differential Galois theory, several special cases for values of constants are identified, which turn out to be integrable (see e.g. [14, 15, 16]). These cases are given by

(i) $b = 0$, all other parameters are arbitrary,
(ii) $a : b : c = 1 : 2 : 1$, λ and μ arbitrary,
(iii) $a : b : c = 1 : 6 : 1$, $\lambda = \mu$,
(iv) $a : b : c = 1 : 12 : 16$, $\lambda = 4\mu$,
(v) $a : b : c = 1 : 6 : 8$, $\lambda = 4\mu$, (proved to be the only non-separable case below)

Cases (ii)–(v) are well-known integrable Hénon-Heiles systems [17]. For each of these cases there exists a second integral of motion K independent of H [17].

Remark 1. One may note that the four cases given above are not symmetric for λ, μ whereas the Hamiltonian is so. Actually there are not only five integrable cases as above, but more. The additional cases

(iv)′ $a : b : c = 16 : 12 : 1, \quad \lambda = \mu/4,$
(v)′ $a : b : c = 8 : 6 : 1, \quad \lambda = \mu/4,$

are symmetric (thus equivalent) to the cases (iv) and (v), respectively. We can make an assumption $a \leq c$ to eliminate these isomorphisms.

Our main result in this subsection is the following

Theorem 2. *For Hamiltonian system with quartic potential (3.5), there exist exact four cases of values of constants, which guarantee the corresponding H-J equation being additively separable. These cases are exactly the first four cases (i)—(iv) in the list above.*

This shows case (v) is the only integrable, but non-separable case.

Proof. Notice now (2.6) reads $\mathbf{K}\,\mathrm{d}V = \sum_{i,j=1}^{2} \mathbf{K}^{ij}\,\partial_j V\,\mathrm{d}q^i$, whose explicit expressions is messy. After taking exterior differentiation

$$\mathrm{d}(\mathbf{K}\,\mathrm{d}V) = Z\;\mathrm{d}x \wedge \mathrm{d}y,$$

its coefficient is a polynomial of x and y, which after collecting the same entries reads

$$\begin{aligned} Z = &(24aC_6 - 12bC_6)\,y^3x + (12bC_6 - 24cC_6)\,yx^3 + (-12aC_4 + 16bC_4)\,y^2x \\ &+(-16bC_5 + 12cC_5)\,yx^2 + (-2bC_4 + 24cC_4)\,x^3 + (-24aC_5 + 2bC_5)\,y^3 \\ &+(-12aC_3 + 2bC_3)\,y^2 + (-4bC_1 + 4bC_2 + 8\mu C_6 - 8\lambda C_6)\,xy + (-2bC_3 + 12cC_3)\,x^2 \\ &+(-2\mu C_4 + 8\lambda C_4)\,x + (-8\mu C_5 + 2\lambda C_5)\,y - 2\mu C_3 + 2\lambda C_3. \end{aligned} \tag{3.6}$$

The vanishing of two-form $\mathrm{d}(\mathbf{K}\mathrm{d}V)$ means its coefficient Z vanishes. All the parameters $a, b, c, \lambda, \mu, C_i$ in (3.6) are constants. In turn Z vanishes identically entails all its coefficients of x, y vanishes. A system of algebraic equations follows,

$$C_3(\lambda - \mu) = C_4(4\lambda - \mu) = C_5(\lambda - 4\mu) = 0 \tag{3.7a}$$
$$C_3(6a - b) = C_3(b - 6c) = 0 \tag{3.7b}$$
$$C_4(b - 12c) = C_4(3a - 4b) = 0 \tag{3.7c}$$
$$C_5(12a - b) = C_5(4b - 3c) = 0 \tag{3.7d}$$
$$C_6(2a - b) = C_6(b - 2c) = 0 \tag{3.7e}$$
$$(C_1 - C_2)b + 2C_6(\lambda - \mu) = 0 \tag{3.7f}$$

This is the system of algebraic equations we want to solve in detail. First we notice when $b = 0$, the parameters $C_1 - C_2 \neq 0$, $C_i = 0, i = 3, 4, 5, 6$, directly solve the above system. The matrix corresponding to Killing tensor $\mathbf{K}$ has distinct eigenvalues C_1, C_2. Hence this is a separable case, which corresponds to case (i) in our list.

To proceed we will always assume $b \neq 0$ below. It can be seen $b \neq 0$ implies $a, c \neq 0$. Otherwise constants C_i are exactly in the non-separable situation (**NS**). To solve the system (3.7), we observe that (3.7b) and (3.7e) imply

$$C_3(a - c) = C_6(a - c) = 0 \tag{3.8}$$

Based on this observation one can make classification as below:

- $\mathbf{a} \neq \mathbf{c}$. Equations (3.8) imply that $C_3 = C_6 = 0$. Substituting this into (3.7), one has

$$\begin{aligned} C_4(4\lambda - \mu) = C_5(\lambda - 4\mu) = 0 \\ C_4(b - 12c) = C_4(3a - 4b) = 0 \\ C_5(12a - b) = C_5(4b - 3c) = 0 \\ C_1 - C_2 = 0 \end{aligned} \tag{3.9}$$

The second equation implies $C_4(a-16c) = 0$. As $a-16c \neq 0$ (otherwise $c/a = 1/16 < 1$), it holds that $C_4 = 0$. We claim $C_5 \neq 0$, otherwise the constants C_k are in non-separable situation (**NS**). Combing all the results above, we arrive at

$$a : b : c = 1 : 12 : 16, \qquad \lambda = 4\mu,$$

which recovers the case (iv) in the list.

- $\mathbf{a = c}$. Again we have $C_4(a-16c) = 0$, which reduces to $C_4 a = 0$. As $a \neq 0$ hence $C_4 = 0$. Similarly $C_5 = 0$. Substituting $C_4 = C_5 = 0, a = c$ into the system (3.7), it reduces to

$$\begin{aligned} &C_3(\lambda-\mu) = 0 \\ &C_3(6a-b) = C_6(2a-b) = 0 \\ &(C_1-C_2)b + 2C_6(\lambda-\mu) = 0 \end{aligned} \tag{3.10}$$

  We discuss its possible solutions:

  - $a = c,\ \lambda = \mu$. The condition $\lambda = \mu$ reduces the system (3.10) further to

  $$C_3(6a-b) = C_6(2a-b) = 0, \qquad C_1 - C_2 = 0.$$

    One of C_3 and C_6 should be non-zero (otherwise the situation (**NS**) arise again), which implies $a : b : c = 1 : 6 : 1$, or $a : b : c = 1 : 2 : 1$. They corresponds to the case (iii) and (ii), respectively.
  - $a = c,\ \lambda \neq \mu$. The system (3.10) is reduced to

  $$C_3 = C_6(2a-b) = (C_1-C_2)b + 2C_6(\lambda-\mu) = 0$$

    Once more $C_6 \neq 0$ to avoid the situation (**NS**), which gives $a : b : c = 1 : 2 : 1$. It is case (ii) in the list. This completes the proof of Theorem 2.

□

Remark 2. One could use the Killing tensor to figure out the coordinate system in which the H-J equation separates. For example, let us consider the case (ii). According to Step 4 in Algorithm, taking $b = 2a, c = a \neq 0$ back into the original system (3.7) one find the following to be a solution of the system (3.7):

$$C_1 = \frac{\mu-\lambda}{a}, \quad C_2 = C_3 = C_4 = C_5 = 0, \quad C_6 = 1$$

Note that $(\mu-\lambda)$ is in general not zero. The characteristic tensor (3.4) turns out to be

$$\mathbf{K} = \begin{pmatrix} y^2 + \frac{\mu-\lambda}{a} & -xy \\ -xy & x^2 \end{pmatrix}. \tag{3.11}$$

Its characteristic equation is

$$\Lambda^2 - (x^2 + y^2 + \frac{\mu-\lambda}{a})\Lambda + \frac{\mu-\lambda}{a}x^2 = 0 \tag{3.12}$$

or in equivalent form

$$\frac{x^2}{\Lambda} + \frac{ay^2}{a\Lambda - (\mu-\lambda)} = 0. \tag{3.13}$$

For the case of $\lambda \neq \mu$, the above equation (3.13) defines just well-known elliptic-hyperbolic coordinates in the Euclidean plane. The eigenvalues λ^1, λ^2, i.e. the solutions of the equation (3.12) or (3.13), are the variables of separation for the dynamical system. Hence we conclude the system is separable in the elliptic coordinates (λ^1, λ^2), determined by $\frac{\mu-\lambda}{a}$.

For the case of $\lambda = \mu$, the solutions of (3.12) are $\lambda^1 = 0$, $\lambda^2 = x^2 + y^2$, one of which is constant. This implies the system separates in degenerated elliptic coordinates. The Hamiltonian is

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \lambda(x^2 + y^2) + a(x^4 + 2x^2y^2 + y^4)$$

with potential $V = \lambda\, r^2 + a\, r^4$ depending on r only. It is easy to see the system separates in the standard polar coordinates (r, θ).

3.2. System with a Homogeneous Quintic Potential

Next we consider a family of Hamiltonian systems with a quintic polynomial potential,

$$H = \frac{1}{2}(p_x^2 + p_y^2) + ay^5 + b\,y^3x^2 + c\,yx^4, \tag{3.14}$$

where a, b, c are scalar constants. Note they are not the most general quintic potential. Our main result regarding this potential is the following

Theorem 3. *For the Hamiltonian (3.14), there exists exact 3 cases of parameters for the corresponding H-J equation to be additively separable. These cases are:*

(i) $b = c = 0$, *a arbitrary;* *(ii)* $a : b : c = 16 : 16 : 3$; *(iii)* $a : b : c = 1 : 10 : 5$.

Remark 3. Case (i) is trivial in that it corresponds to Hamiltonian $H = (p_x^2 + p_y^2)/2 + ay^5$, which trivially separates in Cartesian coordinates. Case (ii) has already appeared in literature [14] where the authors obtained it by using the (weak) Painlevé method. Case (iii) appeared in Perelomov's book [18, p.81]

Proof of Theorem 3. We apply the algorithm again, now to the potential (3.14). Using the general Killing tensor (3.4), it follows that the 2-form $\mathrm{d}(\mathbf{K}\mathrm{d}V) = Z\mathrm{d}x \wedge \mathrm{d}y$, with coefficient Z given by

$$\begin{aligned} Z &= (-6bC_4 + 32cC_4)yx^3 + (-20aC_4 + 20bC_4)y^3x + (21bC_6 - 28cC_6)y^2x^3 \\ &+ (35aC_6 - 14bC_6)y^4x + (-6bC_1 + 6bC_2)y^2x + (-6bC_3 + 12cC_3)yx^2 \\ &+ (-27bC_5 + 12cC_5)y^2x^2 + (-4cC_1 + 4cC_2)x^3 + (-35aC_5 + 2bC_5)y^4 \\ &+ 7cx^5C_6 + (-20aC_3 + 2bC_3)y^3 - 11cx^4C_5. \end{aligned}$$

The form $\mathrm{d}(\mathbf{K}\mathrm{d}V)$ vanishes entails Z also vanishes. Again, Z is a polynomial of variable x, y, hence all of its coefficients are zero. Thus we arrive at a system of algebraic equations

$$\begin{aligned} c(C_1 - C_2) = b(C_1 - C_2) = 0, \\ C_3(b - 2c) = C_3(10a - b) = C_4(3b - 16c) = 0, \\ C_4(a - b) = C_5(9b - 4c) = C_5(35a - 2b) = 0, \\ C_6(3b - 4c) = C_6(5a - 2b) = 0, \qquad cC_5 = cC_6 = 0, \end{aligned} \tag{3.15}$$

We analyze the solution of this family:

- $c = 0$. Substitute this into the system (3.15) to produce

$$\begin{aligned} b(C_1 - C_2) = 0 \\ C_3(10a - b) = C_4(a - b) = 0, \\ C_5(35a - 2b) = C_6(5a - 2b) = 0, \\ C_3b = C_4b = C_5b = C_6b = 0, \end{aligned} \tag{3.16}$$

  One easily sees that b is also zero. (Otherwise the new system leads to the (**NS**) case). Hence the above system (3.16) further reduces to $C_3a = C_4a = C_5a = C_6a = 0$. For any a, it admits a solution $C_1 - C_2 \neq 0$, $C_3 = C_4 = C_5 = C_6 = 0$. Thus we have a separability corresponding to the case (i). In fact we can obviously see this from the original potential (3.14). In the special case of $b = c = 0$, the Hamiltonian is additively itself, implying it separates in the canonical Cartesian system.

- $c \neq 0$. The system (3.15) can be simplified to be

$$\begin{aligned} C_1 - C_2 = C_5 = C_6 = 0, \\ C_3(b-2c) = C_3(10a-b) = 0, \\ C_4(3b-16c) = C_4(a-b) = 0, \end{aligned}$$

  One can see one of C_3, C_4 is not zero. (Otherwise $C_1 - C_2 = C_3 = C_4 = C_5 = C_6 = 0$ — dissatisfies the basic theorem 1). So there exists two possibilities:

  - $C_3 \neq 0$, which gives $a : b : c = 1 : 10 : 5$ corresponding to case (iii).
  - $C_4 \neq 0$, which gives $a : b : c = 16 : 16 : 3$ corresponding to case (ii). We thus reproduce all the cases in the theorem.

□

3.3. Multi-Separable Potentials on Euclidean and Minkowski Planes

We now apply our Algorithm to identify some multi-separable systems which are defined on (pseudo)-Euclidean spaces. We remind that a Hamiltonian system is *multi-separable* if it separates in several distinct coordinate systems.

Theorem 4. *For the system*

$$H = \frac{1}{2}(p_x^2 + p_y^2) + x^2 + ay^2 + \frac{b}{x^2} \tag{3.17}$$

with potential defined on Euclidean plane $\mathbb{E}^2$, where a, b are two constants, there exists exact three cases of parameters such that H is multi-separable. They are given by

(i) $a = 1/4,\ b = 0$; *(ii) $a = 1,\ b$ arbitrary;* *(iii) $a = 4,\ b$ arbitrary.*

Remark 4. Here we consider the multi-separability, i.e. 2^{nd}-order super-integrability for the system (3.17). For any integer $a = k^2, k \in \mathbb{N}$, it admits an additional first integral which is a k^{th}-order polynomial in momenta [19], implying its (higher order) super-integrability. For such potentials there exist much more super-integrable cases than multi-separable ones.

Note that case (iii) is the celebrated Smorodinsky-Winternitz I potential [20], thus by using our Algorithm we reproduce this system quite straightforwardly.

Proof of Theorem 4. According to the algorithm we apply $\mathbf{K}$ (3.4) to $\mathrm{d}V$ where $V = x^2 + ay^2 + b/x^2$. After exterior derivative one has

$$\mathrm{d}\,(\mathbf{K}\,\mathrm{d}V) = Z\,\mathrm{d}x \wedge \mathrm{d}y \tag{3.18}$$

with the coefficient Z given by

$$\begin{aligned} Z = \frac{2}{x^4} \cdot \big(4(a-1)x^5 y\, C_6 + (1-4a)x^4 y\, C_5 \\ +(4-a)x^5 C_4 + (1-a)x^4 C_3 + 3byC_5 + 3bC_3\big) \end{aligned} \tag{3.19}$$

In Z's expression, C_k, a, b are constants. One notice Z is not polynomial in (x, y), but rational functions. Nevertheless, we can transform it to be a polynomial as below. The form $\mathrm{d}(\mathbf{K}\mathrm{d}V)$ vanishes equivalents to $Z \equiv 0$, which, in turn, equivalents to the vanishing of the polynomial $\tilde{Z} = Z \cdot x^4/2$. So we obtain a system of algebraic equations

$$\begin{aligned} &bC_3 = bC_5 = 0, \\ &C_4(a-4) = C_3(a-1) = 0, \\ &C_6(a-1) = C_5(4a-1) = 0 \end{aligned} \tag{3.20}$$

Since C_1, C_2 do not arise in the equations, all of $C_k = 0$ except that $C_1 - C_2 \neq 0$ solves the system above. This implies the Hamiltonian is separable in the Cartesian coordinates. For the system to be multi-separability, it suffices to find another solution linearly independent of the trivial solution given above.

A new solution to equations (3.20) exists if and only if one of the following cases occurs:

- $C_6 \neq 0 \Rightarrow a = 1,\ b$ arbitrary;
- $C_4 \neq 0 \Rightarrow a = 4,\ b$ arbitrary;
- $C_5 \neq 0 \Rightarrow a = \frac{1}{4},\ b = 0$;
- $C_3 \neq 0 \Rightarrow a = 1, b = 0$.

Observe that the last case is only a subcase of the first case. Thus we obtain exact three multi-separable cases, corresponding to cases (ii), (iii), (i) in the theorem, respectively. □

The next configuration space we consider is a Minkowski case $\mathbb{M}^2$ whose metric is $g = \mathrm{d}x^2 - \mathrm{d}y^2$. We compare the two planes $\mathbb{M}^2$ and $\mathbb{E}^2$. Both are of constat curvature (zero), hence the dimensions of vector spaces of their Killing tensor attain the maxima: $\mathcal{K}^1(\mathbb{M}^2)$, $\mathcal{K}^2(\mathbb{M}^2)$ are of dimension three and six, respectively.

Nevertheless, the basis of Killing tensors (hence the entire spaces) are not identical. The Minkowski $\mathbb{M}^2$ has the basis of Killing vectors (compare with (3.1))

$$\partial_x,\ \partial_y \quad \text{(two translations)}, \qquad y\partial_x + x\partial_y \quad \text{(Minkowski “rotation”)} \tag{3.21}$$

The basis of Killing 2-tensors are the following (compare with (3.2))

$$\begin{aligned}
&\mathbf{K}_1 = \partial_x^2, \qquad \mathbf{K}_2 = \partial_y^2, \qquad \mathbf{K}_3 = \partial_x\partial_y + \partial_y\partial_x,\\
&\mathbf{K}_4 = 2y\,\partial_x^2 + x\,\partial_x\partial_y + x\,\partial_y\partial_x, \quad \mathbf{K}_5 = 2x\,\partial_y^2 + y\,\partial_x\partial_y + y\,\partial_y\partial_x,\\
&\mathbf{K}_6 = y^2\partial_x^2 + xy\,\partial_x\partial_y + xy\,\partial_y\partial_x + x^2\partial_y^2,
\end{aligned} \tag{3.22}$$

Carrying out an analysis similar to that for the Euclidean plane (Theorem 4), we arrive at

Theorem 5. *For the system*

$$H = \frac{1}{2}(p_x^2 - p_y^2) + x^2 + ay^2 + \frac{b}{x^2} \tag{3.23}$$

with potential defined on Minkowski $\mathbb{M}^2$, a, b are constants, there exists exact three cases such that H is multi-separable. They are given by

(i) $a = -1/4,\ b = 0$; *(ii) $a = -1,\ b$ arbitrary;* *(iii) $a = -4,\ b$ arbitrary.*

4. Concluding Discussions

Based on Benenti's classical theorem 1, we have suggested an Algorithm and applied it to detect H-J separability of several families of two-dimensional natural systems. This method has the advantage of having a clear procedure and not depending on intricate techniques which can be seen in lots of literatures, thus executable in a computer-like environment.

However, the applications we make here are merely preliminary. There are several directions one can take into account to improve and extend its scope. For example, one may consider some nontrivial (pseudo)-Riemannian spaces such as spaces of constant curvature S^n, H^n etc., or surfaces of revolution. These manifolds are easy to handle as their Killing tensor are much investigated. The crucial task in step 1 in our Algorithm is thus solved.

Another line is to generalize the potentials under discussion to more general ones, which may be involved some arbitrary functions. This can greatly enlarge the families of separable systems. Proceeding the analysis as above may yield some well-known or novel models. It is natural that the calculations are much more complicated, with the aid of computer symbolic system sometimes being a necessity.

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Cross-entropy for generalized hesitant fuzzy sets and their use in multi-criteria decision making

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Abstract

In this paper, the cross-entropy for generalized hesitant fuzzy sets (GHFSs) is developed by integrating the cross-entropy for intuitionistic fuzzy sets (IFSs) and hesitant fuzzy sets (HFSs). First, several measurement formulas are discussed and their properties are studied. Then, two approaches, which are based on the developed generalized hesitant fuzzy cross-entropy, are proposed for solving multi-criteria decision making (MCDM) problems under an generalized hesitant fuzzy environment. Finally, an example is provided to illustrate the practicality and effectiveness of the developed approaches.

1 Introduction

The cross-entropy measures are mainly used to measure the discrimination information, and then it is an important measure in decision making, pattern recognition and other real-world problems. Lots of studies on this issue have been extended and developed to fuzzy and its extended environments. For instance, Vlachos and Sergiadis [14] introduced the concepts of discrimination information and cross-entropy for intuitionistic fuzzy sets (IFSs), and revealed the connection between the notions of entropies for fuzzy sets and IFSs in terms of fuzziness and intuitionism. Hung and Yang [6] constructed J-divergence of IFSs and introduced some useful distance and similarity measures between two IFSs, and applied them to clustering analysis and pattern recognition. Based

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on which, Xia and Xu [17] proposed some cross-entropy and entropy formulas for IFSs and applied them to group decision making. Ye [21] proposed a method of fault diagnosis based on the vague cross-entropy. He [22] also introduced the cross-entropy for IFSs and interval-valued intuitionistic fuzzy sets (IVIFSs) and utilized then to solve multi-criteria decision making (MCDM) problems. Wang and Li [15] provided two improved methods for solving MCDM problems, which were based on the cross-entropy for IFSs. Hung et al. [5] introduced the discrimination information and cross-entropy for IFSs and also used them to improve the fault diagnosis of turbine problems. Mao et al. [9] introduced the cross-entropy and entropy measures for IFSs. Zang and Yu [28] constructed a series of mathematical programming models, which were based on an interval-valued intuitionistic fuzzy cross-entropy, in order to determine the criteria weights and applied them to MCDM problems. Xia and Xu [17] proposed two methods for determining the optimal weights of criteria and developed two pairs of entropy and cross-entropy measures for intuitionistic fuzzy values. The relationships among the entropy, cross-entropy and similarity measures have also attracted many attentions. For example, Liu [8] gave the axiomatic definitions of entropy, distance measure, and similarity measure of fuzzy sets and discussed their basic relations. Zeng and Li [25] discussed the relationship between the similarity measure and the entropy of interval-valued fuzzy sets. Zang and Jiang [27] proposed the entropy and cross-entropy for IVIFSs and discussed the connections among some important information measures. Xu and Xia [19] introduced the concepts of entropy and cross-entropy for hesitant fuzzy sets (HFSs), analyzed the relationships among the entropy, cross-entropy and similarity measures, and developed two multi-attribute decision making methods.

Qjan et al. [10], recently, introduced the concept of generalized hesitant fuzzy sets (GHFSs), extending the element of HFSs from real numbers to intuitionistic fuzzy values, which can arise in group decision making problem. GHFS is fit for the situation when decision maker have a hesitation among several possible memberships with uncertainties. GHFS can reflect the human's hesitance more objectively than other extensions of fuzzy set (IFS, IVIFS and HFS), and thus it is necessary to develop some theories about GHFSs. In this paper, we discuss the cross-entropy for generalized hesitant information. To do this, Section 2 reviews some related preliminaries such as IFSs, HFSs and GHFSs. In Section 3, we propose some cross-entropy formulas for generalized hesitant fuzzy elements, obtain some important conclusions, and provide an example to illustrate the application of cross-entropy in MCDM problem. Finally, Section 4 gives the concluding remarks.

2 Basic concepts

Intuitionistic fuzzy sets introduced by Atanassov [1] have been proven to be highly useful to deal with uncertainty and vagueness.

Definition 1. [1] Let X be ordinary non-empty set. An intuitionistic fuzzy set (IFS) A in X is defined as

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}, \tag{1}$$

where $\mu_A, \nu_A : X \to [0,1]$ denote, respectively, the membership and non-membership functions of A with the condition: $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$.

For an IFS A, $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ represents the degree of hesitation or intuitionistic index of x to A. For a fuzzy set, the degree of hesitation $\pi_A(x) = 0$. Thus for each x, $\mu_A(x)$ and $\nu_A(x)$ define an interval $[\mu_A(x), 1 - \nu_A(x)]$. This interval is the vague value of value set by Gau and Buethrer [4] (Bustince and Burillo [3] proved that vague sets are equivalent to IFSs). Further, the interval can also represent an interval-valued fuzzy set [10]. Hence Xu [18] concluded that IFSs are also equivalent to interval-valued fuzzy sets, and replaced Eq. (1) with

$$A = \{\langle x, [\mu_A(x), 1 - \nu_A(x)]\rangle | x \in X\}. \tag{2}$$

The ordered pair $\alpha(x) = (\mu_\alpha(x), \nu_\alpha(x))$ is referred to an intuitionistic fuzzy value (IFV) [18], where $\mu_\alpha(x), \nu_\alpha(x) \in [0,1]$ and $\mu_\alpha(x) + \nu_\alpha(x) \leq 1$. Associated with the degree of hesitation, an IFV can also be equivalently denoted by $\alpha(x) = (\mu_\alpha(x), \nu_\alpha(x), \pi_\alpha(x))$, where $\mu_\alpha(x), \nu_\alpha(x), \pi_\alpha(x) \in [0,1]$ and $\mu_\alpha(x) + \nu_\alpha(x) + \pi_\alpha(x) = 1$. In the rest of this paper, for a certain x in X, IFV $a = (\mu, \nu, \pi)$ is abbreviated as $a = (\mu, \nu)$ when no misunderstanding raises. Since an IFV represent an interval, an interval $[\mu, 1 - \nu]$ in $[0,1]$ will be directly transformed into (μ, ν).

Definition 2. [5, 6, 14, 17, 22] Let $\alpha_1 = (\mu_{\alpha_1}, \nu_{\alpha_1})$ and $\alpha_2 = (\mu_{\alpha_2}, \nu_{\alpha_2})$ be IFVs, then the cross-entropy α_1 and α_2, denoted as $\mathrm{CE}(\alpha_1, \alpha_2)$, should satisfy the following properties:

(1) $\mathrm{CE}(\alpha_1, \alpha_2) \geq 0$;

(2) $\mathrm{CE}(\alpha_1, \alpha_2) = 0$ if $\alpha_1 = \alpha_2$;

(3) $\mathrm{CE}(\alpha_1^c, \alpha_2^c) = \mathrm{CE}(\alpha_1, \alpha_2)$, where $\alpha_i^c = (\nu_{\alpha_i}, \mu_{\alpha_i})$ is the complement of α_i $(i = 1, 2)$.

In the following, some intuitionistic fuzzy cross-entropy and symmetric intuitionistic fuzzy cross-entropy formulas are reviewed.

Vlachos and Sergiadis [14] developed

$$\mathrm{CE}_1(\alpha_1, \alpha_2) = \mu_{\alpha_1} \ln \frac{2\mu_{\alpha_1}}{\mu_{\alpha_1} + \mu_{\alpha_2}} + \nu_{\alpha_1} \ln \frac{2\nu_{\alpha_1}}{\nu_{\alpha_1} + \nu_{\alpha_2}}, \tag{3}$$

and

$$\begin{aligned}\mathrm{CE}_2(\alpha_1, \alpha_2) = 2 \Bigg(&\frac{\mu_{\alpha_1} \ln \mu_{\alpha_1} + \mu_{\alpha_2} \ln \mu_{\alpha_2}}{2} - \frac{\mu_{\alpha_1} + \mu_{\alpha_2}}{2} \ln \frac{\mu_{\alpha_1} + \mu_{\alpha_2}}{2} \\ &+ \frac{\nu_{\alpha_1} \ln \nu_{\alpha_1} + \nu_{\alpha_2} \ln \nu_{\alpha_2}}{2} - \frac{\nu_{\alpha_1} + \nu_{\alpha_2}}{2} \ln \frac{\nu_{\alpha_1} + \nu_{\alpha_2}}{2} \Bigg). \end{aligned} \tag{4}$$

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Hung and Yang [6] defined

$$\begin{aligned}\mathrm{CE}_3(\alpha_1,\alpha_2) = 2\Bigg(&\frac{\mu_{\alpha_1}\ln\mu_{\alpha_1}+\mu_{\alpha_2}\ln\mu_{\alpha_2}}{2} - \frac{\mu_{\alpha_1}+\mu_{\alpha_2}}{2}\ln\frac{\mu_{\alpha_1}+\mu_{\alpha_2}}{2} \\ &+\frac{\nu_{\alpha_1}\ln\nu_{\alpha_1}+\nu_{\alpha_2}\ln\nu_{\alpha_2}}{2} - \frac{\nu_{\alpha_1}+\nu_{\alpha_2}}{2}\ln\frac{\nu_{\alpha_1}+\nu_{\alpha_2}}{2} \\ &+\frac{\pi_{\alpha_1}\ln\nu_{\alpha_1}+\nu_{\alpha_2}\ln\nu_{\alpha_2}}{2} - \frac{\nu_{\alpha_1}+\nu_{\alpha_2}}{2}\ln\frac{\nu_{\alpha_1}+\nu_{\alpha_2}}{2}\Bigg). \end{aligned} \tag{5}$$

Ye [22] proposed

$$\begin{aligned}\mathrm{CE}_4(\alpha_1,\alpha_2) = &\frac{\mu_{\alpha_1}+1-\nu_{\alpha_1}}{2}\log_2\frac{2(\mu_{\alpha_1}+1-\nu_{\alpha_1})}{2+\mu_{\alpha_1}-\nu_{\alpha_1}+\mu_{\alpha_2}-\nu_{\alpha_2}} \\ &+\frac{\nu_{\alpha_1}+1-\mu_{\alpha_1}}{2}\log_2\frac{2(\nu_{\alpha_1}+1-\mu_{\alpha_1})}{2-\mu_{\alpha_1}+\nu_{\alpha_1}-\mu_{\alpha_2}+\nu_{\alpha_2}}. \end{aligned} \tag{6}$$

Hung et al. [5] developed

$$\begin{aligned}\mathrm{CE}_5(\alpha_1,\alpha_2) = &\mu_{\alpha_1}\log_2\frac{2\mu_{\alpha_1}}{\mu_{\alpha_1}+\mu_{\alpha_2}} + \nu_{\alpha_1}\log_2\frac{2\nu_{\alpha_1}}{\nu_{\alpha_1}+\nu_{\alpha_2}} \\ &+\pi_{\alpha_1}\log_2\frac{2\pi_{\alpha_1}}{\pi_{\alpha_1}+\pi_{\alpha_2}}. \end{aligned} \tag{7}$$

Xia and Xu [17] proposed

$$\begin{aligned}\mathrm{CE}_6(\alpha_1,\alpha_2) = &\frac{1}{1-2^{1-q}}\left(\frac{\mu_{\alpha_1}^q+\mu_{\alpha_2}^q}{2} - \left(\frac{\mu_{\alpha_1}+\mu_{\alpha_2}}{2}\right)^q + \frac{\nu_{\alpha_1}^q+\nu_{\alpha_2}^q}{2}\right. \\ &\left. - \left(\frac{\nu_{\alpha_1}+\nu_{\alpha_2}}{2}\right)^q + \frac{\pi_{\alpha_1}^q+\pi_{\alpha_2}^q}{2} - \left(\frac{\pi_{\alpha_1}+\pi_{\alpha_2}}{2}\right)^q\right), \end{aligned} \tag{8}$$

where $1 < q \leq 2$.

For the symmetric property, it is necessary to modify Eqs. (3)-(8) to obtain a symmetric discrimination information measures for IFVs ([11, 26]):

$$\mathrm{CE}_k^*(\alpha_1,\alpha_2) = \mathrm{CE}_k(\alpha_1,\alpha_2) + \mathrm{CE}_k(\alpha_2,\alpha_1),\ k = 1,2,\ldots,6. \tag{9}$$

The hesitant fuzzy set [12, 13], as a generalization of fuzzy set, permits the membership degree of an element to a set presented as several possible values between 0 and 1, which can better describe the situations where people have hesitancy in providing their preferences over objects in process of decision making.

Definition 3. [12, 13] Given a fixed set X, a hesitant fuzzy set (HFS) on X in terms of function h is that when applied to X returns a subset of $[0,1]$, which can be represented as the following mathematical symbol:

$$E = \{\langle x, h(x)\rangle | x \in X\}, \tag{10}$$

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where $h(x)$ is a set of the some values in $[0, 1]$, denoting the possible membership degrees of the element $x \in X$ to the set E. For convenience, Xia and Xu [16] called $h(x)$ a hesitant fuzzy element (HFE) and the set of all HFEs is denoted by HFES.

Definition 4. [19] Let h_1 and h_2 be two HFEs, then the cross-entropy of h_1 and h_2, denoted as $\mathrm{CE}(h_1, h_2)$, should satisfy the following properties:
(1) $\mathrm{CE}(h_1, h_2) \geq 0$;
(2) $\mathrm{CE}(h_1, h_2) = 0$ if and only if $h_1^{\sigma(i)} = h_2^{\sigma(i)}$ for all $i = 1, 2, \ldots, l$.

Based on Definition 4, $l = l(h_1) = l(h_2)$ and denote the number of elements in h_1 and h_2. The elements are arranged in increasing order in h_1 and h_2, respectively, and $h_1^{\sigma(i)}$ $(i = 1, 2, \ldots, l(h_1))$ and $h_2^{\sigma(i)}$ $(i = 1, 2, \ldots, l(h_2))$ are the ith smallest values in h_1 and h_2, respectively. Xu and Xia [19] constructed several cross-entropy for HFEs:

$$\begin{aligned}
&\mathrm{CE}_1(h_1, h_2)\\
&\quad= \frac{1}{lT} \sum_{i=1}^{l} \left(\frac{(1 + qh_1^{\sigma(i)}) \ln(1 + qh_1^{\sigma(i)}) + (1 + qh_2^{\sigma(i)}) \ln(1 + qh_2^{\sigma(i)})}{2} \right.\\
&\quad - \frac{2 + qh_1^{\sigma(i)} + qh_2^{\sigma(i)}}{2} \ln \frac{2 + qh_1^{\sigma(i)} + qh_2^{\sigma(i)}}{2} + \frac{(1 + q(1 - h_1^{\sigma(l-i+1)}))}{2}\\
&\quad \times \ln(1 + q(1 - h_1^{\sigma(l-i+1)})) + \frac{(1 + q(1 - h_2^{\sigma(l-i+1)})) \ln(1 + q(1 - h_2^{\sigma(l-i+1)}))}{2}\\
&\quad - \frac{2 + q(1 - h_1^{\sigma(l-i+1)} + 1 - h_2^{\sigma(l-i+1)})}{2}\\
&\quad \left. \times \ln \frac{2 + q(1 - h_1^{\sigma(l-i+1)} + 1 - h_2^{\sigma(l-i+1)})}{2} \right),
\end{aligned} \tag{11}$$

where $T = (1 + q) \ln(1 + q) - (2 + q)(\ln(2 + q) - \ln 2)$ and $q > 0$.

$$\begin{aligned}
&\mathrm{CE}_2(h_1, h_2)\\
&\quad= \frac{1}{(1 - 2^{1-p})l} \sum_{i=1}^{l} \left(\frac{(h_1^{\sigma(i)})^p + (h_2^{\sigma(i)})^p}{2} + \frac{(1 - h_1^{\sigma(l-i+1)})^p + (1 - h_2^{\sigma(l-i+1)})^p}{2} \right.\\
&\quad \left. - \left(\frac{h_1^{\sigma(i)} + h_2^{\sigma(i)}}{2} \right)^p + \left(\frac{1 - h_1^{\sigma(l-i+1)} + 1 - h_2^{\sigma(l-i+1)}}{2} \right)^p \right), \quad p > 1.
\end{aligned} \tag{12}$$

For the symmetric property, it is necessary to modify Eqs. (11) and (12) to obtain a symmetric discrimination information measure for HFEs:

$$\mathrm{CE}_k^*(h_1, h_2) = \mathrm{CE}_k(h_1, h_2) + \mathrm{CE}_k(h_2, h_1), \ k = 1, 2. \tag{13}$$

Note that Eqs. (11) and (12) are all defined under the assumption that two HFEs are of the same length. If the corresponding HFEs are not equal in length,

then the shorter one should be extended to be the same size as the longer one by adding the same value repeatedly.

3 Generalized hesitant fuzzy sets and their cross-entropy measures

3.1 Generalized hesitant fuzzy sets

During the evaluating process, several possible memberships of an alternative satisfying a certain criterion may be not only crisp values but also interval values in $[0,1]$. In order to handle this kind of assessment in decision making, Qjan et al. [10] extended HFSs by using IFSs to modify Definition 3.

Definition 5. [10] Given a set of N membership functions:

$$M = \{\alpha_i = (\mu_{\alpha_i}, \nu_{\alpha_i}) | 0 \le \mu_{\alpha_i}, \nu_{\alpha_i} \le 1,\ \mu_{\alpha_i} + \nu_{\alpha_i} \le 1,\ i = 1, 2, \ldots, N\} \quad (14)$$

the generalized hesitant fuzzy set (GHFS) associated with M, that is $\tilde{h}_M$, is defined as follows:

$$\tilde{h}_M(x) = \cup_{(\mu_{\alpha_i}, \nu_{\alpha_i}) \in M} \{(\mu_{\alpha_i}(x), \nu_{\alpha_i}(x))\}. \quad (15)$$

Note that HFSs, IFSs and fuzzy sets are special cases of GHFSs redefined here. In fact, if $\mu_{\alpha_i} + \nu_{\alpha_i} = 1$, for $i = 1, 2, \ldots, N$, then GHFSs reduce to HFSs. If $N = 1$ or union of N IFSs, i.e. $\cup_{i=1}^{N} \alpha_i$, in Eq. (14) is convex set in $[0,1]$, then GHFSs reduce to IFSs. If $N = 1$ and $\mu_{\alpha_N} + \nu_{\alpha_N} = 1$, then GHFSs reduce to FSs. Thus GHFSs are not only the generalization of HFSs but also the generalized representation of fuzzy sets, IFSs and HFSs. For the sake of convenience, given a certain $x \in X$, α represents an IFS in $\tilde{h}(x)$. Notice that α is represented an interval as well. Similar to [16], $\tilde{h}_M(x)$, abbreviated as $\tilde{h}(x)$, is called a generalized hesitant fuzzy element (GHFE) and the set of all GHFEs is denoted by GHFES.

Let $l(\tilde{h})$ be the number of elements of a GHFE $\tilde{h}$. In most cases of two GHFEs $\tilde{h}_1$ and $\tilde{h}_2$, the numbers of elements of $\tilde{h}_1$ and $\tilde{h}_2$ may be different, i.e. $l(\tilde{h}_1) \neq l(\tilde{h}_2)$, and for convenience, let $l = \max\{l(\tilde{h}_1), l(\tilde{h}_2)\}$. To operate correctly, we should extend the shorter ones, until both of them have the same length when we compare them. To extend the shorter one, the best way is to add the same values several times in it. In fact, we can extend the shorter one by adding any values in it. The selection of this value mainly depends on the decision makers' risk preferences. Optimists anticipate desirable outcomes and may add the maximum value, while pessimists expect unfavorable outcomes and may add the minimum value. In this paper, we assume the GHFEs $\tilde{h}_1$ and $\tilde{h}_2$ should have the same length l when we compare them.

Some useful operations on GHFEs are as follows:

Definition 6. [10] Let $\tilde{h}$, $\tilde{h}_1$ and $\tilde{h}_2$ be three GHFEs and $\lambda > 0$, then

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(1) $\tilde{h}_1 \cup \tilde{h}_2 = \cup_{\alpha_1 \in \tilde{h}_1, \alpha_2 \in \tilde{h}_2} \{\alpha_1 \cup \alpha_2\} = \cup_{\alpha_1 \in \tilde{h}_1, \alpha_2 \in \tilde{h}_2} \{(\max\{\mu_{\alpha_1}, \mu_{\alpha_2}\}, \min\{\nu_{\alpha_1}, \nu_{\alpha_2}\})\}$;
(2) $\tilde{h}_1 \cap \tilde{h}_2 = \cup_{\alpha_1 \in \tilde{h}_1, \alpha_2 \in \tilde{h}_2} \{\alpha_1 \cap \alpha_2\} = \cup_{\alpha_1 \in \tilde{h}_1, \alpha_2 \in \tilde{h}_2} \{(\min\{\mu_{\alpha_1}, \mu_{\alpha_2}\}, \max\{\nu_{\alpha_1}, \nu_{\alpha_2}\})\}$;
(3) $\tilde{h}^c = \cup_{\alpha \in \tilde{h}} \{\alpha^c\} = \cup_{\alpha \in \tilde{h}} \{(\nu_\alpha, \mu_\alpha)\}$;
(4) $\tilde{h}_1 \oplus \tilde{h}_2 = \cup_{\alpha_1 \in \tilde{h}_1, \alpha_2 \in \tilde{h}_2} \{\alpha_1 \oplus \alpha_2\} = \cup_{\alpha_1 \in \tilde{h}_1, \alpha_2 \in \tilde{h}_2} \{(\mu_{\alpha_1} + \mu_{\alpha_2} - \mu_{\alpha_1}\mu_{\alpha_2}, \nu_{\alpha_1}\nu_{\alpha_2})\}$;
(5) $\tilde{h}_1 \otimes \tilde{h}_2 = \cup_{\alpha_1 \in \tilde{h}_1, \alpha_2 \in \tilde{h}_2} \{\alpha_1 \otimes \alpha_2\} = \cup_{\alpha_1 \in \tilde{h}_1, \alpha_2 \in \tilde{h}_2} \{(\mu_{\alpha_1}\mu_{\alpha_2}, \nu_{\alpha_1} + \nu_{\alpha_2} - \nu_{\alpha_1}\nu_{\alpha_2})\}$;
(6) $\lambda \tilde{h} = \cup_{\alpha \in \tilde{h}} \{\lambda \alpha\} = \cup_{\alpha \in \tilde{h}} \{(1 - (1 - \mu_\alpha)^\lambda, \nu_\alpha^\lambda)\}$;
(7) $\tilde{h}^\lambda = \cup_{\alpha \in \tilde{h}} \{\alpha^\lambda\} = \cup_{\alpha \in \tilde{h}} \{(\mu_\alpha^\lambda, 1 - (1 - \nu_\alpha)^\lambda)\}$.

Definition 7. Let $\tilde{h}_i$ $(i = 1, 2, \ldots, n)$ be a collection of GHFEs, and let GHFWA : GHFES$^n \to$ GHFES, if

$$\text{GHFWA}_w(\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n) = w_1\tilde{h}_1 \oplus w_2\tilde{h}_2 \oplus \cdots \oplus w_n\tilde{h}_n, \tag{16}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $\tilde{h}_i$ $(i = 1, 2, \ldots, n)$ with $w_i \geq 0$ and $\sum_{i=1}^n w_i = 1$, then GHFWA is called the generalized hesitant fuzzy weighted averaging (GHFWA) operator.

Based on operations (4)-(7) of GHFEs described in Definition 6, we can derive the following result.

Theorem 1. Let $\tilde{h}_i = \cup_{\alpha_i \in \tilde{h}_i} \{\alpha_i\}$ $(i = 1, 2, \ldots, n)$ be a collection of GHFEs, and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $\tilde{h}_i$ $(i = 1, 2, \ldots, n)$ with $w_i \geq 0$ and $\sum_{i=1}^n w_i = 1$. Then the aggregated value, by using the GHFWA operator, is also a GHFE, and

$$\begin{aligned} &\text{GHFWA}_w(\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n) \\ &= \bigcup_{\alpha_1 \in \tilde{h}_1, \alpha_2 \in \tilde{h}_2, \ldots, \alpha_n \in \tilde{h}_n} \left\{ \left(1 - \prod_{i=1}^n (1 - \mu_{\alpha_i})^{w_i}, \prod_{i=1}^n \nu_{\alpha_i}^{w_i} \right) \right\}. \end{aligned} \tag{17}$$

Theorem 1 can be proved by using the mathematical induction and then the process is omitted here.

Definition 8. Let $\tilde{h}_i$ $(i = 1, 2, \ldots, n)$ be a collection of GHFEs, and let GHFWG : GHFES$^n \to$ GHFES, if

$$\text{GHFWG}_w(\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n) = \tilde{h}_1^{w_1} \otimes \tilde{h}_2^{w_2} \otimes \cdots \otimes \tilde{h}_n^{w_n}, \tag{18}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $\tilde{h}_i$ $(i = 1, 2, \ldots, n)$ with $w_i \geq 0$ and $\sum_{i=1}^n w_i = 1$, then GHFWG is called the generalized hesitant fuzzy weighted geometric (GHFWG) operator.

Theorem 2. Let $\tilde{h}_i = \cup_{\alpha_i \in \tilde{h}_i} \{\alpha_i\}$ $(i = 1, 2, \ldots, n)$ be a collection of GHFEs, and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $\tilde{h}_i$ $(i = 1, 2, \ldots, n)$ with $w_i \geq 0$ and $\sum_{i=1}^n w_i = 1$. Then the aggregated value, by using the GHFWG

operator, is also a GHFE, and

$$\begin{aligned}&\mathrm{GHFWG}_w(\tilde{h}_1,\tilde{h}_2,\ldots,\tilde{h}_n)\\&=\bigcup_{\alpha_1\in\tilde{h}_1,\alpha_2\in\tilde{h}_2,\ldots,\alpha_n\in\tilde{h}_n}\left\{\left(\prod_{i=1}^{n}\mu_{\alpha_i}^{w_i},1-\prod_{i=1}^{n}(1-\nu_{\alpha_i})^{w_i}\right)\right\}.\end{aligned}\tag{19}$$

Theorem 2 can be also proved by using the mathematical induction and then the process is omitted here.

3.2 Cross-entropy measures of GHFEs

Definition 9. Let $\tilde{h}_1,\tilde{h}_2\in$ GHFES and CE : GHFES $\times$ GHFES $\to R$, then the cross-entropy of $\tilde{h}_1$ and $\tilde{h}_2$, denoted as $\mathrm{CE}(\tilde{h}_1,\tilde{h}_2)$, should satisfy the following properties:

(1) $\mathrm{CE}(\tilde{h}_1,\tilde{h}_2)\geq 0$;

(2) If $\tilde{h}_1=\tilde{h}_2$, then $\mathrm{CE}(\tilde{h}_1,\tilde{h}_2)=0$;

(3) $\mathrm{CE}(\tilde{h}_1^c,\tilde{h}_2^c)=\mathrm{CE}(\tilde{h}_1,\tilde{h}_2)$, where $\tilde{h}_i^c$ is the complement of $\tilde{h}_i$ defined in Definition 6.

On the basis of Definition 9, we can construct several cross-entropy for GHFEs:

$$\begin{aligned}\mathrm{CE}_1(\tilde{h}_1,\tilde{h}_2)=&\frac{1}{l(\tilde{h}_1)}\sum_{\alpha_1\in\tilde{h}_1}\left(\frac{1}{l(\tilde{h}_2)}\sum_{\alpha_2\in\tilde{h}_2}\left(\mu_{\alpha_1}\log_2\frac{2\mu_{\alpha_1}}{\mu_{\alpha_1}+\mu_{\alpha_2}}\right)\right)\\&+\frac{1}{l(\tilde{h}_1)}\sum_{\alpha_1\in\tilde{h}_1}\left(\frac{1}{l(\tilde{h}_2)}\sum_{\alpha_2\in\tilde{h}_2}\left(\nu_{\alpha_1}\log_2\frac{2\nu_{\alpha_1}}{\nu_{\alpha_1}+\nu_{\alpha_2}}\right)\right);\end{aligned}\tag{20}$$

$$\begin{aligned}\mathrm{CE}_2(\tilde{h}_1,\tilde{h}_2)=&\sqrt[p]{\frac{1}{l(\tilde{h}_1)}\sum_{\alpha_1\in\tilde{h}_1}\left(\frac{1}{l(\tilde{h}_2)}\sum_{\alpha_2\in\tilde{h}_2}\left(\mu_{\alpha_1}\log_2\frac{2\mu_{\alpha_1}}{\mu_{\alpha_1}+\mu_{\alpha_2}}\right)\right)^p}\\&+\sqrt[p]{\frac{1}{l(\tilde{h}_1)}\sum_{\alpha_1\in\tilde{h}_1}\left(\frac{1}{l(\tilde{h}_2)}\sum_{\alpha_2\in\tilde{h}_2}\left(\nu_{\alpha_1}\log_2\frac{2\nu_{\alpha_1}}{\nu_{\alpha_1}+\nu_{\alpha_2}}\right)\right)^p},\end{aligned}\tag{21}$$

where $p\geq 1$;

$$\begin{aligned}&\mathrm{CE}_3(\tilde{h}_1,\tilde{h}_2)\\&=\frac{1}{l(\tilde{h}_1)}\sum_{\alpha_1\in\tilde{h}_1}\left(\frac{1}{l(\tilde{h}_2)}\sum_{\alpha_2\in\tilde{h}_2}\left(\frac{\mu_{\alpha_1}+1-\nu_{\alpha_1}}{2}\log_2\frac{2(\mu_{\alpha_1}+1-\nu_{\alpha_1})}{2+\mu_{\alpha_1}-\nu_{\alpha_1}+\mu_{\alpha_2}-\nu_{\alpha_2}}\right)\right)\\&+\frac{1}{l(\tilde{h}_1)}\sum_{\alpha_1\in\tilde{h}_1}\left(\frac{1}{l(\tilde{h}_2)}\sum_{\alpha_2\in\tilde{h}_2}\left(\frac{1-\mu_{\alpha_1}+\nu_{\alpha_1}}{2}\log_2\frac{2(1-\mu_{\alpha_1}+\nu_{\alpha_1})}{2-\mu_{\alpha_1}+\nu_{\alpha_1}-\mu_{\alpha_2}+\nu_{\alpha_2}}\right)\right);\end{aligned}\tag{22}$$

$$
\begin{aligned}
&\mathrm{CE}_4(\tilde{h}_1,\tilde{h}_2)\\
&=\sqrt[p]{\frac{1}{l(\tilde{h}_1)}\sum_{\alpha_1\in\tilde{h}_1}\left(\frac{1}{l(\tilde{h}_2)}\sum_{\alpha_2\in\tilde{h}_2}\left(\frac{\mu_{\alpha_1}+1-\nu_{\alpha_1}}{2}\log_2\frac{2(\mu_{\alpha_1}+1-\nu_{\alpha_1})}{2+\mu_{\alpha_1}-\nu_{\alpha_1}+\mu_{\alpha_2}-\nu_{\alpha_2}}\right)\right)^p}\\
&+\sqrt[p]{\frac{1}{l(\tilde{h}_1)}\sum_{\alpha_1\in\tilde{h}_1}\left(\frac{1}{l(\tilde{h}_2)}\sum_{\alpha_2\in\tilde{h}_2}\left(\frac{1-\mu_{\alpha_1}+\nu_{\alpha_1}}{2}\log_2\frac{2(1-\mu_{\alpha_1}+\nu_{\alpha_1})}{2-\mu_{\alpha_1}+\nu_{\alpha_1}-\mu_{\alpha_2}+\nu_{\alpha_2}}\right)\right)^p}, \quad (23)
\end{aligned}
$$

where $p \geq 1$;

$$
\begin{aligned}
&\mathrm{CE}_5(\tilde{h}_1,\tilde{h}_2)\\
&=\frac{1}{1-2^{1-q}}\left(\frac{1}{l(\tilde{h}_1)}\sum_{\alpha_1\in\tilde{h}_1}\left(\frac{1}{l(\tilde{h}_2)}\sum_{\alpha_2\in\tilde{h}_2}\left(\frac{\mu_{\alpha_1}^q+\mu_{\alpha_2}^q}{2}-\left(\frac{\mu_{\alpha_1}+\mu_{\alpha_2}}{2}\right)^q\right)\right)\right.\\
&\left.+\frac{1}{l(\tilde{h}_1)}\sum_{\alpha_1\in\tilde{h}_1}\left(\frac{1}{l(\tilde{h}_2)}\sum_{\alpha_2\in\tilde{h}_2}\left(\frac{\nu_{\alpha_1}^q+\nu_{\alpha_2}^q}{2}-\left(\frac{\nu_{\alpha_1}+\nu_{\alpha_2}}{2}\right)^q\right)\right)\right), \qquad (24)
\end{aligned}
$$

where $1 < q \leq 2$;

$$
\begin{aligned}
&\mathrm{CE}_6(\tilde{h}_1,\tilde{h}_2)\\
&=\sqrt[p]{\frac{1}{l(\tilde{h}_1)}\sum_{\alpha_1\in\tilde{h}_i}\frac{1}{1-2^{1-q}}\left(\frac{1}{l(\tilde{h}_2)}\sum_{\alpha_2\in\tilde{h}_2}\left(\frac{\mu_{\alpha_1}^q+\mu_{\alpha_2}^q}{2}-\left(\frac{\mu_{\alpha_1}+\mu_{\alpha_2}}{2}\right)^q\right)\right)^p}\\
&+\sqrt[p]{\frac{1}{l(\tilde{h}_1)}\sum_{\alpha_1\in\tilde{h}_i}\frac{1}{1-2^{1-q}}\left(\frac{1}{l(\tilde{h}_2)}\sum_{\alpha_2\in\tilde{h}_2}\left(\frac{\nu_{\alpha_1}^q+\nu_{\alpha_2}^q}{2}-\left(\frac{\nu_{\alpha_1}+\nu_{\alpha_2}}{2}\right)^q\right)\right)^p}, \quad (25)
\end{aligned}
$$

where $p \geq 1$ and $1 < q \leq 2$.

For the symmetric property, it is necessary to modify Eqs. (20)-(25) to a symmetric discrimination information measure for GHFEs as follows:

$$\mathrm{CE}_k^*(\tilde{h}_1,\tilde{h}_2)=\mathrm{CE}_k(\tilde{h}_1,\tilde{h}_2)+\mathrm{CE}_k(\tilde{h}_2,\tilde{h}_1),\ k=1,2,\ldots,6. \qquad (26)$$

Example 1. Let $\tilde{h}_i$ $(i=1,2,3)$ and $\tilde{h}$ be three patterns and a sample. They are denoted by GHFEs as follows: $\tilde{h}_1=\{(0.5,0.4),(0.6,0.3)\}$, $\tilde{h}_2=\{(0.4,0.5),(0.8,0.1)\}$, $\tilde{h}_3=\{(0.3,0.4),(0.7,0.2)\}$ and $\tilde{h}=\{(0.5,0.4),(0.7,0.2)\}$. Given the sample $\tilde{h}$, which pattern does this sample $\tilde{h}$ most probably belong to ?

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For convenience, let $p = q = 2$. By (20)-(25), we have

$$\begin{aligned}
&\mathrm{CE}_1^*(\tilde{h}_1, \tilde{h}) = 0.0275,\ \mathrm{CE}_1^*(\tilde{h}_2, \tilde{h}) = 0.0976,\ \mathrm{CE}_1^*(\tilde{h}_3, \tilde{h}) = 0.0693;\\
&\mathrm{CE}_2^*(\tilde{h}_1, \tilde{h}) = 0.2601,\ \mathrm{CE}_2^*(\tilde{h}_2, \tilde{h}) = 0.4285,\ \mathrm{CE}_2^*(\tilde{h}_3, \tilde{h}) = 0.3982;\\
&\mathrm{CE}_3^*(\tilde{h}_1, \tilde{h}) = 0.0241,\ \mathrm{CE}_3^*(\tilde{h}_2, \tilde{h}) = 0.0838,\ \mathrm{CE}_3^*(\tilde{h}_3, \tilde{h}) = 0.0541;\\
&\mathrm{CE}_4^*(\tilde{h}_1, \tilde{h}) = 0.2609,\ \mathrm{CE}_4^*(\tilde{h}_2, \tilde{h}) = 0.4298,\ \mathrm{CE}_4^*(\tilde{h}_3, \tilde{h}) = 0.3856;\\
&\mathrm{CE}_5^*(\tilde{h}_1, \tilde{h}) = 0.0300,\ \mathrm{CE}_5^*(\tilde{h}_2, \tilde{h}) = 0.1000,\ \mathrm{CE}_5^*(\tilde{h}_3, \tilde{h}) = 0.0800;\\
&\mathrm{CE}_6^*(\tilde{h}_1, \tilde{h}) = 0.0239,\ \mathrm{CE}_6^*(\tilde{h}_2, \tilde{h}) = 0.0707,\ \mathrm{CE}_6^*(\tilde{h}_3, \tilde{h}) = 0.0620.
\end{aligned}$$

From this data, the proposed symmetric discrimination information measures CE_k^* $(k = 1, 2, 3, 4, 5, 6)$ show the same classification according to the principle of the minimum degree of symmetric discrimination information measure for GHFEs. Thus, the sample $\tilde{h}$ belongs to the pattern $\tilde{h}_1$.

4 Two MCDM approaches based on the cross-entropy measures of GHFEs

For a MCDM problem, let $X = \{x_1, x_2, \ldots, x_m\}$ be a set of m alternatives, and $Y = \{y_1, y_2, \ldots, y_n\}$ be a set of n criteria, whose weight vector is $w = (w_1, w_2, \ldots, w_n)^T$, satisfying $w_j > 0$, $j = 1, 2, \ldots, n$ and $\sum_{j=1}^n w_j = 1$, where w_j denotes the importance degree of the criterion y_j. Decision makers evaluate the performance of alternatives with respect to criteria based on their knowledge and experience. One decision maker could give several evaluation values. However, in the case where two or more decision makers give the same value, it is counted only once. The performance of the alternative x_i with respect to the criterion y_j is measured by a GHFE $\tilde{e}_{ij} = \{\beta_{ij} = (\mu_{\beta_{ij}}, \nu_{\beta_{ij}}) | \beta_{ij} \in \tilde{e}_{ij}\}$, where $\mu_{\beta_{ij}}$ indicates the degree that the alternative x_i satisfies the criterion y_j, $\nu_{\beta_{ij}}$ indicates the degree that the alternative x_i does not satisfy the criterion y_j, such that $\mu_{\beta_{ij}} \in [0, 1]$, $\nu_{\beta_{ij}} \in [0, 1]$, $\mu_{\beta_{ij}} + \nu_{\beta_{ij}} \leq 1$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$. All $\tilde{e}_{ij}$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ are contained in the generalized hesitant fuzzy decision matrix $E = (\tilde{e}_{ij})_{m\times n}$ (see Table 1).

Table 1: The generalized hesitant fuzzy decision matrix

	y_1	y_2	$\cdots$	y_n
x_1	$\tilde{e}_{11}$	$\tilde{e}_{12}$	$\cdots$	$\tilde{e}_{in}$
x_2	$\tilde{e}_{21}$	$\tilde{e}_{22}$	$\cdots$	$\tilde{e}_{2n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_m	$\tilde{e}_{m1}$	$\tilde{e}_{m2}$	$\cdots$	$\tilde{e}_{mn}$

In what follows, we develop two approaches to multi-criteria decision making under generalized hesitant fuzzy environment.

Approach I.

Step 1. Normalize the performance values and then construct the normalized generalized hesitant fuzzy decision matrix.

If all the criteria y_j $(j = 1, 2, \ldots, n)$ are of the same type, then the performance values do not need normalization. Whereas there are, generally, benefit criteria (the bigger the performance values the better) and cost criteria (the smaller the performance values the better) in multi-criteria decision making, in such case, we may transform the performance values of the cost type into the performance values of the benefit type. Then, $E = (\tilde{e}_{ij})_{m\times n}$ can be transformed into the matrix $F = (\tilde{h}_{ij})_{m\times n}$, where

$$\tilde{h}_{ij} = \cup_{\alpha_{ij}\in\tilde{h}_{ij}}\{\alpha_{ij}\} = \begin{cases} \cup_{\beta_{ij}\in\tilde{d}_{ij}}\{\beta_{ij}\}, & \text{for benefit criterion } y_j; \\ \cup_{\beta_{ij}\in\tilde{d}_{ij}}\{\beta_{ij}^c\}, & \text{for cost criterion } y_j, \end{cases}$$
$$i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n, \tag{27}$$

where β_{ij}^c is the complement of $\beta_{ij} = (\mu_{\beta_{ij}}, \nu_{\beta_{ij}})$ such that $\beta_{ij}^c = (\nu_{\beta_{ij}}, \mu_{\beta_{ij}})$.

Step 2. Calculate the separation degree of each component $\tilde{h}_{ij}$ to positive ideal solution and negative ideal solution.

The positive ideal solution (PIS) and negative ideal solution (NIS) can be denoted as $\tilde{h}^+ = \{(1, 0)\}$ and $\tilde{h}^- = \{(0, 1)\}$, respectively, within the generalized hesitant fuzzy environment. The separation between alternatives can be calculated by cross-entropies. For the convenience of both calculation and analysis, only one cross-entropy (21) is selected. The separation degrees, G_{ij}^+ and G_{ij}^-, of each $\tilde{h}_{ij}$ $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ to the PIS $\tilde{h}^+$ and NIS $\tilde{h}^-$, respectively, are derived from Eq. (21):

$$\begin{aligned} G_{ij}^+ &= \mathrm{CE}_2^*(\tilde{h}_{ij}, \tilde{h}^+) = \mathrm{CE}_2(\tilde{h}_{ij}, \tilde{h}^+) + \mathrm{CE}_2(\tilde{h}^+, \tilde{h}_{ij}) \\ &= \sqrt[p]{\frac{1}{l(\tilde{h}_{ij})}\sum_{\alpha_{ij}\in\tilde{h}_{ij}}\left(\mu_{\alpha_{ij}}\log_2\frac{2\mu_{\alpha_{ij}}}{\mu_{\alpha_{ij}}+1}\right)^p} + \sqrt[p]{\frac{1}{l(\tilde{h}_{ij})}\sum_{\alpha_{ij}\in\tilde{h}_{ij}}\nu_{\alpha_{ij}}^p} \\ &\quad + \sqrt[p]{\left(\frac{1}{l(\tilde{h}_{ij})}\sum_{\alpha_{ij}\in\tilde{h}_{ij}}\left(\log_2\frac{2}{1+\mu_{\alpha_{ij}}}\right)\right)^p} \end{aligned} \tag{28}$$

and

$$\begin{aligned} G_{ij}^- &= \mathrm{CE}_1^*(\tilde{h}_{ij}, \tilde{h}^-) = \mathrm{CE}_1(\tilde{h}_{ij}, \tilde{h}^-) + \mathrm{CE}_1(\tilde{h}^-, \tilde{h}_{ij}) \\ &= \sqrt[p]{\frac{1}{l(\tilde{h}_{ij})}\sum_{\alpha_{ij}\in\tilde{h}_{ij}}\mu_{\alpha_{ij}}^p} + \sqrt[p]{\frac{1}{l(\tilde{h}_{ij})}\sum_{\alpha_{ij}\in\tilde{h}_{ij}}\left(\nu_{\alpha_{ij}}\log_2\frac{2\nu_{\alpha_{ij}}}{\nu_{\alpha_{ij}}+1}\right)^p} \\ &\quad + \sqrt[p]{\left(\frac{1}{l(\tilde{h}_{ij})}\sum_{\alpha_{ij}\in\tilde{h}_{ij}}\left(\log_2\frac{2}{1+\nu_{\alpha_{ij}}}\right)\right)^p}. \end{aligned} \tag{29}$$

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Step 3. Calculate the closeness degree of the alternatives to the NIS.

The closeness degree $G(x_i)$ of each alternative x_i $(i = 1, 2, \ldots, m)$ to the NIS can be obtained by following:

$$G(x_i) = \sum_{j=1}^{n} w_j G_{ij}, \text{ where } G_{ij} = \frac{G_{ij}^{-}}{G_{ij}^{+} + G_{ij}^{-}}. \tag{30}$$

Step 4. Rank the alternatives.

The bigger the closeness degree $G(x_i)$, the better the alternative x_i will be, as the alternative x_i is closer to the PIS $\tilde{h}^+$. Therefore, the alternatives x_i $(i = 1, 2, \ldots, m)$ can be ranked in descending order according to the closeness degrees so that the best alternative can be selected.

Approach II.

Step 1. For this step, see Approach I.

Step 2. Calculate the overall aggregated values of each alternative.

Utilize the GHFWA operator (17) (or the GHFWG operator (18)):

$$\begin{aligned} \tilde{h}_i &= \mathrm{GHFWA}_w(\tilde{h}_{i1}, \tilde{h}_{i2}, \ldots, \tilde{h}_{in}) \\ &= \bigcup_{\alpha_{i1} \in \tilde{h}_{i1}, \alpha_{i2} \in \tilde{h}_{i2}, \ldots, \alpha_{in} \in \tilde{h}_{in}} \left\{ \left(1 - \prod_{j=1}^{n} (1 - \mu_{\alpha_{ij}})^{w_j}, \prod_{j=1}^{n} \nu_{\alpha_{ij}}^{w_j} \right) \right\} \end{aligned} \tag{31}$$

or

$$\begin{aligned} \tilde{h}_i &= \mathrm{GHFWG}_w(\tilde{h}_{i1}, \tilde{h}_{i2}, \ldots, \tilde{h}_{in}) \\ &= \bigcup_{\alpha_{i1} \in \tilde{h}_{i1}, \alpha_{i2} \in \tilde{h}_{i2}, \ldots, \alpha_{in} \in \tilde{h}_{in}} \left\{ \left(\prod_{j=1}^{n} \mu_{\alpha_{ij}}^{w_j}, 1 - \prod_{j=1}^{n} (1 - \nu_{\alpha_{ij}})^{w_j} \right) \right\} \end{aligned} \tag{32}$$

to aggregate all the performance values $\tilde{h}_{ij}$ $(j = 1, 2, \ldots, n)$ of the ith line and get the overall performance value $\tilde{h}_i$ corresponding to the alternatives x_i.

Step 3. Calculate the closeness degree of the alternatives to the PIS.

Utilize the cross-entropy (21) between the overall performance value $\tilde{h}_i$ $(i = 1, 2, \ldots, m)$ and the PIS $\tilde{h}^+ = \{(1, 0)\}$ to get closeness degree of each alternative x_i $(i = 1, 2, \ldots, m)$ to the PIS $\tilde{h}^+$:

$$\begin{aligned} G(x_i) &= \mathrm{CE}_2^*(\tilde{h}_i, \tilde{h}^+) = \mathrm{CE}_2(\tilde{h}_i, \tilde{h}^+) + \mathrm{CE}_2^*(\tilde{h}^+, \tilde{h}_i) \\ &= \sqrt[p]{\frac{1}{l(\tilde{h}_i)} \sum_{\alpha_i \in \tilde{h}_i} \left(\mu_{\alpha_i} \log_2 \frac{2\mu_{\alpha_i}}{\mu_{\alpha_i} + 1} \right)^p} + \sqrt[p]{\frac{1}{l(\tilde{h}_i)} \sum_{\alpha_i \in \tilde{h}_i} \nu_{\alpha_i}^p} \\ &\quad + \sqrt[p]{\left(\frac{1}{l(\tilde{h}_i)} \sum_{\alpha_i \in \tilde{h}_i} \left(\log_2 \frac{2}{1 + \mu_{\alpha_i}} \right) \right)^p}. \end{aligned} \tag{33}$$

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Step 4. Rank the alternatives.

The smaller the closeness degree $G(x_i)$, the better the alternative x_i will be, as the alternative x_i is closer to the PIS $\tilde{h}^+$. Therefore, the alternatives x_i $(i = 1, 2, \ldots, m)$ can be ranked in ascending order according to the closeness degrees so that the best alternative can be selected.

5 An illustrative example

In this section, a generalized hesitant fuzzy MCDM problem of selecting an investment is used to illustrate the proposed methods.

A city is planning to build a municipal library. One of the problems facing the city development commissioner is to determine what kind of air-conditioning system should be installed in the library (adapted from [20]). The contractor offers five feasible alternatives x_i $(i = 1, 2, 3, 4, 5)$, which might be adapted to the physical structure of the library. Suppose that three criteria y_1 (economic), y_2 (functional), and y_3 (operational) are taken into consideration in the installation problem, and the weight vector of the criteria y_j $(j = 1, 2, 3)$ is $w = (0.3, 0.5, 0.2)^T$. Assume that the characteristics of the alternatives x_i $(i = 1, 2, 3, 4, 5)$ with respect to the criterion y_j $(j = 1, 2, 3)$ are represented by the GHFEs $\tilde{h}_{ij} = \{\alpha_{ij} = (\mu_{\alpha_{ij}}, \nu_{\alpha_{ij}})|\alpha_{ij} \in \tilde{h}_{ij}\}$, where $\mu_{\alpha_{ij}}$ indicates the degree that the alternative x_i satisfies the criterion y_j and $\nu_{\alpha_{ij}}$ indicates the degree that the alternative x_i does not satisfy the criterion y_j, such that $\mu_{\alpha_{ij}}, \nu_{\alpha_{ij}} \in [0, 1]$ and $\mu_{\alpha_{ij}} + \nu_{\alpha_{ij}} \leq 1$. All $\tilde{h}_{ij} = \{\alpha_{ij} = (\mu_{\alpha_{ij}}, \nu_{\alpha_{ij}})|\alpha_{ij} \in \tilde{h}_{ij}\}$ $(i = 1, 2, 3, 4, 5;\ j = 1, 2, 3)$ are contained in the generalized hesitant fuzzy decision matrix $E = (\tilde{h}_{ij})_{5\times3}$ (see Table 2).

Table 2: The generalized hesitant fuzzy decision matrix

	y_1	y_2	y_3
x_1	$\{(0.3, 0.2), (0.3, 0.4)\}$	$\{(0.7, 0.2), (0.5, 0.2)\}$	$\{(0.5, 0.2), (0.6, 0.3)\}$
x_2	$\{(0.5, 0.2), (0.6, 0.2)\}$	$\{(0.3, 0.1), (0.4, 0.2)\}$	$\{(0.7, 0.1), (0.8, 0.1)\}$
x_3	$\{(0.3, 0.4), (0.4, 0.5)\}$	$\{(0.7, 0.2), (0.8, 0.1)\}$	$\{(0.4, 0.3), (0.4, 0.4)\}$
x_4	$\{(0.2, 0.6), (0.2, 0.7)\}$	$\{(0.8, 0.1), (0.7, 0.2)\}$	$\{(0.7, 0.2), (0.8, 0.1)\}$
x_5	$\{(0.8, 0.1), (0.7, 0.2)\}$	$\{(0.6, 0.3), (0.7, 0.2)\}$	$\{(0.2, 0.5), (0.2, 0.6)\}$

To select the best air-conditioning system, we utilize above-mentioned two approaches to find the decision result(s).

Approach I.

Step 1. Considering that all the attributes y_j $(j = 1, 2, 3)$ are benefit type attributes, the performance values of the alternatives x_i $(i = 1, 2, 3, 4, 5)$ do not need normalization.

Step 2. Utilize Eqs. (28) and (29) (let $p = 2$) to calculate the separation degree G_{ij} of each component $\tilde{h}_{ij}$ $(i = 1, 2, 3, 4, 5;\ j = 1, 2, 3)$ to PIS $\tilde{h}^+ =$

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$\{(1,0)\}$ and NIS $\tilde{h}^- = \{(0,1)\}$ and then we get the following results:

$$\begin{aligned}
&G_{11}^+ = 1.2724, G_{12}^+ = 0.7737, G_{13}^+ = 0.8951, G_{21}^+ = 0.8401, G_{22}^+ = 1.0549,\\
&G_{23}^+ = 0.4620, G_{31}^+ = 1.3496, G_{32}^+ = 0.5201, G_{33}^+ = 1.1911, G_{41}^+ = 1.7059,\\
&G_{42}^+ = 0.5201, G_{43}^+ = 0.5201, G_{51}^+ = 0.5201, G_{52}^+ = 0.7573, G_{53}^+ = 1.6062,\\
&G_{11}^- = 1.2458, G_{12}^- = 1.6622, G_{13}^- = 1.5574, G_{21}^- = 1.6062, G_{22}^- = 1.4369,\\
&G_{23}^- = 1.8601, G_{31}^- = 1.1264, G_{32}^- = 1.8351, G_{33}^- = 1.2969, G_{41}^- = 0.7023,\\
&G_{42}^- = 1.8351, G_{43}^- = 1.8351, G_{51}^- = 1.8351, G_{52}^- = 1.6571, G_{53}^- = 0.8401.
\end{aligned}$$

Step 3. Utilize the weight vector $w = (0.3, 0.5, 0.2)^T$ of the criteria y_j $(j = 1, 2, 3)$ and Eq. (30) to calculate the closeness degree $G(x_i)$ of the alternatives x_i $(i = 1, 2, 3, 4, 5)$ to the NIS:

$$G(x_1) = 0.6166, G(x_2) = 0.6455, G(x_3) = 0.6303, G(x_4) = 0.6329, G(x_5) = 0.6456.$$

Using this, we rank all alternatives x_i $(i = 1, 2, 3, 4, 5)$ in descending order in accordance with the values $G(x_i)$ $(i = 1, 2, 3, 4, 5)$:

$$x_5 \succ x_2 \succ x_4 \succ x_3 \succ x_1.$$

Therefore, the best alternative is x_5.

Approach II.

Step 1. For this step, see Approach I.

Step 2. Utilize the GHFWA operator (31) to aggregate all the performance values $\tilde{h}_{ij}$ $(i = 1, 2, 3, 4, 5; j = 1, 2, 3)$ of the ith line and get the overall performance value $\tilde{h}_i$ corresponding to the alternatives x_i $(i = 1, 2, 3, 4, 5)$;

$$\begin{aligned}
\tilde{h}_1 = \{&(0.5716, 0.2000), (0.5903, 0.2169), (0.4469, 0.2000), (0.4710, 0.2169),\\
&(0.5716, 0.2462), (0.5903, 0.2670), (0.4469, 0.2462), (0.4710, 0.2670)\};\\
\tilde{h}_2 = \{&(0.4658, 0.1231), (0.5075, 0.1231), (0.5055, 0.1741), (0.5440, 0.1741),\\
&(0.5004, 0.1231), (0.5393, 0.1231), (0.5375, 0.1741), (0.5735, 0.1741)\};\\
\tilde{h}_3 = \{&(0.5557, 0.2670), (0.5557, 0.2828), (0.6372, 0.1888), (0.6372, 0.2000),\\
&(0.5757, 0.2855), (0.5757, 0.3024), (0.6536, 0.2018), (0.6536, 0.2138)\};\\
\tilde{h}_4 = \{&(0.6712, 0.1966), (0.6969, 0.1711), (0.5974, 0.2781), (0.6287, 0.2421),\\
&(0.6712, 0.2059), (0.6969, 0.1793), (0.5974, 0.2912), (0.6287, 0.2535)\};\\
\tilde{h}_5 = \{&(0.6268, 0.2390), (0.6268, 0.2479), (0.6768, 0.1951), (0.6768, 0.2024),\\
&(0.5785, 0.2942), (0.5785, 0.3051), (0.6350, 0.2402), (0.6350, 0.2491)\}.
\end{aligned}$$

Step 3. Utilize Eq. (33) to calculate the closeness degree $G(x_i)$ of each alternative x_i $(i = 1, 2, 3, 4, 5)$ to the PIS $\tilde{h}^+ = \{(1,0)\}$:

$$G(x_1) = 0.9146, G(x_2) = 0.8291, G(x_3) = 0.8103, G(x_4) = 0.7350, G(x_5) = 0.7799.$$

Using this, we rank all alternatives x_i $(i = 1, 2, 3, 4, 5)$ in ascending order according to the values $G(x_i)$ $(i = 1, 2, 3, 4, 5)$:

$$x_4 \succ x_5 \succ x_3 \succ x_2 \succ x_1.$$

Therefore, the best alternative is x_4.

If we utilize the GHFWG operator (32) in Step 2 of Approach II, then the closeness degree $G(x_i)$ of each alternative x_i $(i = 1, 2, 3, 4, 5)$ to the PIS is calculated:

$$G(x_1) = 0.9835, G(x_2) = 0.9117, G(x_3) = 0.9700, G(x_4) = 1.0543, G(x_5) = 0.9597.$$

and so the ranking of all alternatives x_i $(i = 1, 2, 3, 4, 5)$ in ascending order according to the values $G(x_i)$ $(i = 1, 2, 3, 4, 5)$ is obtained:

$$x_2 \succ x_5 \succ x_3 \succ x_1 \succ x_4.$$

Therefore, the best alternative is x_2.

From the above analysis, we know that the results obtained by the proposed approaches are different. Each of methods has its advantages and disadvantages and none of them can always perform better than the others in any situations. It perfectly depends on how we look at things, and not on how they are themselves. As we can see, in Approach II, depending on aggregation operators used, the ranking of the alternatives is different. Therefore, the results may lead to different decisions.

6 Conclusions

In this paper, we developed cross-entropy under generalized hesitant fuzzy environment. Axiomatic definition about this information measure have been given for GHFEs. Two approaches, based on the developed generalized hesitant fuzzy cross-entropy, of generalized hesitant fuzzy MCDM are developed which permits the decision maker to provide several possible IFVs for an alternative under the given criterion, which is consistent with humans' hesitant thinking. The illustrative example demonstrated the validity and practicability of the developed approaches.

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ON HARMONIC QUASICONFORMAL MAPPINGS WITH FINITE AREA

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ABSTRACT. In this paper, we study the class of harmonic K-quasiconformal mappings of the unit disk $\mathbf{U}$ with finite Euclidean areas $|f(\mathbf{U})|_{euc}$. We first give the Schwarz-pick lemma (cf. [8]) for this class of mappings as follows:

$$|f_z(z)| \leq \sqrt{\frac{|f(\mathbf{U})|_{euc}}{\pi(1-k^2)}}\frac{1}{1-|z|}, \quad z\in\mathbf{U},$$

where $k=\frac{K-1}{K+1}$. Furthermore, we obtain the sharp coefficient estimates of this class of mappings. As an application, for harmonic mappings $f\in S_H^0$ with finite $|f(\mathbf{U})|_{euc}$ we obtain sharp coefficient estimates.

1. INTRODUCTION

Let $\mathbf{U}=\{z\in\mathbb{C}:|z|<1\}$ be the unit disk in the complex plane $\mathbb{C}$. The classical Schwarz lemma says that if an analytic function φ of $\mathbf{U}$ satisfies that $|\varphi(z)|<1$ and $\varphi(0)=0$. Then $|\varphi(z)|\leq|z|$ and $|\varphi'(0)|\leq 1$. The equality occurs if and only if $\varphi(z)=e^{i\alpha}z$, where α is a real constant. The classical Schwarz lemma is a cornerstone in complex analysis and attracts one to give various versions of its generalization.

A complex-valued function $f(z)$ of class C^2 is said to be a harmonic mapping if it satisfies $f_{z\bar{z}}=0$. It is known that every harmonic mapping $f(z)$ defined in $\mathbf{U}$ admits a canonical decomposition $f(z)=h(z)+\overline{g(z)}$, where $h(z)$ and $g(z)$ are analytic in $\mathbf{U}$ with $g(0)=0$. One can refer to [5] and the references therein for more details about harmonic mappings.

For $z\in\mathbf{U}$, let

$$\Lambda_f(z)=\max_{0\leq\theta\leq 2\pi}|f_z(z)+e^{-2i\theta}f_{\bar{z}}(z)|=|f_z(z)|+|f_{\bar{z}}(z)|, \tag{1}$$

and

$$\lambda_f(z)=\min_{0\leq\theta\leq 2\pi}|f_z(z)+e^{-2i\theta}f_{\bar{z}}(z)|=||f_z(z)|-|f_{\bar{z}}(z)||. \tag{2}$$

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The Lewy theorem [7] tells us that a harmonic mapping f is locally univalent and sense-preserving in $\mathbf{U}$ if and only if its Jacobian satisfies the following condition

$$J_f(z) = \lambda_f(z)\Lambda_f(z) = |f_z(z)|^2 - |f_{\bar{z}}(z)|^2 > 0 \quad \text{for} \quad z \in \mathbf{U}.$$

Suppose that $f(z)$ is a sense-preserving univalent harmonic mapping of $\mathbf{U}$. Then $f(z)$ is a K-quasiconformal mapping if and only if

$$K(f) := \sup_{z\in\mathbf{U}} \frac{|f_z(z)| + |f_{\bar{z}}(z)|}{|f_z(z)| - |f_{\bar{z}}(z)|} \leq K.$$

Harmonic quasiconformal mappings are natural generalizations of conformal mappings. Recently, many mathematicians have studied such an active topic and obtained many interesting results (cf.[1], [6], [10], [8], [12], [13], [14], [15]).

In 2007, M.Knežević and M.Mateljević [8] proved the following Schwarz-Pick lemma for harmonic quasiconformal mappings.

Theorem A. *Let f be a harmonic K-quasiconformal mapping of $\mathbf{U}$ into itself. Then*

$$|f_z(z)| \leq \frac{(K+1)(1-|f(z)|^2)}{2(1-|z|^2)}$$

holds for all $z \in \mathbf{U}$, and

$$d_\lambda(f(z_1), f(z_2)) \leq K d_\lambda(z_1, z_2)$$

holds for any $z_1, z_2 \in \mathbf{U}$, where d_λ is the hyperbolic distance.

Furthermore, they obtained the opposite inequalities in Theorem A as $|f_z(z)| \geq \frac{(K+1)(1-|f(z)|^2)}{2K(1-|z|^2)}$ and $d_\lambda(f(z_1), f(z_2)) \geq \frac{1}{K} d_\lambda(z_1, z_2)$ by assuming that f is onto. Such an assumption is necessary since that $|f_z(z)|$ will bounded below by a positive constant (see [8] for more details).

In 2010, X. Chen and A. Fang [2] improved the above results as follows.

Theorem B. *Let $\Omega \subset \mathbb{C}$ be a simply connected convex domain of hyperbolic type and λ_Ω be its hyperbolic metric density with the Gaussian curvature -4. If f is a harmonic K-quasiconformal mapping of $\mathbf{U}$ onto Ω, then the inequalities*

$$\frac{(K+1)\lambda_{\mathbf{U}}(z)}{2K\lambda_\Omega(f(z))} \leq |f_z(z)| \leq \frac{(K+1)\lambda_{\mathbf{U}}(z)}{2\lambda_\Omega(f(z))}$$

hold for all $z \in \mathbf{U}$. Moreover, the above estimates are sharp.

We point out that the composition of a harmonic mapping f with a conformal mapping φ is still harmonic. Hence we can fix the defined domain as $\mathbf{U}$ for harmonic mappings. However, $\varphi \circ f$ is not harmonic in general. This implies that the Schwarz-Pick lemma for harmonic quasiconformal mappings will closely relate to its target domain. Instead of the assumption that harmonic quasiconformal mappings have convex or bounded ranges, we study the class of harmonic mappings with finite Euclidean areas. Example 1 shows there exists a harmonic mapping with an unbounded range but finite Euclidean area.

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Assume that $f(z) = h(z) + \overline{g(z)}$ is a harmonic K-quasiconformal mapping of $\mathbf{U}$ with a finite Euclidean area $|f(\mathbf{U})|_{euc}$, where

$$h(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n$$

are analytic in $\mathbf{U}$. Under the assumption of finite Euclidean areas, we obtain a new version of the Schwarz-Pick lemma for the class of harmonic K-quasiconformal mappings as follows

$$(3) \qquad |f_z(z)| \le \sqrt{\frac{|f(\mathbf{U})|_{euc}}{\pi(1-k^2)}} \frac{1}{1-|z|}, \quad z \in \mathbf{U},$$

where $k = \frac{K-1}{K+1}$. See Theorem 1 for details.

Furthermore, we obtain the sharp coefficient estimates for $f(z)$

$$(4) \qquad |a_n|^2 + |b_n|^2 \le \frac{(K + 1/K)|f(\mathbf{U})|_{euc}}{2n\pi} \quad (n = 1, 2, \ldots).$$

This result is given at Theorem 2.

Denote by S_H the family of all sense-preserving univalent harmonic mappings defined in $\mathbf{U}$ which admit a canonical representation $f = h + \overline{g}$, where

$$(5) \qquad h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n$$

are analytic in $\mathbf{U}$. The class S_H^0 is the subclass of S_H with $g'(0) = 0$.

A well-known result of the classical analytic univalent functions is the Bieberbach conjecture which was posed by Ludwig Bieberbach in 1916 and was finally proven by Louis de Branges [4]. This result has many important geometric applications. In 1984, T.Sheil-Small [3] published a landmark paper which pointed out that many classical results of conformal mappings have analogues of harmonic mappings. One of the famous results is the coefficients conjecture of S_H^0. As an application of Theorem 2, we obtain the coefficients estimate for $f \in S_H^0$ which is given by (9).

2. Main results and their proofs

Theorem 1. *Let $K \ge 1$ be a constant. If $f(z) = h(z) + \overline{g(z)}$ is a harmonic K-quasiconformal mapping of the unit disk $\mathbf{U}$ such that its Euclidean area $|f(\mathbf{U})|_{euc}$ is finite, then*

$$|f_z(z)| \le \sqrt{\frac{|f(\mathbf{U})|_{euc}}{\pi(1-k^2)}} \frac{1}{1-|z|}, \quad z \in \mathbf{U},$$

where $k = \frac{K-1}{K+1}$.

Proof. Since $f(z)$ is a harmonic K-quasiconformal mapping, we obtain that

$$\sup_{z\in\mathbf{U}} \left| \frac{g'(z)}{h'(z)} \right| \le k = \frac{K-1}{K+1}.$$

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Then

$$\begin{aligned}|f(\mathbf{U})|_{euc} &= \int\int_{\mathbf{U}} (|h'(z)|^2 - |g'(z)|^2)\, d\sigma \\ &= \int\int_{\mathbf{U}} |h'(z)|^2 \left(1 - \frac{|g'(z)|^2}{|h'(z)|^2}\right) d\sigma \\ &\geq (1-k^2) \int\int_{\mathbf{U}} |h'(z)|^2\, d\sigma.\end{aligned}$$

This implies that

$$\int\int_{\mathbf{U}} |h'(z)|^2\, d\sigma \leq \frac{|f(\mathbf{U})|_{euc}}{1-k^2}. \tag{6}$$

By [11, Corollary 2.6.4 and Theorem 2.4.1], we obtain $|h'(z)|^2$ is subharmonic in $\mathbf{U}$. Thus, for $r \in [0, 1-|z|)$, it follows

$$|h'(z)|^2 \leq \frac{1}{2\pi} \int_0^{2\pi} |h'(z+re^{i\theta})|^2\, d\theta. \tag{7}$$

Utilizing the inequality (6), we get

$$\begin{aligned}\pi(1-|z|)^2 |h'(z)|^2 &\leq \int_0^{2\pi} \int_0^{1-|z|} r|h'(z+re^{i\theta})|^2\, dr\, d\theta \\ &= \int\int_{D(z)} |h'(\zeta)|^2\, d\sigma \\ &\leq \int\int_{\mathbf{U}} |h'(\zeta)|^2\, d\sigma \leq \frac{|f(\mathbf{U})|_{euc}}{1-k^2},\end{aligned}$$

where $D(z) := \{\zeta \in \mathbb{C}, |\zeta - z| < 1 - |z|\} \subseteq \mathbf{U}$. Then $|h'(z)|^2 \leq \frac{|f(\mathbf{U})|_{euc}}{\pi(1-|z|)^2(1-k^2)}$.

This completes the proof. □

Remark 1. *The Euclidean area of $f(\mathbf{U})$ is finite doesn't imply that f is bounded. The following Example 1 shows that Theorem 1 is not covered by Theorem A and Theorem B. Furthermore, let $f(z) = e^{i\alpha}z$ be a conformal mapping of $\mathbf{U}$ onto itself, where α is a real constant. Then $|f_z(z)| = 1$ and $|f(\mathbf{U})|_{euc} = \pi$. This shows that (3) is sharp at $z = 0$.*

Example 1. *Let $\Omega_1 = \{\zeta \in \mathbb{C} : 0 \leq \operatorname{Im}\zeta \leq 1, \operatorname{Re}\zeta \geq 0\}$ and $\varphi_1(z) : \mathbf{U} \mapsto \Omega_1$ be a conformal mapping. Let $\varphi_2(\zeta) := \frac{1}{2i}\ln\frac{1+\zeta i}{1-\zeta i}$ be a conformal mapping of Ω_1 onto Ω_2. Then $w(z) = \varphi_2 \circ \varphi_1(z)$ is a conformal mapping of $\mathbf{U}$ onto Ω_2. Here Ω_2 is an unbounded (and not convex) domain with the boundary curves $\{c_1 : \operatorname{Im} w = 0, 0 \leq \operatorname{Re} w \leq \frac{\pi}{2}\}$, $\{c_2 : \operatorname{Re} w = 0, \operatorname{Im} w \in \mathbb{R}\}$ and $\{c_3 : w(t+i) = \frac{1}{2i}\ln\frac{ti}{2-ti},\ t \in [0,\infty)\}$ which is shown by figure 1. Then $|w(\mathbf{U})|_{euc} = \frac{\pi \ln 2}{8}$ is finite. This shows that Theorem 1 is not covered by Theorem A and Theorem B.*

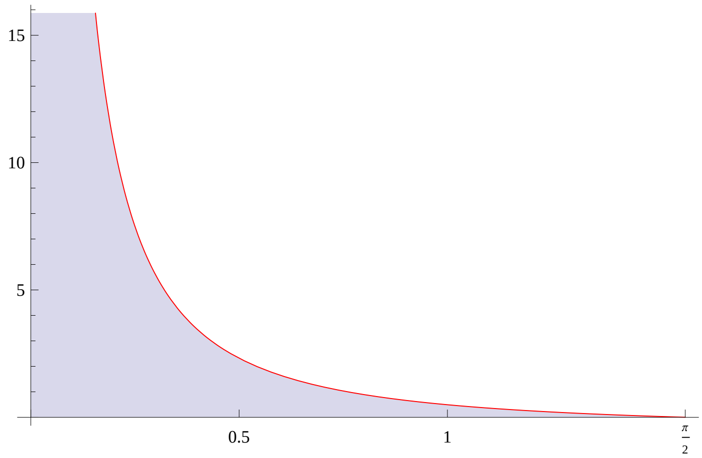

FIGURE 1. Image of the domain Ω_2

Theorem 2. *Let $K \geq 1$ be a constant. If $f(z) = h(z) + \overline{g(z)}$ is a harmonic K-quasiconformal mapping of $\mathbf{U}$ such that $|f(\mathbf{U})|_{euc}$ is finite, where*

$$h(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \tag{8}$$

are analytic in $\mathbf{U}$. Then

$$|a_n|^2 + |b_n|^2 \leq \frac{(K+1/K)|f(\mathbf{U})|_{euc}}{2n\pi} \quad (n = 1, 2, \ldots).$$

The above coefficient estimates are sharp for all $n = 1, 2, \ldots,$ with the extremal functions $f_n(z) = \frac{a}{\sqrt{n}} z^n + \frac{ka}{\sqrt{n}} \overline{z^n}$ where $k = \frac{K-1}{K+1}$ and $a \in \mathbb{C}$ is a constant.

Proof. For every $z = re^{i\theta} \in \mathbf{U}$,

$$f(re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta} + \sum_{n=1}^{\infty} \overline{b_n} r^n e^{-in\theta}.$$

Hence $h'(re^{i\theta}) = \sum\limits_{n=1}^{\infty} n a_n r^{n-1} e^{i(n-1)\theta}$ and $g'(re^{i\theta}) = \sum\limits_{n=1}^{\infty} n b_n r^{n-1} e^{i(n-1)\theta}$. Applying the Parseval identity, we obtain

$$\int\int_{\mathbf{U}} (|h'(z)|^2 + |g'(z)|^2) d\sigma = \pi \sum_{n=1}^{\infty} n(|a_n|^2 + |b_n|^2).$$

Since $f(z)$ is a K-quasiconformal mapping, we have

$$|h'(z)|^2 + |g'(z)|^2 \leq \frac{1}{2}(K + 1/K)\left(|h'(z)|^2 - |g'(z)|^2\right).$$

This implies that

$$\begin{aligned}\pi\sum_{n=1}^{\infty}n(|a_n|^2+|b_n|^2) &\leq \int\int_{\mathbf{U}}\frac{1}{2}(K+1/K)(|h'(z)|^2-|g'(z)|^2)d\sigma\\ &= \frac{1}{2}(K+1/K)|f(\mathbf{U})|_{euc}.\end{aligned}$$

Then

$$|a_n|^2+|b_n|^2\leq\frac{(K+1/K)|f(\mathbf{U})|_{euc}}{2n\pi}\quad(n=1,2,\ldots).$$

Let $f_n(z)=\frac{a}{\sqrt{n}}z^n+\frac{ka}{\sqrt{n}}\overline{z^n}$, then $|f_n(z)|_{euc}=\int\int_{\mathbf{U}}J_{f_n}(z)d\sigma=(1-k^2)\pi|a|^2$. This shows that $|a_n|^2+|b_n|^2=\frac{(1+k^2)|a|^2}{n}=\frac{(K+1/K)|f(\mathbf{U})|_{euc}}{2n\pi}$. Hence, the estimates are sharp.

This completes the proof. □

Theorem 3. *If $f=h+\bar{g}\in S_H^0$ satisfies that $|f(\mathbf{U})|_{euc}$ is finite, where h, g are given by* (5) *with $b_1=0$. Then*

$$|a_n|^2+|b_n|^2\leq s(n,t_0),\quad(n=2,3,\ldots),$$

where $s(n,t_0)$ is given by (10).

Proof. Let $F(\zeta):=\frac{f(t\zeta)}{t}$, where $f\in S_H^0$, $\zeta\in\mathbf{U}$ and $0<t<1$. Then $\Lambda_F(\zeta)=\Lambda_f(t\zeta)$ holds for all $\zeta\in\mathbf{U}$. According to (5) we see that

$$\begin{aligned}F(\zeta) &= \zeta+\sum_{n=2}^{\infty}a_nt^{n-1}\zeta^n+\sum_{n=2}^{\infty}\overline{b_n}t^{n-1}\overline{\zeta^n}\\ &= \zeta+\sum_{n=2}^{\infty}A_n\zeta^n+\sum_{n=2}^{\infty}\overline{B_n\zeta^n},\end{aligned}$$

where $A_n=t^{n-1}a_n$ and $B_n=t^{n-1}b_n$. Let $\omega(z)=\frac{\overline{f_{\bar{z}}(z)}}{f_z(z)}$. Then $\omega(z)$ is holomorphic in $\mathbf{U}$ satisfying $\omega(0)=0$ and $|\omega(z)|<1$. By the Schwarz lemma we know that $|\omega(z)|\leq|z|$ for $z\in\mathbf{U}$. Therefore, for any $0<t<1$ and $\mathbf{U}_t:=\{z:|z|<t\}$ we have

$$\frac{\Lambda_f(z)}{\lambda_f(z)}=\frac{1+|\omega(z)|}{1-|\omega(z)|}\leq\frac{1+t}{1-t}:=K_t.$$

This implies that F is a K_t-quasiconformal mapping of $\mathbf{U}$. Furthermore,

$$|F(\zeta)|_{euc}=\int\int_{\mathbf{U}_t}J_f(\zeta)d\sigma\leq|f(\mathbf{U})|_{euc}.$$

Applying Theorem 2, we have

$$|A_n|^2+|B_n|^2\leq\frac{(1+t^2)|f(\mathbf{U})|_{euc}}{n\pi(1-t^2)},\quad(n=2,3,\ldots).$$

Hence,

$$|a_n|^2+|b_n|^2 \leq \frac{(1+t^2)|f(\mathbf{U})|_{euc}}{n\pi t^{2n-2}(1-t^2)} := s(n,t), \quad (n=2,3,\ldots). \tag{9}$$

Since $\lim_{t\to 0} s(n,t) = \infty = \lim_{t\to 1} s(n,t)$, we see that $\min_{0<t<1} s(n,t)$ exists. Choose the minimal point $t_0 = \sqrt{\frac{n-1}{\sqrt{(n-1)^2+1}+1}}$, then

$$\begin{aligned} |a_n|^2+|b_n|^2 &\leq s(n,t_0) \\ &= \left(\frac{1+\sqrt{(n-1)^2+1}}{n-1}\right)^{n-1}\left(\frac{\sqrt{(n-1)^2+1}+n}{\sqrt{(n-1)^2+1}+2-n}\right)\frac{|f(\mathbf{U})|_{euc}}{n\pi}. \end{aligned} \tag{10}$$

The proof is completed. □

Remark 2. *By direct calculation, we see that $s(n,t_0)$ is an increasing function of n and $\lim_{n\to\infty} s(n,t_0) = \frac{2e|f(\mathbf{U})|_{euc}}{\pi}$. This implies that $s(n,t_0) \leq \frac{2e|f(\mathbf{U})|_{euc}}{\pi}$. Therefore,*

$$|a_n|+|b_n| \leq \sqrt{2(|a_n|^2+|b_n|^2)} \leq 2\sqrt{\frac{e|f(\mathbf{U})|_{euc}}{\pi}}.$$

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Weak Estimates of the Multidimensional Finite Element and Their Applications

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In this article we first introduce interpolation operator of projection type in multidimensional spaces. Then we derive weak estimates for tensor-product block finite elements. Finally, the applications of the weak estimates in superconvergent properties are discussed.

1 Introduction

Superconvergence of the finite element approximation for second order elliptic boundary value problems has been an active research topic (see [1–7]). It is well known that the weak estimates for the finite element and the estimates for the discrete Green's function play important roles in the superconvergence study (see [8–15]). In this article we focus on the study of the weak estimates and we will derive the weak estimates for the multidimensional finite element.

we shall use the symbol C to denote a generic constant, which is independent from the discretization parameter h and which may not be the same in each occurrence and also use the standard notations for the Sobolev spaces and their norms.

We consider the following Poisson equation:

$$\mathcal{L}u \equiv -\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where $\Omega \subset \mathcal{R}^d$ $(d \geq 2)$ is a bounded polytopic domain. The weak formulation of (1.1) reads,

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ satisfying} \\ a(u\,,\,v) = (f\,,\,v) \text{ for all } v \in H_0^1(\Omega). \end{cases}$$

where

$$a(u\,,\,v) \equiv \int_\Omega \nabla u \cdot \nabla v \, dX,$$

and

$$(f\,,\,v) \equiv \int_\Omega f v \, dX.$$

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Let $\{\mathcal{T}^h\}$ be a regular family of rectangular partitions of $\bar{\Omega}$. Denote by $S^h(\Omega)$ a continuous piecewise tensor-product m-degree polynomials space regarding this kind of partitions and let $S_0^h(\Omega) = S^h(\Omega) \cap H_0^1(\Omega)$. Discretizing the above weak formulation using $S_0^h(\Omega)$ as approximating space means,

$$\begin{cases} \text{Find } u_h \in S_0^h(\Omega) \text{ satisfying} \\ a(u_h\,,\, v) = (f\,,\, v) \text{ for all } v \in S_0^h(\Omega). \end{cases}$$

Thus, the following Galerkin orthogonality relation holds.

$$a(u - u_h\,,\, v) = 0 \ \ \forall\, v \in S_0^h(\Omega). \tag{1.2}$$

2 Weak Estimates for the Finite Element

In this section, we first introduce an interpolation operator of projection type in multidimensional spaces, and then derive the weak estimates for the finite element by using the interpolation operator of projection type.

Let element

$$\begin{aligned} e &= (x_{1,e} - h_{1,e}, x_{1,e} + h_{1,e}) \times (x_{2,e} - h_{2,e}, x_{2,e} + h_{2,e}) \\ &\quad \times \cdots \times (x_{d,e} - h_{d,e}, x_{d,e} + h_{d,e}) \\ &\equiv I_1 \times I_2 \times \cdots \times I_d, \end{aligned} \tag{2.1}$$

and let $\{l_{1,j}(x_1)\}_{j=0}^{\infty}$, $\{l_{2,j}(x_2)\}_{j=0}^{\infty}$,$\cdots$, $\{l_{d,j}(x_d)\}_{j=0}^{\infty}$ be the normalized orthogonal Legendre polynomial systems on $L^2(I_1)$, $L^2(I_2)$,$\cdots$, $L^2(I_d)$, respectively. Now let $\partial_{x_1}\partial_{x_2}\cdots\partial_{x_d}u \in L^2(e)$. Then we have the following expansion:

$$\partial_{x_1}\partial_{x_2}\cdots\partial_{x_d}u = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_d=0}^{\infty} \alpha_{i_1 i_2 \cdots i_d} l_{1,i_1}(x_1) l_{2,i_2}(x_2)\cdots l_{d,i_d}(x_d), \tag{2.2}$$

where

$$\alpha_{i_1 i_2 \cdots i_d} = \int_e \partial_{x_1}\partial_{x_2}\cdots\partial_{x_d}u\, l_{1,i_1}(x_1) l_{2,i_2}(x_2)\cdots l_{d,i_d}(x_d)\, dX. \tag{2.3}$$

Set

$$\omega_{k,0}(x_k) = 1,\ \omega_{k,j+1}(x_k) = \int_{x_{k,e}-h_{k,e}}^{x_k} l_{k,j}(\xi)\, d\xi,\ k = 1, \cdots, d,\ j \geq 0.$$

By the Parseval equality, we have for $X = (x_1, x_2, \cdots, x_d) \in e$

$$u(X) = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_d=0}^{\infty} \beta_{i_1 i_2 \cdots i_d} \omega_{1,i_1}(x_1)\omega_{2,i_2}(x_2)\cdots\omega_{d,i_d}(x_d), \tag{2.4}$$

where

$$\beta_{00\cdots 0} = u(x_{1,e} - h_{1,e}, x_{2,e} - h_{2,e}, \cdots, x_{d,e} - h_{d,e}),$$

$$\beta_{i_1 00\cdots 0} = \int_{I_1} \partial_{x_1} u(x_1, x_{2,e} - h_{2,e}, \cdots, x_{d,e} - h_{d,e}) l_{1,i_1-1}(x_1)\, dx_1,$$

$$\begin{aligned}\beta_{i_1 i_2 0\cdots 0} &= \int_{I_1\times I_2} \partial_{x_1}\partial_{x_2} u(x_1, x_2, x_{3,e} - h_{3,e}, \cdots, x_{d,e} - h_{d,e}) \\ &\quad l_{1,i_1-1}(x_1) l_{2,i_2-1}(x_2)\, dx_1 dx_2,\end{aligned}$$

$$\begin{aligned}\beta_{i_1 i_2\cdots i_d} &= \int_e \partial_{x_1}\partial_{x_2}\cdots\partial_{x_d} u(X) \\ &\quad l_{1,i_1-1}(x_1) l_{2,i_2-1}(x_2)\cdots l_{d,i_d-1}(x_d)\, dX,\end{aligned}$$

where $i_k \geq 1,\ k = 1, \cdots, d$. Similarly, the other coefficients can also be given.

We introduce a standard tensor-product polynomial spaces of degree $m \geq 1$ denoted by T_m, i.e.,

$$q(X) = \sum_{(i_1,i_2,\cdots,i_d)\in I} a_{i_1 i_2\cdots i_d} x_1^{i_1} x_2^{i_2}\cdots x_d^{i_d},\ q \in T_m,$$

where the indexing set I is as follows:

$$I = \{(i_1, i_2, \cdots, i_d) | 0 \leq i_1, i_2, \cdots, i_d \leq m\}.$$

Define the tensor-product interpolation operator of projection type by Π_m^e: $H^d(e) \to T_m(e)$ such that

$$\Pi_m^e u(X) = \sum_{(i_1,i_2,\cdots,i_d)\in I} \beta_{i_1 i_2\cdots i_d} \omega_{1,i_1}(x_1)\omega_{2,i_2}(x_2)\cdots\omega_{d,i_d}(x_d). \tag{2.5}$$

By the definitions of the finite element space $S_0^h(\Omega)$ and Π_m^e, we have the interpolation operator of project type

$$\Pi_m : H^d(\Omega) \cap H_0^1(\Omega) \to S_0^h(\Omega),$$

where $(\Pi_m u)|_e = \Pi_m^e u$.

In addition, the function $\omega_{k,i}(x_k)$ has the following properties (see [5]):

$$\begin{array}{ll} a. & \omega_{k,i}(x_{k,e} \pm h_{k,e}) = 0,\ i \geq 2, \\ b. & \omega_{k,i}(x_{k,e} - (x_k - x_{k,e})) = (-1)^i \omega_{k,i}(x_{k,e} + (x_k - x_{k,e})),\ i \geq 2, \\ c. & (\omega_{k,i},\, p_m) = 0,\ \forall\, p_m \in P_m(I_k),\ i \geq m+3, \\ d. & (\omega_{k,i},\, \omega_{k,j}) = 0,\ i, j \geq 2,\ i \neq j,\ \text{and } |i - j| \neq 2, \end{array} \tag{2.6}$$

where $k = 1, \cdots, d$. For simplicity, we write

$$\lambda_{i_1 i_2\cdots i_d} = \beta_{i_1 i_2\cdots i_d} \omega_{1,i_1}(x_1)\omega_{2,i_2}(x_2)\cdots\omega_{d,i_d}(x_d).$$

From (2.4) and (2.5),

$$\begin{aligned} R &= u - \Pi_m^e u \\ &= \Big(\sum_{i_1=0}^m \sum_{i_2=0}^m \cdots \sum_{i_{d-1}=0}^m \sum_{i_d=m+1}^\infty \\ &\quad + \sum_{i_1=0}^m \sum_{i_2=0}^m \cdots \sum_{i_{d-1}=m+1}^\infty \sum_{i_d=0}^\infty \\ &\quad + \cdots + \sum_{i_1=0}^m \sum_{i_2=m+1}^\infty \sum_{i_3=0}^\infty \cdots \sum_{i_d=0}^\infty \\ &\quad + \sum_{i_1=m+1}^\infty \sum_{i_2=0}^\infty \cdots \sum_{i_{d-1}=0}^\infty \sum_{i_d=0}^\infty\Big) \lambda_{i_1 i_2 \cdots i_d}, \end{aligned} \tag{2.7}$$

which is called remainder of interpolation. Next, we will derive the weak estimates for the finite element.

Theorem 2.1 *Let $\{\mathcal{T}^h\}$ be a regular family of rectangular partitions of $\bar{\Omega}$, $u \in W^{m+2,\infty}(\Omega) \cap H_0^1(\Omega)$, and $v \in S_0^h(\Omega)$. Then, the m-degree interpolation operator of projection type Π_m satisfies the following weak estimates:*

$$|a(u - \Pi_m u, v)| \le Ch^{m+1}\|u\|_{m+2,\,\infty,\,\Omega}|v|_{1,\,1,\,\Omega}, \;\; m \ge 1, \tag{2.8}$$

and

$$|a(u - \Pi_m u, v)| \le Ch^{m+2}\|u\|_{m+2,\,\infty,\,\Omega}|v|_{2,\,1,\,\Omega}^h, \;\; m \ge 2. \tag{2.9}$$

where $|v|_{2,\,1,\,\Omega}^h = \sum_{e\in\mathcal{T}^h} |v|_{2,\,1,\,e}$.

Proof. By the properties of $\omega_{k,i}(x_k)$ (see (2.6), c) as well as the orthogonality of the Legendre polynomial system, we have

$$\int_e \nabla R \cdot \nabla v \, dX = \int_e \nabla r \cdot \nabla v \, dX \equiv I_e \;\; \forall e \in \mathcal{T}^h,$$

where

$$\begin{aligned} r = &\Big(\sum_{i_1=0}^m \sum_{i_2=0}^m \cdots \sum_{i_{d-1}=0}^m \sum_{i_d=m+1}^{m+2} \\ &+ \sum_{i_1=0}^m \sum_{i_2=0}^m \cdots \sum_{i_{d-1}=m+1}^{m+2} \sum_{i_d=0}^{m+2} \\ &+ \cdots + \sum_{i_1=0}^m \sum_{i_2=m+1}^{m+2} \sum_{i_3=0}^{m+2} \cdots \sum_{i_d=0}^{m+2} \\ &+ \sum_{i_1=m+1}^{m+2} \sum_{i_2=0}^{m+2} \cdots \sum_{i_{d-1}=0}^{m+2} \sum_{i_d=0}^{m+2}\Big) \lambda_{i_1 i_2 \cdots i_d}. \end{aligned} \tag{2.10}$$

Obviously, r only contains finite terms.

Among the indices i_k, $k = 1, 2, \cdots, d$, when some $i_k = m+1$ or $m+2$, and the others are zero, we have by the orthogonality of the Legendre polynomial system

$$\int_e \nabla \lambda_{i_1 i_2 \cdots i_d} \cdot \nabla v \, dX = 0. \tag{2.11}$$

When only two of the indices i_k, $k = 1, 2, \cdots, d$ are nonzero, and the others are zero, without loss of generality, we assume $i_1 \ne 0$, $i_2 \ne 0$, and $i_3 = i_4 = \cdots = i_d = 0$. It is easy to see that $i_1 + i_2 \ge m+2$. Thus, the integration by parts yields

$$\beta_{i_1 i_2 0 \cdots 0} \;\; = \;\; \int_{I_1\times I_2} \partial_{x_1}\partial_{x_2} u(x_1, x_2, x_{3,e} - h_{3,e}, \cdots, x_{d,e} - h_{d,e})$$

$$
\begin{aligned}
& l_{1,i_1-1}(x_1)l_{2,i_2-1}(x_2)\,dx_1dx_2 \\
= \; & (-1)^{s+t}\int_{I_1\times I_2}\partial_{x_1}^{s+1}\partial_{x_2}^{t+1}u(x_1,x_2,x_{3,e}-h_{3,e},\cdots,x_{d,e}-h_{d,e}) \\
& D^{-s}l_{1,i_1-1}(x_1)D^{-t}l_{2,i_2-1}(x_2)\,dx_1dx_2,
\end{aligned}
$$

where $0\le s\le i_1-1,\ 0\le t\le i_2-1,\ s+t=m,$. The operator $D^{-n}(n\ge 1)$ denotes the integration operator of order n such that

$$\frac{d^n}{dx_i^n}(D^{-n}\varphi(x_i))=\varphi(x_i).$$

In particular, when $n=0$,

$$D^{-n}\varphi(x_i)=\varphi(x_i).$$

Thus,

$$|\beta_{i_1i_20\cdots0}|\le Ch^{m+1}\|u\|_{m+2,\,\infty,\,e}. \tag{2.12}$$

In addition

$$
\begin{aligned}
\left|\int_e\nabla\lambda_{i_1i_20\cdots0}\cdot\nabla v\,dX\right| &\le |\beta_{i_1i_20\cdots0}|\left|\int_e\nabla\left(\omega_{1,i_1}(x_1)\omega_{2,i_2}(x_2)\right)\cdot\nabla v\,dX\right| \\
&\le C|\beta_{i_1i_20\cdots0}|\int_e|\nabla v|\,dX
\end{aligned}
$$

Further, from (2.12), we have

$$\left|\int_e\nabla\lambda_{i_1i_20\cdots0}\cdot\nabla v\,dX\right|\le Ch^{m+1}\|u\|_{m+2,\,\infty,\,e}|v|_{1,\,1,\,e}. \tag{2.13}$$

Similar to the arguments as above, without loss of generality, when $i_k\ne 0,\ k=1,2,\cdots,j$ and $i_{j+1}=i_{j+2}=\cdots=i_d=0$, we have

$$\left|\int_e\nabla\lambda_{i_1i_2\cdots i_j0\cdots0}\cdot\nabla v\,dX\right|\le Ch^{m+1}\|u\|_{m+2,\,\infty,\,e}|v|_{1,\,1,\,e}. \tag{2.14}$$

Finally, we consider the case of $i_k\ne 0,\ k=1,2,\cdots,d$. Obviously, $\sum_{k=1}^d i_k\ge m+d$. We have by the integration by parts

$$
\begin{aligned}
\beta_{i_1i_2\cdots i_d} = \; & \int_e\partial_{x_1}\partial_{x_2}\cdots\partial_{x_d}u(X) \\
& l_{1,i_1-1}(x_1)l_{2,i_2-1}(x_2)\cdots l_{d,i_d-1}(x_d)\,dX \\
= \; & (-1)^{s_1+s_2+\cdots+s_d}\int_e\partial_{x_1}^{s_1+1}\partial_{x_2}^{s_2+1}\cdots\partial_{x_d}^{s_d+1}u(X) \\
& D^{-s_1}l_{1,i_1-1}(x_1)D^{-s_2}l_{2,i_2-1}(x_2)\cdots D^{-s_d}l_{d,i_d-1}(x_d)\,dX,
\end{aligned}
$$

where $0\le s_k\le i_k-1,\ k=1,\cdots,d$ and $\sum_{k=1}^d s_k=m+2-d$. Thus,

$$|\beta_{i_1i_2\cdots i_d}|\le Ch^{m+2-\frac{d}{2}}\|u\|_{m+2,\,\infty,\,e}. \tag{2.15}$$

Obviously,

$$\begin{aligned}\left|\int_e \nabla\lambda_{i_1 i_2\cdots i_d}\cdot\nabla v\,dX\right| &\le |\beta_{i_1 i_2\cdots i_d}|\left|\int_e \nabla\left(\omega_{1,i_1}(x_1)\omega_{2,i_2}(x_2)\cdots\omega_{d,i_d}(x_d)\right)\cdot\nabla v\,dX\right|\\ &\le Ch^{\frac{d-2}{2}}|\beta_{i_1 i_2\cdots i_d}|\int_e|\nabla v|\,dX\end{aligned}$$

Further, from (2.15), we have

$$\left|\int_e \nabla\lambda_{i_1 i_2\cdots i_d}\cdot\nabla v\,dX\right| \le Ch^{m+1}\|u\|_{m+2,\,\infty,\,e}|v|_{1,\,1,\,e}. \tag{2.16}$$

Combining (2.10), (2.11), (2.13), (2.14), and (2.16) yields

$$|I_e| \le Ch^{m+1}\|u\|_{m+2,\,\infty,\,e}|v|_{1,\,1,\,e}. \tag{2.17}$$

Summing over all elements proves the result (2.8). In the following, we will prove the result (2.9).

If $m\ge 2$, without loss of generality, we assume $i_k\ne 0$, $k=1,2,\cdots,j$ and $i_{j+1}=i_{j+2}=\cdots=i_d=0$.

$$\begin{aligned}I_{i_1 i_2\cdots i_j 0\cdots 0} &\equiv \int_e \nabla\lambda_{i_1 i_2\cdots i_j 0\cdots 0}\cdot\nabla v\,dX\\ &= \beta_{i_1 i_2\cdots i_j 0\cdots 0}\int_e \nabla\left(\omega_{1,i_1}(x_1)\omega_{2,i_2}(x_2)\cdots\omega_{j,i_j}(x_j)\right)\cdot\nabla v\,dX\\ &= \beta_{i_1 i_2\cdots i_j 0\cdots 0}\int_e \partial_{x_1}\left(\omega_{1,i_1}(x_1)\omega_{2,i_2}(x_2)\cdots\omega_{j,i_j}(x_j)\right)\partial_{x_1}v\,dX\\ &\quad+\beta_{i_1 i_2\cdots i_j 0\cdots 0}\int_e \partial_{x_2}\left(\omega_{1,i_1}(x_1)\omega_{2,i_2}(x_2)\cdots\omega_{j,i_j}(x_j)\right)\partial_{x_2}v\,dX\\ &\quad+\cdots+\beta_{i_1 i_2\cdots i_j 0\cdots 0}\int_e \partial_{x_j}\left(\omega_{1,i_1}(x_1)\omega_{2,i_2}(x_2)\cdots\omega_{j,i_j}(x_j)\right)\partial_{x_j}v\,dX\\ &= I_1+I_2+\cdots+I_j.\end{aligned} \tag{2.18}$$

We assume $i_1\ge m+1$, thus $i_1\ge m+1\ge 3$. By the orthogonality of the Legendre polynomial system,

$$I_1=\beta_{i_1 i_2\cdots i_j 0\cdots 0}\int_e l_{1,i_1-1}(x_1)\omega_{2,i_2}(x_2)\cdots\omega_{j,i_j}(x_j)\partial_{x_1}v\,dX=0. \tag{2.19}$$

In addition

$$\begin{aligned}I_2 &= \beta_{i_1 i_2\cdots i_j 0\cdots 0}\int_e \omega_{1,i_1}(x_1)l_{2,i_2-1}(x_2)\cdots\omega_{j,i_j}(x_j)\partial_{x_2}v\,dX\\ &= -\beta_{i_1 i_2\cdots i_j 0\cdots 0}\int_e D^{-1}\omega_{1,i_1}(x_1)l_{2,i_2-1}(x_2)\cdots\omega_{j,i_j}(x_j)\partial_{x_1}\partial_{x_2}v\,dX.\end{aligned} \tag{2.20}$$

Similar to the arguments of (2.12), we get

$$|\beta_{i_1 i_2\cdots i_j 0\cdots 0}| \le Ch^{m+2-\frac{j}{2}}\|u\|_{m+2,\,\infty,\,e}. \tag{2.21}$$

In fact

$$D^{-1}\omega_{1,i_1}(x_1)l_{2,i_2-1}(x_2)\cdots\omega_{j,i_j}(x_j)=\mathcal{O}(h^{\frac{j}{2}}). \tag{2.22}$$

Combining (2.20)–(2.22) yields

$$|I_2| \le Ch^{m+2}\|u\|_{m+2,\,\infty,\,e}|v|_{2,\,1,\,e}. \tag{2.23}$$

Similarly, we have

$$|I_k| \leq Ch^{m+2}\|u\|_{m+2,\,\infty,\,e}|v|_{2,\,1,\,e},\; k=3,\cdots,j. \tag{2.24}$$

From (2.18), (2.19), (2.23), and (2.24),

$$|I_{i_1 i_2 \cdots i_j 0 \cdots 0}| \leq Ch^{m+2}\|u\|_{m+2,\,\infty,\,e}|v|_{2,\,1,\,e},\; k=3,\cdots,j. \tag{2.25}$$

When each $i_k \neq 0$, $k=1,\cdots,d$, similar to the above arguments, we easily get

$$\left|\int_e \nabla\lambda_{i_1 i_2\cdots i_d}\cdot\nabla v\,dX\right| \leq Ch^{m+2}\|u\|_{m+2,\,\infty,\,e}|v|_{2,\,1,\,e}. \tag{2.26}$$

From (2.10), (2.11), (2.25), and (2.26),

$$|I_e| \leq Ch^{m+2}\|u\|_{m+2,\,\infty,\,e}|v|_{2,\,1,\,e}.$$

Summing over all elements proves the result (2.9).

3 Superconvergence of the Finite Element

In this section, we will give applications of the weak estimates. Some applications may be found in the published literatures. First we need to give the definitions of the discrete Green's function and the discrete derivative Green's function. For every $Z\in\Omega$, we define the discrete derivative Green's function $\partial_{Z,\ell}G_Z^h \in S_0^h$ and the discrete Green's function $G_Z^h \in S_0^h$ such that (see [7])

$$a(\partial_{Z,\ell}G_Z^h,\, v) = \partial_\ell v(Z) \quad \forall\, v\in S_0^h(\Omega), \tag{3.1}$$

$$a(G_Z^h,\, v) = v(Z) \quad \forall\, v\in S_0^h(\Omega), \tag{3.2}$$

where $\ell\in\mathcal{R}^d$ and $|\ell|=1$. $\partial_\ell v(Z)$ stands for the onesided directional derivative

$$\partial_\ell v(Z) = \lim_{|\Delta Z|\to 0}\frac{v(Z+\Delta Z)-v(Z)}{|\Delta Z|},\;\; \Delta Z = |\Delta Z|\ell.$$

As for $\partial_{Z,\ell}G_Z^h$ and G_Z^h, we have obtained some estimates (see [8–15]).

Using the weak estimates (see (2.8) and (2.9)) and the estimates for $\partial_{Z,\ell}G_Z^h$ and G_Z^h, we give superconvergent estimates of the multidimensional tensor-product m-degree finite element as following:

- In the case of $d=3$, we have (see [8–10])

$$\left|\partial_{Z,\ell}G_Z^h\right|_{1,1} = \mathcal{O}(|\ln h|^{\frac{4}{3}}), \tag{3.3}$$

$$\left|\partial_{Z,\ell}G_Z^h\right|_{2,1}^h = \mathcal{O}(h^{-1}), \tag{3.4}$$

$$\left|G_Z^h\right|_{2,1}^h = \mathcal{O}(|\ln h|^{\frac{2}{3}}). \tag{3.5}$$

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Thus, from (1.2), (2.8), (2.9), and (3.1)–(3,5), we get superconvergent estimates

$$|u_h - \Pi_m u|_{1,\infty,\Omega} \le Ch^{m+1}|\ln h|^{\frac{4}{3}}\|u\|_{m+2,\infty,\Omega},\ \ m=1,\ \ \text{(see [9])}$$

$$|u_h - \Pi_m u|_{1,\infty,\Omega} \le Ch^{m+1}\|u\|_{m+2,\infty,\Omega},\ \ m\ge 2,\ \ \text{(see [9])}$$

and

$$|u_h - \Pi_m u|_{0,\infty,\Omega} \le Ch^{m+2}|\ln h|^{\frac{2}{3}}\|u\|_{m+2,\infty,\Omega},\ \ m\ge 2.\ \ \text{(see [10])}$$

• In the case of $d=4$, we have

$$\left|\partial_{Z,\ell}G_Z^h\right|_{1,1} = \mathcal{O}(|\ln h|^{\frac{5}{4}}),\ \ \text{(see [11])} \tag{3.6}$$

$$\left|\partial_{Z,\ell}G_Z^h\right|_{2,1}^h = \mathcal{O}(h^{-1}|\ln h|^{\frac{1}{2}}),\ \ \text{(see [12])} \tag{3.7}$$

$$\left|G_Z^h\right|_{2,1}^h = \mathcal{O}(|\ln h|^{\frac{1}{2}}). \tag{3.8}$$

Thus, from (1.2), (2.8), (2.9), (3.1), (3.2), and (3.6)–(3,8), we get superconvergent estimates

$$|u_h - \Pi_m u|_{1,\infty,\Omega} \le Ch^{m+1}|\ln h|^{\frac{5}{4}}\|u\|_{m+2,\infty,\Omega},\ \ m=1,$$

$$|u_h - \Pi_m u|_{1,\infty,\Omega} \le Ch^{m+1}|\ln h|^{\frac{1}{2}}\|u\|_{m+2,\infty,\Omega},\ \ m\ge 2,\ \ \text{(see [12])}$$

and

$$|u_h - \Pi_m u|_{0,\infty,\Omega} \le Ch^{m+2}|\ln h|^{\frac{1}{2}}\|u\|_{m+2,\infty,\Omega},\ \ m\ge 2.$$

• In the case of $d=5$, we have (see [13, 14])

$$\left|\partial_{Z,\ell}G_Z^h\right|_{1,1} = \mathcal{O}(|\ln h|^{\frac{7}{5}}), \tag{3.9}$$

$$\left|G_Z^h\right|_{2,1}^h = \mathcal{O}(|\ln h|^{\frac{9}{5}}). \tag{3.10}$$

Thus, from (1.2), (2.8), (2.9), (3.1), (3.2), (3.9), and (3,10), we get superconvergent estimates

$$|u_h - \Pi_m u|_{1,\infty,\Omega} \le Ch^{m+1}|\ln h|^{\frac{7}{5}}\|u\|_{m+2,\infty,\Omega},\ \ m\ge 1,$$

and

$$|u_h - \Pi_m u|_{0,\infty,\Omega} \le Ch^{m+2}|\ln h|^{\frac{9}{5}}\|u\|_{m+2,\infty,\Omega},\ \ m\ge 2.$$

• In the case of $d=6$, we have

$$\left|\partial_{Z,\ell}G_Z^h\right|_{1,1} = \mathcal{O}(|\ln h|^{\frac{4}{3}}),\ \ \text{(see [15])} \tag{3.11}$$

$$\left|G_Z^h\right|_{2,1}^h = \mathcal{O}(|\ln h|^{\frac{4}{3}}). \tag{3.12}$$

Remark 1. The result (3.12) was submitted in JOCAAA.

Thus, from (1.2), (2.8), (2.9), (3.1), (3.2), (3.11), and (3,12), we get superconvergent estimates

$$|u_h - \Pi_m u|_{1,\infty,\Omega} \le Ch^{m+1}|\ln h|^{\frac{4}{3}}\|u\|_{m+2,\infty,\Omega}, \ \ m \ge 1, \quad \text{(see [15])}$$

and

$$|u_h - \Pi_m u|_{0,\infty,\Omega} \le Ch^{m+2}|\ln h|^{\frac{4}{3}}\|u\|_{m+2,\infty,\Omega}, \ \ m \ge 2.$$

- In the case of $d \ge 7$, we only have

$$\left|\partial_{Z,\ell} G_Z^h\right|_{1,1} = \mathcal{O}(h^{\frac{2-d}{2}}), \tag{3.13}$$

$$\left|G_Z^h\right|_{2,1}^h = \mathcal{O}(h^{\frac{4-d}{2}}). \tag{3.14}$$

Remark 2. According to the results (3.13) and (3.14), we can not obtain the pointwise superconvergent estimates in the case of $d \ge 7$.

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NEW INTEGRAL INEQUALITIES OF HERMITE-HADAMARD TYPE FOR OPERATOR m-CONVEX AND (α, m)-CONVEX FUNCTIONS

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ABSTRACT: In this paper, authors introduce the concepts of operator m-convex function and operator (α, m)-convex function, and establish some new integral inequalities of Hermite-Hadamard type for operator m-convex and (α, m)-convex functions.

KEY WORDS: Integral inequality; operator m-convex function; operator (α, m)-convex function.

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1. Introduction

Throughout this paper, we adopt the notations: $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_0 = [0, \infty)$.
We firstly list the definition of convex functions.

Definition 1.1. The function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex function if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

One of the most important integral inequalities for convex functions is the Hadamard inequality (or the Hermite-Hadamard inequality). The following double inequality is well known as the Hadamard inequality in the literature. If any f is convex function on $[a, b] \subseteq \mathbb{R}$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \frac{f(a)+f(b)}{2}. \tag{1.2}$$

Both inequalities hold in the reversed direction if f is concave on $[a, b]$. We note that the Hermite-Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality.

In the literature, the concepts of m-convexity and $(\alpha; m)$-convexity are well known. The concept of m-convexity was first introduced by G. Toader in [11] (see also [1]) and it is defined as follows:

Definition 1.2 ([11])**.** The function $f : [0, b] \to \mathbb{R}$, $b > 0$ is said to be m-convex, where $m \in [0, 1]$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$, we have

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y). \tag{1.3}$$

The class of (α, m)-convex functions was also first introduced in [8] and it is defined as follows:

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Definition 1.3 ([8]). The function $f : [0, b] \to \mathbb{R}$, $b > 0$ is said to be (α, m)-convex, where $(\alpha, m) \in [0, 1]^2$, if we have

$$f(tx + m(1-t)y) \le t^\alpha f(x) + m(1-t^\alpha)f(y) \tag{1.4}$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

In [1], S. S. Dragomir and G. Toader proved the following Hadamard type inequalities for m-convex functions.

Theorem 1.1 ([1]). *Let $f : \mathbb{R}_0 \to \mathbb{R}$ be an m-convex function with $m \in (0, 1]$. if $0 \le a < b < \infty$ and $f \in L_1[a, b]$, then the following inequality holds*

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \min\left\{\frac{f(a) + mf\left(\frac{b}{m}\right)}{2}, \frac{f(b) + mf\left(\frac{a}{m}\right)}{2}\right\}. \tag{1.5}$$

In [2], S. S. Dragomir established new Hadamard-type inequalities for m-convex functions.

Theorem 1.2 ([2]). *Let $f : \mathbb{R}_0 \to \mathbb{R}$ be an m-convex function with $m \in (0, 1]$. if $0 \le a < b < \infty$ and $f \in L_1[am, b]$, then the following inequality holds*

$$\frac{1}{m+1}\left[\frac{1}{mb-a}\int_a^{mb} f(x)\,\mathrm{d}x + \frac{1}{b-ma}\int_{ma}^b f(x)\,\mathrm{d}x\right] \le \frac{f(a)+f(b)}{2}. \tag{1.6}$$

In [10], E. Set et al. proved the following Hadamard type inequalities for (α, m)-convex functions.

Theorem 1.3 ([10]). *Let $f : \mathbb{R}_0 \to \mathbb{R}$ be an (α, m)-convex function with $(\alpha, m) \in (0, 1]^2$. if $0 \le a < b < \infty$ and $f \in L_1[a, b]$, then the following inequality holds*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) \le & \frac{1}{2^\alpha(b-a)}\int_a^b \left[f(x) + m(2^\alpha - 1)f\left(\frac{x}{m}\right)\right]\mathrm{d}x \\ \le & \frac{1}{2^{\alpha+1}(\alpha+1)}\left\{f(a) + f(b) + m(\alpha + 2^\alpha - 1)\left[f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right)\right]\right. \\ & \left. + \alpha m^2(2^\alpha - 1)\left[f\left(\frac{a}{m^2}) + f(\frac{b}{m^2}\right)\right]\right\}\mathrm{d}x. \end{aligned} \tag{1.7}$$

Some generalizations of this result can be found in [12] and [13].

In recent years, several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of operator convex functions introduced by S. S. Dragomir in [5].

Now we review the operator order in $B(H)$ and the continuous functional calculus for a bounded self-adjoint operator. For self-adjoint operators $A, B \in B(H)$, we write $A \le B$ if $\langle Ax, x\rangle \le \langle Bx, x\rangle$ for every vector $x \in H$, we call it the operator order.

Let A be a bounded self-adjoint linear operator on a complex Hilbert space $(H; \langle ., .\rangle)$. The Gelfand map establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all continuous complex-valued functions defined on the spectrum of A, denoted $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [6], p.3). For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$, we have

(*i*) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;

(*ii*) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(f^*) = \Phi(f)^*$;

(*iii*) $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f|$;

(*iv*) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$ for $t \in Sp(A)$.

With this notation, we define

$$f(A) := \Phi(f) \quad \text{for all} \quad f \in C(Sp(A)) \tag{1.8}$$

and we call it the continuous functional calculus for a bounded self-adjoint operator A.

If A is a bounded self-adjoint operator and f is a real-valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a positive operator on H. Moreover, if both f and g are real-valued functions on $Sp(A)$ such that $f(t) \leq g(t)$ for any $t \in Sp(A)$, then $f(A) \leq f(B)$ in the operator order in $B(H)$.

We denoted by $B(H)^+$ the set of all positive operators in $B(H)$ and

$$C(H) := \{A \in B(H)^+ : AB + BA \geq 0 \quad \text{for all} \quad B \in B(H)^+\}. \tag{1.9}$$

It is obvious that $C(H)$ is a closed convex cone in $B(H)$.

A real valued continuous function f on an interval $I \subseteq \mathbb{R}$ is said to be operator convex (operator concave) if the operator inequality

$$f((1-t)A + tB) \leq (\geq)(1-t)f(A) + tf(B) \tag{1.10}$$

holds in the operator order in $B(H)$, for all $t \in [0,1]$ and for every bounded self-adjoint operators A and B in $B(H)$ whose spectra are contained in I.

In [5], S. S. Dragomir gave the operator version of the Hermite-Hadamard inequality for operator convex functions.

Theorem 1.4. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be an operator convex function on the interval I. Then for any self-adjoint operators A and B with spectra in I, we have the inequality*

$$\begin{aligned} &f\left(\frac{A+B}{2}\right) \leq \frac{1}{2}\left[f\left(\frac{3A+B}{4}\right) + f\left(\frac{A+3B}{4}\right)\right] \\ &\leq \int_0^1 f(tA + (1-t)B)\,\mathrm{d}t \leq \frac{1}{2}\left[f\left(\frac{A+B}{2}\right) + \frac{f(A)+f(B)}{2}\right] \leq \frac{f(A)+f(B)}{2}. \end{aligned} \tag{1.11}$$

For recent results related to Hermite-Hadamard type inequalities are given in [3], [4], [6], [7], and plenty of references therein.

The goal of this paper is to obtain new inequalities like those given in Theorems 1.1, 1.2, 1.3, but now for operator m-convex and (α, m)-convex functions.

2. Operator m-convex and (α, m)-convex functions

In order to verify our main results, the following preliminary definitions and lemmas are necessary.

Definition 2.1. Let $[0, b] \subseteq \mathbb{R}_0$ with $b > 0$ and K be a convex set of $B(H)^+$. A continuous function $f : [0, b] \to \mathbb{R}$ is said to be operator m-convex on $[0, b]$ for operators in K, if

$$f(tA + m(1-t)B) \leq tf(A) + m(1-t)f(B) \tag{2.1}$$

in the operator order in $B(H)$, for all $t \in [0,1]$ and every positive operators A and B in K whose spectra are contained in $[0, b]$ and for some fixed $m \in [0,1]$.

Remark 2.1. For $m = 1$, we recapture the concept of operator convex functions defined on $[0, b]$ and for $m = 0$ we get the concept of operator starshaped functions on $[0, b]$, namely, we call $f : [0, b] \to \mathbb{R}$ to be operator starshaped if

$$f(tA) \leq tf(A) \tag{2.2}$$

for all $t \in [0,1]$ and every positive operators A in $B(H)^+$ whose spectra are contained in $[0, b]$.

Lemma 2.1. *If f is operator m-convex, then $f(0) \leq 0$, where 0 is the zero operator on H.*

Proof. Taking $A = 0$ and $B = 0$ in the inequality (2.1), then

$$(1-t)(1-m)f(0) \le 0.$$

Also by $t, m \in [0,1]$, we get $f(0) \le 0$. □

Lemma 2.2. *If f is operator m-convex, then f is operator starshaped.*

Proof. For all $t \in [0,1]$ and positive operators $A \in B(H)^+$ whose spectra is contained in $[0,b]$, we write

$$f(tA) = f(tA + m(1-t)0) \le tf(A) + m(1-t)f(0) \le tf(A).$$

□

Lemma 2.3. *If f is operator m_1-convex and $0 \le m_2 < m_1 \le 1$, then f is operator m_2-convex.*

Proof. For all $t \in [0,1]$ and positive operators $A, B \in B(H)^+$ whose spectra are contained in $[0,b]$, we drive

$$\begin{aligned} f(tA + m_2(1-t)B) &= f\Big(tA + m_1(1-t)\Big(\frac{m_2}{m_1}\Big)B\Big) \le tf(A) + m_1(1-t)f\Big(\frac{m_2}{m_1}B\Big) \\ &\le tf(A) + m_1(1-t)\frac{m_2}{m_1}f(B) = tf(A) + m_2(1-t)f(B). \end{aligned}$$

□

Definition 2.2. Let $[0,b] \subseteq \mathbb{R}_0$ with $b > 0$ and K be a convex set of $B(H)^+$. A continuous function $f : [0,b] \to \mathbb{R}$ is said to be operator (α, m)-convex on $[0,b]$ for operators in K, if

$$f(tA + m(1-t)B) \le t^\alpha f(A) + m(1-t^\alpha)f(B) \tag{2.3}$$

in the operator order in $B(H)$, for all $t \in [0,1]$ and every positive operators A and B in K whose spectra are contained in $[0,b]$ and for some fixed $(\alpha, m) \in [0,1]^2$.

Remark 2.2. It can be easily seen that for $(\alpha, m) \in \{(0,0), (1,1), (1,m)\}$ one obtains the following classes of functions: operator increasing, operator convex and operator m-convex functions respectively.

Similarly to the proof of Lemma 2.1, the following result is valid.

Lemma 2.4. *If f is operator (α, m)-convex, then $f(0) \le 0$, where 0 is the zero operator on H.*

Lemma 2.5 ([9])**.** *Let $A, B \in B(H)^+$. Then $AB + BA$ is positive if and only if $f(A+B) \le f(A) + f(B)$ for all non-negative operator monotone functions f on $\mathbb{R}_0$.*

Now, we give an example of operator m-convex function.

Example 2.1. Since for every positive operator $A, B \in C(H)$, $AB + BA \ge 0$. Utilizing Lemma 2.5, we obtain

$$[tA + m(1-t)B]^s \le t^s A^s + m^s (1-t)^s B^s \le tA^s + m(1-t)B^s.$$

Therefore, the continuous function $f(t) = t^s (0 < s \le 1)$ is operator m-convex on $\mathbb{R}_0$ for operators in $C(H)$.

Remark 2.3. We can consider the same continuous function $f(t) = t^s (0 < s \le 1)$ as an example of operator (α, m)-convex function for $\alpha = 1$.

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3. Some new Hermite-Hadamard type inequalities

We will now point out some new results of the Hermite-Hadamard type.

Theorem 3.1. *Let the continuous function $f:\mathbb{R}_0\to\mathbb{R}$ be operator (α,m)-convex for operators in $K\subseteq B(H)^+$ with $(\alpha,m)\in[0,1]\times(0,1]$. Then for all positive operator $A,B\in K$ with spectra in $\mathbb{R}_0$, the following inequality holds:*

$$\int_0^1 f(tA+(1-t)B)\,\mathrm{d}t \le min\left\{\frac{f(A)+\alpha m f\left(\frac{B}{m}\right)}{\alpha+1},\frac{f(B)+\alpha m f\left(\frac{A}{m}\right)}{\alpha+1}\right\}. \tag{3.1}$$

Proof. For $x\in H$ with $\|x\|=1$ and $m,t\in(0,1]$, we have

$$\big\langle (tA+m(1-t)B)x,x\big\rangle = t\langle Ax,x\rangle + m(1-t)\langle Bx,x\rangle\in\mathbb{R}_0, \tag{3.2}$$

since $\langle Ax,x\rangle\in Sp(A)\subseteq\mathbb{R}_0$ and $\langle Bx,x\rangle\in Sp(B)\subseteq\mathbb{R}_0$.

Continuity of f and (3.2) imply that the operator-valued integral $\int_0^1 f(tA+m(1-t)B)\,\mathrm{d}t$ exists.

Since f is operator (α,m)-convex, therefore for $(\alpha,m)\in[0,1]\times(0,1]$ and $A,B\in K$, we show

$$f(tA+(1-t)B)\le t^\alpha f(A)+m(1-t^\alpha)f\left(\frac{B}{m}\right)$$

and

$$f(tB+(1-t)A)\le t^\alpha f(B)+m(1-t^\alpha)f\left(\frac{A}{m}\right)$$

for all $t\in[0,1]$.

Integrating over t on $[0,1]$, we obtain

$$\int_0^1 f(tA+(1-t)B)\,\mathrm{d}t\le\frac{f(A)+\alpha m f\left(\frac{B}{m}\right)}{\alpha+1}$$

and

$$\int_0^1 f(tB+(1-t)A)\,\mathrm{d}t\le\frac{f(B)+\alpha m f\left(\frac{A}{m}\right)}{\alpha+1}.$$

However

$$\int_0^1 f(tA+(1-t)B)\,\mathrm{d}t=\int_0^1 f(tB+(1-t)A)\,\mathrm{d}t,$$

and the inequality (3.1) is obtained, which completes the proof. □

Corollary 3.1.1. *Under the assumptions of Theorem 3.1, choosing $\alpha=1$, we get the inequality for operator m-convex functions:*

$$\int_0^1 f(tA+(1-t)B)\,\mathrm{d}t\le min\left\{\frac{f(A)+mf\left(\frac{B}{m}\right)}{2},\frac{f(B)+mf\left(\frac{A}{m}\right)}{2}\right\}. \tag{3.3}$$

Furthermore, for $\alpha,m=1$ we have

$$\int_0^1 f(tA+(1-t)B)\,\mathrm{d}t\le\frac{f(A)+f(B)}{2}. \tag{3.4}$$

Theorem 3.2. *Let the continuous function $f:\mathbb{R}_0\to\mathbb{R}$ be operator (α,m)-convex for operators in $K\subseteq B(H)^+$ with $(\alpha,m)\in[0,1]\times(0,1]$. Then for all positive operator $A,B\in K$ with spectra in $\mathbb{R}_0$, the following inequalities hold:*

$$f\left(\frac{A+B}{2}\right)\le\frac{1}{2^\alpha}\int_0^1\left[f(tA+(1-t)B)+m(2^\alpha-1)f\left(\frac{(1-t)A+tB}{m}\right)\right]\mathrm{d}t$$

$$\le \frac{1}{2^{\alpha+1}(\alpha+1)}\left\{f(A)+f(B)+m(\alpha+2^\alpha-1)\left[f\left(\frac{A}{m}\right)+f\left(\frac{B}{m}\right)\right]\right.$$
$$\left.+\alpha m^2(2^\alpha-1)\left[f\left(\frac{A}{m^2}\right)+f\left(\frac{B}{m^2}\right)\right]\right\}. \tag{3.5}$$

Proof. By operator (α, m)-convexity of f, we give

$$\begin{aligned} f\left(\frac{A+B}{2}\right) &\le \frac{1}{2^\alpha}f(tA+(1-t)B)+m\left(1-\frac{1}{2^\alpha}\right)f\left(\frac{(1-t)A+tB}{m}\right)\\ &= \frac{1}{2^\alpha}\left[f(tA+(1-t)B)+m(2^\alpha-1)f\left(\frac{(1-t)A+tB}{m}\right)\right], \end{aligned} \tag{3.6}$$

where $(\alpha, m)\in[0,1]\times(0,1], t\in[0,1]$ and $A, B\in K$ with spectra in $\mathbb{R}_0$.

Integrating over $t\in[0,1]$, we drive the first inequality in (3.5).

Next, from operator (α, m)-convexity of f, we also deduce

$$\begin{aligned} &\frac{1}{2^\alpha}\left[f(tA+(1-t)B)+m(2^\alpha-1)f\left(\frac{(1-t)A+tB}{m}\right)\right]\\ \le&\frac{1}{2^\alpha}\left\{t^\alpha f(A)+m(1-t^\alpha)f\left(\frac{B}{m}\right)+m(2^\alpha-1)\left[t^\alpha f\left(\frac{B}{m}\right)+m(1-t^\alpha)f\left(\frac{A}{m^2}\right)\right]\right\}. \end{aligned} \tag{3.7}$$

Integrating over t on $[0,1]$, we get

$$\begin{aligned} &\frac{1}{2^\alpha}\int_0^1\left[f(tA+(1-t)B)+m(2^\alpha-1)f\left(\frac{(1-t)A+tB}{m}\right)\right]\mathrm{d}t\\ =&\frac{1}{2^\alpha(\alpha+1)}\left\{f(A)+m(\alpha+2^\alpha-1)f\left(\frac{B}{m}\right)+\alpha m^2(2^\alpha-1)f\left(\frac{A}{m^2}\right)\right]. \end{aligned} \tag{3.8}$$

Similarly, taking into account that

$$\int_0^1 f(tA+(1-t)B)\,\mathrm{d}t=\int_0^1 f(tB+(1-t)A)\,\mathrm{d}t$$

and changing the roles of A and B, we obtain

$$\begin{aligned} &\frac{1}{2^\alpha}\int_0^1\left[f(tA+(1-t)B)+m(2^\alpha-1)f\left(\frac{(1-t)A+tB}{m}\right)\right]\mathrm{d}t\\ =&\frac{1}{2^\alpha(\alpha+1)}\left\{f(B)+m(\alpha+2^\alpha-1)f\left(\frac{A}{m}\right)+\alpha m^2(2^\alpha-1)f\left(\frac{B}{m^2}\right)\right]. \end{aligned} \tag{3.9}$$

Summing the inequalities (3.8) and (3.9) and dividing by 2, we get the second inequality in (3.5). The proof thus is complete. □

Corollary 3.2.1. *With the conditions of Theorem 3.2, taking $\alpha=1$, we obtain the inequalities for operator m-convex functions:*

$$\begin{aligned} f\left(\frac{A+B}{2}\right) \le&\frac{1}{2}\int_0^1\left[f(tA+(1-t)B)+mf\left(\frac{(1-t)A+tB}{m}\right)\right]\mathrm{d}t\\ \le&\frac{1}{8}\left\{f(A)+f(B)+2m\left[f\left(\frac{A}{m}\right)+f\left(\frac{B}{m}\right)\right]+m^2\left[f\left(\frac{A}{m^2}\right)+f\left(\frac{B}{m^2}\right)\right]\right\}. \end{aligned} \tag{3.10}$$

In addition, if $\alpha, m=1$, we have

$$f\left(\frac{A+B}{2}\right)\le\int_0^1 f(tA+(1-t)B)\,\mathrm{d}t\le\frac{f(A)+f(B)}{2}. \tag{3.11}$$

Theorem 3.3. *Let the continuous function $f : \mathbb{R}_0 \to \mathbb{R}$ be operator (α, m)-convex for operators in $K \subseteq B(H)^+$ with $(\alpha, m) \in [0, 1] \times (0, 1]$. Then for all positive operator $A, B \in K$ with spectra in $\mathbb{R}_0$, the following inequality holds:*

$$\int_0^1 f(tA + (1-t)B)\,\mathrm{d}t \le \frac{f(A) + f(B) + \alpha m\big[f\big(\frac{A}{m}\big) + f\big(\frac{B}{m}\big)\big]}{2(\alpha+1)}. \tag{3.12}$$

Proof. Using operator (α, m)-convexity of f, we can write

$$f(tA + (1-t)B) \le t^\alpha f(A) + m(1-t^\alpha) f\left(\frac{B}{m}\right)$$

and

$$f(tB + (1-t)A) \le t^\alpha f(B) + m(1-t^\alpha) f\left(\frac{A}{m}\right)$$

for all $t \in [0, 1]$ and some fixed $(\alpha, m) \in [0, 1] \times (0, 1]$.

Adding the above inequalities and integrating over t on $[0, 1]$, we have

$$\int_0^1 f(tA + (1-t)B)\,\mathrm{d}t + \int_0^1 f(tB + (1-t)A)\,\mathrm{d}t \le \frac{f(A) + f(B) + \alpha m\big[f\big(\frac{A}{m}\big) + f\big(\frac{B}{m}\big)\big]}{\alpha+1}.$$

As it is easy to see that

$$\int_0^1 f(tA + (1-t)B)\,\mathrm{d}t = \int_0^1 f(tB + (1-t)A)\,\mathrm{d}t,$$

we deduce the desired result. The proof of Theorem 3.3 is complete. □

Corollary 3.3.1. *Under the assumptions of Theorem 3.3, letting $\alpha = 1$, we get the inequality for operator m-convex functions:*

$$\int_0^1 f(tA + (1-t)B)\,\mathrm{d}t \le \frac{f(A) + f(B) + m\big[f\big(\frac{A}{m}\big) + f\big(\frac{B}{m}\big)\big]}{4}. \tag{3.13}$$

In addition, for $\alpha, m = 1$, we have

$$\int_0^1 f(tA + (1-t)B)\,\mathrm{d}t \le \frac{f(A) + f(B)}{2}. \tag{3.14}$$

Theorem 3.4. *Let the continuous function $f : \mathbb{R}_0 \to \mathbb{R}$ be operator (α, m)-convex for operators in $K \subseteq B(H)^+$ with $(\alpha, m) \in [0, 1] \times (0, 1]$. Then for all positive operator $A, B \in K$ with spectra in $\mathbb{R}_0$, the following inequality holds:*

$$\int_0^1 \big[f(tA + m(1-t)B) + f(tB + m(1-t)A)\big]\,\mathrm{d}t \le \frac{(1+m\alpha)[f(A) + f(B)]}{\alpha+1}. \tag{3.15}$$

Proof. By operator (α, m)-convexity of f, we can obtain

$$\begin{gathered} f(tA + m(1-t)B) \le t^\alpha f(A) + m(1-t^\alpha) f(B), \\ f((1-t)A + mtB) \le (1-t)^\alpha f(A) + m(1-(1-t)^\alpha) f(B), \\ f(tB + m(1-t)A) \le t^\alpha f(B) + m(1-t^\alpha) f(A), \end{gathered}$$

and

$$f((1-t)B + mtA) \le (1-t)^\alpha f(B) + m(1-(1-t)^\alpha) f(A)$$

for all $t \in [0, 1]$ and some fixed $(\alpha, m) \in [0, 1] \times (0, 1]$.

Adding the above inequalities with each other, we get

$$\begin{aligned} &f(tA + m(1-t)B) + f((1-t)A + mtB) + f(tB + m(1-t)A) + f((1-t)B + mtA) \\ &\le \big[t^\alpha + (1-t)^\alpha + m(1-t^\alpha) + m(1-(1-t)^\alpha)\big]\big[f(A) + f(B)\big]. \end{aligned}$$

Now integrating over $t \in [0,1]$ and taking into account that

$$\int_0^1 f(tA + m(1-t)B)\,\mathrm{d}t = \int_0^1 f((1-t)A + mtB)\,\mathrm{d}t$$

and

$$\int_0^1 f(tB + m(1-t)A)\,\mathrm{d}t = \int_0^1 f((1-t)B + mtA)\,\mathrm{d}t$$

we obtain the inequality (3.15). The proof of Theorem 3.4 is complete. □

Corollary 3.4.1. *Under the assumptions of Theorem 3.4, choosing $\alpha = 1$, we get the inequality for operator m-convex functions:*

$$\int_0^1 \big[f(tA + m(1-t)B) + f(tB + m(1-t)A)\big]\,\mathrm{d}t \le \frac{(1+m)[f(A)+f(B)]}{2}. \tag{3.16}$$

Moreover, for $\alpha, m = 1$, we have

$$\int_0^1 f(tA + (1-t)B)\,\mathrm{d}t \le \frac{f(A)+f(B)}{2}. \tag{3.17}$$

Theorem 3.5. *Let the continuous function $f : \mathbb{R}_0 \to \mathbb{R}$ be operator (α, m)-convex for operators in $K \subseteq B(H)^+$ with $(\alpha, m) \in [0,1] \times (0,1]$. Then for all positive operator $A, B \in K$ with spectra in $\mathbb{R}_0$, the following inequalities hold:*

$$\begin{aligned}
&f\left(\frac{2-m}{2}B + \frac{m}{2}(mA)\right)\\
\le&\frac{1}{2^\alpha}\int_0^1 \left[f\big(t(2-m)B + (1-t)m^2A\big) + m(2^\alpha - 1) f\left(\frac{(1-t)(2-m)B + tm^2A}{m}\right)\right]\mathrm{d}t\\
\le&\frac{1}{2^\alpha(\alpha+1)}\Big[f((2-m)B) + m\big(\alpha + 2^\alpha - 1\big)f(mA)\\
&+ m^2\alpha(2^\alpha - 1) f\left(\frac{(2-m)B}{m^2}\right)\Big].
\end{aligned} \tag{3.18}$$

Proof. From operator (α, m)-convexity of f, we can deduce

$$\begin{aligned}
&f\left(\frac{2-m}{2}B + \frac{m}{2}(mA)\right)\\
\le&\frac{1}{2^\alpha} f\big(t(2-m)B + (1-t)m^2A\big) + m\left(1 - \frac{1}{2^\alpha}\right) f\left(\frac{(1-t)(2-m)B + tm^2A}{m}\right)\\
=&\frac{1}{2^\alpha}\left[f\big(t(2-m)B + (1-t)m^2A\big) + m(2^\alpha - 1) f\left(\frac{(1-t)(2-m)B + tm^2A}{m}\right)\right],
\end{aligned} \tag{3.19}$$

where $(\alpha, m) \in [0,1] \times [0,1], t \in (0,1]$ and $A, B \in K$ with spectra in $\mathbb{R}_0$.

Integrating over $t \in [0,1]$, we drive the first inequality in (3.18).

Next, by operator (α, m)-convexity of f, we also write

$$f\big(t(2-m)B + (1-t)m^2A\big) \le t^\alpha f((2-m)B) + m(1-t^\alpha) f(mA) \tag{3.20}$$

and

$$f\left(\frac{(1-t)(2-m)B + tm^2A}{m}\right) \le t^\alpha f(mA) + m(1-t^\alpha) f\left(\frac{(2-m)B}{m^2}\right). \tag{3.21}$$

Submitting the inequalities (3.20) and (3.21) into the inequality (3.19), we get

$$\frac{1}{2^\alpha}\left[f\big(t(2-m)B + (1-t)m^2A\big) + m(2^\alpha - 1) f\left(\frac{(1-t)(2-m)B + tm^2A}{m}\right)\right]$$

$$\leq \frac{1}{2^\alpha}\bigg\{t^\alpha f((2-m)B) + m\big[1-t^\alpha+t^\alpha(2^\alpha-1)\big]f(mA) + m^2(2^\alpha-1)(1-t^\alpha)f\left(\frac{(2-m)B}{m^2}\right)\bigg\}. \tag{3.22}$$

Integrating over t on $[0,1]$, we deduce the second inequality in (3.18). This completes the proof of the Theorem 3.5. □

Corollary 3.5.1. *Under the assumptions of Theorem 3.5, letting $\alpha = 1$, we get the inequalities for operator m-convex functions:*

$$\begin{aligned} & f\left(\frac{2-m}{2}B + \frac{m}{2}(mA)\right) \\ \leq & \frac{1}{2}\int_0^1 \left[f\big(t(2-m)B + (1-t)m^2A\big) + mf\left(\frac{(1-t)(2-m)B + tm^2A}{m}\right)\right] \mathrm{d}t \\ \leq & \frac{1}{4}\left[f((2-m)B) + 2mf(mA) + m^2 f\left(\frac{(2-m)B}{m^2}\right)\right]. \end{aligned} \tag{3.23}$$

In addition, for $\alpha, m = 1$, we drive

$$f\left(\frac{A+B}{2}\right) \leq \int_0^1 f(tA + (1-t)B)\,\mathrm{d}t \leq \frac{f(A)+f(B)}{2}. \tag{3.24}$$

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Non-periodic Multivariate Stochastic Fourier Sine Approximation and Uncertainty Analysis *

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Abstract. In data analysis, one needs to study Fourier sine analysis on the unit cube. However, for this kind of non-periodic case, no exact result is available. In this paper, firstly, based on our multivariate function decomposition, we deduce an asymptotic formula of Fourier sine coefficients of continuously differentiable functions f on $[0,1]^d$. Secondly, we deduce an asymptotic formula of hyperbolic cross approximations of Fourier sine series of f on $[0,1]^d$. By this way we can reconstruct high-dimensional signals by using fewest Fourier sine coefficients. Thirdly, we extend these results to Fourier sine analysis of stochastic processes and give uncertainty of stochastic Fourier sine approximation, i.e., we obtain expectations and variances of stochastic Fourier sine coefficients and stochastic Fourier sine approximation errors. Finally, we discuss some known stochastic processes.

Key words: asymptotic behavior, multivariate decomposition, stochastic approximation, hyperbolic cross truncation

1. Introduction

It is well known that Fourier sine analysis on $[0,1]^d$ is an important tool for signal processing. Based our decomposition of multivariate continuous functions on the cube [11], we first deduce an asymptotic formula of Fourier sine coefficients of continuously differentiable function f on $[0,1]^d$ and obtain a necessary and sufficient condition:

$$c_{\mathbf{n}}(f) = o\left(\frac{1}{n_1 \cdots n_d}\right)$$

for each $n_k \to \infty$. Next we deduce an asymptotic formula of hyperbolic cross truncations of the Fourier sine series of f. Thirdly, we extend these results to the case of stochastic processes. In detail, we will obtain the following three asymptotic behaviors of stochastic Fourier sine analysis.

Suppose that ξ is a continuously differentiable stochastic process on $[0,1]^d$.

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(i) The expectation of Fourier sine coefficients $c_{\mathbf{n}}(\xi)$ satisfy

$$E[\,c_{\mathbf{n}}(\xi)\,] = \left(\prod_{j=1}^{d} \frac{2}{\pi n_j}\right)(\alpha_{\mathbf{n}}(\xi) + o(1)) \qquad (\mathbf{n} = (n_1, ..., n_d) \in \mathbb{Z}_+^d)$$

for each $n_k \to \infty$, where $\alpha_{\mathbf{n}}(\xi)$ is an algebraic sum of expectation of ξ at vertexes of the cube $[0,1]^d$.

(ii) The variance of Fourier sine coefficients $c_{\mathbf{n}}(\xi)$ satisfy

$$\mathrm{Var}\,(c_{\mathbf{n}}(\xi)) = \left(\prod_{j=1}^{d} \frac{4}{\pi^2 n_j^2}\right)(\theta_{\mathbf{n}}(\xi) + o(1))$$

for each $n_k \to \infty$, where $\theta_{\mathbf{n}}(\xi)$ is an algebraic sum of covariance of $\xi(\lambda)$ and $\xi(\lambda')$, where λ and λ' are any two vertexes of $[0,1]^d$.

(iii) The mean square error of hyperbolic cross truncations $S_N^{(h)}(\xi)$ (see (6.1)) of the Fourier sine series of ξ satisfies

$$E[\| \, S_N^{(h)}(\xi) - \xi \, \|_2^2] = W_N(1 + o(1)) \qquad (N \to \infty),$$

the principal part W_N is equal to

$$W_N = \left(\frac{1}{\pi^2}\right)^d \left(\sum_{\mathbf{p}\in\Theta_N} \frac{1}{p_1^2 p_2^2 \cdots p_d^2}\right)\left(\sum_{\lambda\in\{0,1\}^d} E[|\xi(\lambda)|^2]\right)$$

and

$$W_N \sim \frac{\log^{d-1} N}{N},$$

where $\{0,1\}^d$ is the set of vertexes of the cube $[0,1]^d$. The number of Fourier sine coefficients in the Nth hyperbolic cross truncation $S_N^{(h)}(\xi)$ satisfies $N_c \sim N \log^{d-1} N$. So

$$E[\| \, \xi - S_N^{(h)}(\xi) \, \|_2^2] \sim \frac{\log^{2d-2} N_c}{N_c}.$$

However, the number of Fourier sine coefficients in the Nth ordinary partial sum satisfies $N_c = N^d$. So

$$E[\| \, \xi - S_N(\xi) \, \|_2^2] \sim \frac{1}{N_c^{\frac{1}{d}}}.$$

Finally, we discuss some known stochastic processes.

2. Preliminaries

Denote the set of vertexes of the unit cube $[0,1]^d$ by $\{0,1\}^d$ and the boundary of $[0,1]^d$ by $\partial([0,1]^d)$, and $\mathbb{Z}_+^d = \{(n_1, ..., n_d)|$ each $n_k \in \mathbb{Z}_+\}$ and $\mathbb{Z}_+$ is the set of natural numbers.

If f is a function defined on $[0,1]^d$ and $\frac{\partial^d f}{\partial t_1 \cdots \partial t_d}$ continuous on $[0,1]^d$, we say $f \in W([0,1]^d)$. If ξ is a stochastic process defined on $[0,1]^d$ and $\frac{\partial^d \xi}{\partial t_1 \cdots \partial t_d}$ continuous on $[0,1]^d$, we say $\xi \in SW([0,1]^d)$. Denote the expectation and variance of a stochastic variable η by $E[\eta]$ and $\mathrm{Var}(\eta)$, respectively. Denote the covariance and correlation of two stochastic variable ξ, η by $\mathrm{Cov}(\xi, \eta)$ and $R(\xi, \eta)$, respectively.

2.1. Projection operators and fundamental polynomials

We always assume e_1 and e_2 are two disjoint subsets of the set $\{1,2,...,d\}$. Define a projection operator Q_{e_1,e_2} from $[0,1]^d$ to $\partial([0,1]^d)$ as

$$Q_{e_1,e_2}(t_1,...,t_d) = (v_1,...,v_d), \tag{2.1}$$

where

$$v_k = \begin{cases} 0, & k \in e_1, \\ 1, & k \in e_2, \\ t_k, & k \in e \quad (e = \{1,...,d\} \setminus (e_1 \bigcup e_2)). \end{cases}$$

The fundamental polynomial $P^{(e_1,e_2)}(\mathbf{t})$ is defined as

$$P^{(e_1,e_2)}(\mathbf{t}) = \prod_{k\in e_1}(1-t_k) \prod_{k\in e_2} t_k. \tag{2.2}$$

For example, consider the case $d=3$. If $e_1 = \{1,3\}$ and $e_2 = 2$, then

$$Q_{e_1,e_2}(\mathbf{t}) = (0,1,0),$$

$$P^{(e_1,e_2)}(\mathbf{t}) = t_2(1-t_1)(1-t_3),$$

where $\mathbf{t} = (t_1,t_2,t_3) \in [0,1]^3$. If $e_1 = \emptyset$ and $e_2 = \{1,2\}$, then

$$Q_{e_1,e_2}(\mathbf{t}) = (1,1,t_3),$$

$$P^{(e_1,e_2)}(\mathbf{t}) = t_1 t_2,$$

where $\mathbf{t} = (t_1,t_2,t_3) \in [0,1]^3$.

2.2. Decompositions of continuous functions on $[0,1]^d$

Any continuous function f on the cube $[0,1]^d$ can be decomposed into [11]

$$f = \sum_{\nu=1}^{d+1} h_\nu, \tag{2.3}$$

where

$$h_1 = \sum_{|e_1|+|e_2|=d} f(Q_{e_1,e_2}) P^{(e_1,e_2)}.$$

$$h_\nu = \sum_{|e_1|+|e_2|=d-\nu+1} f_{\nu-1}(Q_{e_1,e_2}) P^{(e_1,e_2)} \qquad (2 \le \nu \le d),$$

$$h_{d+1} = f - h_1 - \cdots - h_d,$$

and

$$f_0 = f,$$

$$f_{\nu-1} = f_{\nu-2} - h_{\nu-1} \qquad (2 \le \nu \le d),$$

$$f_d = f_{d-1} - h_d.$$

and the cardinality of a set F is denoted by $|F|$, and

$$\sum_{|e_1|+|e_2|=k} A_{e_1,e_2} := \sum_{\substack{e_1,e_2 \in \{1,...,d\} \\ e_1 \bigcap e_2 = \emptyset \\ |e_1|+|e_2|=k}} A_{e_1,e_2}.$$

3

The following proposition shows the structure of each h_ν.

Proposition 2.1 [11]. If f is a $d-$variate continuous function on the unit cube $[0,1]^d$, then

(i) h_1 is a $d-$variate polynomial and each $f(Q_{e_1,e_2}\mathbf{t})$ is a constant and it is the value of f at a vertex of the cube $[0,1]^d$. Precisely say,

$$f(Q_{e_1,e_2}\mathbf{t}) = f(\lambda_1, ..., \lambda_d),$$

where

$$\lambda_k = \begin{cases} 0, & k \in e_1, \\ 1, & k \in e_2 \quad (e_1 \bigcup e_2 = \{1, ..., d\}). \end{cases}$$

(ii) for each $2 \le \nu \le d$, h_ν is a sum of products of a $(\nu-1)-$variate function $f_{\nu-1}(Q_{e_1,e_2}\cdot)$ on $[0,1]^{\nu-1}$ and $(d-\nu+1)-$variate polynomial $P^{(e_1,e_2)}$, where each product is of separation of variables. Moreover,

$$f_{\nu-1}(Q_{e_1,e_2}\cdot) = 0 \quad \text{on} \quad \partial([0,1]^{\nu-1});$$

(iii) h_{d+1} is a $d-$variate function on $[0,1]^d$ and $h_{d+1}(\cdot) = 0$ on $\partial([0,1]^d)$.

If ξ is a d-variate continuous stochastic process on $[0,1]^d$, then the above decomposition and Proposition 2.1 are still valid.

For example, if ξ is a bivariate continuous function on $[0,1]^2$, then $\xi = h_1 + h_2 + h_3$ and

$$h_1(\mathbf{t}) = \xi(0,0)(1-t_1)(1-t_2) + \xi(0,1)(1-t_1)t_2 + \xi(1,0)t_1(1-t_2) + \xi(1,1)t_1t_2,$$

$$h_2(\mathbf{t}) = \xi_1(1,t_2)t_1 + \xi_1(0,t_2)(1-t_1) + \xi_1(t_1,1)t_2 + \xi_1(t_1,0)(1-t_2) \qquad (\xi_1 = \xi - h_1),$$

$$h_3(\mathbf{t}) = \xi_1(\mathbf{t}) - h_2(\mathbf{t}).$$

We see from this decomposition that h_1 is a polynomial determined by values of f at four vertexes of $[0,1]^2$, h_2 is a sum of products of separation of variables, and $h_2(\mathbf{t}) = 0$ at four vertexes of $[0,1]^2$, and the bivariate function $h_3(\mathbf{t})$ vanishes on the boundary of $[0,1]^2$.

2.3. Fourier sine series of stochastic processes

Let a probability space (Ω, F, P) be given. A stochastic variable ξ is defined as a function ξ from Ω to $\mathbb{R}$ or $\mathbb{C}$. In this paper we always assume that ξ satisfies $E[|\xi|^2] < \infty$, i.e., assume that ξ is a second-order stochastic variable. For a stochastic process $\xi(\mathbf{t})$ on $[0,1]^d$, its autocorrelation function and covariance function are defined respectively as:

$$R_\xi(\mathbf{t},\mathbf{s}) := E[\xi(\mathbf{t})\xi(\mathbf{s})]$$

$$\mathrm{Cov}(\xi(\mathbf{t}),\ \xi(\mathbf{s})) := \mathbf{E}[(\xi(\mathbf{t}) - \mathbf{E}[\xi(\mathbf{t})])(\xi(\mathbf{s}) - \mathbf{E}[\xi(\mathbf{s})])]$$

We recall some known concepts in stochastic calculus [14, 15].

Let $\{\xi_n\}_1^\infty$ be a sequence of second-order stochastic variables and ξ be a second-order stochastic variable. If $\lim\limits_{n\to\infty} E[|\xi_n - \xi|^2] = 0$, then we say $\{\xi_n\}_1^\infty$ converges to ξ in the mean square sense. Based on the above concepts, one can derive the concept of continuous and the concepts of the derivatives and the integrals of stochastic processes [3].

For a stochastic process ξ on $[0,1]^d$, the derivative and the expectation can be exchanged, the integral and the expectation can be exchanged, and Newton-Leibnitz formula holds. For a product of a stochastic process and a deterministic function, differential formula of products holds and the integration by parts also holds.

Let $\xi(\mathbf{t})$ be a stochastic process on $[0,1]^d$ and

$$\int_{[0,1]^d} E[\xi^2(\mathbf{t})]\mathrm{d}\mathbf{t} < \infty.$$

Then $\xi(\mathbf{t})$ can be expanded into the stochastic Fourier sine series

$$\xi(\mathbf{t}) = \sum_{\mathbf{n}\in\mathbb{Z}_+^d} c_{\mathbf{n}}(\xi)\, T_{\mathbf{n}}(\mathbf{t}) \qquad (\, T_{\mathbf{n}}(\mathbf{t}) = \prod_{j=1}^d \sin \pi n_j t_j\,) \tag{2.4}$$

in mean square sense, where the Fourier sine coefficients are stochastic variables and

$$c_{\mathbf{n}}(\xi) = 2^d \int\limits_{[0,1]^d} \xi(\mathbf{t})\, T_{\mathbf{n}}(\mathbf{t})\,\mathrm{d}\mathbf{t} \;\; (\mathbf{n}\in\mathbb{Z}_+^d).$$

3. Asymptotic behavior of Fourier sine coefficients

Let f be a continuous function on the unit cube $[0,1]^d$. By (2.3), f can be decomposed as

$$f = \sum_{\nu=1}^{d+1} h_\nu,$$

where $\{h_\nu\}_1^{d+1}$ are stated in Section 2.1 and

$$f_0 = f,$$

$$f_{\nu-1} = f_{\nu-2} - h_{\nu-1} \qquad (2\le\nu\le d).$$

If $f\in W([0,1]^d)$, we easily prove

$$f_{\nu-1}\in W([0,1]^d) \qquad (\nu=2,...,d). \tag{3.1}$$

Based on this decomposition, we give an asymptotic representation of Fourier sine coefficients of f.

Theorem 3.1. If f is a continuous function on $[0,1]^d$ and $f\in W([0,1]^d)$, then its Fourier sine coefficients $c_{\mathbf{n}}(f)$ possess the asymptotic behavior:

$$c_{\mathbf{n}}(f) = \left(\prod_{j=1}^d \frac{2}{\pi n_j}\right)(K_{\mathbf{n}}^d(f) + \eta_1 + \cdots + \eta_d),$$

where $\eta_k\to 0$ as $n_k\to\infty\,(k=1,.,,,d)$ and

$$K_{\mathbf{n}}^d(f) = \sum_{\lambda\in\{0,1\}^d} f(\lambda)\epsilon_{\mathbf{n}}(\lambda),$$

$$\epsilon_{\mathbf{n}}(\lambda) = \begin{cases} \prod\limits_{j\in G_\lambda} (-1)^{n_j+1}, & G_\lambda\ne\emptyset, \\ 1, & G_\lambda=\emptyset, \end{cases} \tag{3.2}$$

where $\lambda=(\lambda_1,...,\lambda_d)\in\{0,1\}^d$ and $\mathbf{n}=(n_1,...,n_d)$, and $G_\lambda=\{j\in\{1,...,d\},\ \lambda_j=1\}$.

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From this, we see that $\epsilon_{\mathbf{n}}(\lambda) = \pm 1$ and $K^d_{\mathbf{n}}(f)$ is an algebraic sum of values of f at vertexes of $[0,1]^d$. From Theorem 3.1, we deduce the following corollary. This corollary plays an important role in the proof of Theorem 4.1.

Corollary 3.2. If f is a continuous function on $[0,1]^d$ and $f \in W([0,1]^d)$, then its Fourier sine coefficients $c_{\mathbf{n}}(f)$ satisfy

(i) $\sum_{\mathbf{q}\in\{0,1\}^d} |c_{2\mathbf{p}+\mathbf{q}}(f)|^2 = \left(\frac{2}{\pi^2}\right)^d \frac{1}{p_1^2 p_2^2 \cdots p_d^2} \left(\sum\limits_{\lambda\in\{0,1\}^d} |f(\lambda)|^2 + \eta_1' + \cdots + \eta_d' \right)$;

(ii) $c_{\mathbf{n}}(f) = \left(\prod\limits_{j=1}^{d} \frac{1}{n_j} \right) (\eta_1 + \cdots + \eta_d)$ if and only if $f(\lambda) = 0$ $(\lambda \in \{0,1\}^d)$ $(\eta_k \to 0$ as $n_k \to \infty)$.

From Corollary 3.2, we deduce immediately that $c_{\mathbf{n}}(f) = o\left(\frac{1}{n_1 \cdots n_d}\right)$ for each $n_k \to \infty$ if and only if $f(\lambda) = 0$ $(\lambda \in \{0,1\}^d)$.

For example, consider the case $d = 3$. If $f \in W([0,1]^3)$, then Fourier sine coefficients $c_{n_1,n_2,n_3}(f)$ possess the asymptotic behavior:

$$\begin{aligned} c_{n_1,n_2,n_3}(f) = \frac{8}{n_1 n_2 n_3 \pi^3} \big(& f(0,0,0) - (-1)^{n_1} f(1,0,0) - (-1)^{n_2} f(0,1,0) \\ & -(-1)^{n_3} f(0,0,1) + (-1)^{n_1+n_2} f(1,1,0) + (-1)^{n_1+n_3} f(1,0,1) \\ & +(-1)^{n_2+n_3} f(0,1,1) - (-1)^{n_1+n_2+n_3} f(1,1,1) + \eta_1 + \eta_2 + \eta_3 \big), \end{aligned}$$

where $\eta_k \to 0$ as $n_k \to \infty$ $(k = 1,2,3)$.

Proof of Theorem 3.1. By (2.3), the Fourier sine coefficients $c_{\mathbf{n}}(f)$ satisfy

$$c_{\mathbf{n}}(f) = \sum_{\nu=1}^{d+1} c_{\mathbf{n}}(h_\nu), \tag{3.3}$$

where $c_{\mathbf{n}}(h_\nu) = 2^d \int\limits_{[0,1]^d} h_\nu(\mathbf{t}) T_{\mathbf{n}}(\mathbf{t}) \mathrm{d}\mathbf{t}$.

First, we compute $c_{\mathbf{n}}(h_1)$. By (2.4), we have

$$c_{\mathbf{n}}(h_1) = 2^d \sum_{|e_1|+|e_2|=d} \int\limits_{[0,1]^d} f(Q_{e_1,e_2}\mathbf{t}) P^{(e_1,e_2)}(\mathbf{t}) T_{\mathbf{n}}(\mathbf{t}) \mathrm{d}\mathbf{t}.$$

By Proposition 2.1 (i), $f(Q_{e_1,e_2}\mathbf{t}) = f(\lambda)$ is a constant independent of $\mathbf{t}$. So

$$c_{\mathbf{n}}(h_1) = 2^d \sum_{|e_1|+|e_2|=d} f(\lambda_1, ..., \lambda_d) \int\limits_{[0,1]^d} P^{(e_1,e_2)}(\mathbf{t}) T_{\mathbf{n}}(\mathbf{t}) \mathrm{d}\mathbf{t}, \tag{3.4}$$

where $\lambda_k = \begin{cases} 0, & k \in e_1, \\ 1, & k \in e_2 \ (e_1 \bigcup e_2 = \{1, ..., d\}). \end{cases}$ Since

$$P^{(e_1,e_2)}(\mathbf{t}) = \prod_{j\in e_1} (1 - t_j) \prod_{j\in e_2} t_j,$$

$$T_{\mathbf{n}}(\mathbf{t}) = \prod_{j=1}^{d} \sin(\pi n_j t_j), \qquad \mathbf{t} = (t_1, ..., t_d),$$

a direct computation shows that

$$\begin{aligned}
\int\limits_{[0,1]^d} P^{(e_1,e_2)}(\mathbf{t})T_{\mathbf{n}}(\mathbf{t})\mathrm{d}\mathbf{t} &= \left(\prod_{j\in e_1}\int\limits_0^1(1-t_j)\sin(\pi n_j t_j)\mathrm{d}t_j\right)\left(\prod_{j\in e_2}\int\limits_0^1 t_j\sin(\pi n_j t_j)\mathrm{d}t_j\right) \\
&= \left(\prod_{j\in e_1}\tfrac{1}{\pi n_j}\right)\left(\prod_{j\in e_2}\tfrac{(-1)^{n_j+1}}{\pi n_j}\right) \\
&= \left(\prod_{j\in(e_1\bigcup e_2)}\tfrac{1}{\pi n_j}\right)\prod_{j\in e_2}(-1)^{n_j+1} \\
&= \left(\prod_{j=1}^d\tfrac{1}{\pi n_j}\right)\prod_{j\in e_2}(-1)^{n_j+1} \qquad (e_1\bigcup e_2=\{1,...,d\}).
\end{aligned}$$

From this and (3.4), we get

$$c_{\mathbf{n}}(h_1)=2^d\left(\sum_{|e_1|+|e_2|=d} f(\lambda_1,...,\lambda_d)\prod_{j\in e_2}(-1)^{n_j+1}\right)\prod_{j=1}^d\frac{1}{\pi n_j}.$$

Since $e_1\bigcup e_2=\{1,...,d\}$ and $e_1=\{j\in\{1,...,d\},\ \lambda_j=0\}$, and $e_2=\{j\in\{1,...,d\},\ \lambda_j=1\}=:G_\lambda$,

$$c_{\mathbf{n}}(h_1)=2^d K_{\mathbf{n}}^d(f)\left(\prod_{j=1}^d\frac{1}{\pi n_j}\right), \tag{3.5}$$

where $K_{\mathbf{n}}^d(f)=\sum\limits_{\lambda\in\{0,1\}^d} f(\lambda)\prod\limits_{j\in G_\lambda}(-1)^{n_j+1}$.

Next, we compute $c_{\mathbf{n}}(h_\nu)$ $(2\le\nu\le d)$. By (2.3) and (2.4),

$$c_{\mathbf{n}}(h_\nu)=2^d\sum_{|e_1|+|e_2|=d-\nu+1}\int\limits_{[0,1]^d} f_{\nu-1}(Q_{e_1,e_2}\mathbf{t})\,P^{(e_1,e_2)}(\mathbf{t})T_{\mathbf{n}}(\mathbf{t})\mathrm{d}\mathbf{t}. \tag{3.6}$$

Since $|e_1|+|e_2|=d-\nu+1$, we may denote $e_1\bigcup e_2=\{\beta_1,...,\beta_{d-\nu+1}\}$. By (2.2), the fundamental polynomial $P^{(e_1,e_2)}(\mathbf{t})$ only depends on $t_{\beta_1},...,t_{\beta_{d-\nu+1}}$, write

$$P^{(e_1,e_2)}(\mathbf{t})=P^{(e_1,e_2)}(t_{\beta_1},...,t_{\beta_{d-\nu+1}}).$$

Since $e=\{1,...,d\}\setminus(e_1\bigcup e_2)$ and $|e|=\nu-1$, we may denote $e=\{\alpha_1,...,\alpha_{\nu-1}\}$. Then $f_{\nu-1}(Q_{e_1,e_2}\mathbf{t})$ only depends on $t_{\alpha_1},...,t_{\alpha_{\nu-1}}$, write

$$f_{\nu-1}(Q_{e_1,e_2}\mathbf{t})=f_{\nu-1}^{e_1,e_2}(t_{\alpha_1},...,t_{\alpha_{\nu-1}}). \tag{3.7}$$

So each product $f_{\nu-1}(Q_{e_1,e_2}\mathbf{t})P^{(e_1,e_2)}(\mathbf{t})$ in (3.6) is of separated variable type, and so

$$\int\limits_{[0,1]^d} f_{\nu-1}(Q_{e_1,e_2}\mathbf{t})\,P^{(e_1,e_2)}(\mathbf{t})T_{\mathbf{n}}(\mathbf{t})\mathrm{d}\mathbf{t}=L_{\nu,\mathbf{n}}^{(1)}(e_1,e_2)L_{\nu,\mathbf{n}}^{(2)}(e_1,e_2), \tag{3.8}$$

where

$$L_{\nu,\mathbf{n}}^{(1)}(e_1,e_2)=\int\limits_{[0,1]^{d-\nu+1}} P^{(e_1,e_2)}(t_{\beta_1},...,t_{\beta_{d-\nu+1}})\prod_{j=1}^{d-\nu+1}\sin(\pi n_{\beta_j}t_{\beta_j})\mathrm{d}t_{\beta_1}\cdots\mathrm{d}t_{\beta_{d-\nu+1}},$$

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$$L_{\nu,\mathbf{n}}^{(2)}(e_1,e_2)=\int\limits_{[0,1]^{\nu-1}} f_{\nu-1}^{e_1,e_2}(t_{\alpha_1},...,t_{\alpha_{\nu-1}})\prod_{j=1}^{\nu-1}\sin(\pi n_{\alpha_j}t_{\alpha_j})\mathrm{d}t_{\alpha_1}\cdots\mathrm{d}t_{\alpha_{\nu-1}}.$$

We compute $L_{\nu,\mathbf{n}}^{(2)}(e_1,e_2)$. We rewrite it as follows:

$$L_{\nu,\mathbf{n}}^{(2)}(e_1,e_2)=\int\limits_{[0,1]^{\nu-2}}\left(\int\limits_0^1 f_{\nu-1}^{e_1,e_2}(t_{\alpha_1},...,t_{\alpha_{\nu-1}})\sin(\pi n_{\alpha_1}t_{\alpha_1})\mathrm{d}t_{\alpha_1}\right)\prod_{j=2}^{\nu-1}\sin(\pi n_{\alpha_j}t_{\alpha_j})\mathrm{d}t_{\alpha_2}\cdots\mathrm{d}t_{\alpha_{\nu-1}}. \quad (3.9)$$

From (3.1) and (3.7), we know that $f_{\nu-1}^{e_1,e_2}\in W([0,1]^{\nu-1})$. Using integration by parts, the part inside brackets:

$$\int\limits_0^1 f_{\nu-1}^{e_1,e_2}(t_{\alpha_1},...,t_{\alpha_{\nu-1}})\sin(\pi n_{\alpha_1}t_{\alpha_1})\mathrm{d}t_{\alpha_1} = -f_{\nu-1}^{e_1,e_2}(t_{\alpha_1},...,t_{\alpha_{\nu-1}})\frac{\cos(\pi n_{\alpha_1}t_{\alpha_1})}{\pi n_{\alpha_k}}\Bigg|_{t_{\alpha_1}=0}^{1}$$

$$+\frac{1}{\pi n_{\alpha_1}}\int\limits_0^1\frac{\partial f_{\nu-1}^{e_1,e_2}(t_{\alpha_1},...,t_{\alpha_{\nu-1}})}{\partial t_{\alpha_1}}\cos(\pi n_{\alpha_1}t_{\alpha_1})\mathrm{d}t_{\alpha_1}.$$

By (3.7) and Proposition 2.1 (ii), we have

$$f_{\nu-1}^{e_1,e_2}(t_{\alpha_1},...,t_{\alpha_{\nu-1}})\big|_{t_{\alpha_k}=0}^{1}=0 \quad (k=1,...,\nu-1). \quad (3.10)$$

Therefore,

$$\int\limits_0^1 f_{\nu-1}^{e_1,e_2}(t_{\alpha_1},...,t_{\alpha_{\nu-1}})\sin(\pi n_{\alpha_1}t_{\alpha_1})\mathrm{d}t_{\alpha_1}=\frac{1}{\pi n_{\alpha_1}}\int\limits_0^1\frac{\partial f_{\nu-1}^{e_1,e_2}(t_{\alpha_1},...,t_{\alpha_{\nu-1}})}{\partial t_{\alpha_1}}\cos(\pi n_{\alpha_1}t_{\alpha_1})\mathrm{d}t_{\alpha_1}.$$

From this and (3.9), we get

$$L_{\nu,\mathbf{n}}^{(2)}(e_1,e_2)=\frac{1}{\pi n_{\alpha_1}}\int\limits_{[0,1]^{\nu-2}}\left(\int\limits_0^1\frac{\partial f_{\nu-1}^{e_1,e_2}(t_{\alpha_1},...,t_{\alpha_{\nu-1}})}{\partial t_{\alpha_1}}\sin(\pi n_{\alpha_2}t_{\alpha_2})\,\mathrm{d}t_{\alpha_2}\right)\prod_{j=3}^{\nu-1}\sin(\pi n_{\alpha_j}t_{\alpha_j})$$

$$\cos(\pi n_{\alpha_1}t_{\alpha_1})\,\mathrm{d}t_{\alpha_1}\mathrm{d}t_{\alpha_3}\cdots\mathrm{d}t_{\alpha_{\nu-1}}. \quad (3.11)$$

Using integration by parts, the part inside brackets:

$$\int\limits_0^1\frac{\partial}{\partial t_{\alpha_1}}f_{\nu-1}^{e_1,e_2}(t_{\alpha_1},...,t_{\alpha_{\nu-1}})\,\sin(\pi n_{\alpha_2}t_{\alpha_2})\,\mathrm{d}t_{\alpha_2}$$

$$=-\frac{\partial f_{\nu-1}^{e_1,e_2}(t_{\alpha_1},...,t_{\alpha_{\nu-1}})}{\partial t_{\alpha_1}}\frac{\cos(\pi\alpha_2 t_{\alpha_2})}{\pi n_{\alpha_2}}\Bigg|_{t_{\alpha_2}=0}^{1}$$

$$+\frac{1}{\pi n_{\alpha_2}}\int\limits_{[0,1]^{\nu-1}}\frac{\partial^2 f_{\nu-1}^{e_1,e_2}(t_{\alpha_1},...,t_{\alpha_{\nu-1}})}{\partial t_{\alpha_1}\partial t_{\alpha_2}}\cos(\pi n_{\alpha_2}t_{\alpha_2})\,\mathrm{d}t_{\alpha_2}.$$

By Proposition 2.1 (ii),

$$f_{\nu-1}(t_{\alpha_1},0,t_{\alpha_3},...,t_{\alpha_{\nu-1}})=f_{\nu-1}(t_{\alpha_1},1,t_{\alpha_3},...,t_{\alpha_{\nu-1}})=0,$$

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and so
$$\frac{\partial f_{\nu-1}^{e_1,e_2}(t_{\alpha_1},0,...,t_{\alpha_{\nu-1}})}{\partial t_{\alpha_1}}=\frac{\partial f_{\nu-1}^{e_1,e_2}(t_{\alpha_1},1,...,t_{\alpha_{\nu-1}})}{\partial t_{\alpha_1}}=0.$$
Therefore,
$$\int_0^1 \frac{\partial}{\partial t_{\alpha_1}} f_{\nu-1}^{e_1,e_2}(t_{\alpha_1},...,t_{\alpha_{\nu-1}})\,\sin(\pi n_{\alpha_2}t_{\alpha_2})\,\mathrm{d}t_{\alpha_2}$$
$$=\frac{1}{\pi n_{\alpha_2}}\int_{[0,1]^{\nu-1}}\frac{\partial^2 f_{\nu-1}^{e_1,e_2}(t_{\alpha_1},...,t_{\alpha_{\nu-1}})}{\partial t_{\alpha_1}\partial t_{\alpha_2}}\,\cos(\pi n_{\alpha_2}t_{\alpha_2})\,\mathrm{d}t_{\alpha_2}.$$
From this and (3.11), we get
$$L_{\nu,\mathbf{n}}^{(2)}(e_1,e_2)=\frac{1}{(\pi n_{\alpha_1})(\pi n_{\alpha_2})}\int_{[0,1]^{\nu-2}}\left(\int_0^1\frac{\partial^2 f_{\nu-1}^{e_1,e_2}(t_{\alpha_1,...,t_{\alpha_{\nu-1}}})}{\partial t_{\alpha_1}\partial t_{\alpha_2}}\,\sin(\pi n_{\alpha_3}t_{\alpha_3})\,\mathrm{d}t_{\alpha_3}\right)$$
$$\bullet\prod_{j=4}^{\nu-1}\sin(\pi n_{\alpha_j}t_{\alpha_j})\cos(\pi n_{\alpha_1}t_{\alpha_1})\,\cos(\pi n_{\alpha_2}t_{\alpha_2})\,\mathrm{d}t_{\alpha_1}\mathrm{d}t_{\alpha_2}\mathrm{d}t_{\alpha_4}\cdots\mathrm{d}t_{\alpha_{\nu-1}}.$$
Continuing this procedure, we deduce finally that
$$L_{\nu,\mathbf{n}}^{(2)}(e_1,e_2)=\left(\prod_{j=1}^{\nu-1}\frac{1}{\pi n_{\alpha_j}}\right)\int_{[0,1]^{\nu-1}}\frac{\partial^{\nu-1}f_{\nu-1}^{e_1,e_2}(t_{\alpha_1},...,t_{\alpha_{\nu-1}})}{\partial t_{\alpha_1}\cdots\partial t_{\alpha_{\nu-1}}}\left(\prod_{j=1}^{\nu-1}\cos(\pi n_{\alpha_j}t_{\alpha_j})\right)\,\mathrm{d}t_{\alpha_1}\cdots\mathrm{d}t_{\alpha_{\nu-1}}.$$
Since
$$\frac{\partial^{\nu-1}f_{\nu-1}^{e_1,e_2}(t_{\alpha_1},\cdots,t_{\alpha_{\nu-1}})}{\partial t_{\alpha_1}\cdots\partial t_{\alpha_{\nu-1}}}\in C([0,1]^{\nu-1})$$
and $e=\{\alpha_1,...,\alpha_{\nu-1}\}$, applying the Riemann-Lebesgue lemma, we get
$$L_{\nu,\mathbf{n}}^{(2)}(e_1,e_2)=o\left(\prod_{j=1}^{\nu-1}\frac{1}{n_{\alpha_j}}\right)=\left(\prod_{j\in e}\frac{1}{n_j}\right)\epsilon_e,\tag{3.12}$$
where $\epsilon_e\to 0$ as $n_j\to\infty$ $(j\in e)$.

We compute $L_{\nu,\mathbf{n}}^{(1)}(e_1,e_2)$. Notice that $e_1\bigcup e_2=\{\beta_1,...,\beta_{d-\nu+1}\}$. We assume in which
$$e_1=\{\gamma_1,...,\gamma_{m_1}\},$$
$$e_2=\{\delta_1,...,\delta_{m_2}\},$$
where $m_1+m_2=d-\nu+1$. By (2.2), we get
$$L_{\nu,\mathbf{n}}^{(1)}(e_1,e_2)$$
$$=\int_{[0,1]^{d-\nu+1}}P^{(e_1,e_2)}(t_{\beta_1},...,t_{\beta_{d-\nu+1}})\prod_{j=1}^{d-\nu+1}\sin(\pi n_{\beta_j}t_{\beta_j})\mathrm{d}t_{\beta_1}\cdots\mathrm{d}t_{\beta_{d-\nu+1}},$$
$$=\int_{[0,1]^{m_1}}\prod_{j=1}^{m_1}(1-t_{\gamma_j})\,\sin(\pi n_{\gamma_j}t_{\gamma_j})\,\mathrm{d}t_{\gamma_1}\cdots\mathrm{d}t_{\gamma_{m_1}}\int_{[0,1]^{m_2}}\prod_{j=1}^{m_2}t_{\delta_j}\,\sin(\pi n_{\delta_j}t_{\delta_j})\,\mathrm{d}t_{\delta_1}\cdots\mathrm{d}t_{\delta_{m_2}}.$$

A direct computation shows that

$$L_{\nu,\mathbf{n}}^{(1)}(e_1,e_2)=\left(\prod_{j=1}^{m_1}\frac{1}{\pi n_{\gamma_j}}\right)\left(\prod_{j=1}^{m_2}\frac{(-1)^{n_{\delta_j}+1}}{\pi n_{\delta_j}}\right)=O\left(\prod_{j\in e_1\bigcup e_2}\frac{1}{n_j}\right).$$

By (3.12) and (3.6), we have

$$\int\limits_{[0,1]^d} f_{\nu-1}(Q_{e_1,e_2}\mathbf{t})\,P^{(e_1,e_2)}(\mathbf{t})T_{\mathbf{n}}(\mathbf{t})\mathrm{d}\mathbf{t}=\left(\prod_{j=1}^{d}\frac{1}{n_j}\right)\epsilon_e$$

and

$$c_{\mathbf{n}}(h_\nu)=2^d\left(\prod_{j=1}^{d}\frac{1}{n_j}\right)\left(\sum_{|e_1|+|e_2|=d-\nu+1}\epsilon_e\right),$$

where $e=\{1,...,d\}\setminus(e_1\bigcup e_2)$ and $\epsilon_e\to 0$ as $n_j\to\infty\,(j\in e)$. From this, we can deduce that

$$c_{\mathbf{n}}(h_\nu)=\left(\prod_{j=1}^{d}\frac{1}{n_j}\right)(\eta_1^\nu+\cdots+\eta_d^\nu)\qquad(\nu=2,...,d), \tag{3.13}$$

where $\eta_k^{\prime\prime}\to 0$ as $n_k\to\infty$ $(k=1,...,d)$. Finally, we compute $c_{\mathbf{n}}(h_{d+1})$. Since

$$c_{\mathbf{n}}(h_{d+1})=2^d\int\limits_{[0,1]^d} h_{d+1}(t_1,...,t_d)\prod_{j=1}^{d}\sin(\pi n_j t_j)\,\mathrm{d}t_1\cdots\mathrm{d}t_d,$$

by Proposition 2.1 (iii) and (3.3), applying the integration by parts and the Riemann-Lebesgue lemma, we get

$$\begin{aligned} c_{\mathbf{n}}(h_{d+1}) &= 2^d\prod_{j=1}^{d}\tfrac{1}{\pi n_j}\int\limits_{[0,1]^d}\tfrac{\partial^d h_{d+1}(t_1,...,t_d)}{\partial t_1\cdots\partial t_d}\prod_{j=1}^{d}\cos(\pi n_j t_j)\,\mathrm{d}t_1\cdots\mathrm{d}t_d\\ &= o\left(\prod_{j=1}^{d}\tfrac{1}{n_j}\right)=\left(\prod_{j=1}^{d}\tfrac{1}{n_j}\right)\epsilon,\end{aligned}$$

where $\epsilon\to 0$ as $n_j\to\infty$ $(j=1,...,d)$. From this and (3.5), and (3.13), it follows by (3.3) that

$$c_{\mathbf{n}}(f)=c_{\mathbf{n}}(h_1)+\sum_{\nu=2}^{d}c_{\mathbf{n}}(h_\nu)+c_{\mathbf{n}}(h_{d+1})=2^d\left(\prod_{j=1}^{d}\frac{1}{\pi n_j}\right)\left(K_{\mathbf{n}}^d(f)+\eta_1+\cdots+\eta_d\right),$$

where $K_{\mathbf{n}}^d(f)$ is stated in (3.5) and $\eta_k\to 0$ as $n_k\to\infty\,(k=1,...,d)$. Theorem 3.1 is proved. □

Proof of Corollary 3.2. Let $n=2\,\mathbf{p}+\mathbf{q}$ ($\mathbf{p}\in Z_+^d$, $\mathbf{q}\in\{0,1\}^d$). Then, for each $\mathbf{q}=(q_1,...,q_d)$ and $\mathbf{p}=(p_1,...,p_d)$, by Theorem 3.1, we have

$$c_{2\mathbf{p}+\mathbf{q}}(f)=2^d\left(\prod_{j=1}^{d}\frac{1}{\pi(2p_j+q_j)}\right)(K_{2\mathbf{p}+\mathbf{q}}^d(f)+o(\eta_1+\cdots+\eta_d))$$

and

$$K^d_{2\mathbf{p}+\mathbf{q}}(f) = \sum_{\lambda\in\{0,1\}^d} f(\lambda) \prod_{j\in G_\lambda} (-1)^{q_j+1} =: K^d_{\mathbf{q}}(f).$$

From this, we see that $K^d_{2\mathbf{p}+\mathbf{q}}$ only depends on $\mathbf{q}$. So, for $\mathbf{q} \in \{0,1\}^d$, we have

$$c_{2\mathbf{p}+\mathbf{q}}(f) = \left(\prod_{j=1}^d \frac{1}{\pi p_j}\right) (K^d_{\mathbf{q}}(f) + \hat{\eta}_1 + \cdots + \hat{\eta}_d, \tag{3.14}$$

where $\hat{\eta}_k \to 0$ as $p_k \to \infty\,(k = 1, ..., d)$ and

$$\sum_{\mathbf{q}\in\{0,1\}^d} |c^2_{2\mathbf{p}+\mathbf{q}}(f)| = \left(\prod_{j=1}^d \frac{1}{\pi^2 p_j^2}\right) \left(\sum_{\mathbf{q}\in\{0,1\}^d} \left(K^d_{\mathbf{q}}(f)\right)^2 + \eta'_1 + \cdots + \eta'_d\right), \tag{3.15}$$

where $\eta'_k \to 0$ as $p_k \to \infty\,(k = 1, ..., d)$. By (3.2), we have

$$\sum_{q\in\{0,1\}^d} (K^d_{\mathbf{q}}(f))^2 = \sum_{\lambda\in\{0,1\}^d} \sum_{\lambda'\in\{0,1\}^d} f(\lambda)f(\lambda') \sum_{q\in\{0,1\}^d} \epsilon_q(\lambda)\epsilon_q(\lambda').$$

where

$$\epsilon_q(\lambda)\epsilon_q(\lambda') = \prod_{j\in G_\lambda} (-1)^{q_j+1} \prod_{j\in G_{\lambda'}} (-1)^{q_j+1} = (-1)^{\sum\limits_{j\in G_\lambda} q_j + \sum\limits_{j\in G_{\lambda'}} q_j + |G_\lambda| + |G_{\lambda'}|}.$$

When $\lambda \neq \lambda'$, without loss of generality, we assume that $i \in G_\lambda,\ i \notin G_{\lambda'}$. So we have

$$\sum_{q\in\{0,1\}^d} \epsilon_q(\lambda)\epsilon_q(\lambda') = \sum_{q_1=0}^1 \cdots \sum_{q_{i-1}=0}^1 \sum_{q_{i+1}=0}^1 \cdots \sum_{q_d=0}^1 (-1)^{\sum\limits_{\substack{j\in G_\lambda\\ j\neq i}} q_j + \sum\limits_{j\in G_{\lambda'}} q_j + |G_\lambda| + |G_{\lambda'}|} \sum_{q_i=0}^1 (-1)^{q_i} = 0. \tag{3.16}$$

When $\lambda = \lambda'$. Then

$$\sum_{q\in\{0,1\}^d} \epsilon_q(\lambda)\epsilon_q(\lambda') = \sum_{q_1=0}^1 \cdots \sum_{q_d=0}^1 (-1)^{2\sum\limits_{j\in G_\lambda} q_j + 2|G_\lambda|} = 2^d. \tag{3.17}$$

From this, we get

$$\sum_{\mathbf{q}\in\{0,1\}^d} (K^d_{\mathbf{q}}(f))^2 = \left(\sum_{\substack{\lambda,\lambda'\in\{0,1\}^d\\ \lambda=\lambda'}} + \sum_{\substack{\lambda,\,\lambda'\in\{0,1\}^d\\ \lambda\neq\lambda'}}\right) f(\lambda)f(\lambda') \sum_{q\in\{0,1\}^d} \epsilon_q(\lambda)\epsilon_q(\lambda') = 2^d \sum_{\lambda\in\{0,1\}^d} f^2(\lambda).$$

From this and (3.15), we get (i). If

$$c_{\mathbf{n}}(f) = \left(\prod_{j=1}^d \frac{1}{n_j}\right) (\eta_1 + \cdots + \eta_d), \tag{3.18}$$

we have

$$\sum_{\mathbf{q}\in\{0,1\}^d} |c_{2\mathbf{p}+\mathbf{q}}|^2 = \left(\prod_{j=1}^d \frac{1}{p_j^2}\right) (\eta_1 + \cdots + \eta_d).$$

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By (i), the later is equivalent to

$$\sum_{\lambda\in\{0,1\}^d} f^2(\lambda)=0$$

which is equivalent to $f(\lambda)=0$ ($\lambda\in\{0,1\}^d$). Conversely, if $f(\lambda)=0$ ($\lambda\in\{0,1\}^d$), then, by Theorem 3.1, we deduce (3.18). So we get (ii). Corollary 3.2 is proved. □

4. Asymptotic behaviors of hyperbolic cross truncation approximation

Approximation rate of multivariate functions by partial sum of Fourier sine series deteriorates rapidly as the dimension d increases.

Let f be a continuous function on $[0,1]^d$. Denote the partial sums of its Fourier sine series by $S_N(f)$:

$$S_N(f;\mathbf{t})=\sum_{n_1,...,n_d=1}^{N} c_{\mathbf{n}}(f)\,T_{\mathbf{n}}(\mathbf{t}) \qquad \mathbf{n}=(n_1,...,n_d),$$

where $c_{\mathbf{n}}(f)=2^d\int_{[0,1]^d} f(\mathbf{t})T_{\mathbf{n}}(\mathbf{t})\,\mathrm{d}\mathbf{t}$ and $T_{\mathbf{n}}(\mathbf{t})$ is stated in (2.4).

If f is a continuous function on $[0,1]^d$ and $f\in W([0,1]^d)$, then, by the Parseval identity, the partial sum $S_N(f)$ of the Fourier sine series of f satisfies

$$\begin{aligned}2^d\parallel S_N(f)-f\parallel_2^2 &= \left(\sum_{n_1,...,n_d=1}^{\infty}-\sum_{n_1,...,n_d=1}^{N}\right)c^2_{n_1,...,n_d}(f)\\ &= O(1)\left(\sum_{n_1,...,n_d=1}^{\infty}-\sum_{n_1,...,n_d=1}^{N}\right)\frac{1}{n_1^2n_2^2\cdots n_d^2}\\ &= O(1)\left(\sum_{\nu=1}^{d}\frac{d!}{\nu!(d-\nu)!}\left(\sum_{n_1,...,n_\nu=N+1}^{\infty}\frac{1}{n_1^2\cdots n_\nu^2}\right)\left(\sum_{n_{\nu+1},...,n_d=1}^{N}\frac{1}{n_{\nu+1}^2\cdots n_d^2}\right)\right)\\ &= O(1)\left(\sum_{\nu=1}^{d}\left(\sum_{k=N+1}^{\infty}\frac{1}{k^2}\right)^{\nu}\left(\sum_{k=1}^{N}\frac{1}{k^2}\right)^{d-\nu+1}\right)=O\left(\frac{1}{N}\right).\end{aligned} \tag{4.1}$$

In the partial sum of Fourier sine series, the number of its Fourier sine coefficients:

$$N_c=N^d.$$

So, by (4.1), it follows that for $f\in W([0,1]^d)$, the partial sums $S_N(f)$ satisfy

$$\parallel f-S_N(f)\parallel_2^2=O\left(\frac{1}{N}\right)=O\left(\frac{1}{N_c^{\frac{1}{d}}}\right).$$

Consider hyperbolic cross truncations of the Fourier sine series of f on $[0,1]^d$. The Fourier sine series of f can be rewritten in the form

$$f(\mathbf{t})=\sum_{p_1,...,p_d=1}^{\infty}\sum_{\mathbf{q}\in\{0,1\}^d} c_{2\mathbf{p}+\mathbf{q}}T_{2\mathbf{p}+\mathbf{q}}(\mathbf{t}) \qquad (\mathbf{p}=(p_1,...,p_d)\in\mathbb{Z}_+).$$

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Define the hyperbolic cross truncations of Fourier sine series of f are

$$S_N^{(h)}(f;\mathbf{t}) = \sum_{\substack{1\le|\mathbf{p}|\le N-1 \\ 1\le p_1,\dots,p_d\le N-1}} \sum_{\mathbf{q}\in\{0,1\}^d} c_{2\mathbf{p}+\mathbf{q}} T_{2\mathbf{p}+\mathbf{q}}(\mathbf{t}),$$

where $|\mathbf{p}| = \prod_{k=1}^{d} p_k$.

Based on asymptotic formula, we deduce that the asymptotic formula of hyperbolic cross truncations of Fourier sine series

Denote

$$\Theta_N = \left\{ \mathbf{p} = (p_1, ..., p_d) \in \mathbb{Z}_+^d \quad p_1 \in \mathbb{Z}_+, \quad 1 \le p_2, ..., p_d \le N-1, \quad \prod_{k=1}^{d} p_k \ge N \right\}. \tag{4.2}$$

The difference $f(\mathbf{t}) - S_N^{(h)}(f;\mathbf{t})$ is equal to

$$\sum_{\mathbf{p}\in\Theta_N} \sum_{\mathbf{q}\in\{0,1\}^d} c_{2\mathbf{p}+\mathbf{q}} T_{2\mathbf{p}+\mathbf{q}}(\mathbf{t}) + \left(\sum_{p_1,\dots,p_d=1}^{\infty} - \sum_{p_1=1}^{\infty} \sum_{p_2,\dots,p_d=1}^{N} \right) \sum_{\mathbf{q}\in\{0,1\}^d} c_{2\mathbf{p}+\mathbf{q}} T_{2\mathbf{p}+\mathbf{q}}(\mathbf{t}).$$

By the Parseval identity, we deduce that

$$\begin{aligned} 2^d \parallel f - S_N^{(h)}(f) \parallel_2^2 &= \sum_{\mathbf{p}\in\Theta_N} \sum_{\mathbf{q}\in\{0,1\}^d} c_{2\mathbf{p}+\mathbf{q}}^2 \\ &+ \left(\sum_{p_1,\dots,p_d=1}^{\infty} - \sum_{p_1=1}^{\infty} \sum_{p_2,\dots,p_d=1}^{N} \right) \sum_{\mathbf{q}\in\{0,1\}^d} c_{2\mathbf{p}+\mathbf{q}}^2 \\ &= P_N(f) + Q_N(f). \end{aligned} \tag{4.3}$$

By Corollary 3.2 and (4.1),

$$\begin{aligned} Q_N(f) &\le \left(\sum_{p_1,\dots,p_d=1}^{\infty} - \sum_{p_1,\dots,p_d=1}^{N} \right) \sum_{\mathbf{q}\in\{0,1\}^d} c_{2\mathbf{p}+\mathbf{q}}^2 \\ &= O(1) \left(\sum_{p_1,\dots,p_d=1}^{\infty} - \sum_{p_1,\dots,p_d=1}^{N} \right) \tfrac{1}{p_1^2\cdots p_d^2} = O\left(\tfrac{1}{N}\right). \end{aligned} \tag{4.4}$$

By Corollary 3.2, it follows that

$$\begin{aligned} P_N(f) &= \left(\tfrac{2}{\pi^2}\right)^d \sum_{\mathbf{p}\in\Theta_N} \tfrac{1}{p_1^2\cdots p_d^2} \left(\sum_{\lambda\in\{0,1\}^d} |f(\lambda)|^2 + \eta_1' + \cdots + \eta_d' \right) \\ &= P_N^{(1)} + P_N^{(2)}, \end{aligned} \tag{4.5}$$

where

$$P_N^{(1)} = \left(\frac{2}{\pi^2}\right)^d \sum_{\mathbf{p}\in\Theta_N} \frac{1}{p_1^2\cdots p_d^2} \sum_{\lambda\in\{0,1\}^d} |f(\lambda)|^2,$$

$$P_N^{(2)} = O(1) \sum_{\mathbf{p}\in\Theta_N} \frac{1}{p_1^2\cdots p_d^2}(\eta_1' + \cdots \eta_d'),$$

and $\eta_k \to 0$ as $p_k \to \infty$ $(k = 1, ..., d)$.

We estimate the order of $P_N^{(1)} \to 0$ as $N \to \infty$.

Notice that

$$P_N^{(1)} \sim \sum_{\mathbf{p}\in\Theta_N} \frac{1}{p_1^2\cdots p_d^2} \sim \int_{x_1,...,x_d \geq N} \frac{\mathrm{d}x_1\cdots \mathrm{d}x_d}{x_1^2\cdots x_d^2} =: R_N$$

and

$$R_N = \int_1^N \mathrm{d}x_1 \int_1^{\frac{N}{x_1}} \mathrm{d}x_2 \cdots \int_1^{\frac{N}{x_1\cdots x_k}} \mathrm{d}x_{k+1} \cdots \int_{\frac{N}{x_1\cdots x_{d-1}}}^{\infty} \frac{\mathrm{d}x_d}{x_1^2\cdots x_d^2}.$$

A direct computation shows that

$$\begin{aligned}\int_{\frac{N}{x_1\cdots x_{d-1}}}^{\infty} \frac{\mathrm{d}x_d}{x_1^2\cdots x_d^2} &= \frac{1}{x_1^2\cdots x_{d-1}^2} \int_{\frac{N}{x_1\cdots x_{d-1}}}^{\infty} \frac{\mathrm{d}x_d}{x_d^2} \\ &= \frac{1}{Nx_1\cdots x_{d-1}},\end{aligned}$$

$$\begin{aligned}\int_1^{\frac{N}{x_1\cdots x_{d-2}}} \mathrm{d}x_{d-1} \int_{\frac{N}{x_1\cdots x_{d-1}}}^{\infty} \frac{\mathrm{d}x_d}{x_1^2\cdots x_d^2} &= \frac{1}{Nx_1\cdots x_{d-2}} \int_1^{\frac{N}{x_1\cdots x_{d-2}}} \frac{\mathrm{d}x_{d-1}}{x_{d-1}} \\ &= \frac{1}{Nx_1\cdots x_{d-2}} \log \frac{N}{x_1\cdots x_{d-2}},\end{aligned}$$

and

$$\begin{aligned}&\int_1^{\frac{N}{x_1\cdots x_{d-3}}} \mathrm{d}x_{d-2} \int_1^{\frac{N}{x_1\cdots x_{d-2}}} \mathrm{d}x_{d-1} \int_{\frac{N}{x_1\cdots x_{d-1}}}^{\infty} \frac{\mathrm{d}x_d}{x_1^2\cdots x_d^2} \\ &= \frac{1}{Nx_1\cdots x_{d-3}} \int_1^{\frac{N}{x_1\cdots x_{d-3}}} \frac{1}{x_{d-2}} \log \frac{N}{x_1\cdots x_{d-2}} \mathrm{d}x_{d-2} \\ &= \frac{1}{Nx_1\cdots x_{d-3}} \int_1^{\frac{N}{x_1\cdots x_{d-3}}} \frac{\log u}{u} \mathrm{d}u \\ &\sim \frac{1}{Nx_1\cdots x_{d-3}} \log^2 \frac{N}{x_1\cdots x_{d-3}}.\end{aligned}$$

Continuing this procedure, we deduce that

$$P_N^{(1)} \sim R_N \sim \frac{1}{N} \int_1^N \frac{1}{x_1} \log^{d-2} \frac{N}{x_1} \mathrm{d}x_1 \sim \frac{\log^{d-1} N}{N}. \tag{4.6}$$

We estimate $P_N^{(2)}$.

Let

$$S_N^{(k)} = \sum_{\mathbf{p}\in\Theta_N} \frac{\eta_k'}{p_1^2\cdots p_d^2} \qquad (k = 1, ..., d).$$

From this and (4.6), we deduce that

$$S_N^{(1)} = \sum_{\mathbf{p}\in\Theta_N} \frac{\eta_1'}{p_1^2\cdots p_d^2} = O\left(\frac{1}{N}\right) \int_1^N \frac{1}{x_1} \log^{d-2} \frac{N}{x_1} \eta_1' \,\mathrm{d}x_1.$$

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Since $\eta_1' \to 0$ as $x_1 \to \infty$ and $\int_1^\infty \frac{1}{x_1} \log^{d-2} \frac{N}{x_1} \mathrm{d}x_1 = \infty$, by a known result in Calculus and (4.6),

$$S_N^{(1)} = o\left(\frac{1}{N}\right) \int_1^N \frac{1}{x_1} \log^{d-2} \frac{N}{x_1} \,\mathrm{d}x_1 = o\left(\frac{\log^{d-1} N}{N}\right).$$

An argument similar to $S_N^{(1)}$ shows that for each k,

$$S_N^{(k)} = o\left(\frac{\log^{d-1} N}{N}\right).$$

From this and (4.2)-(4.6), we get

$$2^d \parallel f - S_N^{(h)} \parallel_2^2 = P_N^{(1)}(1 + o(1)).$$

Theorem 4.1. Let $f \in W([0,1]^d)$. Then hyperbolic cross truncations $S_N^{(h)}(f; \mathbf{t})$ satisfy

$$\parallel f - S_N^{(h)}(f) \parallel_2^2 = \widetilde{P}_N^{(1)}(1 + o(1)) \qquad (N \to \infty),$$

where

$$\widetilde{P}_N^{(1)} = \left(\frac{1}{\pi^2}\right)^d \left(\sum_{\mathbf{p} \in \Theta_N} \frac{1}{p_1^2 \cdots p_d^2}\right) \left(\sum_{\lambda \in \{0,1\}^d} |f(\lambda)|^2\right)$$

and

$$\Theta_N = \{\mathbf{p} = (p_1, ..., p_d): \quad p_1 \in \mathbb{Z}_+, \quad 1 \le p_2, ..., p_d \le N-1, \quad p_1, ..., p_d \ge N\} \tag{4.7}$$

and

$$P_N^{(1)} \sim \frac{\log^{d-1} N}{N}.$$

We easily see that the number of Fourier sine coefficients in hyperbolic cross truncations satisfies

$$N_c \sim N \log^{d-1} N.$$

In fact,

$$\begin{aligned} N_c &= \sum_{\mathbf{p} \in \Theta_N} \sum_{\mathbf{q} \in \{0,1\}^d} 1 \\ &\sim \int_1^N \mathrm{d}x_1 \int_1^{\frac{N}{x_1}} \mathrm{d}x_2 \cdots \int_1^{\frac{N}{x_1 \cdots x_{d-1}}} \mathrm{d}x_d \\ &\sim \int_1^N \mathrm{d}x_1 \int_1^{\frac{N}{x_1}} \mathrm{d}x_2 \cdots \int_1^{\frac{N}{x_1 \cdots x_{d-2}}} \frac{N}{x_1 \cdots x_{d-1}} \mathrm{d}x_{d-1} \\ &\sim N \log^{d-1} N. \end{aligned}$$

Therefore, by Theorem 4.1,

$$\parallel f - S_N^{(h)}(f) \parallel_2^2 \sim \frac{\log^{2d-2} N_c}{N_c}.$$

From this, we see that for $f \in W([0,1]^d)$, the hyperbolic cross approximation of Fourier sine series is a better approximation tool than ordinary partial sum approximation.

5. Asymptotic behaviors of stochastic Fourier sine coefficients

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We extend the results in Sections 3-4 to stochastic processes. Let $\xi(\mathbf{t})$ be a continuous stochastic process on $[0,1]^d$. Then $\xi(\mathbf{t})$ can be expanded into the stochastic Fourier sine series

$$\xi(\mathbf{t}) = \sum_{\mathbf{n}\in\mathbb{Z}_+^d} c_{\mathbf{n}}(\xi)\, T_{\mathbf{n}}(\mathbf{t}) \tag{5.1}$$

in mean square sense, where

$$c_{\mathbf{n}}(\xi) = 2^d \int\limits_{[0,1]^d} \xi(\mathbf{t}) T_{\mathbf{n}}(\mathbf{t})\,\mathrm{d}\mathbf{t}$$

and $T_{\mathbf{n}}(\mathbf{t})$ is stated in (2.4). The Fourier sine coefficients are stochastic variables. We discuss their expectations, second-order moments, and variances.

Theorem 5.1. If ξ is a stochastic process on $[0,1]^d$ and $\xi \in SW([0,1]^d)$, then the expectations, second-order moments, and variances of its Fourier sine coefficients possess the following asymptotic behaviors:

(i) $E[\,c_{\mathbf{n}}(\xi)\,] = \left(\prod_{j=1}^{d} \frac{2}{\pi n_j}\right)(\alpha_{\mathbf{n}}(\xi) + r_1 + \cdots + r_d)$ and

$$\alpha_{\mathbf{n}}(\xi) = \sum_{\lambda\in\{0,1\}^d} (\,E[\,\xi(\lambda)\,]\epsilon_{\mathbf{n}}(\lambda)\,)\,,$$

where $\alpha_{\mathbf{n}}(\xi)$ is an algebraic sum of expectation of ξ at vertexes of the cube $[0,1]^d$ and $r_k \to 0$ as $n_k \to \infty$, and $\epsilon_{\mathbf{n}}(\lambda)$ is stated in (3.2).

(ii) $E[\,c^2_{\mathbf{n}}(\xi)\,] = \left(\prod_{j=1}^{d} \frac{4}{\pi^2 n_j^2}\right)(\beta_{\mathbf{n}}(\xi) + r'_1 + \cdots + r'_d)$ and

$$\beta_{\mathbf{n}}(\xi) = \sum_{\lambda\in\{0,1\}^d}\ \sum_{\lambda'\in\{0,1\}^d} E[\,\xi(\lambda)\xi(\lambda')\,]\epsilon_{\mathbf{n}}(\lambda)\epsilon_{\mathbf{n}}(\lambda'),$$

where $r'_k \to 0$ as $n_k \to \infty$.

(iii) $\mathrm{Var}\,[\,c_{\mathbf{n}}(\xi)\,] = \left(\prod_{j=1}^{d} \frac{4}{\pi^2 n_j^2}\right)(\theta_{\mathbf{n}}(\xi) + r''_1 + \cdots + r''_d)$, where

$$\theta_{\mathbf{n}}(\xi) = \sum_{\lambda\in\{0,1\}^d}\ \sum_{\lambda'\in\{0,1\}^d} \mathrm{Cov}(\xi(\lambda),\ \xi(\lambda'))\,\epsilon_{\mathbf{n}}(\lambda)\,\epsilon_{\mathbf{n}}(\lambda').$$

and $r''_k \to 0$ as $n_k \to \infty$.

For example, consider the case $d = 2$. Assume that a stochastic process $\xi \in SW([0,1]^2)$. Then

$$E[\,c_{\mathbf{n}}(\xi)\,] = \frac{4}{n_1 n_2 \pi^2}\,(\,E[\,\xi(0,0)\,] - (-1)^{n_1}E[\,\xi(1,0)\,] - (-1)^{n_2}E[\,\xi(0,1)\,] + E[\,\xi(1,1)\,] + r_1 + r_2\,)$$

and

$$\begin{aligned}
\mathrm{Var}\,(c_{\mathbf{n}}(\xi)) \quad &= \quad \tfrac{16}{n_1^2 n_2^2 \pi^4}(\eta_{0,0} - (-1)^{n_1}\eta_{0,1} - (-1)^{n_2}\eta_{0,2} + \eta_{0,3} - (-1)^{n_1}\eta_{1,0} \\
&+ \quad \eta_{1,1} + (-1)^{n_1+n_2}\eta_{1,2} - (-1)^{n_1}\eta_{1,3} - (-1)^{n_2}\eta_{2,0} + (-1)^{n_1+n_2}\eta_{2,1} \\
&+ \quad \eta_{2,2} - (-1)^{n_2}\eta_{2,3} + \eta_{3,0} - (-1)^{n_1}\eta_{3,1} - (-1)^{n_2}\eta_{3,2} + \eta_{3,3} + r'_1 + r'_2),
\end{aligned}$$

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where

$$\eta_{\lambda_1+2\lambda_2,\lambda_1'+2\lambda_2'} = \mathrm{Cov}\,(\xi(\lambda_1,\lambda_2),\ \xi(\lambda_1',\ \lambda_2')) \qquad (\lambda_1,\lambda_2,\lambda_1',\lambda_2' = 0 \text{ or } 1)$$

and $r_1,\, r_1' \to 0$ as $n_1 \to \infty$ and $r_2,\, r_2' \to 0$ as $n_2 \to \infty$.

Proof of Theorem 5.1. Exchanging the expectation and integral, we deduce from (5.1) that

$$E[\, c_{\mathbf{n}}(\xi)\,] = 2^d \int\limits_{[0,1]^d} E[\,\xi(\mathbf{t})\,]\, \mathbf{T}_{\mathbf{n}}(\mathbf{t})\,\mathrm{d}\mathbf{t} = c_{\mathbf{n}}(\, E[\,\xi(\mathbf{t})\,]\,),$$

i.e., $E[\, c_{\mathbf{n}}(\xi)\,]$ is the Fourier sine coefficients of the deterministic function $E[\,\xi(\mathbf{t})\,]$. Exchanging the expectation and partial derivative, we deduce from $\xi \in SW([0,1]^d)$ that

$$E[\,\xi(\mathbf{t})\,] \in W([0,1]^d).$$

Using Theorem 3.1, we get (i).

By (5.1), we get $|c_{\mathbf{n}}(\xi)|^2 = 2^{2d} \int\limits_{[0,1]^d} \int\limits_{[0,1]^d} \xi(\mathbf{t})\xi(\mathbf{s})\, T_{\mathbf{n}}(\mathbf{t}) T_{\mathbf{n}}(\mathbf{s}) \mathrm{d}\mathbf{t}\,\mathrm{d}\mathbf{s}$, and so

$$E[\,|c_{\mathbf{n}}(\xi)|^2\,] = 2^{2d} \int\limits_{[0,1]^d} \int\limits_{[0,1]^d} R_\xi(\widetilde{\mathbf{t}})\, T_{\mathbf{n}}(\mathbf{t}) T_{\mathbf{n}}(\mathbf{s})\,\mathrm{d}\mathbf{t}\,\mathrm{d}\mathbf{s}, \tag{5.2}$$

where the autocorrelation function $R_\xi(\widetilde{\mathbf{t}}) = E[\,\xi(\mathbf{t})\xi(\mathbf{s})\,]$ is a $2d-$variate deterministic function and

$$\mathbf{t} = (t_1, ..., t_d), \qquad\qquad \mathbf{s} = (s_1, ..., s_d),$$

$$\widetilde{\mathbf{t}} = (t_1, ..., t_{2d}) \qquad (t_{d+i} = s_i,\ i = 1, ..., d).$$

Let $n_{d+j} = n_j$ $(j = 1, ..., d)$. Then (5.2) can be rewritten into

$$E[\,|c_{n_1,...,n_d}(\xi)|^2\,] = 2^{2d} \int\limits_{[0,1]^{2d}} R_\xi(\widetilde{\mathbf{t}}) \prod_{j=1}^{2d} \sin(\pi n_j t_j) \mathrm{d}\widetilde{\mathbf{t}}.$$

From the definition of Fourier sine coefficients, we see that $E[\,|c_{n_1,...,n_d}(\xi)|^2\,]$ is the Fourier sine coefficient of $2d-$variate function R_ξ, that is,

$$E[\,|c_{n_1,...,n_d}(\xi)|^2\,] = c_{n_1,...,n_{2d}}(R_\xi) \qquad (n_{d+j} = n_j,\ j = 1, ..., d). \tag{5.3}$$

By the assumption $\xi \in SW([0,1]^d)$, we deduce that $R_\xi \in W([0,1]^{2d})$. In Theorem 3.1, replacing f by R_ξ and d by $2d$ and letting $n_{d+j} = n_j$ $(j = 1, ..., d)$, we obtain

$$c_{n_1,...,n_{2d}}(R_\xi) = \left(\prod_{j=1}^{2d} \frac{2}{\pi n_j} \right) (\, \beta_{n_1,...,n_d}(\xi) + r_1 + \cdots + r_d\,), \tag{5.4}$$

where

$$\beta_{n_1,...,n_d}(\xi) = K^{2d}_{n_1,...,n_{2d}}(R_\xi) = \sum_{\widetilde{\lambda} \in \{0,1\}^{2d}} R_\xi(\widetilde{\lambda}) \left(\prod_{j \in G_{\widetilde{\lambda}}} (-1)^{n_j+1} \right)$$

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and

$$\widetilde{\lambda} = (\lambda_1, ..., \lambda_{2d}) = (\lambda_1, ..., \lambda_d, \lambda'_1, ..., \lambda'_d),$$
$$G_{\widetilde{\lambda}} = \{j \in \{1, ..., 2d\}, \ \ \lambda_j = 1\}.$$

By $n_{d+j} = n_j \ (j = 1, ..., d)$, we have

$$\begin{aligned}\prod_{j\in G_{\widetilde{\lambda}}} (-1)^{n_j+1} &= \left(\prod_{j\in(G_{\widetilde{\lambda}}\bigcap\{1,...,d\})} (-1)^{n_j+1}\right)\left(\prod_{j\in(G_{\widetilde{\lambda}}\bigcap\{d+1,...,2d\})} (-1)^{n_j+1}\right) \\ &= \prod_{j\in G_\lambda} (-1)^{n_j+1} \prod_{j\in G_{\lambda'}} (-1)^{n_j+1},\end{aligned} \tag{5.5}$$

where

$$G_\lambda = \{j \in \{1, ..., d\}, \ \ \lambda_j = 1\},$$
$$G_{\lambda'} = \{j \in \{1, ..., d\}, \ \ \lambda'_j = 1\}.$$

Since

$$R_\xi(\lambda_1, ..., \lambda_{2d}) = E[\xi(\lambda_1, ..., \lambda_d)\xi(\lambda'_1, ..., \lambda'_d)] \qquad (\lambda_{d+j} = \lambda'_j),$$

by (5.5), we deduce by (5.4) that

$$\beta_{n_1,...,n_d}(\xi) = \sum_{\lambda\in\{0,1\}^d} \sum_{\lambda'\in\{0,1\}^d} E[\,\xi(\lambda_1, ..., \lambda_d)\xi(\lambda'_1, ..., \lambda'_d)\,] \prod_{j\in G_\lambda} (-1)^{n_j+1} \prod_{j\in G_{\lambda'}} (-1)^{n_j+1}. \tag{5.6}$$

From this and (5.3)-(5.4), we get (ii). From (i),

$$E[c^2_{\mathbf{n}}(\xi)] = \prod_{j=1}^{d} \frac{4}{\pi^2 n_j^2}(\alpha^2_{\mathbf{n}}(\xi) + \widetilde{r}_1 + \cdots + \widetilde{r}_d),$$

where each $\widetilde{r}_k \to 0$ as $n_k \to \infty$. Again, by (ii),

$$\mathrm{Var}(c_{\mathbf{n}}(\xi)) = E[c^2_{\mathbf{n}}(\xi)] - |E[c_{\mathbf{n}}(\xi)]|^2 = \prod_{j=1}^{d} \frac{4}{\pi^2 n_j^2}(\beta_{\mathbf{n}}(\xi) - \alpha^2_{\mathbf{n}}(\xi) + r''_1 + \cdots + r''_d),$$

where $r''_k \to 0$ as $n_k \to \infty$. Noticing that

$$\begin{aligned}\beta_{\mathbf{n}}(\xi) - \alpha^2_{\mathbf{n}}(\xi) &= \sum_{\lambda,\lambda'\in\{0,1\}^d} (E[\xi(\lambda)\xi(\lambda')] - E[\xi(\lambda)]E[\xi(\lambda')])\,\epsilon_{\mathbf{n}}(\lambda)\epsilon_{\mathbf{n}}(\lambda') \\ &= \sum_{\lambda,\lambda'\in\{0,1\}^d} \mathrm{Cov}(\xi(\lambda),\, \xi(\lambda'))\epsilon_{\mathbf{n}}(\lambda)\epsilon_{\mathbf{n}}(\lambda'),\end{aligned}$$

we get (iii). Theorem 5.1 is proved. □

6. Asymptotic behavior of hyperbolic cross hyperbolic cross approximations of stochastic Fourier sine series

Let $\xi(\mathbf{t})$ be a continuous stochastic process on $[0, 1]^d$. The hyperbolic cross truncation of its Fourier sine series is

$$S_N^{(h)}(\xi, \mathbf{t}) = \sum_{|\mathbf{p}|\le N-1} \sum_{\mathbf{q}\in\{0,1\}^d} c_{2\mathbf{p}+\mathbf{q}} T_{2\mathbf{p}+\mathbf{q}}(\mathbf{t}), \tag{6.1}$$

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where $|\mathbf{p}| = \prod_{k=1}^{d} p_k$ ($\mathbf{p} = (p_1, ..., p_d) \in \mathbb{Z}_+^d$), is a stochastic sine polynomial. We give an asymptotic behavior of the hyperbolic cross approximation.

Theorem 6.1. Let ξ be a stochastic process on $[0,1]^d$. If $\xi \in SW([0,1]^d)$. Then the hyperbolic cross truncations $S_N^{(h)}(\xi)$ of the stochastic Fourier sine series of ξ satisfy

$$E[\parallel S_N^{(h)}(\xi) - \xi \parallel_2^2] = W_N(\xi)(1 + o(1)) \qquad (N \to \infty),$$

where

$$W_N(\xi) = \left(\frac{1}{\pi^2}\right)^d \sum_{\mathbf{p}\in\Theta_N} \frac{1}{p_1^2 \cdots p_d^2} \sum_{\lambda\in\{0,1\}^d} E[|\xi(\lambda)|^2]$$

and Θ_N is stated in (4.2) and

$$P_N^{(1)}(\xi) \sim \frac{\log^{d-1} N}{N}.$$

Proof. By using an argument similar to Section 4, we deduce that (4.3) is still valid when f is replaced by ξ. Taking expectation on both sides,

$$\begin{aligned}
&2^d E[\parallel \xi - S_N^{(h)}(\xi) \parallel_2^2] \\
&= \sum_{\mathbf{p}\in\Theta_N} \sum_{\mathbf{q}\in\{0,1\}^d} E[c_{2\mathbf{p}+\mathbf{q}}^2(\xi)] \\
&\quad + \left(\sum_{p_1,...,p_d=1}^{\infty} - \sum_{p_1=1}^{\infty} \sum_{p_2,...,p_d=1}^{N}\right) \sum_{\mathbf{q}\in\{0,1\}^d} E[c_{2\mathbf{p}+\mathbf{q}}^2(\xi)] \\
&= P_N(\xi) + Q_N(\xi).
\end{aligned}$$

By Theorem 5.1,

$$E[c_{2\mathbf{p}+\mathbf{q}}^2(\xi)] = O\left(\frac{1}{p_1^2 \cdots p_d^2}\right).$$

This implies that $Q_N(\xi) = O\left(\frac{1}{N}\right)$. By Theorem 5.1 (ii), it follows that

$$\begin{aligned}
&2^d E[\parallel \xi - S_N^{(h)}(\xi) \parallel_2^2] \\
&= \frac{1}{\pi^{2d}} \sum_{\mathbf{p}\in\Theta_N} \frac{1}{p_1^2\cdots p_d^2} \left(\sum_{\mathbf{q}\in\{0,1\}^d} \beta_{2\mathbf{p}+\mathbf{q}} + r_1'' + \cdots + r_d''\right) \qquad (6.2)\\
&=: P_{\mathbf{n}}^{(1)}(\xi) + P_N^{(2)}(\xi) + O\left(\tfrac{1}{N}\right),
\end{aligned}$$

where Θ_N is stated in (4.7) and each $r_k'' \to 0$ as $n_k \to \infty$, and

$$P_N^{(1)}(\xi) = \frac{1}{\pi^{2d}} \sum_{\mathbf{p}\in\Theta_N} \frac{1}{p_1^2 \cdots p_d^2} \left(\sum_{\mathbf{q}\in\{0,1\}^d} \beta_{2\mathbf{p}+\mathbf{q}}\right),$$

$$P_N^{(2)}(\xi) = \frac{1}{\pi^{2d}} \sum_{\mathbf{p}\in\Theta_N} \frac{1}{p_1^2 \cdots p_d^2} \left(\sum_{\mathbf{q}\in\{0,1\}^d} (r_1'' + \cdots + r_d'')\right).$$

By (3.17),

$$\begin{aligned}\sum_{\mathbf{q}\in\{0,1\}^d}\beta_{2\mathbf{p}+\mathbf{q}} &= \tfrac{1}{\pi^{2d}}\sum_{\lambda,\lambda'\in\{0,1\}^d}E[\xi(\lambda)\xi(\lambda')]\left(\sum_{\mathbf{q}\in\{0,1\}^d}\epsilon_{2\mathbf{p}+\mathbf{q}}(\lambda)\epsilon_{2\mathbf{p}+\mathbf{q}}(\lambda')\right)\\ &= \tfrac{2^d}{\pi^{2d}}\sum_{\lambda\in\{0,1\}^d}E[\xi^2(\lambda)].\end{aligned}$$

So

$$P_N^{(1)}(\xi)=\left(\frac{2}{\pi^2}\right)^d\sum_{\mathbf{p}\in\Theta_N}\frac{1}{p_1^2\cdots p_d^2}\sum_{\lambda\in\{0,1\}^d}E[|\xi(\lambda)|^2].$$

Similar to the argument of Theorem 4.1,

$$P_N^{(1)}(\xi)\sim\sum_{\mathbf{p}\in\Theta_N}\frac{1}{p_1^2\cdots p_d^2}\sim\frac{\log^{d-1}N}{N}$$

and

$$P_N^{(2)}(\xi)=o\left(\frac{\log^{d-1}N}{N}\right).$$

From this and (6.2), we deduce the desired result. □

7. Examples

In data analysis, the following three stochastic processes are often used [16].

(i) Gaussian stochastic process $\xi_{SE}(\mathbf{t})$ with mean $\mathbf{0}$ and square exponential covariance function:

$$K_{SE}(\mathbf{t},\mathbf{t}')=e^{\left(-\frac{\|\mathbf{t}-\mathbf{t}'\|_2^2}{2l^2}\right)}, \tag{7.1}$$

where $\mathbf{t}=(t_1,...,t_d)$ and $\mathbf{t}'=(t_1',...,t_d')$, and $\|\ \mathbf{t}\ \|_2^2=\sum\limits_{k=1}^d t_k^2$, and $l>0$.

(ii) Gaussian stochastic process $\xi_{RQ}(\mathbf{t})$ with mean $\mathbf{0}$ and rational quadratic covariance function:

$$K_{RQ}(\mathbf{t},\mathbf{t}')=(1+\|\ \mathbf{t}-\mathbf{t}'\ \|^2)^{-\alpha},$$

where $\mathbf{t}=(t_1,...,t_d)$ and $\mathbf{t}'=(t_1',...,t_d')$, and $\alpha\ge 0$.

(iii) Gaussian stochastic process $\xi_L(\mathbf{t})$ with mean $\mathbf{0}$ and linear covariance function:

$$K_L(\mathbf{t},\mathbf{t}')=<\mathbf{t},\ \mathbf{t}'>,$$

where $\mathbf{t}=(t_1,...,t_d)$ and $\mathbf{t}'=(t_1',...,t_d')$, and $<\mathbf{t},\ \mathbf{t}'>=\sum\limits_{k=1}^d t_kt_k'$.

Since these stochastic processes are differentiable [14, 15], we can use the theorems in Sections 5-6 to research their Fourier sine expansions, including variance estimates of Fourier sine coefficients and asymptotic formulas of hyperbolic cross truncation approximation.

We expand ξ_{SE} into a Fourier sine series on $[0,1]^d$ as follows:

$$\xi_{SE}(\mathbf{t})=\sum_{\mathbf{n}\in\mathbb{Z}_+^d}c_{\mathbf{n}}(\xi_{SE})T_{\mathbf{n}}(\mathbf{t}),$$

where $c_{\mathbf{n}}(\xi_{SE}) = 2^d \int\limits_{[0,1]^d} \xi_{SE}(\mathbf{t}) T_{\mathbf{n}}(\mathbf{t}) \mathrm{d}\mathbf{t}$.

Since $E[\xi_{SE}(\mathbf{t})] = 0\,(\mathbf{t} \in [0,1]^d)$, by Theorem 5.1 (i), we have

$$E[c_{\mathbf{n}}(\xi_{SE})] = o\left(\frac{1}{n_1 \cdots n_d}\right) \quad \text{as each} \quad n_k \to \infty \qquad (\mathbf{n} = (n_1, ..., n_d)).$$

Let $\mathbf{t} = (t_1, ..., t_d)$ and $\mathbf{t}' = (t'_1, ..., t'_d)$. Then

$$\| \mathbf{t} - \mathbf{t}' \|_2^2 = \sum_{k=1}^{d} (t_k - t'_k)^2,$$

and by (7.1),

$$\mathrm{Cov}(\xi_{SE}(\mathbf{t}),\ \xi_{SE}(\mathbf{t}')) = \prod_{k=1}^{d} e^{-\frac{1}{2l^2}(t_k - t'_k)^2}.$$

By Theorem 5.1 (iii), we obtain that the Fourier sine coefficients $c_{\mathbf{n}}(\xi_{SE})$ satisfy,

$$\mathrm{Var}(c_{\mathbf{n}}(\xi_{SE})) = \prod_{j=1}^{d} \frac{4}{\pi^2 n_j^2} (\theta_{\mathbf{n}}(\xi_{SE}) + r''_1 + \cdots + r''_d) \qquad \text{and} \quad r''_k \to 0 \text{ as } n_k \to \infty.$$

where

$$\theta_{\mathbf{n}}(\xi_{SE}) = \sum_{\lambda \in \{0,1\}^d} \sum_{\lambda' \in \{0,1\}^d} \left(\prod_{k=1}^{d} e^{-\frac{1}{2l^2}(\lambda_k - \lambda'_k)^2} \right) \varepsilon_{\mathbf{n}}(\lambda) \varepsilon_{\mathbf{n}}(\lambda'),$$

and $\varepsilon_{\mathbf{n}}(\lambda)$ is stated in (3.2) and $\lambda = (\lambda_1, ..., \lambda_d)$ and $\lambda' = (\lambda'_1, ..., \lambda'_d)$.

Next we find the asymptotic formula of the hyperbolic cross truncation approximation of Fourier sine series of ξ_{SE}. The hyperbolic cross truncation is

$$S_N^{(h)}(\xi_{SE}, \mathbf{t}) = \sum_{p_1 \cdots p_d \le N-1} \sum_{\mathbf{q} \in \{0,1\}^d} c_{2\mathbf{p}+\mathbf{q}}(\xi_{SE}) T_{2\mathbf{p}+\mathbf{q}}(\mathbf{t}) \qquad (p_i \in \mathbb{Z}_+,\ i = 1, ..., d).$$

Note that $E[|\xi_{SE}(\mathbf{t})|^2] = R_{SE}(0) = 1$ and $\sum\limits_{\lambda \in \{0,1\}^d} 1 = 2^d$. By theorem 6.1, we have

$$E[\| S_N^{(h)}(\xi_{SE}) - \xi_{SE} \|_2^2] = W_N(\xi_{SE})(1 + o(1)) \qquad (N \to \infty),$$

where

$$W_N(\xi_{SE}) = \left(\frac{1}{\pi^2}\right)^d \sum_{\mathbf{p} \in \Theta_N} \frac{1}{p_1^2 \cdots p_d^2} \sum_{\lambda \in \{0,1\}^d} 1 = \left(\frac{4}{\pi^2}\right)^d \sum_{\mathbf{p} \in \Theta_N} \frac{1}{p_1^2 \cdots p_d^2}.$$

So we get the asymptotic formula as follows:

$$E[\| S_N^{(h)}(\xi_{SE}) - \xi_{SE} \|_2^2] = \left(\frac{2}{\pi^2}\right)^d \left(\sum_{\mathbf{p} \in \Theta_N} \frac{1}{p_1^2 \cdots p_d^2} \right) (1 + o(1)) \qquad (N \to \infty),$$

where Θ_N is stated in (4.2) and the number N_c of Fourier sine coefficients in the hyperbolic cross truncation $S_N^{(h)}(\xi_{SE})$ is equivalent to $N \log^{d-1} N$.

Similarly, for the stochastic processes ξ_{RQ} and ξ_L, using the same method as above, we can give the variance estimates of their Fourier sine coefficients and asymptotic formulas of hyperbolic cross truncation approximations.

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HIGHER-ORDER DEGENERATE BERNOULLI POLYNOMIALS

DAE SAN KIM AND TAEKYUN KIM

ABSTRACT. Carlitz introduced the degenerate Bernoulli polynomials and derived, among other things, the so-called degenerate Staudt-Clausen theorem for the degenerate Bernoulli numbers as an analogue of the classical Staudt-Clausen theorem. In this paper, we consider the higher-order Carlitz's degenerate Bernoulli polynomials with umbral calculus viewpoint and derive new identities and properties of those polynomials associated with special polynomials which are derived from umbral calculus.

1. INTRODUCTION

The degenerate Bernoulli polynomials $\beta_n(\lambda, x)\,(\lambda \neq 0)$ are defined by Carlitz to be

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n(\lambda, x)\frac{t^n}{n!}, \quad (\lambda \neq 0), \quad (\text{see } [3, 4]). \tag{1.1}$$

Ustinov rediscovered these polynomials in [18], which are called Korobov polynomials of the second kind and denoted by $k_n^{(\lambda)}(x)$.

When $x = 0$, $\beta_n(\lambda) = \beta_n(\lambda, 0)$ are called the degenerate Bernoulli numbers. Now, we observe that

$$\lim_{\lambda\to 0} \beta_n(\lambda, x) = \beta_n(0, x) = B_n(x), \quad \lim_{\lambda\to\infty} \lambda^{-n}\beta_n(\lambda, \lambda x) = b_n(x), \tag{1.2}$$

where $B_n(x)$ and $b_n(x)$ are the Bernoulli polynomials of the first kind and of the second kind.

The first few degenerate Bernoulli polynomials are given by $\beta_0(\lambda, x) = 1$, $\beta_1(\lambda, x) = x - \frac{1}{2} + \frac{1}{2}\lambda$, $\beta_2(\lambda, x) = x^2 - x + \frac{1}{6} - \frac{1}{6}\lambda^2$, $\beta_3(\lambda, x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x - \frac{3}{2}\lambda x^2 + \frac{3}{2}\lambda x + \frac{1}{4}\lambda^3 - \frac{1}{4}\lambda, \ldots$. As an analogue of the classical Staudt-Clausen theorem for Bernoulli numbers, Carlitz proved the so called degenerate Staudt-Clausen theorem for $\beta_n(\lambda)$, (λ a rational number) (see [3, 19, 20]). The generalized falling factorials $(x|\lambda)_n$ for any $\lambda \in \mathbb{C}$ are defined as

$$(x|\lambda)_0 = 1, \quad (x|\lambda)_n = x(x-\lambda)\cdots(x-\lambda(n-1)), \quad (\text{for } n > 0).$$

Carlitz also found in [4] the following relation expressing sums of generalized falling factorals in terms of degenerate Bernoulli polynomials: for integers l, m with $l \geq 1, m \geq 0$,

$$\sum_{i=0}^{l-1} (i|\lambda)_m = \frac{1}{m+1}\left(\beta_{m+1}(\lambda, l) - \beta_{m+1}(\lambda)\right), \tag{1.3}$$

2000 *Mathematics Subject Classification.* 05A19, 05A40, 11B83.

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which, by letting $\lambda \to 0$, becomes the familiar relation

$$\sum_{i=0}^{l-1} i^m = \frac{1}{m+1}\left(B_{m+1}(l) - B_{m+1}\right). \tag{1.4}$$

For $r \in \mathbb{N}$, the Bernoulli polynomials of the second kind of order r are defined by the generating function to be

$$\left(\frac{t}{\log(1+t)}\right)^r (1+t)^x = \sum_{n=0}^{\infty} b_n^{(r)}(x)\frac{t^n}{n!}, \quad (\text{see [16]}), \tag{1.5}$$

and the Bernoulli polynomials of order r are given by

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!}, \quad (\text{see [2, 5–7, 9]}). \tag{1.6}$$

When $x = 0$, $B_n^{(r)} = B_n^{(r)}(0)$, $b_n^{(r)} = b_n^{(r)}(0)$ are called the Bernouli numbers of the first kind of order r and of the second kind of order r. For $\mu \in \mathbb{C}$ with $\mu \neq 1$, the Frobenius-Euler polynomials with order $s \in \mathbb{N}$ are defined by the generating function to be

$$\left(\frac{1-\mu}{e^t-\mu}\right)^s e^{xt} = \sum_{n=0}^{\infty} H_n^{(s)}(x|\mu)\frac{t^n}{n!}, \quad (\text{see [1, 10–12]}). \tag{1.7}$$

When $x = 0$, $H_n^{(s)}(\mu) = H_n^{(s)}(0|\mu)$ are called the Frobenius-Euler numbers of order s. As is well known, the Stirling number of the second kind is defined by the generating function to be

$$\left(e^t-1\right)^n = n!\sum_{l=n}^{\infty} S_2(l,m)\frac{t^l}{l!}, \quad (n \in \mathbb{Z}_{\geq 0}), \quad (\text{see [16, 17]}). \tag{1.8}$$

For $n \geq 0$, the Stirling number of the first kind is given by

$$(x)_n = x(x-1)\cdots(x-(n-1)) = \sum_{l=0}^{n} S_1(n,l)\,x^l, \quad (\text{see [13, 15, 16, 21]}).$$

Let $\mathcal{F}$ be the set of all formal power series in the variable t:

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \,\middle|\, a_k \in \mathbb{C} \right\}. \tag{1.9}$$

Let $\mathbb{P} = \mathbb{C}[x]$ and let $\mathbb{P}^*$be the vector space of all linear functionals on $\mathbb{P}$.$\langle L|\,p(x)\rangle$ denotes the action of the linear functional L on $p(x)$ which satisfies $\langle L+M|\,p(x)\rangle = \langle L|\,p(x)\rangle + \langle M|\,p(x)\rangle$, and $\langle cL|\,p(x)\rangle = c\,\langle L|\,p(x)\rangle$, where c is a complex constant. The linear functional $\langle f(t)|\,\cdot\rangle$ on $\mathbb{P}$ is defined as

$$\langle f(t)|\,x^n\rangle = a_n, \quad (n \geq 0), \quad \text{where } f(t) \in \mathcal{F}. \tag{1.10}$$

Thus, by (1.9) and (1.10), we get

$$\left\langle t^k\middle|\,x^n\right\rangle = n!\delta_{n,k}, \quad (n,k \geq 0), \quad (\text{see [14, 16]}), \tag{1.11}$$

where $\delta_{n,k}$ is the Kronecker symbol.

Let $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} t^k$. Then, by (1.11), we get $\langle f_L(t)|\,x^n\rangle = \langle L|\,x^n\rangle$. So, the map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all

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linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the umbral algebra and the umbral calculus is the study of umbral algebra. The order $o(f(t))$ of a power series $f(t) \neq 0$ is the smallest integer k for which the coefficient of t^k does not vanish (see [14, 16]). If $o(f(t)) = 0$, then $f(t)$ is called an invertible series; if $o(f(t)) = 1$, then $f(t)$ is called a delta series. Let $f(t), g(t)$ be a delta series and an invertible series, respectively. Then there exists a unique sequence $s_n(x)(\deg s_n(x) = n)$ such that $\left\langle g(t) f(t)^k \middle| s_n(x) \right\rangle = n!\delta_{n,k}$, for $k \geq 0$. Such a sequence $s_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$which is denoted by $s_n(x) \sim (g(t), f(t))$ (see [14, 16]). The sequence $s_n(x)$ is Sheffer for $(g(t), f(t))$ if and only if

$$\frac{1}{g\left(\overline{f}(t)\right)} e^{y\overline{f}(t)} = \sum_{k=0}^{\infty} \frac{s_k(y)}{k!} t^k, \quad (y \in \mathbb{C}), \quad (\text{see } [11, 17]), \tag{1.12}$$

where $\overline{f}(t)$ is the compositional inverse of $f(t)$ with $\overline{f}(f(t)) = f\left(\overline{f}(t)\right) = t$.

Let $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$. Then we see that

$$f(t) = \sum_{k=0}^{\infty} \left\langle f(t) \middle| x^k \right\rangle \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \left\langle t^k \middle| p(x) \right\rangle \frac{x^k}{k!}. \tag{1.13}$$

From (1.13), we have

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, \quad e^{yt} p(x) = p(x+y). \tag{1.14}$$

By (1.14), we get $\langle e^{yt} | p(x) \rangle = p(y)$.

For $s_n(x) \sim (g(t), f(t))$, we have the following equations ([16]):

$$f(t) s_n(x) = n s_{n-1}(x), \ (n \geq 1), \quad s_n(x+y) = \sum_{j=0}^{n} \binom{n}{j} s_j(x) p_{n-j}(y), \tag{1.15}$$

where $p_n(x) = g(t) s_n(x)$,

(1.16)

$$s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)}\right) \frac{1}{f'(t)} s_n(x), \quad s_n(x) = \sum_{j=0}^{n} \frac{1}{j!} \left\langle g\left(\overline{f}(t)\right)^{-1} \overline{f}(t)^j \middle| x^n \right\rangle x^j,$$

and

$$\begin{aligned} &\langle f(t) | x p(x) \rangle = \langle \partial_t f(t) | p(x) \rangle, \\ &\frac{d}{dx} s_n(x) = \sum_{l=0}^{n-1} \binom{n}{l} \left\langle \overline{f}(t) \middle| x^{n-l} \right\rangle s_l(x), \quad (n \geq 1). \end{aligned} \tag{1.17}$$

In particular, for $p_n(x) \sim (1, f(t))$, $q_n(x) \sim (1, g(t))$, we note that

$$q_n(x) = x \left(\frac{f(t)}{g(t)}\right)^n x^{-1} p_n(x), \quad (n \geq 1). \tag{1.18}$$

Let us assume that $s_n(x) \sim (g(t), f(t))$, $r_n(x) \sim (h(t), l(t))$. Then we have

$$s_n(x) = \sum_{m=0}^{n} C_{n,m} r_m(x), \quad (n \geq 0), \tag{1.19}$$

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where

$$C_{n,m}=\frac{1}{m!}\left\langle \frac{h\left(\overline{f}\left(t\right)\right)}{g\left(\overline{f}\left(t\right)\right)}\left(l\left(\overline{f}\left(t\right)\right)\right)^{m}\middle| x^{n}\right\rangle ,\quad \text{(see [16])}. \tag{1.20}$$

In this paper, we consider, for any positive integer r, the degenerate Bernoulli polynomials $\beta_n^{(r)}(\lambda,x)$ of order r which are defined by the generating function to be

$$\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{r}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\beta_n^{(r)}(\lambda,x)\frac{t^n}{n!},\quad (r\in\mathbb{Z}_{\geq 0}). \tag{1.21}$$

From (1.20) and (1.21), we note that

$$\beta_n^{(r)}(\lambda,x)\sim\left(\left(\frac{\lambda\left(e^t-1\right)}{e^{\lambda t}-1}\right)^{r},\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right). \tag{1.22}$$

That is, $\beta_n^{(r)}(\lambda,x)$ is the Sheffer polynomial for the pair

$$\left(g(t)=\left(\frac{\lambda\left(e^t-1\right)}{e^{\lambda t}-1}\right)^{r},f(t)=\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right).$$

The purpose of this paper is to give new identities and properties of the higher-order degenerate Bernoulli polynomials associated with special polynomials which are derived from umbral calculus.

2. Higher-order degenerate Bernoulli polynomials

For $n\geq 0$, we note that

$$x^n\sim(1,t),\quad \left(\frac{\lambda\left(e^t-1\right)}{e^{\lambda t}-1}\right)^{r}\beta_n^{(r)}(\lambda,x)\sim\left(1,\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right),$$

From (1.18), we can derive the following equation:

$$\begin{aligned}\left(\frac{\lambda\left(e^t-1\right)}{e^{\lambda t}-1}\right)^{r}\beta_n^{(r)}(\lambda,x)&=x\left(\frac{\lambda t}{e^{\lambda t}-1}\right)^{n}x^{-1}x^n=x\left(\frac{\lambda t}{e^{\lambda t}-1}\right)^{n}x^{n-1}\\&=\sum_{l=0}^{n-1}\binom{n-1}{l}\lambda^l B_l^{(n)}x^{n-l},\quad (n\geq 1).\end{aligned} \tag{2.1}$$

Thus, by (2.1), we get

$$\begin{aligned}&\beta_n^{(r)}(\lambda,x)\\&=\sum_{l=0}^{n-1}\binom{n-1}{l}\lambda^l B_l^{(n)}\left(\frac{e^{\lambda t}-1}{\lambda\left(e^t-1\right)}\right)^{r}x^{n-l}\\&=\sum_{l=0}^{n-1}\binom{n-1}{l}\lambda^l B_l^{(n)}\left(\frac{t}{e^t-1}\right)^{r}\left(\frac{e^{\lambda t}-1}{\lambda t}\right)^{r}x^{n-l}\\&=\sum_{l=0}^{n-1}\binom{n-1}{l}\lambda^l B_l^{(n)}\left(\frac{t}{e^t-1}\right)^{r}\left(r!\sum_{k=0}^{\infty}S_2(k+r,r)\frac{\lambda^k}{(k+r)!}t^k\right)x^{n-l}\\&=\sum_{l=0}^{n-1}\binom{n-1}{l}\lambda^l B_l^{(n)}\left(\frac{t}{e^t-1}\right)^{r}\sum_{k=0}^{n-l}\frac{\binom{n-l}{k}}{\binom{k+r}{r}}S_2(k+r,r)\lambda^k x^{n-l-k}\end{aligned} \tag{2.2}$$

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$$=\sum_{l=0}^{n-1}\sum_{k=0}^{n-l}\frac{\binom{n-1}{l}\binom{n-l}{k}}{\binom{k+r}{r}}S_2(k+r,r)\lambda^{k+l}B_l^{(n)}\left(\frac{t}{e^t-1}\right)^r x^{n-l-k}$$
$$=\sum_{l=0}^{n-1}\sum_{k=0}^{n-l}\frac{\binom{n-1}{l}\binom{n-l}{k}}{\binom{k+r}{r}}S_2(k+r,r)\lambda^{k+l}B_l^{(n)}B_{n-l-k}^{(r)}(x).$$

Therefore, by (2.2), we obtain the following theorem.

Theorem 2.1. *For $n \geq 1$, we have*

$$\beta_n^{(r)}(\lambda,x)=\sum_{l=0}^{n-1}\sum_{k=0}^{n-l}\frac{\binom{n-1}{l}\binom{n-l}{k}}{\binom{k+r}{r}}S_2(k+r,r)\lambda^{k+l}B_l^{(n)}B_{n-k-l}^{(r)}(x).$$

Remark. When $x=0$ and $r=1$, we get

$$\beta_n(\lambda)=\sum_{l=0}^{n-1}\sum_{k=0}^{n-l}\frac{1}{k+1}\binom{n-1}{l}\binom{n-l}{k}\lambda^{k+l}B_l^{(n)}B_{n-k-l}. \tag{2.3}$$

From (1.18), (1.22) and

$$(x|\lambda)_n=\lambda^n\left(\frac{x}{\lambda}\right)_n=\sum_{m=0}^{n}S_1(n,m)\lambda^{n-m}x^m\sim\left(1,\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right).$$

We note that

$$\begin{aligned}\beta_n^{(r)}(\lambda,x)&=\sum_{m=0}^{n}S_1(n,m)\lambda^{n-m}\left(\frac{e^{\lambda t}-1}{\lambda(e^t-1)}\right)^r x^m \\ &=\sum_{m=0}^{n}S_1(n,m)\lambda^{n-m}\left(\frac{t}{e^t-1}\right)^r\left(\frac{e^{\lambda t}-1}{\lambda t}\right)^r x^m \\ &=\sum_{m=0}^{n}S_1(n,m)\lambda^{n-m}\left(\frac{t}{e^t-1}\right)^r\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{k+r}{r}}S_2(k+r,r)\lambda^k x^{m-k} \\ &=\lambda^n\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{k+r}{r}}S_1(n,m)S_2(k+r,r)\lambda^{k-m}B_{m-k}^{(r)}(x).\end{aligned} \tag{2.4}$$

Therefore, by (2.4), we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$\beta_n^{(r)}(\lambda,x)=\lambda^n\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{k+r}{r}}S_1(n,m)S_2(k+r,r)\lambda^{k-m}B_{m-k}^{(r)}(x).$$

Remark. For $r=1$ and $x=0$, we get an expression for the degenerate Bernoulli numbers:

$$\beta_n(\lambda)=\lambda^n\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{1}{k+1}\binom{m}{k}S_1(n,m)\lambda^{k-m}B_{m-k}. \tag{2.5}$$

Here we use the conjugation representation.

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For $\beta_n^{(r)}(\lambda,x) \sim \left(g(t)=\left(\frac{\lambda(e^t-1)}{e^{\lambda t}-1}\right)^r, f(t)=\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right)$, we observe that

$$
\begin{aligned}
(2.6) \qquad & \left\langle g\left(\overline{f}(t)\right)^{-1}\overline{f}(t)^j \middle| x^n\right\rangle \\
&=\left\langle \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r\left(\frac{1}{\lambda}\log(1+\lambda t)\right)^j \middle| x^n\right\rangle \\
&=\lambda^{-j}\left\langle \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r \middle| j!\sum_{l=j}^{\infty}S_1(l,j)\frac{\lambda^l}{l!}t^l x^n\right\rangle \\
&=j!\lambda^{-j}\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\lambda^l\left\langle \sum_{m=0}^{\infty}\beta_m^{(r)}(\lambda)\frac{t^m}{m!} \middle| x^{n-l}\right\rangle \\
&=j!\lambda^{-j}\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\lambda^l\beta_{n-l}^{(r)}(\lambda).
\end{aligned}
$$

Therefore, by (1.16) and (2.6), we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, $r \geq 1$, we have*

$$\beta_n^{(r)}(\lambda,x)=\sum_{j=0}^{n}\lambda^{-j}\left(\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\lambda^l\beta_{n-l}^{(r)}(\lambda)\right)x^j.$$

Remark. Recall that

$$(2.7) \quad \left(\frac{\lambda(e^t-1)}{e^{\lambda t}-1}\right)^r\beta_n^{(r)}(\lambda,x)\sim\left(1,\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right), \quad (x|\lambda)_n\sim\left(1,\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right).$$

Thus, by (2.7), we get

$$(2.8) \quad \left(\frac{\lambda(e^t-1)}{e^{\lambda t}-1}\right)^r\beta_n^{(r)}(\lambda,x)=(x|\lambda)_n, \quad \text{and } \frac{e^{\lambda t}-1}{\lambda}(x|\lambda)_n=n(x|\lambda)_{n-1}.$$

From (2.8), we have

$$
\begin{aligned}
(2.9) \qquad \left(e^t-1\right)^r\beta_n^{(r)}(\lambda,x) &= \left(\frac{e^{\lambda t}-1}{\lambda}\right)^r(x|\lambda)_n \\
&=\begin{cases}(n)_r(x|\lambda)_{n-r} & , \text{ if } r\leq n\\ 0 & , \text{ if } r>n.\end{cases}
\end{aligned}
$$

By (2.9), we get

$$
\begin{aligned}
(2.10) \quad t^r\beta_n^{(r)}(\lambda,x) &= \begin{cases}(n)_r\lambda^{n-r}\left(\frac{t}{e^t-1}\right)^r\left(\frac{x}{\lambda}\right)_{n-r} & , \text{ if } r\leq n\\ 0 & , \text{ if } r>n\end{cases} \\
&=\begin{cases}(n)_r\lambda^{n-r}\sum_{m=0}^{n-r}S_1(n-r,m)\lambda^{-m}B_m^{(r)}(x) & , \text{ if } r\leq n\\ 0 & , \text{ if } r>n.\end{cases}
\end{aligned}
$$

Therefore, from (1.14) and (2.10), we have

(2.11)

$$\left(\frac{d}{dx}\right)^r\beta_n^{(r)}(\lambda,x)=\begin{cases}(n)_r\lambda^{n-r}\sum_{m=0}^{n-r}S_1(n-r,m)\lambda^{-m}B_m^{(r)}(x) & , \text{ if } r\leq n\\ 0 & , \text{ if } r>n.\end{cases}$$

In particular,

$$\frac{d}{dx}\beta_n(\lambda,x)=\begin{cases} n\lambda^{n-1}\sum_{m=0}^{n-1}S_1(n-1,m)\lambda^{-m}B_m(x) & \text{, if } r\le n\\ 0 & \text{, if } r>n. \end{cases} \tag{2.12}$$

To proceed further, we recall that the λ-Daehee polynomials $D_{n,\lambda}^{(r)}(x)$ of order r are given by

$$\left(\frac{\lambda\log(1+t)}{(1+t)^{\lambda}-1}\right)^{r}(1+t)^{x}=\sum_{n=0}^{\infty}D_{n,\lambda}^{(r)}(x)\frac{t^n}{n!},\quad (\text{see } [12,\ 16])\,. \tag{2.13}$$

From (1.5), (1.11) and (2.13), we have

$$\begin{aligned}
&\beta_n^{(r)}(\lambda,y)\\
=&\left\langle \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{r}(1+\lambda t)^{\frac{y}{\lambda}}\middle| x^n\right\rangle\\
=&\sum_{l=0}^{n}\binom{n}{l}\lambda^l b_l^{(r)}\left(\frac{y}{\lambda}\right)\left\langle \left(\frac{\log(1+\lambda t)}{\lambda\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}\right)^{r}\middle| x^{n-l}\right\rangle\\
=&\sum_{l=0}^{n}\binom{n}{l}\lambda^l b_l^{(r)}\left(\frac{y}{\lambda}\right)\left\langle \sum_{m=0}^{\infty}D_{m,\frac{1}{\lambda}}^{(r)}\lambda^m\frac{t^m}{m!}\middle| x^{n-l}\right\rangle\\
=&\sum_{l=0}^{n}\binom{n}{l}\lambda^l b_l^{(r)}\left(\frac{y}{\lambda}\right)D_{n-l,\frac{1}{\lambda}}^{(r)}\lambda^{n-l}\\
=&\lambda^n\sum_{l=0}^{n}\binom{n}{l}D_{n-l,\frac{1}{\lambda}}^{(r)}b_l^{(r)}\left(\frac{y}{\lambda}\right),
\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}
&\beta_n^{(r)}(\lambda,y)\\
=&\left\langle \sum_{l=0}^{\infty}\beta_l^{(r)}(\lambda,y)\frac{t^l}{l!}\middle| x^n\right\rangle\\
=&\left\langle \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{r}(1+\lambda t)^{\frac{y}{\lambda}}\middle| x^n\right\rangle\\
=&\left\langle \left(\frac{\lambda t}{\log(1+\lambda t)}\right)^{r}\middle| \left(\frac{\log(1+\lambda t)}{\lambda\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}\right)^{r}(1+\lambda t)^{\frac{y}{\lambda}}x^n\right\rangle\\
=&\left\langle \left(\frac{\lambda t}{\log(1+\lambda t)}\right)^{r}\middle| \sum_{l=0}^{\infty}D_{l,\lambda^{-1}}^{(r)}\left(\frac{y}{\lambda}\right)\lambda^l\frac{t^l}{l!}x^n\right\rangle\\
=&\sum_{l=0}^{n}\binom{n}{l}\lambda^l D_{l,\lambda^{-1}}^{(r)}\left(\frac{y}{\lambda}\right)\left\langle \sum_{m=0}^{\infty}b_m^{(r)}\lambda^m\frac{t^m}{m!}\middle| x^{n-l}\right\rangle\\
=&\sum_{l=0}^{n}\binom{n}{l}\lambda^l D_{l,\lambda^{-1}}^{(r)}\left(\frac{y}{\lambda}\right)b_{n-l}^{(r)}\lambda^{n-l}
\end{aligned} \tag{2.15}$$

$$=\lambda^n \sum_{l=0}^{n} \binom{n}{l} b_{n-l}^{(r)} D_{l,\lambda^{-1}}^{(r)} \left(\frac{y}{\lambda}\right).$$

Therefore, by (2.14) and (2.15), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$\sum_{l=0}^{n} \binom{n}{l} D_{n-l,\frac{1}{\lambda}}^{(r)} b_l^{(r)} \left(\frac{x}{\lambda}\right) = \sum_{l=0}^{n} \binom{n}{l} b_{n-l}^{(r)} D_{l,\lambda^{-1}}^{(r)} \left(\frac{x}{\lambda}\right) = \lambda^{-n} \beta_n^{(r)} (\lambda, x).$$

Recalling that

$$\beta_n^{(r)} (\lambda, x) \sim \left(g(t) = \left(\frac{\lambda (e^t - 1)}{e^{\lambda t} - 1}\right)^r, f(t) = \frac{1}{\lambda} \left(e^{\lambda t} - 1\right) \right),$$

we observe that

(2.16)

$$\begin{aligned}
&\left\langle g\left(\overline{f}(t)\right)^{-1} \overline{f}(t)^j \middle| x^n \right\rangle \\
=&j!\lambda^{-j} \sum_{l=j}^{n} S_1(l,j) \binom{n}{l} \lambda^l \left\langle \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^r \middle| x^{n-l} \right\rangle \\
=&j!\lambda^{-j} \sum_{l=j}^{n} S_1(l,j) \binom{n}{l} \lambda^l \left\langle \left(\frac{\log(1+\lambda t)}{\lambda \left((1+\lambda t)^{\frac{1}{\lambda}} - 1 \right)} \right)^r \middle| \left(\frac{\lambda t}{\log(1+\lambda t)} \right)^r x^{n-l} \right\rangle \\
=&j!\lambda^{-j} \sum_{l=j}^{n} S_1(l,j) \binom{n}{l} \lambda^l \left\langle \left(\frac{\log(1+\lambda t)}{\lambda \left((1+\lambda t)^{\frac{1}{\lambda}} - 1 \right)} \right)^r \middle| \sum_{m=0}^{\infty} b_m^{(r)} \frac{\lambda^m}{m!} t^m x^{n-l} \right\rangle \\
=&j!\lambda^{-j} \sum_{l=j}^{n} S_1(l,j) \binom{n}{l} \lambda^l \sum_{m=0}^{n-l} \binom{n-l}{m} \lambda^m b_m^{(r)} \left\langle \left(\frac{\log(1+\lambda t)}{\lambda \left((1+\lambda t)^{\frac{1}{\lambda}} - 1 \right)} \right)^r \middle| x^{n-l-m} \right\rangle \\
=&j!\lambda^{-j} \sum_{l=j}^{n} S_1(l,j) \binom{n}{l} \lambda^l \sum_{m=0}^{n-l} \binom{n-l}{m} \lambda^m b_m^{(r)} D_{n-l-m,\lambda^{-1}}^{(r)} \lambda^{n-l-m} \\
=&j!\lambda^n \sum_{l=j}^{n} \sum_{m=0}^{n-l} \binom{n}{l} \binom{n-l}{m} S_1(l,j) \lambda^{-j} b_m^{(r)} D_{n-l-m,\lambda^{-1}}^{(r)}.
\end{aligned}$$

From (1.16) and (2.16), we have

(2.17)

$$\beta_n^{(r)} (\lambda, x) = \lambda^n \sum_{j=0}^{n} \left\{ \sum_{l=j}^{n} \sum_{m=0}^{n-l} \binom{n}{l} \binom{n-l}{m} S_1(l,j) \lambda^{-j} b_m^{(r)} D_{n-l-m,\lambda^{-1}}^{(r)} \right\} x^j.$$

Remark. We have

$$\begin{aligned}
&\lim_{\lambda \to 0} \beta_n^{(r)} (\lambda, x) = \beta_n^{(r)} (0, x) = B_n^{(r)} (x), \\
&\lim_{\lambda \to 0} D_{n,\lambda}^{(r)} (x) = (x)_n, \\
&\lim_{\lambda \to \infty} \lambda^{-n} \beta_n^{(r)} (\lambda, \lambda x) = b_n^{(r)} (x),
\end{aligned}$$

$$\lim_{\lambda\to\infty}\lambda^{-n}D_{n,\lambda}^{(r)}(\lambda x)=B_n^{(r)}(x).$$

where $r>0$.

From (1.22), we note that

$$p_n(x)=\left(\frac{\lambda(e^t-1)}{e^{\lambda t}-1}\right)^r\beta_n^{(r)}(\lambda,x)=(x|\lambda)_n\sim\left(1,\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right). \tag{2.18}$$

By (2.18) and (1.15), we get

$$\beta_n^{(r)}(\lambda,x+y)=\sum_{j=0}^n\binom{n}{j}\beta_j^{(r)}(\lambda,x)(y|\lambda)_{n-j}, \tag{2.19}$$

and, by (1.14) and (1.15), we get

$$\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\beta_n^{(r)}(\lambda,x)=n\beta_{n-1}^{(r)}(\lambda,x). \tag{2.20}$$

From (2.20), we have

$$\beta_n^{(r)}(\lambda,x+\lambda)-\beta_n^{(r)}(\lambda,x)=n\lambda\beta_{n-1}^{(r)}(\lambda,x). \tag{2.21}$$

Therefore, by (2.17), (2.19) and (2.20), we obtain the following theorem.

Theorem 2.5. *For $n\geq 0$, we have*

$$\beta_n^{(r)}(\lambda,x)=\lambda^n\sum_{j=0}^n\left\{\sum_{l=j}^n\sum_{m=0}^{n-l}\binom{n}{l}\binom{n-l}{m}S_1(l,j)\lambda^{-j}b_m^{(r)}D_{n-l-m,\lambda^{-1}}^{(r)}\right\}x^j,$$

and

$$\begin{aligned}\beta_n^{(r)}(\lambda,x+\lambda)&=\sum_{j=0}^n\binom{n}{j}\beta_j^{(r)}(\lambda,x)(\lambda|\lambda)_{n-j}\\&=n\lambda\beta_{n-1}^{(r)}(\lambda,x)+\beta_n^{(r)}(\lambda,x).\end{aligned}$$

For $\beta_n^{(r)}(\lambda,x)\sim\left(g(t)=\left(\frac{\lambda\left(e^t-1\right)}{e^{\lambda t}-1}\right)^r,f(t)=\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right)$, we note that

$$\begin{aligned}&\frac{g'(t)}{g(t)}\\&=(\log g(t))'\\&=r\left(\log\lambda+\log\left(e^t-1\right)-\log\left(e^{\lambda t}-1\right)\right)'\\&=\frac{r}{t}\left(\sum_{l=0}^\infty B_l(1)\frac{t^l}{l!}-\sum_{l=0}^\infty B_l(1)\frac{\lambda^lt^l}{l!}\right)\\&=\frac{r}{t}\sum_{l=1}^\infty B_l(1)\left(1-\lambda^l\right)\frac{t^l}{l!}\\&=r\sum_{l=0}^\infty B_{l+1}(1)\left(1-\lambda^{l+1}\right)\frac{t^l}{(l+1)!}.\end{aligned} \tag{2.22}$$

By (2.22), we get

$$\frac{g'(t)}{g(t)}\beta_n^{(r)}(\lambda,x) \tag{2.23}$$

$$
\begin{aligned}
=&r\sum_{l=0}^{\infty}B_{l+1}\left(1\right)\left(1-\lambda^{l+1}\right)\frac{t^{l}}{\left(l+1\right)!}\lambda^{n}\\
&\times\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{k+r}{r}}S_{1}\left(n,m\right)S_{2}\left(k+r,r\right)\lambda^{k-m}B_{m-k}^{(r)}\left(x\right)\\
=&\lambda^{n}r\sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{l=0}^{m-k}S_{1}\left(n,m\right)S_{2}\left(k+r,r\right)\lambda^{k-m}\\
&\times\sum_{l=0}^{m-k}B_{l+1}\left(1\right)\left(1-\lambda^{l+1}\right)\frac{1}{\left(l+1\right)!}\left(m-k\right)_{l}B_{m-k-l}^{(r)}\left(x\right)\\
=&\lambda^{n}r\sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{l=0}^{m-k}\frac{1}{m-k-l+1}\frac{\binom{m}{k}\binom{m-k}{l}}{\binom{k+r}{r}}\left(\lambda^{k-m}-\lambda^{1-l}\right)\\
&\times S_{1}\left(n,m\right)S_{2}\left(k+r,r\right)B_{m-k-l+1}\left(1\right)B_{l}^{(r)}\left(x\right)\\
=&r\sum_{l=0}^{n}\sum_{m=l}^{n}\sum_{k=l}^{m}\frac{1}{k-l+1}\frac{\binom{m}{k}\binom{k}{l}}{\binom{m-k+r}{r}}\left(\lambda^{n-k}-\lambda^{n-l+1}\right)S_{1}\left(n,m\right)\\
&\times S_{2}\left(m-k+r,r\right)B_{k-l+1}\left(1\right)B_{l}^{(r)}\left(x\right).
\end{aligned}
$$

From (1.16) and (2.23), we have

$$
\begin{aligned}
(2.24)\qquad &\beta_{n+1}^{(r)}\left(\lambda,x\right)\\
=&x\beta_{n}^{(r)}\left(\lambda,x-\lambda\right)-e^{-\lambda t}\frac{g'\left(t\right)}{g\left(t\right)}\beta_{n}^{(r)}\left(\lambda,x\right)\\
=&x\beta_{n}^{(r)}\left(\lambda,x-\lambda\right)-r\sum_{l=0}^{n}\left(\sum_{m=l}^{n}\sum_{k=l}^{m}\frac{1}{k-l+1}\frac{\binom{m}{k}\binom{k}{l}}{\binom{m-k+r}{r}}\left(\lambda^{n-k}-\lambda^{n-l+1}\right)\right.\\
&\left.\times\ S_{1}\left(n,m\right)S_{2}\left(m-k+r,r\right)B_{k-l+1}\left(1\right)\right)B_{l}^{(r)}\left(x-\lambda\right).
\end{aligned}
$$

Therefore, by (2.24), we obtain the following theorem.

Theorem 2.6. *For $n\geq 0$, we have*

$$
\begin{aligned}
&\beta_{n+1}^{(r)}\left(\lambda,x\right)\\
=&x\beta_{n}^{(r)}\left(\lambda,x-\lambda\right)-r\sum_{l=0}^{n}\left(\sum_{m=l}^{n}\sum_{k=l}^{m}\frac{1}{k-l+1}\frac{\binom{m}{k}\binom{k}{l}}{\binom{m-k+r}{r}}\left(\lambda^{n-k}-\lambda^{n-l+1}\right)\right.\\
&\left.\times S_{1}\left(n,m\right)S_{2}\left(m-k+r,r\right)B_{k-l+1}\left(1\right)\right)B_{l}^{(r)}\left(x-\lambda\right).
\end{aligned}
$$

By $\beta_{n}^{(r)}\left(\lambda,x\right)\sim\left(g\left(t\right)=\left(\frac{\lambda\left(e^{t}-1\right)}{e^{\lambda t}-1}\right)^{r},f\left(t\right)=\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right)$, we get

$$
\begin{aligned}
(2.25)\qquad &\left\langle \overline{f}\left(t\right)\middle|\,x^{n-l}\right\rangle\\
=&\left\langle \frac{1}{\lambda}\log\left(1+\lambda t\right)\middle|\,x^{n-l}\right\rangle\\
=&\lambda^{-1}\left\langle \sum_{m=1}^{\infty}\left(-1\right)^{m-1}\lambda^{m}\left(m-1\right)!\frac{t^{m}}{m!}\middle|\,x^{n-l}\right\rangle
\end{aligned}
$$

$$=\lambda^{-1}(-1)^{n-l-1}\lambda^{n-l}(n-l-1)!$$
$$=(-\lambda)^{n-l-1}(n-l-1)!.$$

From (1.17) and (2.25), we have

$$\frac{d}{dx}\beta_n^{(r)}(\lambda,x)=n!\sum_{l=0}^{n-1}\frac{(-\lambda)^{n-l-1}}{l!\,(n-l)}\beta_l^{(r)}(\lambda,x)\,. \tag{2.26}$$

Let $n\geq 1$. Then, by (1.11) and (1.17), we get

$$\begin{aligned}
&\beta_n^{(r)}(\lambda,y) \\
&=\left\langle \sum_{l=0}^{\infty}\beta_l^{(r)}(\lambda,y)\frac{t^l}{l!}\middle| x^n\right\rangle \\
&=\left\langle \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r(1+\lambda t)^{\frac{y}{\lambda}}\middle| x^n\right\rangle \\
&=\left\langle \partial_t\left(\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r(1+\lambda t)^{\frac{y}{\lambda}}\right)\middle| x^{n-1}\right\rangle \\
&=\left\langle \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r\partial_t(1+\lambda t)^{\frac{y}{\lambda}}\middle| x^{n-1}\right\rangle \\
&\quad+\left\langle \left(\partial_t\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r\right)(1+\lambda t)^{\frac{y}{\lambda}}\middle| x^{n-1}\right\rangle .
\end{aligned} \tag{2.27}$$

The first term of (2.27) is

$$y\left\langle \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r(1+\lambda t)^{\frac{y-\lambda}{\lambda}}\middle| x^{n-1}\right\rangle = y\beta_{n-1}^{(r)}(\lambda,y-\lambda)\,. \tag{2.28}$$

For the second term of (2.27), we observe that

$$\partial_t\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r = r\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{r-1}\partial_t\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right), \tag{2.29}$$

where

$$\begin{aligned}
&\partial_t\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right) \\
&=\frac{(1+\lambda t)^{\frac{1}{\lambda}}-1-t(1+\lambda t)^{\frac{1}{\lambda}-1}}{\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)^2} \\
&=\frac{(1+\lambda t)^{\frac{1}{\lambda}}-1-t\left\{(1+\lambda t)^{\frac{1}{\lambda}-1}-(1+\lambda t)^{-1}\right\}-t(1+\lambda t)^{-1}}{\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)^2}
\end{aligned} \tag{2.30}$$

$$=-\frac{1}{1+\lambda t}\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}+\frac{1}{t} \tag{2.31}$$

$$\times\left\{\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}-\frac{1}{(1+\lambda t)}\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^2\right\}.$$

Thus, by (2.29) and (2.30), we get

$$\begin{aligned}(2.32)\qquad &\partial_t\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r\\ &=-\frac{r}{(1+\lambda t)}\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r\\ &\quad+\frac{r}{t}\left\{\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r-\frac{1}{1+\lambda t}\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{r+1}\right\}.\end{aligned}$$

From (2.32), we note that the second term of (2.27) is

(2.33)

$$\begin{aligned}&-r\left\langle\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r(1+\lambda t)^{\frac{y-\lambda}{\lambda}}\middle|x^{n-1}\right\rangle\\ &+r\left\langle(1+\lambda t)^{\frac{y}{\lambda}}\middle|\frac{1}{t}\left\{\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r-\frac{1}{1+\lambda t}\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{r+1}\right\}x^{n-1}\right\rangle\\ =&-r\beta_{n-1}^{(r)}(\lambda,y-\lambda)\\ &+\frac{r}{n}\left\langle(1+\lambda t)^{\frac{y}{\lambda}}\middle|\left\{\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r-\frac{1}{1+\lambda t}\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{r+1}\right\}x^n\right\rangle\\ =&-r\beta_{n-1}^{(r)}(\lambda,y-\lambda)\\ &+\frac{r}{n}\left\langle\left(\frac{t}{(1+t)^{\frac{1}{\lambda}}-1}\right)^r(1+\lambda t)^{\frac{y}{\lambda}}\middle|x^n\right\rangle\\ &-\frac{r}{n}\left\langle\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{r+1}(1+\lambda t)^{\frac{y-\lambda}{\lambda}}\middle|x^n\right\rangle\\ =&-r\beta_{n-1}^{(r)}(\lambda,y-\lambda)+\frac{r}{n}\beta_n^{(r)}(\lambda,y)-\frac{r}{n}\beta_n^{(r+1)}(\lambda,y-\lambda).\end{aligned}$$

By (2.27), (2.28) and (2.33), we get

$$(2.34)\qquad\left(1-\frac{r}{n}\right)\beta_n^{(r)}(\lambda,x)=(x-r)\,\beta_{n-1}^{(r)}(\lambda,x-\lambda)-\frac{r}{n}\beta_n^{(r+1)}(\lambda,x-\lambda).$$

Therefore, by (2.34), we obtain the following theorem.

Theorem 2.7. *For $n\geq 1$, we have*

$$\beta_n^{(r+1)}(\lambda,x-\lambda)=\left(1-\frac{n}{r}\right)\beta_n^{(r)}(\lambda,x)+\left(\frac{n}{r}x-n\right)\beta_{n-1}^{(r)}(\lambda,x-\lambda).$$

Here we compute

$$(2.35)\qquad\left\langle\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r\left(\frac{1}{\lambda}\log(1+\lambda t)\right)^m\middle|x^n\right\rangle$$

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in two different ways.

On one hand, it is equal to

$$\begin{aligned}
(2.36)\qquad &\lambda^{-m}\left\langle \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r (\log(1+\lambda t))^m \middle| x^n\right\rangle \\
=&\lambda^{-m}\left\langle \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r \middle| m!\sum_{l=m}^{\infty} S_1(l,m)\frac{\lambda^l}{l!}t^l x^n\right\rangle \\
=&m!\lambda^{-m}\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)\lambda^l\left\langle \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r \middle| x^{n-l}\right\rangle \\
=&m!\lambda^{-m}\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)\lambda^l\left\langle \sum_{k=0}^{\infty}\beta_k^{(r)}(\lambda)\frac{t^k}{k!} \middle| x^{n-l}\right\rangle \\
=&m!\lambda^{-m}\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)\lambda^l\beta_{n-l}^{(r)}(\lambda).
\end{aligned}$$

On the other hand, it is equal to

$$\begin{aligned}
(2.37)\qquad &\left\langle \partial_t\left(\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r\left(\frac{1}{\lambda}\log(1+\lambda t)\right)^m\right) \middle| x^{n-1}\right\rangle \\
=&\left\langle \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r \partial_t\left(\frac{1}{\lambda}\log(1+\lambda t)\right)^m \middle| x^{n-1}\right\rangle \\
&+\left\langle \left(\partial_t\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r\right)\left(\frac{1}{\lambda}\log(1+\lambda t)\right)^m \middle| x^{n-1}\right\rangle.
\end{aligned}$$

The first term of (2.37) is

(2.38)

$$\begin{aligned}
&m\left\langle \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r\left(\frac{1}{\lambda}\log(1+\lambda t)\right)^{m-1}(1+\lambda t)^{-1} \middle| x^{n-1}\right\rangle \\
=&m\lambda^{-(m-1)}\left\langle \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r (1+\lambda t)^{-1} \middle| (\log(1+\lambda t))^{m-1}x^{n-1}\right\rangle \\
=&m\lambda^{-(m-1)}\left\langle \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r (1+\lambda t)^{-1} \middle| (m-1)!\sum_{l=m-1}^{\infty}S_1(l,m-1)\frac{\lambda^l t^l}{l!}x^{n-1}\right\rangle \\
=&m!\lambda^{-(m-1)}\sum_{l=m-1}^{n-1}\binom{n-1}{l}S_1(l,m-1)\lambda^l\left\langle \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r (1+\lambda t)^{-\frac{\lambda}{\lambda}} \middle| x^{n-1-l}\right\rangle \\
=&m!\lambda^{-(m-1)}\sum_{l=m-1}^{n-1}\binom{n-1}{l}S_1(l,m-1)\lambda^l\beta_{n-1-l}^{(r)}(\lambda,-\lambda).
\end{aligned}$$

For the second term of (2.37), we recall that

$$(2.39)\qquad \partial_t\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r$$

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$$=-\frac{r}{(1+\lambda t)}\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{r}$$
$$+\frac{r}{t}\left\{\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{r}-\frac{1}{1+\lambda t}\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{r+1}\right\}.$$

Now, the second term of (2.37) is

(2.40)
$$\lambda^{-m}\left\langle \partial_t\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{r}\middle|\left(\log(1+\lambda t)\right)^{m}x^{n-1}\right\rangle$$
$$=m!\lambda^{-m}\sum_{l=m}^{n-1}\binom{n-1}{l}S_1(l,m)\lambda^{l}\left\langle \partial_t\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{r}\middle|x^{n-1-l}\right\rangle$$
$$=m!\lambda^{-m}\sum_{l=m}^{n-1}\binom{n-1}{l}S_1(l,m)\lambda^{l}\left\{-r\left\langle\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{r}(1+\lambda t)^{\frac{-\lambda}{\lambda}}\middle|x^{n-l-1}\right\rangle\right.$$
$$+\frac{r}{n-l}\left\langle\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{r}\middle|x^{n-l}\right\rangle$$
$$\left.-\frac{r}{n-l}\left\langle\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{r+1}(1+\lambda t)^{-\frac{\lambda}{\lambda}}\middle|x^{n-l}\right\rangle\right\}$$
$$=m!\lambda^{-m}\sum_{l=m}^{n-1}\binom{n-1}{l}S_1(l,m)\lambda^{l}$$
$$\times\left\{-r\beta_{n-1-l}^{(r)}(\lambda,-\lambda)+\frac{r}{n-l}\beta_{n-l}^{(r)}(\lambda)-\frac{r}{n-l}\beta_{n-l}^{(r+1)}(\lambda,-\lambda)\right\}.$$

From (2.35), (2.36), (2.37), and (2.40), we have

$$m!\lambda^{-m}\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)\lambda^{l}\beta_{n-l}^{(r)}(\lambda)$$
$$=m!\lambda^{-(m-1)}\sum_{l=m-1}^{n-1}\binom{n-1}{l}S_1(l,m-1)\lambda^{l}\beta_{n-l-1}^{(r)}(\lambda,-\lambda)$$
$$+m!\lambda^{-m}\sum_{l=m}^{n-1}\binom{n-1}{l}S_1(l,m)\lambda^{l}\left(-r\beta_{n-1-l}^{(r)}(\lambda,-\lambda)+\frac{r}{n-l}\beta_{n-l}^{(r)}(\lambda)\right.$$
$$\left.-\frac{r}{n-l}\beta_{n-l}^{(r+1)}(\lambda,-\lambda)\right),$$

where $n-1\geq m\geq 1$.

After simplification and modification, we get: for $n-1\geq m\geq 1$,

$$\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)\lambda^{n-l}\beta_{l}^{(r)}(\lambda) \tag{2.41}$$

$$
\begin{aligned}
=&\lambda \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1,m-1)\lambda^{n-l-1}\beta_l^{(r)}(\lambda,-\lambda) \\
&+\sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1(n-l-1,m)\lambda^{n-l-1} \\
&\times\left(-r\beta_l^{(r)}(\lambda,-\lambda)+\frac{r}{l+1}\beta_{l+1}^{(r)}(\lambda)-\frac{r}{l+1}\beta_{l+1}^{(r+1)}(\lambda,-\lambda)\right) \\
=&\lambda \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1,m-1)\lambda^{n-l-1}\beta_l^{(r)}(\lambda,-\lambda) \\
&-r\sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1(n-l-1,m)\lambda^{n-l-1}\beta_l^{(r)}(\lambda,-\lambda) \\
&+\frac{r}{n}\sum_{l=0}^{n-m-1} \binom{n}{l+1} S_1(n-l-1,m)\lambda^{n-l-1}\beta_{l+1}^{(r)}(\lambda) \\
&-\frac{r}{n}\sum_{l=0}^{n-m-1} \binom{n}{l+1} S_1(n-l-1,m)\lambda^{n-l-1}\beta_{l+1}^{(r+1)}(\lambda,-\lambda) \\
=&\lambda \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1,m-1)\lambda^{n-l-1}\beta_l^{(r)}(\lambda,-\lambda) \\
&-r\sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1(n-l-1,m)\lambda^{n-l-1}\beta_l^{(r)}(\lambda,-\lambda) \\
&+\frac{r}{n}\sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l,m)\lambda^{n-l}\beta_l^{(r)}(\lambda) \\
&-\frac{r}{n}\sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l,m)\lambda^{n-l}\beta_l^{(r+1)}(\lambda,-\lambda).
\end{aligned}
$$

Therefore, by (2.41), we obtain the following theorem.

Theorem 2.8. *For $n-1 \geq m \geq 1$, we have*

$$
\begin{aligned}
&\left(1-\frac{r}{n}\right)\sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l,m)\lambda^{n-l}\beta_l^{(r)}(\lambda) \\
=&\lambda \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1,m-1)\lambda^{n-l-1}\beta_l^{(r)}(\lambda,-\lambda) \\
&-r\sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1(n-l-1,m)\lambda^{n-l-1}\beta_l^{(r)}(\lambda,-\lambda) \\
&-\frac{r}{n}\sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l,m)\lambda^{n-l}\beta_l^{(r+1)}(\lambda,-\lambda).
\end{aligned}
$$

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For $r > s \geq 1$, by (1.19), (1.20) and (1.22), we get

$$\beta_n^{(r)}(\lambda, x) = \sum_{m=0}^{n} C_{n,m} \beta_m^{(s)}(\lambda, x), \tag{2.42}$$

where

$$\begin{aligned} C_{n,m} &= \frac{1}{m!} \left\langle \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^{r-s} t^m \middle| x^n \right\rangle \\ &= \frac{1}{m!} \left\langle \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^{r-s} \middle| t^m x^n \right\rangle \\ &= \binom{n}{m} \left\langle \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^{r-s} \middle| x^{n-m} \right\rangle \\ &= \binom{n}{m} \left\langle \sum_{l=0}^{\infty} \beta_l^{(r-s)}(\lambda) \frac{t^l}{l!} \middle| x^{n-m} \right\rangle \\ &= \binom{n}{m} \beta_{n-m}^{(r-s)}(\lambda). \end{aligned} \tag{2.43}$$

Therefore, by (2.42)and (2.43), we obtain the following theorem.

Theorem 2.9. *For $r > s \geq 1$, we have*

$$\beta_n^{(r)}(\lambda, x) = \sum_{m=0}^{n} \binom{n}{m} \beta_{n-m}^{(r-s)}(\lambda) \beta_m^{(s)}(\lambda, x).$$

Remark. Replacing x by $x + \lambda$ in Theorem 2.7, we have

$$\beta_n^{(r+1)}(\lambda, x) = \left(1 - \frac{n}{r}\right) \beta_n^{(r)}(\lambda, x + \lambda) + \left(\frac{n}{r}x + \frac{n}{r}\lambda - n\right) \beta_{n-1}^{(r)}(\lambda, x). \tag{2.44}$$

From Theorem 2.5, we note that

$$\begin{aligned} \beta_n^{(r)}(\lambda, x + \lambda) &= \sum_{j=0}^{n} \binom{n}{j} \beta_j^{(r)}(\lambda, x) (\lambda|\lambda)_{n-j} \\ &= n\lambda \beta_{n-1}^{(r)}(\lambda, x) + \beta_n^{(r)}(\lambda, x). \end{aligned} \tag{2.45}$$

Substituting (2.45) into (2.44), we get

$$\beta_n^{(r+1)}(\lambda, x) = \left(1 - \frac{n}{r}\right) \beta_n^{(r)}(\lambda, x) + \frac{n}{r}(x + (\lambda - 1) r - (n-1)\lambda) \beta_{n-1}^{(r)}(\lambda, x). \tag{2.46}$$

By using this and induction on r, it is shown in [[19]] that

$$\beta_n^{(r)}(\lambda, x) = r \binom{n}{r} \sum_{k=0}^{r-1} (-1)^{r-1-k} \sigma_{r-1,k}(\lambda, x, n) \frac{\beta_{n-k}(\lambda, x)}{n-k}, \tag{2.47}$$

where

$$\sigma_{r,k}(\lambda, x, n) = \sum_{1 \leq i_k < i_{k-1} < \cdots < i_1 \leq r} \prod_{j=1}^{k} (x + (\lambda - 1) i_j - (n-j)\lambda).$$

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For $r > s \geq 1$, by (1.19), (1.20) and (1.22), we have

$$\beta_n^{(s)}(\lambda, x) = \sum_{m=0}^{n} C_{n,m} \beta_m^{(r)}(\lambda, x), \tag{2.48}$$

where

$$\begin{aligned} C_{n,m} &= \frac{1}{m!} \left\langle \left(\frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} \right)^{r-s} t^m \middle| x^n \right\rangle \\ &= \binom{n}{m} \left\langle \left(\frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} \right)^{r-s} \middle| x^{n-m} \right\rangle. \end{aligned} \tag{2.49}$$

Observe here that

$$\begin{aligned} &\left(\frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} \right)^{r-s} \\ =& \left(\frac{e^{\frac{1}{\lambda}\log(1+\lambda t)} - 1}{t} \right)^{r-s} \\ =& \frac{1}{t^{r-s}} \left(e^{\frac{1}{\lambda}\log(1+\lambda t)} - 1 \right)^{r-s} \\ =& \frac{1}{t^{r-s}} (r-s)! \sum_{l=r-s}^{\infty} S_2(l, r-s) \frac{(\log(1+\lambda t))^l}{\lambda^l l!} \\ =& (r-s)! \sum_{l=r-s}^{\infty} \frac{S_2(l, r-s)}{l!} \left(\frac{\log(1+\lambda t)}{\lambda t} \right)^l t^{l-(r-s)} \\ =& (r-s)! \sum_{l=0}^{\infty} \frac{S_2(l+r-s, r-s)}{(l+r-s)!} t^l \left(\frac{\log(1+\lambda t)}{\lambda t} \right)^{l+r-s} \\ =& (r-s)! \sum_{l=0}^{\infty} \frac{S_2(l+r-s, r-s)}{(l+r-s)!} t^l (l+r-s)! \\ &\times \sum_{k=0}^{\infty} S_1(k+l+r-s, l+r-s) \frac{(\lambda t)^k}{(k+l+r-s)!} \\ =& (r-s)! \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} S_1(k+l+r-s, l+r-s) \\ &\times S_2(l+r-s, r-s) \frac{\lambda^k}{(k+l+r-s)!} t^{k+l} \\ =& (r-s)! \sum_{j=0}^{\infty} \sum_{k+l=j} S_1(k+l+r-s, l+r-s) \\ &\times S_2(l+r-s, r-s) \frac{\lambda^k}{(k+l+r-s)!} t^{k+l} \\ =& (r-s)! \sum_{j=0}^{\infty} \sum_{k=0}^{j} S_1(j+r-s, j-k+r-s) \end{aligned} \tag{2.50}$$

$$\times S_2\left(j-k+r-s,r-s\right)\frac{\lambda^k}{(j+r-s)!}t^j.$$

From (2.49) and (2.50), we have

(2.51)

$$\begin{aligned}
&C_{n,m}\\
=&\binom{n}{m}(r-s)!\\
&\times\left\langle\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}S_1\left(j+r-s,j-k+r-s\right)S_2\left(j-k+r-s,r-s\right)\frac{\lambda^k j!}{(j+r-s)!}\right)\frac{t^j}{j!}\middle| x^{n-m}\right\rangle\\
=&\binom{n}{m}(r-s)!\sum_{k=0}^{n-m}S_1\left(n-m+r-s,n-m-k+r-s\right)\\
&\times S_2\left(n-m-k+r-s,r-s\right)\frac{\lambda^k(n-m)!}{(n-m+r-s)!}\\
=&\frac{\binom{n}{m}}{\binom{n-m+r-s}{r-s}}\sum_{k=0}^{n-m}S_1\left(n-m+r-s,n-m-k+r-s\right)S_2\left(n-m-k+r-s,r-s\right)\lambda^k.
\end{aligned}$$

Therefore, by (2.48) and (2.51), we obtain the following theorem.

Theorem 2.10. *For $r>s\geq 1$, we have*

$$\begin{aligned}
&\beta_n^{(s)}(\lambda,x)\\
=&\sum_{m=0}^{n}\left\{\frac{\binom{n}{m}}{\binom{n-m+r-s}{r-s}}\sum_{k=0}^{n-m}S_1\left(n-m+r-s,n-m-k+r-s\right)\right.\\
&\left.\times S_2\left(n-m-k+r-s,r-s\right)\lambda^k\right\}\beta_m^{(r)}(\lambda,x)\\
=&\sum_{m=0}^{n}\left\{\frac{\binom{n}{m}}{\binom{n-m+r-s}{r-s}}\sum_{k=0}^{n-m}S_1\left(n-m+r-s,k+r-s\right)\right.\\
&\left.\times S_2\left(k+r-s,r-s\right)\lambda^{n-m-k}\right\}\beta_m^{(r)}(\lambda,x).
\end{aligned}$$

For $(x|\lambda)_n\sim\left(1,\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right)$, by (1.19), (1.20) and (1.22), we get

$$\beta_n^{(r)}(\lambda,x)=\sum_{m=0}^{n}C_{n,m}\,(x|\lambda)_m\,, \tag{2.52}$$

where

$$\begin{aligned}
C_{n,m}&=\frac{1}{m!}\left\langle\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r t^m\middle| x^n\right\rangle\\
&=\frac{1}{m!}\left\langle\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r\middle| t^m x^n\right\rangle\\
&=\binom{n}{m}\left\langle\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r\middle| x^{n-m}\right\rangle
\end{aligned} \tag{2.53}$$

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$$= \binom{n}{m} \left\langle \sum_{k=0}^{\infty} \beta_k^{(r)}(\lambda) \frac{t^k}{k!} \middle| x^{n-m} \right\rangle$$
$$= \binom{n}{m} \beta_{n-m}^{(r)}(\lambda).$$

Therefore, by (2.52) and (2.53), we obtain the following theorem.

Theorem 2.11. *For $n \geq 0$, we have*

$$\beta_n^{(r)}(\lambda, x) = \sum_{m=0}^{n} \binom{n}{m} \beta_{n-m}^{(r)}(\lambda) (x|\lambda)_m .$$

Remark. For $n \geq 0$, we get

(2.54)

$$(x|\lambda)_n = \sum_{m=0}^{n} \left\{ \frac{\binom{n}{m}}{\binom{n-m+r}{r}} \sum_{k=0}^{n-m} S_1(n-m+r, k+r) S_2(k+r, r) \lambda^{n-m-k} \right\} \beta_m^{(r)}(\lambda, x).$$

By (1.6) and (1.12), we easily get

$$B_n^{(s)}(x) \sim \left(\left(\frac{e^t - 1}{t} \right)^s, t \right), \quad (s \in \mathbb{N}). \tag{2.55}$$

From (1.19), (1.20), (1.22) and (2.55), we have

$$B_n^{(s)}(x) = \sum_{m=0}^{n} C_{n,m} \beta_m^{(r)}(\lambda, x), \tag{2.56}$$

where

(2.57)

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{\left(\frac{\lambda(e^t-1)}{e^{\lambda t}-1} \right)^r}{\left(\frac{e^t-1}{t} \right)^s} \left(\frac{1}{\lambda} \left(e^{\lambda t} - 1 \right) \right)^m \middle| x^n \right\rangle$$
$$= \frac{1}{m! \lambda^m} \left\langle \left(\frac{\lambda t}{e^{\lambda t} - 1} \right)^r \left(\frac{e^t - 1}{t} \right)^r \left(\frac{t}{e^t - 1} \right)^s \left(e^{\lambda t} - 1 \right)^m \middle| x^n \right\rangle$$
$$= \frac{1}{m! \lambda^m} \left\langle \left(\frac{\lambda t}{e^{\lambda t} - 1} \right)^r \left(\frac{e^t - 1}{t} \right)^r \left(\frac{t}{e^t - 1} \right)^s \middle| m! \sum_{l=m}^{\infty} S_2(l, m) \frac{\lambda^l}{l!} t^l x^n \right\rangle$$
$$= \lambda^{-m} \sum_{l=m}^{n} \binom{n}{l} S_2(l, m) \lambda^l \left\langle \left(\frac{\lambda t}{e^{\lambda t} - 1} \right)^r \left(\frac{e^t - 1}{t} \right)^r \left(\frac{t}{e^t - 1} \right)^s \middle| x^{n-l} \right\rangle$$
$$= \lambda^{-m} \sum_{l=m}^{n} \binom{n}{l} S_2(l, m) \lambda^l \left\langle \left(\frac{e^t - 1}{t} \right)^r \left(\frac{t}{e^t - 1} \right)^s \middle| \sum_{k=0}^{\infty} B_k^{(r)} \frac{\lambda^k}{k!} t^k x^{n-l} \right\rangle$$
$$= \lambda^{-m} \sum_{l=m}^{n} \binom{n}{l} S_2(l, m) \lambda^l \sum_{k=0}^{n-l} \binom{n-l}{k} B_k^{(r)} \lambda^k \left\langle \left(\frac{e^t - 1}{t} \right)^r \left(\frac{t}{e^t - 1} \right)^s \middle| x^{n-l-k} \right\rangle .$$

Case 1. For $r > s \geq 1$, we have

$$\left\langle \left(\frac{e^t - 1}{t} \right)^r \left(\frac{t}{e^t - 1} \right)^s \middle| x^{n-l-k} \right\rangle \tag{2.58}$$

$$
\begin{aligned}
&= \left\langle \left(\frac{e^t-1}{t} \right)^{r-s} \middle| x^{n-l-k} \right\rangle \\
&= \left\langle (r-s)! \sum_{j=0}^{\infty} \frac{S_2(j+r-s, r-s)\, j!}{(j+r-s)!} \frac{t^j}{j!} \middle| x^{n-l-k} \right\rangle \\
&= (r-s)! \frac{S_2(n-l-k+r-s, r-s)(n-l-k)!}{(n-l-k+r-s)!} \\
&= S_2(n-l-k+r-s, r-s) \Big/ \binom{n-l-k+r-s}{r-s}
\end{aligned}
$$

Case 2. For $r = s \geq 1$, we get

$$\left\langle \left(\frac{e^t-1}{t} \right)^{r} \left(\frac{t}{e^t-1} \right)^{s} \middle| x^{n-l-k} \right\rangle = \left\langle 1 \middle| x^{n-l-k} \right\rangle = \delta_{0,n-l-k} = \delta_{k,n-l}. \tag{2.59}$$

Case 3. For $s > r \geq 1$, we have

$$\left\langle \left(\frac{e^t-1}{t} \right)^{r} \left(\frac{t}{e^t-1} \right)^{s} \middle| x^{n-l-k} \right\rangle = \left\langle \left(\frac{t}{e^t-1} \right)^{s-r} \middle| x^{n-l-k} \right\rangle = B_{n-l-k}^{(s-r)}. \tag{2.60}$$

Therefore, by (2.56), (2.57), (2.58), (2.59) and (2.60), we obtain the following theorem.

Theorem 2.12. *Let $n \geq 0$. Then we have*

$$
B_n^{(s)}(x) = \begin{cases}
\sum_{m=0}^{n} \left\{ \lambda^{-m} \sum_{l=m}^{n} \sum_{k=0}^{n-l} \frac{\binom{n}{l}\binom{n-l}{k}}{\binom{n-l-k+r-s}{r-s}} S_2(l,m) \right. \\
\left. \times\, S_2(n-l-k+r-s, r-s) \lambda^{k+l} B_k^{(r)} \right\} \beta_m^{(r)}(\lambda, x), & \text{if } r > s \geq 1, \\
\lambda^n \sum_{m=0}^{n} \left\{ \lambda^{-m} \sum_{l=m}^{n} \binom{n}{l} S_2(l,m) B_{n-l}^{(r)} \right\} \beta_m^{(r)}(\lambda, x), & \text{if } r = s \geq 1, \\
\sum_{m=0}^{n} \lambda^{-m} \left\{ \sum_{l=m}^{n} \sum_{k=0}^{n-l} \binom{n}{l} \binom{n-l}{k} S_2(l,m) \right. \\
\left. \times \lambda^{k+l} B_k^{(r)} B_{n-l-k}^{(s-r)} \right\} \beta_m^{(r)}(\lambda, x), & \text{if } s > r \geq 1.
\end{cases}
$$

Remark. Let $r > s \geq 1$. Then we get

$$\beta_n^{(r)}(\lambda, x) = \sum_{m=0}^{n} \left\{ \lambda^{-m} \sum_{l=m}^{n} \sum_{k=0}^{n-l} \binom{n}{l} \binom{n-l}{k} S_1(l,m) \lambda^{k+l} b_k^{(s)} \beta_{n-l-k}^{(r-s)}(\lambda) \right\} B_m^{(s)}(x). \tag{2.61}$$

For $r = s \geq 1$, we have

$$\beta_n^{(r)}(\lambda, x) = \lambda^n \sum_{m=0}^{n} \lambda^{-m} \left\{ \sum_{l=m}^{n} \binom{n}{l} S_1(l,m) b_{n-l}^{(s)} \right\} B_m^{(s)}(x). \tag{2.62}$$

If $s > r \geq 1$, then we note that

$$
\begin{aligned}
&\beta_n^{(r)}(\lambda, x) \qquad (2.63) \\
&= \lambda^n \sum_{m=0}^{n} \left\{ \lambda^{-m} \sum_{l=m}^{n} \sum_{k=0}^{n-l} \sum_{i=0}^{n-l-k} \frac{\binom{n}{l}\binom{n-l}{k}}{\binom{n-l-k+s-r}{s-r}} S_1(l,m) \right.
\end{aligned}
$$

$$\times S_1(n-l-k+s-r,i+s-r)\,S_2(i+s-r,s-r)\,\lambda^{-i}b_k^{(s)}\Big\}\,B_m^{(s)}(x).$$

From (1.7) and (1.12), we get

$$H_n^{(s)}(x|\mu)\sim\left(\left(\frac{e^t-\mu}{1-\mu}\right)^s,t\right). \tag{2.64}$$

By (1.19), (1.20), (1.22) and (2.64), we have

$$\beta_n^{(r)}(\lambda,x)=\sum_{m=0}^{n}C_{n,m}H_m^{(s)}(x|\mu), \tag{2.65}$$

where

(2.66)

$$\begin{aligned}
&C_{n,m}\\
&=\frac{1}{m!}\left\langle\frac{\left(e^{\frac{1}{\lambda}\log(1+\lambda t)}-\mu/1-\mu\right)^s}{\left(\lambda\left(e^{\frac{1}{\lambda}\log(1+\lambda t)}-1\right)/e^{\log(1+\lambda t)}-1\right)^r}\left(\frac{1}{\lambda}\log(1+\lambda t)\right)^m\middle|x^n\right\rangle\\
&=\frac{1}{m!\lambda^m(1-\mu)^s}\left\langle\left((1+\lambda t)^{\frac{1}{\lambda}}-\mu\right)^s\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r(\log(1+\lambda t))^m\middle|x^n\right\rangle\\
&=\frac{1}{\lambda^m(1-\mu)^s}\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)\lambda^l\left\langle\left((1+\lambda t)^{\frac{1}{\lambda}}-\mu\right)^s\middle|\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r x^{n-l}\right\rangle\\
&=\frac{1}{\lambda^m(1-\mu)^s}\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)\lambda^l\sum_{k=0}^{n-l}\binom{n-l}{k}\beta_k^{(r)}(\lambda)\left\langle\left((1+\lambda t)^{\frac{1}{\lambda}}-\mu\right)^s\middle|x^{n-l-k}\right\rangle.
\end{aligned}$$

It is easy to show that

$$\begin{aligned}
&\left\langle\left((1+\lambda t)^{\frac{1}{\lambda}}-\mu\right)^s\middle|x^{n-l-k}\right\rangle\\
&=\sum_{i=0}^{s}\binom{s}{i}(-\mu)^{s-i}\left\langle\sum_{j=0}^{\infty}(i|\lambda)_j\frac{t^j}{j!}\middle|x^{n-l-k}\right\rangle\\
&=\sum_{i=0}^{s}\binom{s}{i}(-\mu)^{s-i}(i|\lambda)_{n-l-k}.
\end{aligned} \tag{2.67}$$

From (2.66) and (2.67), we have

(2.68)

$$\begin{aligned}
&C_{n,m}\\
=&\frac{1}{\lambda^m(1-\mu)^s}\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)\lambda^l\sum_{k=0}^{n-l}\binom{n-l}{k}\beta_k^{(r)}(\lambda)\sum_{i=0}^{s}\binom{s}{i}(-\mu)^{s-i}(i|\lambda)_{n-l-k}\\
=&\frac{1}{\lambda^m(1-\mu)^s}\sum_{l=m}^{n}\sum_{k=0}^{n-l}\sum_{i=0}^{s}\binom{n}{l}\binom{n-l}{k}\binom{s}{i}S_1(l,m)\lambda^l(-\mu)^{s-i}\beta_k^{(r)}(\lambda)(i|\lambda)_{n-l-k}.
\end{aligned}$$

Therefore, by (2.65) and (2.68), we obtain the following theorem.

Theorem 2.13. *For $\mu \in \mathbb{C}$ with $\mu \neq 1$, $n \geq 0$, we have*

$$\begin{aligned}
&\beta_n^{(r)}(\lambda, x) \\
=&\frac{1}{(1-\mu)^s} \sum_{m=0}^{n} \left\{ \lambda^{-m} \sum_{l=m}^{n} \sum_{k=0}^{n-l} \sum_{i=0}^{s} \binom{n}{l} \binom{n-l}{k} \binom{s}{i} S_1(l,m) \lambda^l (-\mu)^{s-i} \right. \\
&\left. \times \beta_k^{(r)}(\lambda) (i|\lambda)_{n-l-k} \right\} H_m^{(s)}(x|\mu).
\end{aligned}$$

Remark. For $n \geq 0$, we have

$$\begin{aligned}
&H_n^{(s)}(x|\mu) \\
=&\sum_{m=0}^{n} \left\{ \frac{1}{\lambda^m} \sum_{l=0}^{n-m} \sum_{k=m}^{n-l} \binom{n}{l} \binom{n-l}{k} S_2(k,m) \lambda^{k+l} B_l^{(r)} H_{n-l-k-j}^{(r)}(\mu) \right\} \beta_m^{(r)}(\lambda, x).
\end{aligned}$$

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KOROBOV POLYNOMIALS OF THE SEVENTH KIND AND OF THE EIGHTH KIND

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ABSTRACT. In this paper, we consider the Korobov polynomials of the seventh kind and of the eighth kind. We present several explicit formulas and recurrence relations for these polynomials. In addition, we establish connections between our polynomials and several known families of polynomials.

1. INTRODUCTION

The *degenerate Bernoulli polynomials* are the degenerate version of Bernoulli polynomials introduced by Calitz [3, 4]. On the other hand, the Korobov polynomials of the first kind are the first degenerate version of the Bernoulli polynomials of the second kind, see [13, 14].

In recent years, many researchers studied various kinds of degenerate versions of families polynomials like Bernoulli polynomials, Euler polynomials, falling factorial polynomials, Bell polynomials and their variants, see [6–10] and references therein. Along this line of research, we introduced in [8, 9] four kinds of new degenerate versions of Bernoulli polynomials of the second kind, called the *Korobov polynomials of the third, fourth, fifth, and sixth kind.*

Here, we will discuss two other degenerate versions of Bernoulli polynomials of the second kind, namely, the *Korobov polynomials of the seventh and eighth kind.* We will investigate some properties, explicit expressions, recurrence relations, and connections with other families polynomials with the help of umbral calculus (see [10, 15, 16]). To do that, we recall some families polynomials. The *Bernoulli polynomials of the second kind* $b_n(x)$ are given by the generating function

$$\frac{t}{\log(1+t)}(1+t)^x = \sum_{n\geq 0} b_n(x)\frac{t^n}{n!}. \tag{1.1}$$

For $x = 0$, $b_n = b_n(0)$ are called the *Bernoulli numbers of the second kind.* The *Daehee polynomials* $D_n(x)$ are defined by the generating function

$$\frac{\log(1+t)}{t}(1+t)^x = \sum_{n\geq 0} D_n(x)\frac{t^n}{n!}. \tag{1.2}$$

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When $x = 0$, $D_n = D_n(0)$ are called the *Daehee numbers*. The *Krobov polynomials* $K_n(\lambda, x)$ of the first kind are given by

$$\frac{\lambda t}{(1+t)^\lambda - 1}(1+t)^x = \sum_{n\geq 0} K_n(\lambda, x)\frac{t^n}{n!}. \tag{1.3}$$

When $x = 0$, $K_n(\lambda) = K_n(\lambda, 0)$ are called the *Korobov numbers of the first kind*. The *degenerate falling factorial polynomials* $(x)_{n,\lambda}$ were defined in [7] by the generating function

$$(1+\lambda)^{\frac{x}{\lambda}\frac{(1+t)^\lambda-1}{\lambda}} = \sum_{n\geq 0}(x)_{n,\lambda}\frac{t^n}{n!}. \tag{1.4}$$

Clearly, $\lim_{\lambda\to 0}(x)_{n,\lambda} = (x)_n$, the nth falling factorial polynomial. These polynomials can be defined as $(x)_{n,\lambda} \sim (1, f(t))$, where

$$f(t) = \left(1 + \frac{\lambda^2 t}{\log(1+\lambda)}\right)^{\frac{1}{\lambda}} - 1 \text{ and } \bar{f}(t) = \frac{\log(1+\lambda)}{\lambda}\frac{(1+t)^\lambda - 1}{\lambda}. \tag{1.5}$$

Note that we write $s_n(x) \sim (g(t), f(t))$ if $\sum_{n\geq 0} s_n(x)\frac{t^n}{n!} = \frac{1}{g(\bar{f}(t))}e^{x\bar{f}(t)}$, where $\bar{f}(t)$ is the compositional inverse of $f(t)$, see [15,16]. The *degenerate Stirling numbers of the first kind* $S_1(n, k \mid \lambda)$, $n \geq k \geq 0$, were given in [7] by the generating function

$$\frac{1}{k!}\left(\frac{(1+t)^\lambda - 1}{\lambda}\right)^k = \sum_{n\geq k} S_1(n, k \mid \lambda)\frac{t^n}{n!}, \tag{1.6}$$

so that, in the notation of umbral calculus, $S_1(n, k \mid \lambda) = \frac{1}{k!}\left\langle \left(\frac{(1+t)^\lambda-1}{\lambda}\right)^k |x^n \right\rangle$. Then, it was shown in [7] that

$$(x)_{n,\lambda} = \sum_{k=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^k S_1(n, k \mid \lambda)x^k$$

with

$$S_1(n, k \mid \lambda) = \sum_{m=k}^{n} S_1(n, m)S_2(m, k)\lambda^{m-k},$$

where $\lim_{\lambda\to 0} S_1(n, k \mid \lambda) = S_1(n, k)$ is the Stirling number of the first kind.

Here, we introduce *Korobov polynomials of the seventh kind* $K_{n,7}(\lambda, x)$ and of the eighth kind $K_{n,9}(\lambda, x)$, respectively given by

$$\frac{\log(1+\lambda t)}{\lambda\log(1+t)}(1+\lambda)^{\frac{x}{\lambda}\frac{(1+t)^\lambda-1}{\lambda}} = \sum_{n\geq 0} K_{n,7}(\lambda, x)\frac{t^n}{n!}, \tag{1.7}$$

$$\frac{\log(1+\lambda t)}{(1+t)^\lambda - 1}(1+\lambda)^{\frac{x}{\lambda}\frac{(1+t)^\lambda-1}{\lambda}} = \sum_{n\geq 0} K_{n,8}(\lambda, x)\frac{t^n}{n!}. \tag{1.8}$$

When $x = 0$, $K_{n,7}(\lambda) = K_{n,7}(\lambda, 0)$ and $K_{n,8}(\lambda) = K_{n,8}(\lambda, 0)$ are called the *Korobov numbers of the seventh kind and of the eighth kind*, respectively. We observe that

$$\lim_{\lambda\to 0}\frac{\lambda t}{(1+t)^\lambda-1}(1+t)^x = \lim_{\lambda\to 0}\frac{\log(1+\lambda t)}{\lambda\log(1+t)}(1+\lambda)^{\frac{x}{\lambda}\frac{(1+t)^\lambda-1}{\lambda}}$$
$$= \lim_{\lambda\to 0}\frac{\log(1+\lambda t)}{(1+t)^\lambda-1}(1+\lambda)^{\frac{x}{\lambda}\frac{(1+t)^\lambda-1}{\lambda}} = \frac{t}{\log(1+t)}(1+t)^x,$$

which implies that $\lim_{\lambda\to 0} K_n(\lambda, x) = \lim_{\lambda\to 0} K_{n,7}(\lambda, x) = \lim_{\lambda\to 0} K_{n,8}(\lambda, x) = b_n(x)$. It is immediate to see that $K_{n,7}(\lambda, x)$ and $K_{n,8}(\lambda, x)$ are Sheffer sequences (see [15,16]) for the respective pairs $\left(\frac{\lambda\log(1+f(t))}{\log(1+\lambda f(t))}, f(t)\right)$ and $\left(\frac{(1+f(t))^\lambda-1}{\log(1+\lambda f(t))}, f(t)\right)$, where $f(t)$ is given in (1.5). Thus, (1.7) and (1.8) can be presented as

$$K_{n,7}(\lambda, x) \sim \left(\frac{\lambda\log(1+f(t))}{\log(1+\lambda f(t))}, f(t)\right) = \left(\frac{\log\left(1+\frac{\lambda^2 t}{\log(1+\lambda)}\right)}{\log(1+\lambda f(t))}, f(t)\right), \tag{1.9}$$

$$K_{n,8}(\lambda, x) \sim \left(\frac{(1+f(t))^\lambda-1}{\log(1+\lambda f(t))}, f(t)\right) = \left(\frac{\frac{\lambda^2 t}{\log(1+\lambda)}}{\log(1+\lambda f(t))}, f(t)\right). \tag{1.10}$$

In the next two sections, we will use umbral calculus in order to study some properties, explicit formulas, recurrence relations and identities about the Korobov polynomials of the seventh kind and of the eighth kind. In last section, we present connections between our polynomials and several known families of polynomials.

2. Explicit expressions

In this section, we present several explicit formulas for the Korobov polynomials of the seventh kind and of the eighth kind, namely $K_{n,7}(\lambda, x)$ and $K_{n,8}(\lambda, x)$.

Theorem 2.1. *For all $n \geq 0$,*

$$K_{n,7}(\lambda, x) = \sum_{k=0}^{n}\sum_{\ell=k}^{n}\binom{n}{\ell}\frac{\log^k(1+\lambda)}{\lambda^k}S_1(\ell, k|\lambda)K_{n-\ell,7}(\lambda)x^k$$
$$= \sum_{k=0}^{n}\sum_{\ell=k}^{n}\sum_{m=0}^{n-\ell}\binom{n}{\ell}\binom{n-\ell}{m}\frac{\log^k(1+\lambda)}{\lambda^k}S_1(\ell, k|\lambda)b_m D_{n-\ell-m}\lambda^{n-\ell-m}x^k,$$
$$K_{n,8}(\lambda, x) = \sum_{k=0}^{n}\sum_{\ell=k}^{n}\binom{n}{\ell}\frac{\log^k(1+\lambda)}{\lambda^k}S_1(\ell, k|\lambda)K_{n-\ell,8}(\lambda)x^k$$
$$= \sum_{k=0}^{n}\sum_{\ell=k}^{n}\sum_{m=0}^{n-\ell}\binom{n}{\ell}\binom{n-\ell}{m}\frac{\log^k(1+\lambda)}{\lambda^k}S_1(\ell, k|\lambda)K_m(\lambda)D_{n-\ell-m}\lambda^{n-\ell-m}x^k.$$

Proof. We proceed the proof by using the conjugation representation for Sheffer sequences (see [15,16]): $s_n(x) = \sum_{k=0}^{n}\frac{1}{k!}\langle g(\bar{f}(t))^{-1}\bar{f}(t)^k|x^n\rangle x^k$, for any $s_n(x) \sim (g(t), f(t))$.

Thus, by (1.9), we have

$$\begin{aligned}K_{n,7}(\lambda,x)&=\sum_{k=0}^{n}\frac{1}{k!}\left\langle\frac{\log(1+\lambda t)}{\lambda\log(1+t)}\frac{\log^k(1+\lambda)((1+t)^\lambda-1)^k}{\lambda^{2k}}|x^n\right\rangle x^k\\&=\sum_{k=0}^{n}\frac{\log^k(1+\lambda)}{\lambda^k}\left\langle\frac{\log(1+\lambda t)}{\lambda\log(1+t)}|\frac{((1+t)^\lambda-1)^k}{k!\lambda^k}x^n\right\rangle x^k,\end{aligned}$$

which, by (1.6) and (1.7), implies

$$\begin{aligned}K_{n,7}(\lambda,x)&=\sum_{k=0}^{n}\frac{\log^k(1+\lambda)}{\lambda^k}\left\langle\frac{\log(1+\lambda t)}{\lambda\log(1+t)}|\sum_{\ell\geq k}S_1(\ell,k|\lambda)\frac{t^\ell}{\ell!}x^n\right\rangle x^k\\&=\sum_{k=0}^{n}\sum_{\ell=k}^{n}\binom{n}{\ell}\frac{\log^k(1+\lambda)}{\lambda^k}S_1(\ell,k|\lambda)\left\langle\frac{\log(1+\lambda t)}{\lambda\log(1+t)}|x^{n-\ell}\right\rangle x^k\qquad(2.1)\\&=\sum_{k=0}^{n}\sum_{\ell=k}^{n}\binom{n}{\ell}\frac{\log^k(1+\lambda)}{\lambda^k}S_1(\ell,k|\lambda)K_{n-\ell,7}(\lambda)x^k.\end{aligned}$$

On the other hand, by (2.1), we have

$$K_{n,7}(\lambda,x)=\sum_{k=0}^{n}\sum_{\ell=k}^{n}\binom{n}{\ell}\frac{\log^k(1+\lambda)}{\lambda^k}S_1(\ell,k|\lambda)\left\langle\frac{\log(1+\lambda t)}{\lambda t}|\frac{t}{\log(1+t)}x^{n-\ell}\right\rangle x^k,$$

which, by (1.1) and (1.2), we obtain

$$\begin{aligned}&K_{n,7}(\lambda,x)\\&=\sum_{k=0}^{n}\sum_{\ell=k}^{n}\binom{n}{\ell}\frac{\log^k(1+\lambda)}{\lambda^k}S_1(\ell,k|\lambda)\left\langle\frac{\log(1+\lambda t)}{\lambda t}|\sum_{m\geq0}b_m\frac{t^m}{m!}x^{n-\ell}\right\rangle x^k\\&=\sum_{k=0}^{n}\sum_{\ell=k}^{n}\sum_{m=0}^{n-\ell}\binom{n}{\ell}\binom{n-\ell}{m}\frac{\log^k(1+\lambda)}{\lambda^k}S_1(\ell,k|\lambda)b_m\left\langle\frac{\log(1+\lambda t)}{\lambda t}|x^{n-\ell-m}\right\rangle x^k\\&=\sum_{k=0}^{n}\sum_{\ell=k}^{n}\sum_{m=0}^{n-\ell}\binom{n}{\ell}\binom{n-\ell}{m}\frac{\log^k(1+\lambda)}{\lambda^k}S_1(\ell,k|\lambda)b_m\left\langle\sum_{j\geq0}D_j\lambda^j\frac{t^j}{j!}|x^{n-\ell-m}\right\rangle x^k\\&=\sum_{k=0}^{n}\sum_{\ell=k}^{n}\sum_{m=0}^{n-\ell}\binom{n}{\ell}\binom{n-\ell}{m}\frac{\log^k(1+\lambda)}{\lambda^k}S_1(\ell,k|\lambda)b_mD_{n-\ell-m}\lambda^{n-\ell-m}x^k,\end{aligned}$$

which completes the proof of formulas for $k_{n,7}(\lambda,x)$.

Now let us deal with the case $K_{n,8}(\lambda,x)$. Similarly, by using the conjugation representation for Sheffer sequences, (1.10) and (1.6), we obtain

$$\begin{aligned}K_{n,8}(\lambda,x)&=\sum_{k=0}^{n}\sum_{\ell=k}^{n}\binom{n}{\ell}\frac{\log^k(1+\lambda)}{\lambda^k}S_1(\ell,k|\lambda)\left\langle\frac{\log(1+\lambda t)}{(1+t)^\lambda-1}|x^{n-\ell}\right\rangle x^k\qquad(2.2)\\&=\sum_{k=0}^{n}\sum_{\ell=k}^{n}\binom{n}{\ell}\frac{\log^k(1+\lambda)}{\lambda^k}S_1(\ell,k|\lambda)K_{n-\ell,8}(\lambda)x^k.\end{aligned}$$

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On the other hand, by (2.2), we have

$$K_{n,8}(\lambda,x)=\sum_{k=0}^{n}\sum_{\ell=k}^{n}\binom{n}{\ell}\frac{\log^k(1+\lambda)}{\lambda^k}S_1(\ell,k|\lambda)\left\langle\frac{\log(1+\lambda t)}{\lambda t}|\frac{\lambda t}{(1+t)^{\lambda}-1}x^{n-\ell}\right\rangle x^k,$$

which, by (1.3) and (1.2), we obtain

$$\begin{aligned}
&K_{n,8}(\lambda,x)\\
&=\sum_{k=0}^{n}\sum_{\ell=k}^{n}\binom{n}{\ell}\frac{\log^k(1+\lambda)}{\lambda^k}S_1(\ell,k|\lambda)\left\langle\frac{\log(1+\lambda t)}{\lambda t}|\sum_{m\geq 0}K_m(\lambda)\frac{t^m}{m!}x^{n-\ell}\right\rangle x^k\\
&=\sum_{k=0}^{n}\sum_{\ell=k}^{n}\sum_{m=0}^{n-\ell}\binom{n}{\ell}\binom{n-\ell}{m}\frac{\log^k(1+\lambda)}{\lambda^k}S_1(\ell,k|\lambda)K_m(\lambda)\left\langle\frac{\log(1+\lambda t)}{\lambda t}|x^{n-\ell-m}\right\rangle x^k
\end{aligned}$$

(2.3)

$$=\sum_{k=0}^{n}\sum_{\ell=k}^{n}\sum_{m=0}^{n-\ell}\binom{n}{\ell}\binom{n-\ell}{m}\frac{\log^k(1+\lambda)}{\lambda^k}S_1(\ell,k|\lambda)K_m(\lambda)D_{n-\ell-m}\lambda^{n-\ell-m}x^k,$$

which completes the proof. □

Now, we express our polynomials in terms of the degenerate falling factorial polynomials.

Theorem 2.2. *For all $n\geq 0$,*

$$\begin{aligned}
K_{n,7}(\lambda,x)&=\sum_{\ell=0}^{n}\binom{n}{\ell}\left(\sum_{m=0}^{n-\ell}\binom{n-\ell}{m}\lambda^{n-\ell-m}b_mD_{n-\ell-m}\right)(x)_{\ell,\lambda},\\
K_{n,8}(\lambda,x)&=\sum_{\ell=0}^{n}\binom{n}{\ell}\left(\sum_{m=0}^{n-\ell}\binom{n-\ell}{m}\lambda^{n-\ell-m}K_m(\lambda)D_{n-\ell-m}\right)(x)_{\ell,\lambda}.
\end{aligned}$$

Proof. By (1.9), we have

$$K_{n,7}(\lambda,y)=\left\langle\frac{\log(1+\lambda t)}{\lambda\log(1+t)}(1+\lambda)^{\frac{y}{\lambda}\frac{(1+t)^{\lambda}-1}{\lambda}}|x^n\right\rangle=\left\langle\frac{\log(1+\lambda t)}{\lambda\log(1+t)}|(1+\lambda)^{\frac{y}{\lambda}\frac{(1+t)^{\lambda}-1}{\lambda}}x^n\right\rangle,$$

which, by (1.4), implies

$$K_{n,7}(\lambda,y)=\left\langle\frac{\log(1+\lambda t)}{\lambda\log(1+t)}|\sum_{\ell\geq 0}(y)_{\ell,\lambda}\frac{t^{\ell}}{\ell!}x^n\right\rangle=\sum_{\ell=0}^{n}\binom{n}{\ell}(y)_{\ell,\lambda}\left\langle\frac{\log(1+\lambda t)}{\lambda\log(1+t)}|x^{n-\ell}\right\rangle.$$

Therefore, by (2.1), we obtain

$$K_{n,7}(\lambda,y)=\sum_{\ell=0}^{n}\binom{n}{\ell}\left(\sum_{m=0}^{n-\ell}\binom{n-\ell}{m}\lambda^{n-\ell-m}b_mD_{n-\ell-m}\right)(y)_{\ell,\lambda},$$

which completes the proof for $K_{n,7}(\lambda,y)$.

By using similar arguments as above together with (1.10) and (1.4), we obtain

$$K_{n,8}(\lambda,y)=\sum_{\ell=0}^{n}\binom{n}{\ell}(y)_{\ell,\lambda}\left\langle\frac{\log(1+\lambda t)}{(1+t)^{\lambda}-1}|x^{n-\ell}\right\rangle.$$

Therefore, by (2.2) and (2.3), we have

$$K_{n,8}(\lambda,y)=\sum_{\ell=0}^{n}\binom{n}{\ell}\left(\sum_{m=0}^{n-\ell}\binom{n-\ell}{m}\lambda^{n-\ell-m}K_m(\lambda)D_{n-\ell-m}\right)(y)_{\ell,\lambda},$$

which completes the proof. □

In the next theorem, we find explicit formulas for the coefficient of x^j in $K_{n,7}(\lambda,x)$ and $K_{n,8}(\lambda,x)$.

Theorem 2.3. *For all $n\geq 0$ and $s=7,8$,*

$$\begin{aligned}&K_{n,s}(\lambda,x)\\&=\sum_{j=0}^{n}\left(\sum_{k=j}^{n}\sum_{\ell=0}^{k}\sum_{m=0}^{\ell}\frac{(-1)^{\ell-m}}{\ell!}\binom{\ell}{m}\binom{k}{j}(m|\lambda)_{k-j}\frac{\log^j(1+\lambda)}{\lambda^j}S_1(n,k|\lambda)K_{\ell,s}(\lambda)\right)x^j.\end{aligned}$$

Proof. By (1.4) and (1.9), we have

$$\frac{\lambda\log(1+f(t))}{\log(1+\lambda f(t))}K_{n,7}(\lambda,x)=(x)_{n,\lambda}=\sum_{k=0}^{n}\frac{\log^k(1+\lambda)}{\lambda^k}S_1(n,k|\lambda)x^k\sim(1,f(t)).$$

Thus,

$$\begin{aligned}K_{n,7}(\lambda,x)&=\sum_{k=0}^{n}\frac{\log^k(1+\lambda)}{\lambda^k}S_1(n,k|\lambda)\frac{\log(1+\lambda f(t))}{\lambda\log(1+f(t))}x^k\\&=\sum_{k=0}^{n}\sum_{\ell=0}^{k}\frac{\log^k(1+\lambda)}{\lambda^k}S_1(n,k|\lambda)\frac{K_{\ell,7}(\lambda)}{\ell!}(f(t))^\ell x^k.\end{aligned}\tag{2.4}$$

Note that

$$\begin{aligned}(f(t))^\ell x^k&=\sum_{m=0}^{\ell}\binom{\ell}{m}(-1)^{\ell-m}\left(1+\lambda^2t/\log(1+\lambda)\right)^{m/\lambda}x^k\\&=\sum_{m=0}^{\ell}\sum_{j=0}^{k}\binom{\ell}{m}(-1)^{\ell-m}(m|\lambda)_j\left(\lambda/\log(1+\lambda)\right)^j\binom{k}{j}x^{k-j}.\end{aligned}$$

Therefore,

$$\begin{aligned}&K_{n,7}(\lambda,x)\\&=\sum_{k=0}^{n}\sum_{\ell=0}^{k}\sum_{m=0}^{\ell}\sum_{j=0}^{k}(-1)^{\ell-m}(m|\lambda)_{k-j}\frac{\log^j(1+\lambda)}{\lambda^j}S_1(n,k|\lambda)\frac{K_{\ell,7}(\lambda)}{\ell!}\binom{\ell}{m}\binom{k}{j}x^j\\&=\sum_{j=0}^{n}\left(\sum_{k=j}^{n}\sum_{\ell=0}^{k}\sum_{m=0}^{\ell}\frac{(-1)^{\ell-m}}{\ell!}\binom{\ell}{m}\binom{k}{j}(m|\lambda)_{k-j}\frac{\log^j(1+\lambda)}{\lambda^j}S_1(n,k|\lambda)K_{\ell,7}(\lambda)\right)x^j.\end{aligned}$$

By (1.4) and (1.10), we have

$$\frac{(1+f(t))^\lambda-1}{\log(1+\lambda f(t))}K_{n,8}(\lambda,x)=(x)_{n,\lambda}=\sum_{k=0}^{n}\frac{\log^k(1+\lambda)}{\lambda^k}S_1(n,k|\lambda)x^k\sim(1,f(t)).$$

Thus, by using the above arguments, we obtain

$$\begin{aligned}
&K_{n,8}(\lambda,x)\\
&=\sum_{j=0}^{n}\left(\sum_{k=j}^{n}\sum_{\ell=0}^{k}\sum_{m=0}^{\ell}\frac{(-1)^{\ell-m}}{\ell!}\binom{\ell}{m}\binom{k}{j}(m|\lambda)_{k-j}\frac{\log^{j}(1+\lambda)}{\lambda^{j}}S_1(n,k|\lambda)K_{\ell,8}(\lambda)\right)x^{j},
\end{aligned}$$

which completes the proof. □

In the next theorem, we express Korobov polynomials of seventh and eighth kinds in terms of Korobov polynomials of fifth and sixth kinds.

Theorem 2.4. *For all $n\geq 0$ and $s=7,8$,*

$$K_{n,s}(\lambda,x)=\sum_{\ell=0}^{n}\binom{n}{\ell}D_{n-\ell}\lambda^{n-\ell}K_{\ell,s-2}(\lambda,x).$$

Proof. Recall that Korobov polynomials of the fifth kind (see [9]) is given by

$$\frac{t}{\log(1+t)}(1+\lambda)^{\frac{x}{\lambda}\frac{(1+t)^{\lambda}-1}{\lambda}}=\sum_{n\geq 0}K_{n,5}(\lambda,x)\frac{t^n}{n!}.$$

So, by (1.7), we have

$$\begin{aligned}
K_{n,7}(\lambda,y)&=\left\langle\frac{\log(1+\lambda t)}{\lambda t}|\frac{t}{\log(1+t)}(1+\lambda)^{\frac{y}{\lambda}\frac{(1+t)^{\lambda}-1}{\lambda}}x^{n}\right\rangle\\
&=\sum_{\ell=0}^{n}\binom{n}{\ell}K_{\ell,5}(\lambda,y)\left\langle\frac{\log(1+\lambda t)}{\lambda t}|x^{n-\ell}\right\rangle,
\end{aligned}$$

which, by (1.2), implies $K_{n,7}(\lambda,y)=\sum_{\ell=0}^{n}\binom{n}{\ell}K_{\ell,5}(\lambda,y)D_{n-\ell}\lambda^{n-\ell}$.

Recall that Korobov polynomials of the sixth kind (see [9]) is defined by

$$\frac{\lambda t}{(1+t)^{\lambda}-1}(1+\lambda)^{\frac{x}{\lambda,y}\frac{(1+t)^{\lambda}-1}{\lambda}}=\sum_{n\geq 0}K_{n,6}(\lambda,x)\frac{t^n}{n!}.$$

Similarly, by (1.2) and (1.8), we obtain $K_{n,8}(\lambda,y)=\sum_{\ell=0}^{n}\binom{n}{\ell}K_{\ell,6}(\lambda,y)D_{n-\ell}\lambda^{n-\ell}$, as claimed. □

In the next theorem, we express our polynomials $K_{n,7}(\lambda,x)$ and $K_{n,8}(\lambda,x)$ in terms of *degenerate Bernoulli numbers* $\beta_{\ell}^{(n)}(\lambda)$ of order n, which are given by the generating function

$$\frac{t^n}{((1+\lambda t)^{1/\lambda}-1)^n}=\sum_{\ell\geq 0}\beta_{\ell}^{(n)}(\lambda)\frac{t^{\ell}}{\ell!}. \tag{2.5}$$

Theorem 2.5. *For all $n\geq 1$ and $s=7,8$,*

$$\begin{aligned}
&K_{n,s}(\lambda,x)\\
&\sum_{j=0}^{n}\left(\sum_{k=j}^{n}\sum_{\ell=0}^{k}\sum_{m=0}^{\ell}\frac{(-1)^{\ell-m}\binom{n-1}{k-1}\binom{\ell}{m}\binom{k}{j}}{\ell!}(m|\lambda)_{k-j}\left(\frac{\log(1+\lambda)}{\lambda}\right)^{j}\beta_{n-k}^{(n)}(\lambda)K_{\ell,s}(\lambda)\right)x^{j}.
\end{aligned}$$

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Proof. It is not hard to see that $\frac{\log^n(1+\lambda)x^n}{\lambda^n} \sim (1, \lambda t/\log(1+\lambda))$. Thus, by (1.9), we have

$$\frac{\lambda\log(1+f(t))}{\log(1+\lambda f(t))}K_{n,7}(\lambda,x) = x\left(\frac{\frac{\lambda t}{\log(1+\lambda)}}{(1+\lambda^2 t/\log(1+\lambda))^{1/\lambda}-1}\right)^n x^{-1}\frac{\log^n(1+\lambda)x^n}{\lambda^n}$$
$$= \frac{\log^n(1+\lambda)}{\lambda^n}x\left(\frac{r}{(1+\lambda r)^{1/\lambda}-1}\right)^n |_{r=\lambda t/\log(1+\lambda)}\ x^{n-1},$$

which, by (2.5), implies

$$\frac{\lambda\log(1+f(t))}{\log(1+\lambda f(t))}K_{n,7}(\lambda,x) = \frac{\log^n(1+\lambda)}{\lambda^n}x\sum_{k\geq 0}\beta_k^{(n)}(\lambda)\frac{1}{k!}\left(\frac{\lambda t}{\log(1+\lambda)}\right)^k x^{n-1}$$
$$= \sum_{k=0}^{n-1}\binom{n-1}{k}\beta_k^{(n)}(\lambda)\left(\frac{\log(1+\lambda)}{\lambda}\right)^{n-k}x^{n-k}$$
$$= \sum_{k=0}^{n}\binom{n-1}{k-1}\beta_{n-k}^{(n)}(\lambda)\left(\frac{\log(1+\lambda)}{\lambda}\right)^{k}x^{k}.$$

On the other hand, by (2.4), we have

$$\frac{\log(1+\lambda f(t))}{\lambda\log(1+f(t))}x^k$$
$$= \sum_{\ell=0}^{k}\sum_{m=0}^{\ell}\sum_{j=0}^{k}\frac{K_{\ell,7}(\lambda)}{\ell!}\binom{\ell}{m}\binom{k}{j}(-1)^{\ell-m}(m|\lambda)_{k--j}\frac{\lambda^{k-j}}{\log^{k-j}(1+\lambda)}x^j.$$

Therefore, the polynomials $K_{n,7}(\lambda,x)$ is given by

$$\sum_{j=0}^{n}\left(\sum_{k=j}^{n}\sum_{\ell=0}^{k}\sum_{m=0}^{\ell}\frac{(-1)^{\ell-m}\binom{n-1}{k-1}\binom{\ell}{m}\binom{k}{j}}{\ell!}(m|\lambda)_{k-j}\left(\frac{\log(1+\lambda)}{\lambda}\right)^j\beta_{n-k}^{(n)}(\lambda)K_{\ell,7}(\lambda)\right)x^j.$$

By using similar argument as above with using (1.10), we obtain the formula for the nth Korobov polynomial $k_{n,8}(\lambda,x)$ of the eighth kind (we leave the details for the interested reader). □

3. Recurrences

In this section, we present several recurrences for the Korobov polynomials of the seventh kind and of the eighth kind. Note that, by (1.9), (1.10) and the fact that $(x)_{n,\lambda} \sim (1, f(t))$, we obtain $K_{n,d}(\lambda, x+y) = \sum_{j=0}^n \binom{n}{j}K_{j,d}(\lambda,x)(y)_{n-j,\lambda}$, for $d = 7, 8$.

Proposition 3.1. *For all $n \geq 1$ and $s = 7, 8$,*

$$K_{n,s}(\lambda,x) + nK_{n-1,s}(\lambda,x)$$
$$= \sum_{m=0}^{n}\left(\sum_{k=m}^{n}\sum_{\ell=k}^{n}\binom{n}{\ell}\binom{k}{m}(1|\lambda)_{k-m}\frac{\log^m(1+\lambda)}{\lambda^m}S_1(\ell,k|\lambda)K_{n-\ell,s}(\lambda)\right)x^m.$$

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Proof. It is well-known that if $s_n(x) \sim (g(t), f(t))$, then we have $f(t)s_n(x) = ns_{n-1}(x)$ (see [15, 16]). Thus, by (1.9) and (1.10), we obtain $\left(\left(1 + \frac{\lambda^2 t}{\log(1+\lambda)}\right)^{\frac{1}{\lambda}} - 1\right) K_{n,s}(\lambda, x) = nK_{n-1,s}(\lambda, x)$, which implies $K_{n,s}(\lambda, x) + nK_{n-1,s}(\lambda, x) = \left(1 + \frac{\lambda^2 t}{\log(1+\lambda)}\right)^{\frac{1}{\lambda}} K_{n,s}(\lambda, x)$. By Theorem 2.1 we have

$$\begin{aligned}
&K_{n,s}(\lambda, x) + nK_{n-1,s}(\lambda, x)\\
&= \sum_{k=0}^{n}\sum_{\ell=k}^{n} \binom{n}{\ell} \frac{\log^k(1+\lambda)}{\lambda^k} S_1(\ell, k|\lambda) K_{n-\ell,s}(\lambda) \left(1 + \frac{\lambda^2 t}{\log(1+\lambda)}\right)^{\frac{1}{\lambda}} x^k\\
&= \sum_{k=0}^{n}\sum_{\ell=k}^{n}\sum_{m=0}^{k} \binom{n}{\ell} \frac{\log^{k-m}(1+\lambda)}{\lambda^{k-m}} S_1(\ell, k|\lambda) K_{n-\ell,s}(\lambda) (1|\lambda)_m \frac{t^m}{m!} x^k\\
&= \sum_{k=0}^{n}\sum_{\ell=k}^{n}\sum_{m=0}^{k} \binom{n}{\ell}\binom{k}{m} \frac{\log^{k-m}(1+\lambda)}{\lambda^{k-m}} S_1(\ell, k|\lambda) K_{n-\ell,s}(\lambda) (1|\lambda)_m x^{k-m}\\
&= \sum_{m=0}^{n}\left(\sum_{k=m}^{n}\sum_{\ell=k}^{n} \binom{n}{\ell}\binom{k}{m} (1|\lambda)_{k-m} \frac{\log^m(1+\lambda)}{\lambda^m} S_1(\ell, k|\lambda) K_{n-\ell,s}(\lambda)\right) x^m.
\end{aligned}$$

which completes the proof. □

In the next result, we express $\frac{d}{dx}K_{n,7}(\lambda, x)$ and $\frac{d}{dx}K_{n,8}(\lambda, x)$ in terms of $K_{n,7}(\lambda, x)$ and $K_{n,8}(\lambda, x)$, respectively.

Proposition 3.2. *For all $n \geq 0$ and $s = 7, 8$,*

$$\frac{d}{dx}K_{n,s}(\lambda, x) = \frac{\log(1+\lambda)}{\lambda^2} \sum_{\ell=0}^{n-1} \binom{n}{\ell} (\lambda)_{n-\ell} K_{\ell,s}(\lambda, x).$$

Proof. Note that $\frac{d}{dx}s_n(x) = \sum_{\ell=0}^{n-1}\binom{n}{\ell}\langle \bar{f}(t)|x^{n-\ell}\rangle s_\ell(x)$, for all $s_n(x) \sim (g(t), f(t))$, see [15, 16]. So, for $s_n(x) = K_{n,s}(\lambda, x)$, it remains to compute $A = \langle \bar{f}(t)|x^{n-\ell}\rangle$. By (1.9) and (1.10), we have $A = \frac{\log(1+\lambda)}{\lambda^2}\langle \sum_{j\geq 1}(\lambda)_j \frac{t^j}{j!}|x^{n-\ell}\rangle = \frac{\log(1+\lambda)}{\lambda^2}(\lambda)_{n-\ell}$, which completes the proof. □

Theorem 3.3. *For all $n \geq 1$ and $s = 7, 8$,*

$$\begin{aligned}
K_{n,s}(\lambda, x) &= \frac{x\log(1+\lambda)}{\lambda} \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} (\lambda - 1)_{n-1-\ell} K_{\ell,s}(\lambda, x)\\
&\quad + \frac{1}{n}\sum_{\ell=0}^{n}\sum_{m=0}^{n-\ell} \binom{n}{\ell} (n-\ell)_{n-\ell-m} u_s(\ell, m),
\end{aligned}$$

where

$$\begin{aligned}
u_7(\ell) &= b_\ell \left\{(-\lambda)^{n-\ell-m}(x)_{m,\lambda} - (-1)^{n-\ell-m}K_{m,7}(\lambda, x)\right\},\\
u_8(\ell) &= K_\ell(\lambda) \left\{(-\lambda)^{n-\ell-m}(x)_{m,\lambda} - \binom{\lambda - 1}{n-\ell-m} K_{m,8}(\lambda, x)\right\}.
\end{aligned}$$

Proof. Since the similarity between $K_{n,7}(\lambda,x)$ and $K_{n,8}(\lambda,x)$ (see (1.9) and (1.10)), we omit the proof of the case $K_{n,8}(\lambda,x)$ and give only the details of the case $K_{n,7}(\lambda,x)$. By (1.9), we have

$$K_{n,7}(\lambda,y)=\left\langle \frac{d}{dt}\left(\frac{\log(1+\lambda t)}{\lambda\log(1+t)}(1+\lambda)^{\frac{y}{\lambda}\frac{(1+t)^\lambda-1}{\lambda}}\right)|x^{n-1}\right\rangle = A+B, \tag{3.1}$$

where $B=\left\langle \frac{d}{dt}\frac{\log(1+\lambda t)}{\lambda\log(1+t)}(1+\lambda)^{\frac{y}{\lambda}\frac{(1+t)^\lambda-1}{\lambda}}|x^{n-1}\right\rangle$ and $A=\left\langle \frac{\log(1+\lambda t)}{\lambda\log(1+t)}\frac{d}{dt}(1+\lambda)^{\frac{y}{\lambda}\frac{(1+t)^\lambda-1}{\lambda}}|x^{n-1}\right\rangle$. First, we compute the term B.

$$\begin{aligned}
B&=\left\langle \frac{\log(1+\lambda t)}{\lambda\log(1+t)}(1+\lambda)^{\frac{y}{\lambda}\frac{(1+t)^\lambda-1}{\lambda}}\frac{\log(1+\lambda)}{\lambda}y(1+t)^{\lambda-1}|x^{n-1}\right\rangle\\
&=\frac{y\log(1+\lambda)}{\lambda}\left\langle \frac{\log(1+\lambda t)}{\lambda\log(1+t)}(1+\lambda)^{\frac{y}{\lambda}\frac{(1+t)^\lambda-1}{\lambda}}|(1+t)^{\lambda-1}x^{n-1}\right\rangle\\
&=\frac{y\log(1+\lambda)}{\lambda}\sum_{\ell=0}^{n-1}\binom{n-1}{\ell}(\lambda-1)_\ell\left\langle \frac{\log(1+\lambda t)}{\lambda\log(1+t)}(1+\lambda)^{\frac{y}{\lambda}\frac{(1+t)^\lambda-1}{\lambda}}|x^{n-1-\ell}\right\rangle\\
&=\frac{y\log(1+\lambda)}{\lambda}\sum_{\ell=0}^{n-1}\binom{n-1}{\ell}(\lambda-1)_\ell K_{n-1-\ell,7}(\lambda,y)\\
&=\frac{y\log(1+\lambda)}{\lambda}\sum_{\ell=0}^{n-1}\binom{n-1}{\ell}(\lambda-1)_{n-1-\ell} K_{\ell,7}(\lambda,y).
\end{aligned}$$

Now, we compute the first term A,

$$\begin{aligned}
A&=\left\langle \frac{t}{\log(1+t)}\frac{1}{t}\left\{\frac{1}{1+\lambda t}-\frac{\log(1+\lambda t)}{\lambda\log(1+t)}(1+t)^{-1}\right\}(1+\lambda)^{\frac{y}{\lambda}\frac{(1+t)^\lambda-1}{\lambda}}|x^{n-1}\right\rangle\\
&=\frac{1}{n}\left\langle \left\{\frac{1}{1+\lambda t}-\frac{\log(1+\lambda t)}{\lambda\log(1+t)}(1+t)^{-1}\right\}(1+\lambda)^{\frac{y}{\lambda}\frac{(1+t)^\lambda-1}{\lambda}}|\frac{t}{\log(1+t)}x^{n}\right\rangle\\
&=\frac{1}{n}\left\langle \left\{\frac{1}{1+\lambda t}-\frac{\log(1+\lambda t)}{\lambda\log(1+t)}(1+t)^{-1}\right\}(1+\lambda)^{\frac{y}{\lambda}\frac{(1+t)^\lambda-1}{\lambda}}|\sum_{\ell\ge0}b_\ell\frac{t^\ell}{\ell!}x^{n}\right\rangle.
\end{aligned}$$

Note that $\frac{1}{1+\lambda t}-\frac{\log(1+\lambda t)}{\lambda\log(1+t)}(1+t)^{-1}$ has order at least one. Thus,

$$\begin{aligned}
A&=\frac{1}{n}\sum_{\ell=0}^{n}\binom{n}{\ell}b_\ell\Big\{\left\langle(1+\lambda)^{\frac{y}{\lambda}\frac{(1+t)^\lambda-1}{\lambda}}|\frac{1}{1+\lambda t}x^{n-\ell}\right\rangle\\
&\qquad-\left\langle\frac{\log(1+\lambda t)}{\lambda\log(1+t)}(1+\lambda)^{\frac{y}{\lambda}\frac{(1+t)^\lambda-1}{\lambda}}|\frac{1}{1+t}x^{n-\ell}\right\rangle\Big\}\\
&=\frac{1}{n}\sum_{\ell=0}^{n}\binom{n}{\ell}b_\ell\Big\{\sum_{m=0}^{n-\ell}(-\lambda)^m(n-\ell)_m\left\langle\sum_{k\ge0}(y)_{k,\lambda}\frac{t^k}{k!}|x^{n-\ell-m}\right\rangle\\
&\qquad-\sum_{m=0}^{n-\ell}(-1)^m(n-\ell)_m\left\langle\sum_{k\ge0}K_{k,7}(\lambda,y)\frac{t^k}{k!}|x^{n-\ell-m}\right\rangle\Big\},
\end{aligned}$$

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which implies

$$A = \frac{1}{n}\sum_{\ell=0}^{n}\sum_{m=0}^{n-\ell}\binom{n}{\ell}b_\ell\Big\{(-\lambda)^m(n-\ell)_m(y)_{n-\ell-m,\lambda}-(-1)^m(n-\ell)_mK_{n-\ell-m,7}(\lambda,y)\Big\},$$

Hence, by substituting the expressions of A and B in (3.1), we complete the proof. □

4. Connections with families of polynomials

In this section, we present some examples on the connections with families of polynomials. To do that, we recall for any two Sheffer sequences $s_n(x)\sim(g(t),f(t))$ and $r_n(x)\sim(h(t),\ell(t))$, we have that $s_n(x)=\sum_{m=0}^{n}C_{n,m}r_m(x)$, where (see [15, 16])

$$C_{n,m}=\frac{1}{m!}\left\langle \frac{h(\bar f(t))}{g(\bar f(t))}(\ell(\bar f(t)))^m|x^n\right\rangle. \tag{4.1}$$

We start with the connection to *Bernoulli polynomials* $B_n^{(s)}(x)$ *of order* s. Recall that the *Bernoulli polynomials* $B_n^{(s)}(x)$ *of order* s are defined by the generating function $\left(\frac{t}{e^t-1}\right)^s e^{xt}=\sum_{n\geq0}B_n^{(s)}(x)\frac{t^n}{n!}$, equivalently,

$$B_n^{(s)}(x)\sim\left(\left(\frac{e^t-1}{t}\right)^s,t\right) \tag{4.2}$$

(see [2, 5, 15]). In the next result, we express our polynomials in terms of Bernoulli polynomials of order s.

Theorem 4.1. *Let $d=7,8$. For all $n\geq0$, $K_{n,d}(\lambda,x)=\sum_{k=0}^{n}C_{n,m}B_m^{(s)}(x)$, where*

$$C_{n,m}=\sum_{\ell=0}^{n-m}\sum_{k=0}^{n-\ell-m}\frac{\binom{n}{\ell}\binom{k+m}{m}}{\binom{k+s}{s}}K_{\ell,d}(\lambda)S_2(k+s,s)\frac{\log^{k+m}(1+\lambda)}{\lambda^{k+m}}S_1(n-\ell,k+m|\lambda).$$

Proof. Since the similarity between $K_{n,7}(\lambda,x)$ and $K_{n,8}(\lambda,x)$ (see (1.9) and (1.10)), we omit the proof of the case $K_{n,8}(\lambda,x)$ and give only the details of the case $K_{n,7}(\lambda,x)$. Let $K_{n,7}(\lambda,x)=\sum_{m=0}^{n}C_{n,m}B_m^{(s)}(x)$. So, by (1.9) and (4.2), we have

$$\begin{aligned}
C_{n,m}&=\frac{1}{m!}\left\langle\frac{(e^{\bar f(t)}-1)^s}{\bar f^s(t)}\frac{\log(1+\lambda t)}{\lambda\log(1+t)}\bar f^m(t)|x^n\right\rangle\\
&=\frac{1}{m!}\left\langle\frac{(e^{\bar f(t)}-1)^s}{\bar f^s(t)}\bar f^m(t)|\frac{\log(1+\lambda t)}{\lambda\log(1+t)}x^n\right\rangle\\
&=\frac{1}{m!}\left\langle\frac{(e^{\bar f(t)}-1)^s}{\bar f^s(t)}\bar f^m(t)|\sum_{\ell\geq0}K_{\ell,7}(\lambda)\frac{t^\ell}{\ell!}x^n\right\rangle\\
&=\frac{1}{m!}\sum_{\ell=0}^{n}\binom{n}{\ell}K_{\ell,7}(\lambda)\left\langle s!\sum_{k\geq0}S_2(k+s,s)\frac{\bar f^{k+m}(t)}{(k+s)!}|x^{n-\ell}\right\rangle.
\end{aligned}$$

Thus,

$$\begin{aligned}
C_{n,m} &= \frac{s!}{m!}\sum_{\ell=0}^{n-m}\sum_{k=0}^{n-\ell-m}\binom{n}{\ell}K_{\ell,7}(\lambda)S_2(k+s,s)\left\langle \frac{\bar{f}^{k+m}(t)}{(k+s)!}|x^{n-\ell}\right\rangle \\
&= \frac{s!}{m!}\sum_{\ell=0}^{n-m}\sum_{k=0}^{n-\ell-m}\binom{n}{\ell}K_{\ell,7}(\lambda)S_2(k+s,s)\frac{\log^{k+m}(1+\lambda)}{(k+s)!\lambda^{k+m}}\left\langle \frac{((1+t)^{\lambda}-1)^{k+m}}{\lambda^{k+m}}|x^{n-\ell}\right\rangle \\
&= \frac{s!}{m!}\sum_{\ell=0}^{n-m}\sum_{k=0}^{n-\ell-m}\binom{n}{\ell}K_{\ell,7}(\lambda)S_2(k+s,s)\frac{\log^{k+m}(1+\lambda)}{(k+s)!\lambda^{k+m}}(k+m)!S_1(n-\ell,k+m|\lambda).
\end{aligned}$$

Therefore,

$$C_{n,m} = \sum_{\ell=0}^{n-m}\sum_{k=0}^{n-\ell-m}\frac{\binom{n}{\ell}\binom{k+m}{m}}{\binom{k+s}{s}}K_{\ell,7}(\lambda)S_2(k+s,s)\frac{\log^{k+m}(1+\lambda)}{\lambda^{k+m}}S_1(n-\ell,k+m|\lambda),$$

as required. □

Similar techniques as in the proof of the previous theorem, we can express our polynomials $K_{n,7}(\lambda,x), K_{n,8}(\lambda,x)$ in terms of other families. Below we present two examples, where we leave the proofs to the interested reader. In the first example, we express our polynomials in terms of Frobenius-Euler polynomials. Note that the *Frobenius-Euler polynomials* $H_n^{(s)}(x|\mu)$ of order s are defined by the generating function $\left(\frac{1-\mu}{e^t-\mu}\right)^s e^{xt} = \sum_{n\geq 0} H_n^{(s)}(x|\mu)\frac{t^n}{n!}, (\mu \neq 1)$, or equivalently, $H_n^{(s)}(x|\mu) \sim \left(\left(\frac{e^t-\mu}{1-\mu}\right)^s, t\right)$ (see [1, 11, 12]).

Theorem 4.2. *For all $n \geq 0$ and $d = 7, 8$, $K_{n,d}(\lambda, x) = \sum_{m=0}^{n} C_{n,m}H_m^{(s)}(x|\mu)$, where*

$$C_{n,m} = \sum_{\ell=0}^{n-m}\sum_{k=0}^{s}\sum_{j=k}^{n-\ell-m}\frac{k!\binom{j+m}{m}\binom{n}{\ell}\binom{s}{k}}{(1-\mu)^k}\frac{\log^{j+m}(1+\lambda)}{\lambda^{j+m}}S_1(n-\ell,j+m|\lambda)S_2(j,k)K_{\ell,d}(\lambda).$$

For what follows, we define the associated sequence for $1-(1+\lambda^2 t/\log(1+\lambda))^{-1/\lambda}$, namely $(x)^{(n,\lambda)}$. Thus,

$$(x)^{(n,\lambda)} \sim (1, 1-(1+\lambda^2 t/\log(1+\lambda))^{-1/\lambda}).$$

Recall here that $(x)_n \sim (1, e^t-1)$, $(x)^{(n)} \sim (1, 1-e^{-t})$, $(x)_{n,\lambda} \sim (1, (1+\lambda^2 t/\log(1+\lambda))^{1/\lambda}-1)$ and $(1+\lambda^2 t/\log(1+\lambda))^{1/\lambda}-1 \to e^t-1$, as $\lambda \to 0$. Now, we ready to present our second example.

Theorem 4.3. *For all $n \geq 0$ and $d = 7, 8$, $K_{n,d}(\lambda, x) = \sum_{m=0}^{n} C_{n,m}(x)^{(m,\lambda)}$, where*

$$C_{n,m} = \sum_{\ell=0}^{n}(-1)^{n-\ell-m}\binom{n}{\ell}\binom{n-\ell}{m}(n-1-\ell)_{n-\ell-m}K_{\ell,d}(\lambda).$$

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Some identities on the higher-order twisted q-Euler numbers and polynomials

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Abstract : In this paper we investigate some interesting symmetric identities for twisted q-Euler polynomials of higher order in complex field.

Key words : Symmetric properties, power sums, Euler numbers and polynomials, twisted q-Euler numbers and polynomials.

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1. Introduction

Euler numbers and polynomials possess many interesting properties and arising in many areas of mathematics, mathematical physics and statistical physics. Many mathematicians have studied in the area of the q- extension of Euler numbers and polynomials(see [1, 2, 3, 5, 6, 7, 8, 9, 11, 13]). Recently, Y. Hu studied several identities of symmetry for Carlitz's q-Bernoulli numbers and polynomials in complex field(see [3]). D. Kim *et al.*[4] derived some identities of symmetry for (h, q)-extension of higher-order Euler numbers and polynomials. D. V. Dolgy *et al.*[2] derived some identities of symmetry for higher-order generalized q-Euler polynomials. In this paper, we establish some interesting symmetric identities for twisted q-Euler polynomials of higher order in complex field. The purpose of this paper is to present a systemic study of the twisted q-Euler numbers and polynomials of higher-order by using the multiple q-Euler zeta function. Throughout this paper, the notations $\mathbb{N}, \mathbb{Z}, \mathbb{R}$, and $\mathbb{C}$ denote the sets of positive integers, integers, real numbers, and complex numbers, respectively, and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. We assume that $q \in \mathbb{C}$ with $|q| < 1$. Throughout this paper we use the notation:

$$[x]_q = \frac{1-q^x}{1-q} \text{ (cf. [1, 2, 3, 5]) }.$$

Note that $\lim_{q\to 1}[x] = x$. Let ε be the p^N-th root of unity(see [10, 12, 13]).

In [5], T. Kim introduced the multiple q-Euler zeta function which interpolates higher-order q-Euler polynomials at negative integers as follows:

$$\zeta_{q,r}(s,x) = [2]_q^r \sum_{m_1,\cdots,m_r=0}^{\infty} \frac{(-1)^{\sum_{j=1}^r m_j} q^{\sum_{j=1}^r m_j}}{[m_1+\cdots+m_r+x]_q^s}, \tag{1}$$

where $s \in \mathbb{C}$ and $x \in \mathbb{R}$, with $x \neq 0, -1, -2, \ldots$.

Recently, D. V. Dolgy *et al.*[2] considered some symmetric identities for higher-order generalized q-Euler polynomials. The Euler polynomials of order $r \in \mathbb{N}$ attached to χ are also defined by the generating function:

$$\left(2\sum_{l=0}^{d-1} \frac{\chi(l)(-1)^l e^{(x+l)t}}{e^{dt}+1}\right)^r = \sum_{m=0}^{\infty} E_{m,\chi}^{(r)}(x)\frac{t^m}{m!}. \tag{2}$$

When $x = 0$, $E_{n,\chi}^{(r)} = E_{n,\chi}^{(r)}(0)$ are called the Euler numbers $E_{n,\chi}^{(r)}$ attached to χ(see [2, 4]).

For $h \in \mathbb{Z}, \alpha, k \in \mathbb{N}$, and $n \in \mathbb{Z}_+$, we introduced the higher order twisted q-Euler polynomials with weight α as follows(see [7]):

$$\widetilde{E}_{n,q,\varepsilon}^{(\alpha)}(k|x) = \frac{[2]_q^k}{(1-q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{q^{\alpha l x}}{(1+\varepsilon q^{\alpha l+h}) \cdots (1+\varepsilon q^{\alpha l+h-k+1})}.$$

In the special case, $x = 0$, $\widetilde{E}_{n,q,w}^{(\alpha)}(k|0) = \widetilde{E}_{n,q,w}^{(\alpha)}(k)$ are called the higher-order twisted q-Euler numbers with weight α.

We consider the higher order q-Euler polynomials of order r attached to χ twisted by ramified roots of unity as follows(see [10]):

$$\sum_{n=0}^{\infty} E_{n,\chi,\varepsilon,q}^{(r)}(x) \frac{t^n}{n!} = 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} (-\varepsilon)^{\sum_{j=0}^{r} m_j} \left(\prod_{i=1}^{r} \chi(m_i) \right) e^{[x+\sum_{j=1}^{r} m_j]_q t}.$$

In the special case $x = 0$, the sequence $E_{n,\chi,\varepsilon,q}^{(r)}(0) = E_{n,\chi,\varepsilon,q}^{(r)}$ are called the n-th q-Euler numbers of order r attached to χ twisted by ramified roots of unity.

As is well known, the higher-order twisted q-Euler polynomials $E_{n,q,\varepsilon}^{(k)}(x)$ are defined by the following generating function to be

$$\begin{aligned} \widetilde{F}_{q,\varepsilon}^{(k)}(t,x) &= [2]_q^k \sum_{m_1,\cdots,m_k=0}^{\infty} (-1)^{m_1+\cdots+m_k} \varepsilon^{m_1+\cdots+m_k} e^{[m_1+\cdots+m_k+x]_q t} \\ &= \sum_{n=0}^{\infty} E_{n,q,\varepsilon}^{(k)}(x) \frac{t^n}{n!}, \end{aligned} \tag{3}$$

where $k \in \mathbb{N}$. When $x = 0, E_{n,q,\varepsilon}^{(k)} = E_{n,q,\varepsilon}^{(k)}(0)$ are called the higher-order twisted q-Euler numbers $E_{n,q,\varepsilon}^{(k)}$. Observe that if $q \to 1, \varepsilon \to 1$, then $E_{n,q,\varepsilon}^{(k)} \to E_n^{(k)}$ and $E_{n,q,\varepsilon}^{(k)}(x) \to E_{n,\chi}^{(k)}(x)$.

By using (3) and Cauchy product, we have

$$\begin{aligned} E_{n,q,\varepsilon}^{(k)}(x) &= \sum_{l=0}^{n} \binom{n}{l} q^{lx} E_{l,q,\varepsilon}^{(k)} [x]_q^{n-l} \\ &= (q^x E_{q,\varepsilon}^{(k)} + [x]_q)^n, \end{aligned} \tag{4}$$

with the usual convention about replacing $(E_{q,\varepsilon}^{(k)})^n$ by $E_{n,q,\varepsilon}^{(k)}$.

By using complex integral and (3), we can also obtain the multiple twisted q-l-function as follows:

$$\begin{aligned} l_{q,\varepsilon}^{(k)}(s,x) &= \frac{1}{\Gamma(s)} \int_0^\infty \widetilde{F}_{q,\varepsilon}^{(k)}(-t,x) t^{s-1} dt \\ &= [2]_q^k \sum_{m_1,\cdots,m_k=0}^{\infty} \frac{(-1)^{\sum_{j=1}^{k} m_j} \varepsilon^{\sum_{j=1}^{k} m_j}}{[m_1+\cdots+m_k+x]_q^s}, \end{aligned} \tag{5}$$

where $s \in \mathbb{C}$ and $x \in \mathbb{R}$, with $x \neq 0, -1, -2, \ldots$.

By using Cauchy residue theorem, the value of multiple twisted q-l-function at negative integers is given explicitly by the following theorem:

Theorem 1. Let $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$. We obtain

$$l_{q,\varepsilon}^{(k)}(-n,x) = E_{n,q,\varepsilon}^{(k)}(x).$$

The purpose of this paper is to obtain some interesting identities of the power sums and the higher-order twisted q-Euler polynomials $E_{n,q,\varepsilon}^{(k)}(x)$ using the symmetric properties for multiple

twisted q-l-function. In this paper, if we take $\varepsilon = 1$ in all equations of this article, then [2] are the special case of our results.

2. Symmetry identities for multiple twisted q-l-function

In this section, by using the similar method of [2, 3, 4], expect for obvious modifications, we investigate some symmetric identities for higher-order twisted q-Euler polynomials $E^{(k)}_{n,q,\varepsilon}(x)$. We assume that ε be the p^N-th root of unity. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $h \in \mathbb{Z}, k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain certain symmetry identities for multiple twisted q-l-function.

Observe that $[xy]_q = [x]_{q^y}[y]_q$ for any $x, y \in \mathbb{C}$. In (5), we derive next result by substitute $w_2x + \frac{w_2}{w_1}(j_1 + \cdots + j_k)$ for x in and replace q and ε by q^{w_1} and ε^{w_1}, respectively.

$$
\begin{aligned}
&\frac{1}{[2]^k_{q^{w_1}}} l^{(k)}_{q^{w_1},\varepsilon^{w_1}}(s, w_2x + \frac{w_2}{w_1}(j_1 + \cdots + j_k)) \\
&= \sum_{m_1,\cdots,m_k=0}^{\infty} \frac{(-1)^{\sum_{j=1}^k m_j} \varepsilon^{w_1 \sum_{j=1}^k m_j}}{[m_1 + \cdots + m_k + w_2x + \frac{w_2}{w_1}(j_1 + \cdots + j_k)]^s_{q^{w_1}}} \\
&= \sum_{m_1,\cdots,m_k=0}^{\infty} \frac{(-1)^{\sum_{j=1}^k m_j} \varepsilon^{w_1 \sum_{j=1}^k m_j}}{\left[\frac{w_1(m_1 + \cdots + m_k) + w_1w_2x + w_2(j_1 + \cdots + j_k)}{w_1}\right]^s_{q^{w_1}}} \\
&= \sum_{m_1,\cdots,m_k=0}^{\infty} \frac{(-1)^{\sum_{j=1}^k m_j} \varepsilon^{w_1 \sum_{j=1}^k m_j}}{\frac{[w_1(m_1 + \cdots + m_k) + w_1w_2x + w_2(j_1 + \cdots + j_k)]^s_q}{[w_1]^s_q}} \\
&= [w_1]^s_q \sum_{m_1,\cdots,m_k=0}^{\infty} \frac{(-1)^{\sum_{j=1}^k m_j} \varepsilon^{w_1 \sum_{j=1}^k m_j}}{[w_1(m_1 + \cdots + m_k) + w_1w_2x + w_2(j_1 + \cdots + j_k)]^s_q} \\
&= [w_1]^s_q \sum_{m_1,\cdots,m_k=0}^{\infty} \sum_{i_1,\cdots,i_k=0}^{w_2-1} \frac{(-1)^{\sum_{j=1}^k m_j} \varepsilon^{w_1 \sum_{j=1}^k m_j}}{[w_1(m_1 + \cdots + m_k) + w_1w_2x + w_2(j_1 + \cdots + j_k)]^s_q} \\
&= [w_1]^s_q \sum_{m_1,\cdots,m_k=0}^{\infty} \sum_{i_1,\cdots,i_k=0}^{w_2-1} (-1)^{\sum_{j=1}^k (dw_2m_j + i_j)} \\
&\quad \times \varepsilon^{w_1 \sum_{j=1}^k (dw_2m_j + i_j)} \\
&\quad \times \left([w_1(dw_2m_1 + i_1) + \cdots + w_1(dw_2m_k + i_k) + w_1w_2x + w_2(j_1 + \cdots + j_k)]^s_q\right)^{-1} \\
&= [w_1]^s_q \sum_{m_1,\cdots,m_k=0}^{\infty} \sum_{i_1,\cdots,i_k=0}^{w_2-1} (-1)^{\sum_{j=1}^k m_j} (-1)^{\sum_{j=1}^k i_j} \\
&\quad \times \varepsilon^{dw_1w_2 \sum_{j=1}^k m_j} \varepsilon^{w_1 \sum_{j=1}^k i_j} \\
&\quad \times \left([w_1w_2(x + dm_1 + \cdots + dm_k) + w_1(i_1 + \cdots + i_k) + w_2(j_1 + \cdots + j_k)]^s_q\right)^{-1}
\end{aligned} \tag{6}
$$

Thus, from (6), we can derive the following equation.

$$
\begin{aligned}
&\frac{[w_2]^s_q}{[2]^k_{q^{w_1}}} \sum_{j_1,\cdots,j_k=0}^{w_1-1} (-1)^{\sum_{l=1}^k j_l} \varepsilon^{w_2 \sum_{l=1}^k j_l} \\
&\qquad \times l^{(k)}_{q^{w_1},\varepsilon^{w_1}}(s, w_2x + \frac{w_2}{w_1}(j_1 + \cdots + j_k)) \\
&= [w_1]^s_q [w_2]^s_q \sum_{m_1,\cdots,m_k=0}^{\infty} \sum_{i_1,\cdots,i_k=0}^{w_2-1} \sum_{j_1,\cdots,j_k=0}^{w_1-1} (-1)^{\sum_{l=1}^k (j_l + i_l + m_l)} \\
&\quad \times \varepsilon^{dw_1w_2 \sum_{l=1}^k m_l} \varepsilon^{w_1 \sum_{l=1}^k i_l} \varepsilon^{w_2 \sum_{l=1}^k j_l} \\
&\quad \times \left([w_1w_2(x + dm_1 + \cdots + dm_k) + w_1(i_1 + \cdots + i_k) + w_2(j_1 + \cdots + j_k)]^s_q\right)^{-1}
\end{aligned} \tag{7}
$$

By using the same method as (7), we have

$$
\begin{aligned}
&\frac{[w_1]_q^s}{[2]_{q^{w_2}}^k} \sum_{j_1,\cdots,j_k=0}^{w_2-1} (-1)^{\sum_{l=1}^k j_l} \varepsilon^{w_1 \sum_{l=1}^k j_l} \\
&\qquad \times l_{q^{w_2},\varepsilon^{w_2}}^{(k)}(s, w_1 x + \frac{w_1}{w_2}(j_1+\cdots+j_k)) \\
&= [w_1]_q^s [w_2]_q^s \sum_{m_1,\cdots,m_k=0}^{\infty} \sum_{j_1,\cdots,j_k=0}^{w_2-1} \sum_{i_1,\cdots,i_k=0}^{w_1-1} (-1)^{\sum_{l=1}^k (j_l+i_l+m_l)} \\
&\quad \times \varepsilon^{dw_1w_2 \sum_{l=1}^k m_l} \varepsilon^{w_2 \sum_{l=1}^k i_l} \varepsilon^{w_1 \sum_{l=1}^k j_l} \\
&\quad \times \left([w_1w_2(x+dm_1+\cdots+dm_k) + w_1(j_1+\cdots+j_k) + w_2(i_1+\cdots+i_k)]_q^s\right)^{-1}
\end{aligned} \tag{8}
$$

Therefore, by (7) and (8), we have the following theorem.

Theorem 2. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $h \in \mathbb{Z}$, we obtain

$$
\begin{aligned}
&[w_2]_q^s [2]_{q^{w_2}}^k \sum_{j_1,\cdots,j_k=0}^{w_1-1} (-1)^{\sum_{l=1}^k j_l} \varepsilon^{w_2 \sum_{l=1}^k j_l} l_{q^{w_1},\varepsilon^{w_1}}^{(k)} \left(s, w_2 x + \frac{w_2}{w_1}(j_1+\cdots+j_k)\right) \\
&= [w_1]_q^s [2]_{q^{w_1}}^k \sum_{j_1,\cdots,j_k=0}^{w_2-1} (-1)^{\sum_{l=1}^k j_l} \varepsilon^{w_1 \sum_{l=1}^k j_l} l_{q^{w_2},\varepsilon^{w_2}}^{(k)} \left(s, w_1 x + \frac{w_1}{w_2}(j_1+\cdots+j_k)\right)
\end{aligned} \tag{9}
$$

By (9) and Theorem 1, we obtain the following theorem.

Theorem 3. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $h \in \mathbb{Z}, k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain

$$
\begin{aligned}
&[w_2]_q^s [2]_{q^{w_2}}^k \sum_{j_1,\cdots,j_k=0}^{w_1-1} (-1)^{\sum_{l=1}^k j_l} \varepsilon^{w_2 \sum_{l=1}^k j_l} E_{n,q^{w_1},\varepsilon^{w_1}}^{(k)} \left(w_2 x + \frac{w_2}{w_1}(j_1+\cdots+j_k)\right) \\
&= [w_1]_q^s [2]_{q^{w_1}}^k \sum_{j_1,\cdots,j_k=0}^{w_2-1} (-1)^{\sum_{l=1}^k j_l} \varepsilon^{w_1 \sum_{l=1}^k j_l} E_{n,q^{w_2},\varepsilon^{w_2}}^{(k)} \left(w_1 x + \frac{w_1}{w_2}(j_1+\cdots+j_k)\right).
\end{aligned} \tag{10}
$$

From (4), we note that

$$
\begin{aligned}
E_{n,q,\varepsilon}^{(k)}(x+y) &= (q^{x+y} E_{n,q,\varepsilon}^{(k)} + [x+y]_q)^n \\
&= \sum_{i=0}^{n} \binom{n}{i} q^{xi} E_{i,q,\varepsilon}^{(k)}(y) [x]_q^{n-i}.
\end{aligned} \tag{11}
$$

with the usual convention about replacing $(E_{q,\varepsilon}^{(k)})^n$ by $E_{n,q,\varepsilon}^{(k)}$.

By (11), we have

$$
\begin{aligned}
&\sum_{j_1,\cdots,j_k=0}^{w_1-1} (-1)^{\sum_{l=1}^k j_l} \varepsilon^{w_2 \sum_{l=1}^k j_l} E_{n,q^{w_1},\varepsilon^{w_1}}^{(k)} \left(w_2 x + \frac{w_2}{w_1}(j_1+\cdots+j_k)\right) \\
&= \sum_{j_1,\cdots,j_k=0}^{w_1-1} (-1)^{\sum_{l=1}^k j_l} \varepsilon^{w_2 \sum_{l=1}^k j_l} \\
&\times \sum_{i=0}^{n} \binom{n}{i} q^{w_2 i (j_1+\cdots+j_k)} E_{i,q^{w_1},\varepsilon^{w_1}}^{(k)}(w_2 x) \left[\frac{w_2}{w_1}(j_1+\cdots+j_k)\right]_{q^{w_1}}^{n-i} \\
&= \sum_{j_1,\cdots,j_k=0}^{w_1-1} (-1)^{\sum_{l=1}^k j_l} \varepsilon^{w_2 \sum_{l=1}^k j_l} \\
&\times \sum_{i=0}^{n} \binom{n}{i} q^{w_2 (n-i) \sum_{l=1}^k j_l} E_{n-i,q^{w_1},\varepsilon^{w_1}}^{(k)}(w_2 x) \left[\frac{w_2}{w_1}(j_1+\cdots+j_k)\right]_{q^{w_1}}^{i}
\end{aligned} \tag{12}
$$

Hence we have the following theorem.

Theorem 4. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain

$$\begin{aligned}&\sum_{j_1,\cdots,j_k=0}^{w_1-1}(-1)^{\sum_{l=1}^k j_l}\varepsilon^{w_2\sum_{l=1}^k j_l}E_{n,q^{w_1},\varepsilon^{w_1}}^{(k)}\left(w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_k)\right)\\&=\sum_{i=0}^{n}\binom{n}{i}[w_2]_q^i[w_1]_q^{-i}E_{n-i,q^{w_1},\varepsilon^{w_1}}^{(k)}(w_2x)\sum_{j_1,\cdots,j_k=0}^{w_1-1}(-1)^{\sum_{l=1}^k j_l}\varepsilon^{w_2\sum_{l=1}^k j_l}[j_1\cdots+j_k]_{q^{w_2}}^i.\end{aligned}$$

For each integer $n\geq 0$, let

$$\mathcal{S}_{n,i,q,\varepsilon}^{(k)}(w)=\sum_{j_1,\cdots,j_k=0}^{w-1}(-1)^{\sum_{l=1}^k j_l}\varepsilon^{\sum_{l=1}^k j_l}[j_1\cdots+j_k]_q^i.$$

The above sum $\mathcal{S}_{n,i,q,\varepsilon}^{(k)}(w)$ is called the alternating q-power sums.

By Theorem 4, we have

$$\begin{aligned}&[2]_{q^{w_2}}^k[w_1]_q^n\sum_{j_1,\cdots,j_k=0}^{w_1-1}(-1)^{\sum_{l=1}^k j_l}\varepsilon^{w_2\sum_{l=1}^k j_l}E_{n,q^{w_1},\varepsilon^{w_1}}^{(k)}\left(w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_k)\right)\\&=[2]_{q^{w_2}}^k\sum_{i=0}^{n}\binom{n}{i}[w_2]_q^i[w_1]_q^{n-i}E_{n-i,q^{w_1},\varepsilon^{w_1}}^{(k)}(w_2x)\mathcal{S}_{n,i,q^{w_2},\varepsilon^{w_2}}^{(k)}(w_1)\end{aligned}\tag{13}$$

By using the same method as in (13), we have

$$\begin{aligned}&[2]_{q^{w_1}}^k[w_2]_q^n\sum_{j_1,\cdots,j_k=0}^{w_2-1}(-1)^{\sum_{l=1}^k j_l}\varepsilon^{w_1\sum_{l=1}^k j_l}E_{n,q^{w_2},\varepsilon^{w_2}}^{(k)}\left(w_1x+\frac{w_1}{w_2}(j_1+\cdots+j_k)\right)\\&=[2]_{q^{w_1}}^k\sum_{i=0}^{n}\binom{n}{i}[w_1]_q^i[w_2]_q^{n-i}E_{n-i,q^{w_2},\varepsilon^{w_2}}^{(k)}(w_1x)\mathcal{S}_{n,i,q^{w_1},\varepsilon^{w_1}}^{(k)}(w_2)\end{aligned}\tag{14}$$

Therefore, by (13) and (14) and Theorem 3, we have the following theorem.

Theorem 5. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain

$$\begin{aligned}&[2]_{q^{w_2}}^k\sum_{i=0}^{n}\binom{n}{i}[w_2]_q^i[w_1]_q^{n-i}E_{n-i,q^{w_1},\varepsilon^{w_1}}^{(k)}(w_2x)\mathcal{S}_{n,i,q^{w_2},\varepsilon^{w_2}}^{(k)}(w_1)\\&=[2]_{q^{w_1}}^k\sum_{i=0}^{n}\binom{n}{i}[w_1]_q^i[w_2]_q^{n-i}E_{n-i,q^{w_2},\varepsilon^{w_2}}^{(k)}(w_1x)\mathcal{S}_{n,i,q^{w_1},\varepsilon^{w_1}}^{(k)}(w_2).\end{aligned}$$

By Theorem 5, we obtain the interesting symmetric identity for the higher-order twisted q-Euler numbers $E_{n,q,\varepsilon}^{(k)}$ in complex field.

Corollary 6. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain

$$\begin{aligned}&[2]_{q^{w_2}}^k\sum_{i=0}^{n}\binom{n}{i}[w_2]_q^i[w_1]_q^{n-i}\mathcal{S}_{n,i,q^{w_2},\varepsilon^{w_2}}^{(k)}(w_1)E_{n-i,q^{w_1},\varepsilon^{w_1}}^{(k)}\\&=[2]_{q^{w_1}}^k\sum_{i=0}^{n}\binom{n}{i}[w_1]_q^i[w_2]_q^{n-i}\mathcal{S}_{n,i,q^{w_1},\varepsilon^{w_1}}^{(k)}(w_2)E_{n-i,q^{w_2},\varepsilon^{w_2}}^{(k)}.\end{aligned}$$

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UMBRAL CALCULUS ASSOCIATED WITH NEW DEGENERATE BERNOULLI POLYNOMIALS

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ABSTRACT. In this paper, we introduce new degenerate Bernoulli polynomials which are derived from umbral calculus and investigate some interesting properties of those polynomials.

1. INTRODUCTION

The Bernoulli polynomials are defined by the generating function

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\quad (\text{see [1–14]}). \tag{1.1}$$

When $x=0$, $B_n=B_n(0)$ are called the ordinary Bernoulli numbers. From (1.1), we note that

$$B_n(x)=\sum_{l=0}^{n}\binom{n}{l}B_l x^{n-l},\quad (n\geq 0),\quad (\text{see [13]}). \tag{1.2}$$

Thus, by (1.2), we get

$$\frac{d}{dx}B_n(x)=nB_{n-1}(x),\quad (n\in\mathbb{N}). \tag{1.3}$$

In [2], L. Carlitz introduced the degenerate Bernoulli polynomials which are given by the generating function

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\beta_n(x\mid\lambda)\frac{t^n}{n!}. \tag{1.4}$$

When $x=0$, $\beta_n(0\mid\lambda)=\beta_n(\lambda)$ are called Carlitz's degenerate Bernoulli numbers (see [2]).

Thus, by (1.4), we get

$$\beta_n(x\mid\lambda)=\sum_{l=0}^{n}\binom{n}{l}\beta_l(\lambda)(x\mid\lambda)_{n-l},\quad (n\geq 0), \tag{1.5}$$

where $(x\mid\lambda)_n=x(x-\lambda)\cdots(x-\lambda(n-1))$.

Let $\mathbb{C}$ be the field of complex numbers and let $\mathcal{F}$ be the set of all formal power series in the variable t over $\mathbb{C}$ with

$$\mathcal{F}=\left\{f(t)=\sum_{k=0}^{\infty}a_k\frac{t^k}{k!}\,\middle|\, a_k\in\mathbb{C}\right\}.$$

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1

Let $\mathbb{P}=\mathbb{C}[x]$ and $\mathbb{P}^*$ denotes the vector space of all linear functionals on $\mathbb{P}$. The action of the linear functional $L\in\mathbb{P}^*$ on a polynomial $p(x)$ is denoted by $\langle L|\,p(x)\rangle$, and linearly extended as $\langle cL+c'L'|\,p(x)\rangle=c\langle L|\,p(x)\rangle+c'\langle L'|\,p(x)\rangle$, where c and $c'\in\mathbb{C}$.

For $f(t)=\sum_{k=0}^{\infty}a_k\frac{t^k}{k!}\in\mathcal{F}$, we define a linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|\,x^n\rangle=a_n \tag{1.6}$$

for all $n\geq 0$, (see [1, 5, 13]).

Thus, by (1.6), we get

$$\left\langle t^k\middle|\,x^n\right\rangle=n!\delta_{n,k},\quad (n,k\geq 0),\quad (\text{see [7, 13]}), \tag{1.7}$$

where $\delta_{n,k}$ is the Kronecker's symbol.

Let $f_L(t)=\sum_{k=0}^{\infty}\left\langle L|\,x^k\right\rangle\frac{t^k}{k!}$. Then we have $\langle f_L(t)|\,x^n\rangle=\langle L|\,x^n\rangle$ $(n\geq 0)$. The mapping $L\mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ will denote both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power seires and a linear functional. We shall call $\mathcal{F}$ the umbral algebra. The umbral calculus is the study of umbral algebra and can be also desribed as a systematic study of the class of Sheffer sequences. The order $o(f)$ of the non-zero power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish (see [12, 13]).

For $f(t),g(t)\in\mathcal{F}$ with $o(f)=1$ and $o(g)=0$, there exists a unique sequence $s_n(x)$ of polynomials such that

$$\left\langle g(t)\,f(t)^k\middle|\,s_n(x)\right\rangle=n!\delta_{n,k},\quad (n,k\geq 0).$$

The sequence $s_n(x)$ is called the Sheffer sequence for $(g(t),f(t))$ which is denoted by $s_n(x)\sim(g(t),f(t))$(see [10, 13]).

Let $f(t)\in\mathcal{F}$ and $p(x)\in\mathbb{P}$. Then by (1.7), we get

$$\left\langle e^{yt}\middle|\,p(x)\right\rangle=p(y),\quad \langle f(t)\,g(t)|\,p(x)\rangle=\langle g(t)|\,f(t)\,p(x)\rangle, \tag{1.8}$$

and

$$f(t)=\sum_{k=0}^{\infty}\left\langle f(t)|\,x^k\right\rangle\frac{t^k}{k!},\quad p(x)=\sum_{k=0}^{\infty}\left\langle t^k\middle|\,p(x)\right\rangle\frac{x^k}{k!},\quad (\text{see [13]}). \tag{1.9}$$

By (1.9), we easily get

$$p^{(k)}(0)=\left\langle t^k\middle|\,p(x)\right\rangle=\left\langle 1|\,p^{(k)}(x)\right\rangle,\quad (k\geq 0), \tag{1.10}$$

where $p^{(k)}(0)$ denotes the k-th derivative of $p(x)$ with respect to x at $x=0$.

From (1.10), we have

$$t^kp(x)=p^{(k)}(x). \tag{1.11}$$

In [13], it is known that

$$s_n(x)\sim(g(t),f(t))\iff\frac{1}{g\left(\overline{f}(t)\right)}e^{x\overline{f}(t)}=\sum_{n=0}^{\infty}s_n(x)\frac{t^n}{n!}, \tag{1.12}$$

where $\overline{f}(t)$ is the compositional inverse of $f(t)$ such that $f\left(\overline{f}(t)\right)=\overline{f}(f(t))=t$.

From (1.7), we can easily derive

$$e^{yt}p(x)=p(x+y),\quad \text{where } p(x)\in\mathbb{P}=\mathbb{C}[x]. \tag{1.13}$$

For $p(x) \in \mathbb{P}$, we have

$$\left\langle \frac{e^{yt}-1}{t} \middle| p(x) \right\rangle = \int_0^y p(u)\,du, \quad \langle f(t) | xp(x) \rangle = \langle \partial_t f(t) | p(x) \rangle .$$

Let $f_1(t), f_2(t), \dots, f_m(t) \in \mathcal{F}$. Then we have

$$\langle f_1(t) f_2(t) \cdots f_m(t) | x^n \rangle = \sum \binom{n}{i_1, \dots, i_m} \left\langle f_1(t) | x^{i_1} \right\rangle \cdots \left\langle f_m(t) | x^{i_m} \right\rangle \tag{1.14}$$

where the sum is over all nonnegative integers $i_1, \dots, i_m$ such that $i_1 + \cdots + i_m = n$.

In this paper, we introduce new degenerate Bernoulli polynomials which are different Carlitz's degenerate Bernoulli polynomials and investigate some interesting properties of those polynomials.

2. Umbral calculus and degenerate Bernoulli polynomials

From (1.1) and (1.13), we have

$$B_n(x) \sim \left(\frac{e^t - 1}{t}, t \right), \quad (n \geq 0) . \tag{2.1}$$

Now, we introduce the new degenerate Bernoulli polynomials which are derived from Sheffer sequence as follows:

$$\beta_{n,\lambda}(x) \sim \left(\frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t}, t \right), \quad (n \geq 0) . \tag{2.2}$$

From (1.12) and (2.2), we have

$$\sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!} = \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} e^{xt}. \tag{2.3}$$

When $x = 0$, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers.

Note that

$$\begin{aligned} \sum_{n=0}^{\infty} \lim_{\lambda \to 0} \beta_{n,\lambda}(x) \frac{t^n}{n!} &= \lim_{\lambda \to 0} \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} e^{xt} \\ &= \frac{t}{e^t - 1} e^{xt} \\ &= \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \end{aligned} \tag{2.4}$$

Thus, by (2.4), we get

$$\lim_{\lambda \to 0} \beta_{n,\lambda}(x) = B_n(x), \quad (n \geq 0) . \tag{2.5}$$

From (2.3), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!} &= \left(\sum_{l=0}^{\infty} \beta_{l,\lambda} \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} \frac{x^m}{m!} t^m \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} \beta_{l,\lambda} x^{n-l} \right) \frac{t^n}{n!}. \end{aligned} \tag{2.6}$$

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Thus, by (2.6), we get

$$\beta_{n,\lambda}(x) = \sum_{l=0}^{n} \binom{n}{l} \beta_{l,\lambda} x^{n-l}, \quad (n \geq 0),$$

and

$$\begin{aligned} \frac{d}{dx}\beta_{n,\lambda}(x) &= \sum_{l=1}^{n} \binom{n}{l} \beta_{l,\lambda}(n-l)x^{n-l-1} \\ &= n\sum_{l=0}^{n-1} \binom{n-1}{l} \beta_{l,\lambda} x^{n-1-l} \\ &= n\beta_{n-1,\lambda}(x). \end{aligned} \tag{2.7}$$

From (1.11) and (2.3), we have

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} x^n = \beta_{n,\lambda}(x), \quad (n \geq 0), \tag{2.8}$$

and

$$t\beta_{n,\lambda}(x) = \frac{d}{dx}\beta_{n,\lambda}(x) = n\beta_{n-1,\lambda}(x), \quad (n \geq 1). \tag{2.9}$$

Thus, by (2.8) and (2.9), we get

$$\begin{aligned} &\int_{x}^{x+y} \beta_{n,\lambda}(u)\,du \\ &= \frac{1}{n+1}\left\{\beta_{n+1,\lambda}(x+y) - \beta_{n+1,\lambda}(x)\right\} \\ &= \frac{e^{yt}-1}{t}\beta_{n,\lambda}(x) \\ &= \sum_{k=1}^{\infty} \frac{y^k}{k!} t^{k-1} \beta_{n,\lambda}(x). \end{aligned} \tag{2.10}$$

From (2.9), we have

$$\beta_n(x) = t\left\{\frac{1}{n+1}\beta_{n+1,\lambda}(x)\right\}. \tag{2.11}$$

Thus, by (2.11), we get

$$\begin{aligned} \left\langle \frac{e^{yt}-1}{t} \middle| \beta_{n,\lambda}(x) \right\rangle &= \left\langle e^{yt}-1 \middle| \frac{1}{n+1}\beta_{n+1,\lambda}(x) \right\rangle \\ &= \int_0^y \beta_{n,\lambda}(u)\,du. \end{aligned} \tag{2.12}$$

Therefore, by (2.12), we obtain the following theorem.

Theorem 1. *For $n \geq 0$, we have*

$$\left\langle \frac{e^{yt}-1}{t} \middle| \beta_{n,\lambda}(x) \right\rangle = \int_0^y \beta_{n,\lambda}(u)\,du.$$

For $r \in \mathbb{N}$, the degenerate Bernoulli polynomials of order r are defined by the generating function

$$\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{r} e^{xt}=\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(r)}(x)\frac{t^{n}}{n!}. \tag{2.13}$$

When $x=0$, $\beta_{n,\lambda}^{(r)}=\beta_{n,\lambda}^{(r)}(0)$ are called the higher order degenerate Bernoulli numbers.

Indeed, $\lim_{\lambda\to 0}\beta_{n,\lambda}^{(r)}(x)=B_{n}^{(r)}(x)$, where $B_{n}^{(r)}(x)$ are the higher-order Bernoulli polynomials which are defined by the generating function

$$\left(\frac{t}{e^{t}-1}\right)^{r} e^{xt}=\sum_{n=0}^{\infty}B_{n}^{(r)}(x)\frac{t^{n}}{n!}.$$

From (2.13), we have

$$\beta_{n,\lambda}^{(r)}(x)=\sum_{l=0}^{n}\binom{n}{l}\beta_{l,\lambda}^{(r)}x^{n-l},\quad (n\geq 0), \tag{2.14}$$

and

$$\begin{aligned}\frac{d}{dx}\beta_{n,\lambda}^{(r)}(x)&=\sum_{l=0}^{n}\binom{n}{l}\beta_{l,\lambda}^{(r)}(n-l)x^{n-l-1}\\&=n\sum_{l=0}^{n-1}\binom{n-1}{l}\beta_{l,\lambda}^{(r)}x^{n-l-1}\\&=n\beta_{n-1,\lambda}^{(r)}(x),\quad (n\geq 1).\end{aligned} \tag{2.15}$$

By (2.8) and (2.13), we easily get

$$\beta_{n,\lambda}^{(r)}=\sum_{l_1+\cdots+l_r=n}\binom{n}{l_1,\ldots,l_r}\beta_{l_1,\lambda}\cdots\beta_{l_r,\lambda}. \tag{2.16}$$

Thus, by (2.14) and (2.16), we see that $\beta_{n,\lambda}^{(r)}(x)$ is a monic polynomial of degree n with coefficients in $\mathbb{Q}(\lambda)$.

From (2.14) and (2.15), we can derive

$$\begin{aligned}\int_{x}^{x+y}\beta_{n,\lambda}^{(r)}(u)\,du&=\frac{1}{n+1}\left\{\beta_{n+1,\lambda}^{(r)}(x+y)-\beta_{n+1,\lambda}^{(r)}(x)\right\}\\&=\frac{e^{yt}-1}{t}\beta_{n,\lambda}^{(r)}(x).\end{aligned} \tag{2.17}$$

If $s_n(x)\sim(g(t),t)$, then $s_n(x)$ is called an Appell sequence.

From (1.12) and (2.13), we have

$$\beta_{n,\lambda}^{(r)}(x)\sim\left(\left(\frac{(1+\lambda t)^{\frac{1}{\lambda}}-1}{t}\right)^{r},t\right),\quad (n\geq 0). \tag{2.18}$$

Thus, by (2.18), we note that $\beta_{n,\lambda}^{(r)}(x)$ is the Appell sequence for $\left(\frac{(1+\lambda t)^{\frac{1}{\lambda}}-1}{t}\right)^{r}$. From (2.18), we have

$$\left(\frac{(1+\lambda t)^{\frac{1}{\lambda}}-1}{t}\right)^{r}\beta_{n,\lambda}^{(r)}(x)\sim(1,t),\quad x^{n}\sim(1,t),\quad (n\geq 0). \tag{2.19}$$

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Thus, by (2.19), we get

$$x^n = \left(\frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t}\right)^r \beta_{n,\lambda}^{(r)}(x), \quad (n \geq 0). \tag{2.20}$$

We observe that

$$\begin{aligned} &\left(\frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t}\right)^r \\ &= \frac{1}{t^r}\left(e^{\frac{1}{\lambda}\log(1+\lambda t)} - 1\right)^r \\ &= \frac{1}{t^r} r! \sum_{l=r}^{\infty} S_2(l,r)\,\lambda^{-l}\frac{(\log(1+\lambda t))^l}{l!} \\ &= \frac{1}{t^r} r! \sum_{l=0}^{\infty} S_2(l+r,r)\,\lambda^{-(l+r)}\frac{1}{(l+r)!}(l+r)! \sum_{n=l+r}^{\infty} S_1(n,l+r)\frac{\lambda^n t^n}{n!} \\ &= \frac{1}{t^r} r! \sum_{l=0}^{\infty} S_2(l+r,r)\,\lambda^{-(l+r)} \sum_{n=l}^{\infty} S_1(n+r,l+r)\frac{\lambda^{n+r}}{(n+r)!}t^{n+r} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n} S_2(l+r,r)\,S_1(n+r,l+r)\,\lambda^{n-l}\frac{1}{\binom{n+r}{r}}\right)\frac{t^n}{n!} \end{aligned} \tag{2.21}$$

By (2.20) and (2.21), we get

$$x^m = \sum_{n=0}^{m}\sum_{l=0}^{n} S_2(l+r,r)\,S_1(n+r,l+r)\,\lambda^{n-l}\frac{\binom{m}{n}}{\binom{n+r}{r}}\beta_{m-n,\lambda}^{(r)}(x), \quad (m \geq 0). \tag{2.22}$$

Therefore, by (2.22), we obtain the following theorem.

Theorem 2. *For $m \geq 0$, we have*

$$x^m = \sum_{n=0}^{m}\sum_{l=0}^{n} S_2(l+r,r)\,S_1(n+r,l+r)\,\lambda^{n-l}\frac{\binom{m}{n}}{\binom{n+r}{r}}\beta_{m-n,\lambda}^{(r)}(x),$$

where $S_1(m,n)$ and $S_2(m,n)$ are the Stirling numbers of the first kind and of the second kind defined by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)\,x^l,$$
$$x^n = \sum_{l=0}^{n} S_2(n,l)\,(x)_l.$$

From (1.11) and (2.18), we have

$$t\beta_{n,\lambda}^{(r)}(x) = t\left\{\frac{1}{n+1}\beta_{n+1,\lambda}^{(r)}(x)\right\}, \quad (n \geq 0), \tag{2.23}$$

and

$$\left\langle \frac{e^{yt}-1}{t}\middle|\,\beta_{n,\lambda}^{(r)}(x)\right\rangle = \left\langle e^{yt}-1\middle|\,\frac{1}{n+1}\beta_{n+1,\lambda}^{(r)}(x)\right\rangle \tag{2.24}$$

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$$= \int_0^y \beta_{n,\lambda}^{(r)}(u)\,du.$$

Moreover,

$$\begin{aligned}(2.25)\quad \beta_{n,\lambda}^{(r)} &= \left\langle \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^r \middle| x^n \right\rangle \\ &= \sum_{n=i_1+\cdots+i_r} \binom{n}{i_1,\ldots,i_r} \left\langle \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \middle| x^{i_1} \right\rangle \cdots \left\langle \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \middle| x^{i_r} \right\rangle\end{aligned}$$

and

$$\beta_{n,\lambda} = \left\langle \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \middle| x^n \right\rangle, \quad (n \geq 0). \tag{2.26}$$

Therefore, by (2.24), (2.25) and (2.26), we obtain the following theorem.

Theorem 3. *For $n \geq 0$, we have*

$$\left\langle \frac{e^{yt}-1}{t} \middle| \beta_{n,\lambda}^{(r)}(x) \right\rangle = \int_0^y \beta_{n,\lambda}^{(r)}(u)\,du,$$

and

$$\beta_{n,\lambda}^{(r)} = \sum_{n=i_1+\cdots+i_r} \binom{n}{i_1,\ldots,i_r} \beta_{i_1,\lambda} \cdots \beta_{i_r,\lambda}.$$

Let $\mathbb{P}_n = \{ p(x) \in \mathbb{C}[x] |\, \deg p(x) \leq n \}, \quad (n \geq 0)$. For $p(x) \in \mathbb{P}_n$, we assume that

$$p(x) = \sum_{k=0}^{n} b_k \beta_{k,\lambda}(x). \tag{2.27}$$

From (2.2), we have

$$\left\langle \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} t^k \middle| \beta_{n,\lambda}(x) \right\rangle = n!\delta_{n,k}, \quad (n,k \geq 0). \tag{2.28}$$

Thus, by (2.27) and (2.28), we get

$$\begin{aligned}(2.29)\quad \left\langle \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} t^k \middle| p(x) \right\rangle &= \sum_{l=0}^{n} b_l \left\langle \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} t^k \middle| \beta_{l,\lambda}(x) \right\rangle \\ &= \sum_{l=0}^{n} b_l l! \delta_{l,k} = k! b_k.\end{aligned}$$

Hence,

$$\begin{aligned}(2.30)\quad b_k &= \frac{1}{k!} \left\langle \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} t^k \middle| p(x) \right\rangle \\ &= \frac{1}{k!} \left\langle \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} \middle| p^{(k)}(x) \right\rangle,\end{aligned}$$

where $p^{(k)}(x) = \frac{d^k}{dx^k} p(x)$.

Therefore, by (2.27) and (2.30), we obtain the following theorem.

Theorem 4. *Let $p(x) \in \mathbb{P}_n$. Then we have*

$$p(x) = \sum_{k=0}^{n} b_k \beta_{k,\lambda}(x),$$

where

$$b_k = \frac{1}{k!} \left\langle \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} \middle| p^{(k)}(x) \right\rangle.$$

Let $p(x) \in \mathbb{P}_n$ with $p(x) = \beta_{n,\lambda}^{(r)}(x)$. Then, we have

$$p^{(k)}(x) = \left(\frac{d}{dx}\right)^k \beta_{n,\lambda}^{(r)}(x) = k! \binom{n}{k} \beta_{n-k,\lambda}^{(r)}(x). \tag{2.31}$$

Let us assume that

$$p(x) = \beta_{n,\lambda}^{(r)}(x) = \sum_{k=0}^{n} b_k \beta_{k,\lambda}(x). \tag{2.32}$$

Then, by Theorem 5, we get

$$\begin{aligned} b_k &= \frac{1}{k!} \left\langle \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} \middle| p^{(k)}(x) \right\rangle \\ &= \binom{n}{k} \left\langle \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} \middle| \beta_{n-k,\lambda}^{(r)}(x) \right\rangle \\ &= \binom{n}{k} \left\langle \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} \middle| \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}\right)^r x^{n-k} \right\rangle \\ &= \binom{n}{k} \left\langle 1 \middle| \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}\right)^{r-1} x^{n-k} \right\rangle \\ &= \binom{n}{k} \beta_{n-k,\lambda}^{(r-1)}. \end{aligned} \tag{2.33}$$

Therefore, by (2.32) and (2.33), we obtain the following theorem.

Theorem 5. *For $r \in \mathbb{N}$ and $n \geq 0$, we have*

$$\beta_{n,\lambda}^{(r)}(x) = \sum_{k=0}^{n} \binom{n}{k} \beta_{n-k,\lambda}^{(r-1)} \beta_{k,\lambda}(x).$$

Let $p(x) \in \mathbb{P}_n$ with $p(x) = \sum_{k=0}^{n} b_k^{(r)} \beta_{k,\lambda}^{(r)}(x)$. By (2.18), we get

$$\begin{aligned} \left\langle \left(\frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t}\right)^r t^k \middle| p(x) \right\rangle &= \sum_{l=0}^{n} b_l^{(r)} \left\langle \left(\frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t}\right)^r t^k \middle| \beta_{l,\lambda}^{(r)}(x) \right\rangle \\ &= \sum_{l=0}^{n} b_l^{(r)} l! \delta_{l,k} = k! b_k^{(r)}. \end{aligned} \tag{2.34}$$

Thus, by (2.34), we get

$$b_k^{(r)} = \frac{1}{k!}\left\langle \left(\frac{(1+\lambda t)^{\frac{1}{\lambda}}-1}{t}\right)^r t^k \middle| p(x)\right\rangle. \tag{2.35}$$

Theorem 6. *For $p(x) \in \mathbb{P}_n$, we have*

$$p(x) = \sum_{k=0}^{n} b_k^{(r)} \beta_{k,\lambda}^{(r)}(x),$$

where

$$\begin{aligned} b_k^{(r)} &= \frac{1}{k!}\left\langle \left(\frac{(1+\lambda t)^{\frac{1}{\lambda}}-1}{t}\right)^r t^k \middle| p(x)\right\rangle \\ &= \frac{1}{k!}\left\langle \left(\frac{(1+\lambda t)^{\frac{1}{\lambda}}-1}{t}\right)^r \middle| p^{(k)}(x)\right\rangle, \end{aligned}$$

where $p^{(k)}(x) = \left(\frac{d}{dx}\right)^k p(x)$.

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Regularization Smoothing Approximation of Fuzzy Parametric Variational Inequality Constrained Stochastic Optimization

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Abstract. This work is motivated by the fact that very little is known about the fuzzy parametric variational inequalities constrained stochastic optimization problems in finite dimension real numeral spaces, which are studied more difficult because of the existence of random variable and fuzzified version. Based on the notion of quasi-Monte Carlo estimate and method of centres with entropic regularization, we develop a class of new regularization smoothing approximation approaches to discretize the stochastic optimization problem with continuous random variable, and construct a centre iterative algorithm for approximating the optimal solutions of the stochastic optimization problems. Further, we give some comprehensive convergence theorems of optimal solutions for the resulting optimization problem. Finally, a numerical illustration is analyzed.

Key Words and Phrases. Regularization smoothing approximation, fuzzy parametric variational inequality, Stochastic optimization problem, centre iterative algorithm with quasi-Monte Carlo estimate, comprehensive convergence.

AMS Subject Classification. 49J40, 65K05, 90C30, 90C33

1 Introduction

As all we know, mathematical program with equilibrium constraints is a constrained optimization problem in which the essential constraints are defined by a parametric variational inequality. This class of problems can be regarded as a generalization of a bilevel programming problem and it therefore plays an important role in many fields such as transportation, communication networks, structural mechanics, economic equilibrium, multilevel game, and mathematical programming itself. See, for example, [1–7] and the reference therein. Moreover, in order to describe the uncertainties, Monica [5] considered the Bochner integrability setting, a measure space of indices and use random fuzzy mappings, and presented random fixed point theorems with random fuzzy mappings, extensions of the ones with random data.

In this paper, we study approximation of optimal solutions for the following fuzzy parametric variational inequality constrained stochastic optimization problem in n-dimension real numeral set $\mathbb{R}^n$:

$$\begin{array}{ll} \min\limits_{x,y(\cdot)} & E_\omega[f(x,y(\omega),\omega)] \\ \text{s.t.} & x\in U\subset\mathbb{R}^n, \\ & y(\omega)\in C(x,\omega) \\ & \langle F(x,y(\omega),\omega), z(\omega)-y(\omega)\rangle \gtrapprox 0, \text{ for all } z(\omega)\in C(x,\omega) \text{ and a.e. } \omega\in\Omega, \end{array} \tag{1.1}$$

where E_ω denotes the mathematical expectation with respect to the random variable $\omega\in\Omega$ on probability space $(\Omega,\mathcal{A},\Gamma)$, $f:\mathbb{R}^{n+m}\times\Omega\to\mathbb{R}$ and $F:\mathbb{R}^{n+m}\times\Omega\to\mathbb{R}^m$ are two nonlinear random functions, $C:\mathbb{R}^n\times\Omega\to 2^{\mathbb{R}^m}$ is a multi-valued random function, $\langle F(x,y(\omega),\omega), z(\omega)-y(\omega)\rangle\gtrapprox 0$ are fuzzy inequalities (also called fuzzy stochastic variational inequality problems, in short, $\widetilde{VI_\omega}$), "$\gtrapprox$"

denotes the fuzzified version of "$\geq$" with the linguistic interpretation "approximately greater than or equal to", and "a.e." is the abbreviation for almost everywhere".

Remark 1.1. Problem (1.1) is brand new in the literature including and can be thought as a generalized version of some problems, includes a number of stochastic mathematical program with equilibrium constraints (SMPEC), mathematical programs with fuzzy equilibrium constraints (MPFEC) mathematical program with equilibrium constraints (MPEC) and mathematical program with complementarity constraints (MPCC) have been studied by many authors as special cases. See, for example, [1–4, 6, 8–14] and the references therein, and the following examples.

Example 1.1 If Ω is a singleton, then problem (1.1) reduces to the following MPFEC:

$$\begin{array}{ll} \min & g(x,u) \\ \text{s.t.} & x \in U, \\ & u \text{ solves } \widetilde{VI}\,(G(x,\cdot), D(x)), \end{array} \tag{1.2}$$

where $g : \mathbb{R}^{n+m} \to \mathbb{R}$ and $G : \mathbb{R}^{n+m} \to \mathbb{R}^m$ is a continuously differentiable function, $D : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ is a set valued function, and u solves $\widetilde{VI}(G(x,\cdot), D(x))$ if and only if $u \in D(x)$ and $\langle G(x,u), z-u\rangle \gtrsim 0$ for all $z \in D(x)$. Problem (1.2) was introduced and studied by Hu and Liu [12] and Lan et al. [13]. Moreover, Hu and Liu [12] pointed out "although a powerful theory has been developed for variational inequalities, the parameterized setting in MPEC makes these problems very difficult to solve, and due to the vagueness involved in real world problems, the MPEC problem in a fuzzy environment becomes an important problem both in theory and in practice", and "problem (1.2) is a constrained optimization problem whose constraints include some fuzzy parametric variational inequalities.

In 2013, inspired by the works of Hu and Liu [12] and other researchers, Lan et al. [13] constructed an iterative algorithm for finding a solution of a class of mathematical program problems with fuzzy parametric variational inequality constraints by using a new smoothing approach based on a version of the method of centres with entropic regularization techniques. In fact, the tolerance approach and entropic regularization technique have been successfully proposed in solving various problems, which are important numerical methods for solving fuzzy variational inequalities in a fuzzy environment and nonlinear semi-infinite programming problems. See, for example, [3, 8, 14–21] and the references therein.

Example 1.2. Since a solution satisfying a fuzzy inequality system to a membership degree close to 1 is a near optimal solution to the corresponding regular inequality problem [22], if $y(\omega) \equiv v$ for all $\omega \in \Omega$ and the degree for the fuzzy inequalities in (1.1) is close to 1, then problem (1.1) is equivalent to the following SMPEC:

$$\begin{array}{ll} \min & E_\omega[f(x,v,\omega)] \\ \text{s.t.} & x \in U,\, \omega \in \Omega \\ & v \text{ solves } VI(F(x,\cdot,\omega), C(x,\omega)), \end{array} \tag{1.3}$$

where $VI(F(x,\cdot,\omega), C(x,\omega))$ denotes the variational inequality problem defined by the pair $(F(x,\cdot,\omega), C(x,\omega))$ for all $x \in \mathbb{R}^n$ and $\omega \in \Omega$. In 2003, Lin et al. [9] considered problem (1.3) and showed that SMPEC can be thought as a generalization of MPEC, and proposed a smoothing implicit programming method to establish a comprehensive convergence theory for the lower-level wait-and-see model. Further, there are many stochastic formulations of MPEC proposed in the recent discussions. For related works, we refer readers to [1, 3, 6, 8, 10, 11]. However, there has been very little study on applications of these theories and approaches to (1.1).

Over years of development, optimization approaches have become one of the most promising techniques for engineering applications and an MPEC is a hard problem because its constraints fail to satisfy a standard constraint qualification at any feasible point [23]. However, since the existence of the random variable ω and the fuzzified version "$\gtrsim$" mean that (1.1) involves multiple complementarity-type constraints, it is more difficult to solve problem (1.1) than to solve an ordinary MPCC, MPEC, MPFEC or SMPEC generally. Therefore, our focus in this paper is to develop a class of new regularization smoothing approximation approaches to define some parameters of the objective function fuzzy yielded by fuzzy constraints, and consider the approximation-solvability for an equivalent stochastic parametric optimization problem of problem (1.1).

Motivated and inspired by the above works, we shall give some preliminaries needed throughout the whole paper in Section 2. Specially, by using the notion of tolerance approach and the fuzzy set theory, we show that the fuzzy parametric variational inequality constrained stochastic optimization problem (1.1) and a fuzzy complementarity constrained optimization problem can be converted to a regular nonlinear parametric optimization problem. In Sections 3, we will construct a centre iterative algorithm and develop a class of new regularization smoothing approximation approach for solving the stochastic fuzzy optimization based on quasi-Monte Carlo estimate, and establish comprehensive convergence theorems of the solution. We also report some numerical simulation analysis results in Section 4.

2 Preliminaries

Throughout in this paper, we assumption that $(\Omega, \mathcal{A}, \Gamma)$ is a complete σ-finite measure space and the probability measure Γ of our considered space $(\Omega, \mathcal{A}, \Gamma)$ is non-atomic. Let $\mathcal{B}(\mathbb{R}^m)$ be the class of Borel σ-fields in $\mathbb{R}^m$ and $P(U)$ denote the power set of a vector space U.

Definition 2.1. (i) A function $y : \Omega \to \mathbb{R}^m$ is said to be measurable, if for any $B \in \mathcal{B}(\mathbb{R}^m)$, $\{\omega \in \Omega : y(\omega) \in B\} \in \mathcal{A}$).

(ii) The multi-valued function $\Psi : \Omega \to P(U)$ is called said to be measurable, if for any $B \in \mathcal{B}(U)$, $\Psi^{-1}(B) = \{\omega \in \Omega : \Psi(\omega) \cap B \neq \emptyset\} \in \mathcal{A}$.

(iii) A multi-valued random function $\Phi : \mathbb{R}^n \times \Omega \to 2^{\mathbb{R}^m}$ is said to be measurable, if for any $x \in \mathbb{R}^n$, $\Phi(x, \cdot)$ is measurable.

(iv) $F : \mathbb{R}^{n+m} \times \Omega \to \mathbb{R}^m$ is called a random and continuously differentiable function, when $F(x, z, \omega) = \zeta(\omega)$ is measurable for any $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$, and $F(\cdot, \cdot, \omega)$ is continuously differentiable for all $\omega \in \Omega$.

Definition 2.2. Let $C, C^* : \mathbb{R}^n \times \Omega \to 2^{\mathbb{R}^m}$ be two multi-valued random function. Then

(i) $C(x, \omega)$ is said to be convex cone, if $C(x, \cdot)$ is convex cone for every $x \in \mathbb{R}^n$, that is,

$$\alpha y(\cdot) + \beta w(\cdot) \in C(x, \cdot) \text{ for any positive scalars } \alpha, \beta \text{ and all measurable function } y(\cdot), w(\cdot) \in C(x, \cdot);$$

(ii) $C^*(x, \omega)$ is called polar (dual) cone of $C(x, \omega) \subset \mathbb{R}^m$ for $x \in \mathbb{R}^n$ and $\omega \in \Omega$, if $C^*(x, \cdot)$ is polar (dual) cone for every $x \in \mathbb{R}^n$, i.e.

$$\langle \xi, \nu(\cdot) \rangle \geq 0 \quad \forall \xi \in \mathbb{R}^m \text{ and for each measurable function } \nu(\cdot) \in C(x, \cdot).$$

In other words, the polar (dual) cone $C^*(x, \omega)$ can be expressed as follows:

$$C^*(x, \omega) = \{\xi \in \mathbb{R}^m \| \langle \xi, \nu(\omega) \rangle \geq 0 \quad \forall \nu(\omega) \in C(x, \omega)\}.$$

Definition 2.3. Let $\sigma > 0$ and $\varsigma > 0$ be constants. a function $M : \mathbb{R}^n \to \mathbb{R}^m$ is said to be Hölder continuous on $K \subset \mathbb{R}^m$ with order σ and Hölder constant ς if

$$\|M(u) - M(v)\| \leq \varsigma \|u - v\|^\sigma, \quad \forall u, v \in K.$$

holds for all u and v in K.

Remark 2.1. If $\sigma = 1$, then the definition of Hölder continuity reduces to definition of Lipschitz continuity. We note that for two different positive numbers σ and σ', Hölder continuous functions with order σ and those with order σ' constitute different subclasses. For example, the function $M(u) := \sqrt{\|u\|}$ for all $u \in K \subset \mathbb{R}^m$ is Hölder continuous with order $\sigma = \frac{1}{2}$, but not Lipschitz continuous.

In the sequel, we give some preparations needed later to approximating the optimal solutions of problem (1.1). First, we propose discretization of the stochastic objective function in (1.1) with continuous random variable.

Lemma 2.1. Let $\zeta : \Omega \to [0, +\infty)$ be the continuous probability density function of ω. Then the objective function in (1.1) can be represented as

$$E_\omega(f(x, y(\omega), \omega) = \frac{1}{L} \sum_{\omega \in \Omega_L} f(x, y(\omega), \omega)\zeta(\omega), \tag{2.1}$$

where $\Omega_L := \{\omega_1, \omega_2, \cdots, \omega_L\}$ is a uniformly distributed sample set from Ω.

Proof. Let Ω be a sample space, which is usually denoted using set notation, and the possible outcomes are listed as elements in the set. If Ω is unbounded, under some mild conditions, we can approximate the problem by a sequence of programs with bounded sampling spaces (see [11]) for more details. In the sequel, let Ω be a bounded rectangle. In particular, without loss of generality, we assume that $\Omega = [0, 1]^\kappa$. Let $\zeta : \Omega \to [0, +\infty)$ be the continuous probability density function of ω. Then the objective function in (1.1) can be represented as

$$E_\omega(f(x, y(\omega), \omega) = \int_\Omega f(x, y(\omega), \omega)\zeta(\omega)d\omega.$$

Based on quasi-Monte Carlo method in [24], now we estimate numerical integration to the objective function in problem (1.1). Roughly speaking, given a function $\phi : \Omega \to \mathbb{R}$, the quasi-Monte Carlo estimate for $E_\omega[\phi(\omega)]$ is obtained by taking a uniformly distributed sample set $\Omega_L := \{\omega_1, \omega_2, \cdots, \omega_L\}$ from Ω and letting $E_\omega[\phi(\omega)] \approx \frac{1}{L}\sum_{\omega\in\Omega_L} \phi(\omega)$. This implies that (2.1) holds. □

Next, we consider the random membership functions of each fuzzy stochastic inequality and stochastic fuzzy objective yielded by the fuzzy constraints in (1.1).

Let the membership function for each fuzzy stochastic inequality $\langle F(x, y(\omega), \omega), z - y(\omega)\rangle \gtrsim 0$ as follows: for all $x \in \mathbb{R}^n$ and any $z \in C(x, \omega)$,

$$\mu_{\tilde{\Omega}_z}(x, y(\omega), \omega) = \begin{cases} 1, & \text{if } \langle F(x, y(\omega), \omega), z - y(\omega)\rangle \geq 0, \\ \mu_z(\langle F(x, y(\omega), \omega), z - y(\omega)\rangle), & \text{if } \langle F(x, y(\omega), \omega), z - y(\omega)\rangle \in [-t_z, 0), \\ 0, & \text{if } \langle F(x, y(\omega), \omega), z - y(\omega)\rangle < -t_z, \end{cases} \tag{2.2}$$

specify the degree to which the regular inequality $\langle F(x, y(\omega), \omega), z-y(\omega)\rangle \geq 0$ is satisfied, where $\tilde{\Omega}_z$ is a fuzzy set actually determined by the fuzzy stochastic inequality in $\mathbb{R}^{n+m} \times \Omega$, $t_z \geq 0$ is the tolerance level which can be tolerated by decision makers in the accomplishment of the fuzzy stochastic inequality $\langle F(x, y(\omega), \omega), z - y(\omega)\rangle \gtrsim 0$. We usually assume that $\mu_z(\langle F(x, y(\omega), \omega), z - y(\omega)\rangle) \in [0, 1]$ and it is continuous and strictly increasing over $[-t_z, 0)$. Fig. 1 shows different shapes of such membership fu

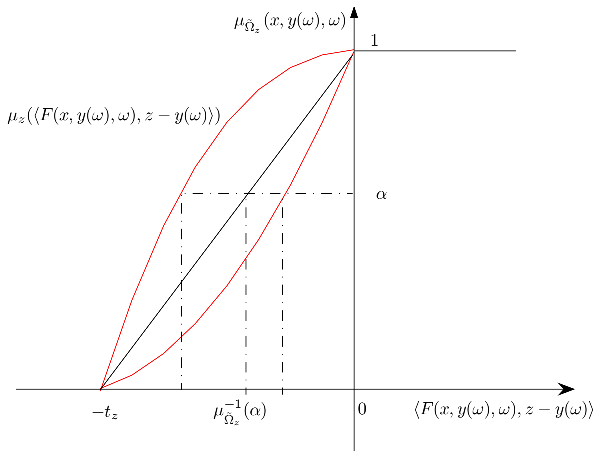

Figure 1: The membership function $\mu_{\tilde{\Omega}_z}(x, y(\omega), \omega)$.

Similarly, the random membership function of the objective, $\mu_{\tilde{S}_0}(x, y(\omega), \omega)$, is defined as follows:

$$\mu_{\tilde{S}_0}(x, y(\omega), \omega) = \begin{cases} 1, & \text{if } E_\omega[f(x, y(\omega), \omega)] < \underline{f}, \\ \mu_0(E_\omega[f(x, y(\omega), \omega)]), & \text{if } E_\omega[f(x, y(\omega), \omega)] \in [\underline{f}, \bar{f}), \\ 0, & \text{if } E_\omega[f(x, y(\omega), \omega)] \geq \bar{f}, \end{cases} \tag{2.3}$$

where $\underline{f}$ and $\bar{f}$ are two parameters defined as follows:

$$\begin{aligned} \bar{f} = \min \quad & E_\omega[f(x, y(\omega), \omega)] \\ \text{s.t.} \quad & x \in U,\, \omega \in \Omega, \\ & \langle F(x, y(\omega), \omega), z - y(\omega)\rangle \geq 0, \quad \forall z \in C(x, \omega) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \underline{f} = \min \quad & E_\omega[f(x, y(\omega), \omega)] \\ \text{s.t.} \quad & x \in U,\, \omega \in \Omega, \\ & \langle F(x, y(\omega), \omega), z - y(\omega)\rangle \geq -t_z, \quad \forall z \in C(x, \omega). \end{aligned} \tag{2.5}$$

By [22, 25], one can know that studying such a problem (1.1) is related to finding "almost optimal" solutions for a general convex minimization problem (see also [13, 14, 17]). Thus, we extend the idea and have the following result.

Lemma 2.2. Let $C(x, \omega)$ be a convex cone for all $x \in \mathbb{R}^n$ and $\omega \in \Omega$. Then the problem $\widetilde{VI_\omega}$, i.e., finding $y(\omega) \in C(x, \omega)$ such that

$$\langle F(x, y(\omega), \omega), z(\omega) - y(\omega)\rangle \gtrsim 0, \quad \forall z(\omega) \in C(x, \omega), \tag{2.6}$$

is equivalent to the fuzzy complementarity problem of finding $y(\omega) \in \mathbb{R}^m$ such that

$$y(\omega) \in C(x, \omega), \quad \langle F(x, y(\omega), \omega), y(\omega)\rangle \overset{=}{\sim} 0, \quad F(x, y(\omega), \omega) \overset{\in}{\sim} C^*(x, \omega), \tag{2.7}$$

where "$\overset{=}{\sim}$" denotes the fuzzified version of "=" with the linguistic interpretation "approximately equal to", "$\overset{\in}{\sim}$" denotes the fuzzified version of "$\in$" with the linguistic interpretation "approximately in" and $C^*(x, \omega)$ is a polar (dual) cone of $C(x, \omega) \subset \mathbb{R}^m$ for all $x \in \mathbb{R}^n$ and $\omega \in \Omega$.

Proof. We start by showing that problem (2.7)$\subset$ problem (2.6). For any $x \in \mathbb{R}^n$ and $\omega \in \Omega$, suppose that $y^*(\omega)$ is a solution of problem (2.7), then we have

$$\langle F(x, y^*(\omega), \omega), y^*(\omega)\rangle \overset{=}{\sim} 0 \tag{2.8}$$

and

$$\langle F(x, y^*(\omega), \omega), \upsilon(\omega)\rangle \gtrsim 0, \quad \forall \upsilon(\omega) \in C(x, \omega). \tag{2.9}$$

Combining (2.8) and (2.9), we have $\langle F(x, y^*(\omega), \omega), \upsilon(\omega) - y^*(\omega)\rangle \gtrsim 0$ for all $\upsilon(\omega) \in C(x, \omega)$. Thus, $y^*(\omega)$ is also a solution of problem (2.6) for all $\omega \in \Omega$.

Now we show that problem (2.6) $\subset$ problem (2.7). Let $y^*(\omega)$ be the solution of problem (2.6) with the membership degree $\alpha \in [0, 1]$ for every $\omega \in \Omega$. According to the tolerance approach [15, 21], by (2.2), we have

$$\langle F(x, y^*(\omega), \omega), \upsilon(\omega) - y^*(\omega)\rangle \geq \mu^{-1}_{\tilde{\Omega}_z}(\alpha) \geq -t_z, \quad \forall \upsilon(\omega) \in C(x, \omega), \tag{2.10}$$

where for all $\upsilon(\omega) \in C(x, \omega)$ and any $\omega \in \Omega$, $\mu^{-1}_{\tilde{\Omega}_z}$ is the inverse functions of $\mu_{\tilde{\Omega}_z}(x, \cdot, \omega)$ and $t_z > 0$ is the tolerance level which a decision maker can tolerate in the accomplishment of the fuzzy inequality $\langle F(x, y(\omega), \omega), \upsilon(\omega) - y(\omega)\rangle \gtrsim 0$. Suppose that for $\bar{t}_z \geq 0$ and $\hat{t}_z < 0$, either

$$\langle F(x, y^*(\omega), \omega), y^*(\omega)\rangle > \bar{t}_z \quad \text{or} \quad \langle F(x, y^*(\omega), \omega), y^*(\omega)\rangle < \hat{t}_z$$

is true. For any $x \in \mathbb{R}^n$ and each $\omega \in \Omega$, since $C(x, \omega)$ is a convex cone, we have $\langle F(x, y^*(\omega), \omega), y^*(\omega)\rangle \geq \frac{-t_z}{\lambda - 1}$, $t_z \geq 0$ when $\upsilon(\omega) = \lambda y^*(\omega)$ with $\lambda > 1$, and $\langle F(x, y^*(\omega), \omega), y^*(\omega)\rangle \leq t_z$, when $\upsilon(\omega) = 0$. If

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$\langle F(x,y^*(\omega),\omega),y^*(\omega)\rangle > \bar{t}_z$ for $\bar{t}_z \geq 0$, This leads to a contradiction. If $\langle F(x,y^*(\omega),\omega),y^*(\omega)\rangle < \hat{t}_z$ for $\hat{t}_z < 0$, then $t_z \leq \hat{t}_z$. There lies a contradiction. Therefore,

$$\hat{t}_z \leq \langle F(x,y^*(\omega),\omega),y^*(\omega)\rangle \leq \bar{t}_z,$$

for $\bar{t}_z \geq 0$ and $\hat{t}_z < 0$, that is, $\langle F(x,y^*(\omega),\omega),y^*(\omega)\rangle \mathrel{\underset{\sim}{=}} 0$. Furthermore, from (2.10), we have for any $v(\omega) \in C(x,\omega)$,

$$\langle F(x,y^*(\omega),\omega),v(\omega)\rangle \geq \langle F(x,y^*(\omega),\omega),y^*(\omega)\rangle - t_z \geq \hat{t}_z - t_z.$$

This implies that $\langle F(x,y^*(\omega),\omega),v(\omega)\rangle \mathrel{\underset{\sim}{\geq}} 0$ for all $v(\omega) \in C(x,\omega)$. Hence, we have $F(x,y^*(\omega),\omega) \mathrel{\underset{\sim}{\in}} C^*(x,\omega)$. Therefore, $y^*(\omega)$ for any $\omega \in \Omega$ is also a solution of problem (2.7). This completes the proof. □

Based on Lemma 2.2 and the work of [21], we have the following results.

Lemma 2.3. Let $C(x,\omega)$ is a convex cone with polar (dual) cone $C^*(x,\omega)$ for all $(x,\omega) \in \mathbb{R}^n \times \Omega$ and α be a new variable. Then the stochastic optimization problem (1.1) can eventually be expressed as the following regular semi-infinite optimization problem with finitely many variables $x, y(\omega), \omega$ and α:

$$\begin{array}{ll} \max & \alpha \\ \text{s.t.} & \mu_{\tilde{S}_0}(x,y(\omega),\omega) \geq \alpha, \\ & \mu_{\tilde{\Omega}_z}(x,y(\omega),\omega) \geq \alpha, \\ & (x,y(\omega),\omega) \in S, \\ & 0 \leq \alpha \leq 1, \end{array} \tag{2.11}$$

where $S = \{(x,y(\omega),\omega) \in \mathbb{R}^{n+m} \times \Omega |\, x \in U,\, \omega \in \Omega,\, F(x,y(\omega),\omega) \mathrel{\underset{\sim}{\in}} C^*(x,\omega)\}$, and the random membership functions $\mu_{\tilde{S}_0}$ and $\mu_{\tilde{\Omega}_z}$ are the same as in (2.3) and (2.2), respectively.

Proof. It follows from Lemma 2.1 that, in order to find a solution to the stochastic optimization problem (1.1) with $C(x,\omega)$ being a convex cone for $(x,\omega) \in \mathbb{R}^n \times \Omega$, we should consider the following stochastic fuzzy complementarity constrained optimization problem:

$$\begin{array}{ll} \min & E_\omega[f(x,y(\omega),\omega)] \\ \text{s.t.} & x \in U,\, \omega \in \Omega,\, y(\omega) \in C(x,\omega), \\ & \langle F(x,y(\omega),\omega),y(\omega)\rangle \mathrel{\underset{\sim}{=}} 0, \\ & F(x,y(\omega),\omega) \mathrel{\underset{\sim}{\in}} C^*(x,\omega). \end{array} \tag{2.12}$$

Since a global minimum is often required for practical problems, by the work of [21] and the description of the fuzzy stochastic inequalities (2.6), and a solution of problem (2.12) can be taken as the solution with the highest membership in the fuzzy decision set and eventually obtained by solving the following regular nonlinear parametric optimization problem:

$$\max_{(x,y(\omega),\omega)\in S} \min\left\{\mu_{\tilde{S}_0}(x,y(\omega),\omega), \mu_{\tilde{\Omega}_z}(x,y(\omega),\omega)\right\},$$

which implies that the result holds for the new variable α. □

Remark 2.2. Moreover, if the membership functions $\mu_{\tilde{S}_0}$ and $\mu_{\tilde{\Omega}_z}$ in Lemma 2.3 are invertible, then from (2.11), we get

$$\begin{array}{ll} \max & \alpha \\ \text{s.t.} & (x,y(\omega),\omega)_0 \geq \mu^{-1}_{\tilde{S}_0}(\alpha), \\ & (x,y(\omega),\omega)_{C^*} \geq \mu^{-1}_{\tilde{\Omega}_z}(\alpha), \\ & (x,y(\omega),\omega) \in S, \\ & 0 \leq \alpha \leq 1, \end{array} \tag{2.13}$$

where $(x,y(\omega),\omega)_0$ and $(x,y(\omega),\omega)_{C^*}$ can be followed by (2.3) and (2.2), respectively.

3 Regularization smoothing approximation algorithms

In this section, based on the "method of centres" with entropic regularization, we develop a class of new smoothing approach and construct a centre iterative algorithm for solving the stochastic fuzzy optimization (1.1), and give the solution theorems.

In the sequel, we first give the following assumption ($\mathbf{H}_C$) for convenience: Define

$$\begin{array}{rl} C(x,\omega) := & \{v(\omega)\in\mathbb{R}^m \,|\, D(x)v(\omega)\geq 0,\, D(x)=[d_i(x)] \text{ is an } l\times m \\ & \text{matrix}, d_i(x) \text{ is the } i\text{th row of } D(x),\ \forall i=1,2,\cdots,l\}. \end{array} \tag{3.1}$$

STEP I. The random membership function of the fuzzy stochastic inequalities in (1.1) can be specified under condition ($\mathbf{H}_C$).

From (3.1), it is easy to see that the multi-valued operator $C(x,\omega)$ is a convex cone for any $(x,\omega)\in\mathbb{R}^n\times\Omega$, and can be shown that $F(x,y(\omega),\omega)\in C^*(x,\omega)$ if and only if there exists a nonnegative random vector $r(\omega)=(r_1(\omega),r_2(\omega),\cdots,r_l(\omega))^T\in\mathbb{R}^l$ such that

$$F(x,y(\omega),\omega)=r_1(\omega)d_1^T(x)+r_2(\omega)d_2^T(x)+\cdots+r_l(\omega)d_l^T(x)=D^T(x)r(\omega), \tag{3.2}$$

that is, for every $i=1,2,\cdots,l$, $d_i'(x)F(x,y(\omega),\omega)\geq 0$, where $d_i'(x)$ is normal to $d_i(x)$ (see [17, 26]).

It follows that the fuzzy stochastic optimization problem (2.12) can be rewritten as the following generalized stochastic optimization problem with fuzzy stochastic inequality constraints:

$$\begin{array}{ll} \min & E_\omega[f(x,y(\omega),\omega)] \\ \text{s.t.} & x\in U,\ \omega\in\Omega, \\ & d_i(x)y(\omega)\geq 0,\ i=1,2,\cdots,l, \\ & \langle F(x,y(\omega),\omega),y(\omega)\rangle \gtrsim 0, \\ & \langle -F(x,y(\omega),\omega),y(\omega)\rangle \gtrsim 0, \\ & d_i'(x)F(x,y(\omega),\omega)\gtrsim 0,\ i=1,2,\cdots,l, \end{array} \tag{3.3}$$

and each fuzzy stochastic inequality in (3.3) can be represented by a fuzzy set $\tilde{S}_j$ (i.e., represent of $\tilde{\Omega}_z$ in (2.11) or (2.13)) with corresponding random membership function $\mu_{\tilde{S}_j}(x,y(\omega),\omega)$ for $j=1,2,\cdots,l+2$. To specify the membership functions $\mu_{\tilde{S}_j}$, $j=1,2,\cdots,l+2$, similar treatment to (2.2), we define the membership functions as follows:

$$\begin{array}{l} \mu_{\tilde{S}_1}(x,y(\omega),\omega)=\begin{cases} 1, & \text{if } \langle F(x,y(\omega),\omega),y(\omega)\rangle\geq 0, \\ \mu_1(\langle F(x,y(\omega),\omega),y(\omega)\rangle), & \text{if } \langle F(x,y(\omega),\omega),y(\omega)\rangle\in[-t_1,0), \\ 0, & \text{if } \langle F(x,y(\omega),\omega),y(\omega)\rangle< -t_1, \end{cases} \\ \mu_{\tilde{S}_2}(x,y(\omega),\omega)=\begin{cases} 1, & \text{if } \langle -F(x,y(\omega),\omega),y(\omega)\rangle\geq 0, \\ \mu_2(\langle -F(x,y(\omega),\omega),y(\omega)\rangle), & \text{if } \langle -F(x,y(\omega),\omega),y(\omega)\rangle\in[-t_2,0), \\ 0, & \text{if } \langle -F(x,y(\omega),\omega),y(\omega)\rangle< -t_2, \end{cases} \\ \mu_{\tilde{S}_{i+2}}(x,y(\omega),\omega)=\begin{cases} 1, & \text{if } d_i'F(x,y(\omega),\omega)\geq 0, \\ \mu_{i+2}(d_i'F(x,y(\omega),\omega)), & \text{if } d_i'F(x,y(\omega),\omega)\in[-t_{i+2},0), \\ 0, & \text{if } d_i'F(x,y(\omega),\omega)< -t_{i+2}, \end{cases} \end{array} \tag{3.4}$$

where $t_{i+2}\geq 0$ for $i=1,2,\cdots,l$, is the tolerance level which one decision maker can tolerate in the accomplishment of the fuzzy stochastic inequalities in (3.3).

STEP II. Discrete approximation of problem (1.1) with condition ($\mathbf{H}_C$) need to be given.

By Lemma 2.1 and **STEP I**, we have the following problem as an appropriate discrete approximation of problem (3.3):

$$\begin{array}{ll} \min & \frac{1}{L}\sum\limits_{\omega\in\Omega_L} f(x,y(\omega),\omega)\zeta(\omega) \\ \text{s.t.} & x\in U\subset\mathbb{R}^n,\ \omega\in\Omega_L, \\ & d_i(x)y(\omega)\geq 0,\ i=1,2,\cdots,l, \\ & \langle F(x,y(\omega),\omega),y(\omega)\rangle \gtrsim 0, \\ & \langle -F(x,y(\omega),\omega),y(\omega)\rangle \gtrsim 0, \\ & d_i'(x)F(x,y(\omega),\omega)\gtrsim 0,\ i=1,2,\cdots,l. \end{array} \tag{3.5}$$

We note that the sample set Ω_L is chosen to be asymptotically dense in Ω. Especially, it follows from (2.4), (2.5) and (3.5) that the appropriate discrete approximation of two parameter $\bar{f}$ and $\underline{f}$ can be shown as follows, respectively:

$$\begin{array}{rl} \bar{f} = \min & \frac{1}{L} \sum\limits_{\omega\in\Omega_L} f(x, y(\omega), \omega)\zeta(\omega) \\ \text{s.t.} & x \in U \subset \mathbb{R}^n,\ \omega \in \Omega_L, \\ & d_i(x)y(\omega) \geq 0,\ i = 1, 2, \cdots, l, \\ & \langle F(x, y(\omega), \omega), y(\omega)\rangle \geq 0, \\ & \langle -F(x, y(\omega), \omega), y(\omega)\rangle \geq 0, \\ & d_i'(x)F(x, y(\omega), \omega) \geq 0,\ i = 1, 2, \cdots, l \end{array} \tag{3.6}$$

and

$$\begin{array}{rl} \underline{f} = \min & \frac{1}{L} \sum\limits_{\omega\in\Omega_L} f(x, y(\omega), \omega)\zeta(\omega) \\ \text{s.t.} & x \in U \subset \mathbb{R}^n,\ \omega \in \Omega_L, \\ & d_i(x)y(\omega) \geq 0,\ i = 1, 2, \cdots, l, \\ & \langle F(x, y(\omega), \omega), y(\omega)\rangle \geq -t_1, \\ & \langle -F(x, y(\omega), \omega), y(\omega)\rangle \geq -t_2, \\ & d_i'(x)F(x, y(\omega), \omega) \geq -t_{i+2},\ i = 1, 2, \cdots, l. \end{array} \tag{3.7}$$

STEP III. A new centre iterative method for solving problem (3.8) should be adopt.

It follows from (2.13), (2.3) and (3.4)-(3.7) that an optimal solution of the stochastic optimization problem (1.1) can be obtained by approximating for the following stochastic parametric optimization problem:

$$\begin{cases} \max & \alpha \\ \text{s.t.} & \mu_{\tilde{S}_0}^{-1}(\alpha) - \frac{1}{L} \sum\limits_{\omega\in\Omega_L} f(x, y(\omega), \omega)\zeta(\omega) \geq 0, \\ & \mu_{\tilde{S}_1}^{-1}(\alpha) - \langle F(x, y(\omega), \omega), y(\omega)\rangle \geq 0, \\ & \mu_{\tilde{S}_2}^{-1}(\alpha) + \langle F(x, y(\omega), \omega), y(\omega)\rangle \geq 0, \\ & -\mu_{\tilde{S}_j}^{-1}(\alpha) + d_{j-2}'F(x, y(\omega), \omega) \geq 0, \quad j = 3, 4, \cdots, l+2, \\ & d_i(x)y(\omega) \geq 0, \quad i = 1, 2, \cdots, l, \\ & 0 \leq \alpha \leq 1, \quad x \in U,\ \omega \in \Omega_L. \end{cases} \tag{3.8}$$

It is interested in developing an efficient algorithm to solve (3.8) based on a framework of centre iterations. This iterative approach can be traced back to Huard's work [28]. The basic concepts are easy to understand and very adaptive to new developments. To describe the approach, we denote the feasible domain of (3.8) by a set V and define some terminologies. A general assumption for this approach is that V is bounded and convex, and the interior of V is nonempty.

Definition 3.1 For any given point $(x, y(\omega), \omega, \alpha)$ in the convex domain V, we define the "distance $\mathcal{L}$ of $(x, y(\omega), \omega, \alpha)$ to the boundary of V" by a continuous function

$$\begin{aligned} \mathcal{L}((x, y(\omega), \omega, \alpha), V) \quad &= \min_{\substack{i=1,2,\cdots,l \\ j=3,4,\cdots,l+2}} \Big\{\alpha,\ 1-\alpha,\ \mu_{\tilde{S}_0}^{-1}(\alpha) - \frac{1}{L}\sum_{\omega\in\Omega_L} f(x, y(\omega), \omega)\zeta(\omega), \\ &\qquad \mu_{\tilde{S}_1}^{-1}(\alpha) - \langle F(x, y(\omega), \omega), y(\omega)\rangle, \mu_{\tilde{S}_2}^{-1}(\alpha) + \langle F(x, y(\omega), \omega), y(\omega)\rangle, \\ &\qquad -\mu_{\tilde{S}_j}^{-1}(\alpha) + d_{j-2}'F(x, y(\omega), \omega),\ d_i(x)y(\omega)\Big\}. \end{aligned}$$

Definition 3.2 Let a distance function $\mathcal{L}((x, y(\omega), \omega, \alpha), V)$ be defined on a convex domain V. Then a point $(\bar{x}, \bar{y}(\omega), \omega, \bar{\alpha}) \in V$ is called the "centre of V", if it maximizes the distance function $\mathcal{L}((x, y(\omega), \omega, \alpha), V)$, i.e.,

$$(\bar{x}, \bar{y}(\omega), \omega, \bar{\alpha}): \quad \mathcal{L}((\bar{x}, \bar{y}(\omega), \omega, \bar{\alpha}), V) = \max\left\{\mathcal{L}((x, y(\omega), \omega, \alpha), V) \,|\, (x, y(\omega), \omega, \alpha) \in V\right\}.$$

Thus, a new centre iterative method for problem (3.8) could be described as follows.

Algorithm 3.1. *Step* 1. Taking a point $(x^k, y^k(\omega), \omega, \alpha^k)$ in V, then we consider the distance $\mathcal{L}$ in convex domain $W_k = V \cap \{(x, y(\omega), \omega, \alpha) | \alpha \geq \alpha^k\}$.

Step 2. Solving the maximal problem $\max\{\mathcal{L}((x, y(\omega), \omega, \alpha), W_k) | \, (x, y(\omega), \omega, \alpha) \in W_k\}$ and denoting the new iterative point $(x^{k+1}, y^{k+1}(\omega), \omega, \alpha^{k+1})$ as a centre of W_k, then we have

$$\begin{aligned}&(x^{k+1}, y^{k+1}(\omega), \omega, \alpha^{k+1}) : \\ &\quad \mathcal{L}((x^{k+1}, y^{k+1}(\omega), \omega, \alpha^{k+1}), W_k) = \max\{\mathcal{L}((x, y(\omega), \omega, \alpha), W_k) | \, (x, y(\omega), \omega, \alpha) \in W_k\},\end{aligned}$$

where

$$\begin{aligned}\mathcal{L}((x, y(\omega), \omega, \alpha), W_k) \quad &= \min_{\substack{i=1,2,\cdots,l \\ j=3,4,\cdots,l+2}} \Big\{\alpha - \alpha^k, \, \alpha, \, 1 - \alpha, \, \mu_{\tilde{S}_0}^{-1}(\alpha) - \frac{1}{L} \sum_{\omega \in \Omega_L} f(x, y(\omega), \omega)\zeta(\omega), \\ &\quad \mu_{\tilde{S}_1}^{-1}(\alpha) - \langle F(x, y(\omega), \omega), y(\omega)\rangle, \, \mu_{\tilde{S}_2}^{-1}(\alpha) + \langle F(x, y(\omega), \omega), y(\omega)\rangle, \\ &\quad -\mu_{\tilde{S}_j}^{-1}(\alpha) + d'_{j-2} F(x, y(\omega), \omega), \, d_i(x) y(\omega)\Big\}\end{aligned}$$

is the distance function defined on the convex domain W_k.

Step 3. Start working again with $(x^{k+1}, y^{k+1}(\omega), \omega, \alpha^{k+1})$ instead of $(x^k, y^k(\omega), \omega, \alpha^k)$ and go to *Step* 1.

It follows from the properties introduced in [28, Lemma 2.2] and Algorithm 3.1 that the major computational work lies in the determination of the centres required, i.e., at the kth iteration, the following "min-max problem" should be solved:

$$\begin{aligned}-\min_{x, y(\omega), \omega, \alpha} \mathcal{L}((x, y(\omega), \omega, \alpha), W_k) \quad &= \min_{x, y(\omega), \omega, \alpha} \max_{\substack{i=1,2,\cdots,l \\ j=3,4,\cdots,l+2}} \Big\{\alpha^k - \alpha, \, -\alpha, \, \alpha - 1, \\ &\quad \tfrac{1}{L} \sum_{\omega \in \Omega_L} f(x, y(\omega), \omega)\zeta(\omega) - \mu_{\tilde{S}_0}^{-1}(\alpha), \\ &\quad \langle F(x, y(\omega), \omega), y(\omega)\rangle - \mu_{\tilde{S}_1}^{-1}(\alpha), \\ &\quad -\langle F(x, y(\omega), \omega), y(\omega)\rangle - \mu_{\tilde{S}_2}^{-1}(\alpha), \\ &\quad -d'_{j-2} F(x, y(\omega), \omega) + \mu_{\tilde{S}_j}^{-1}(\alpha), \, -d_i(x) y(\omega)\Big\}.\end{aligned} \tag{3.9}$$

STEP IV A class of new regularization smoothing approximation algorithms is developed under condition $(\mathbf{H}_C)$.

Since the maximal membership function (see [12]) in the "min-max" problem (3.9) is non-differentiability, it is easy to see that one major difficulty encountered is to develop a class of new smoothing approximation methods, which are based on the notion of newly proposed "entropic regularization procedure" (see [18]).

Algorithm 3.2.

Step 1. Set $k = 0$, give the initial iterate $(x^0, y^0(\omega), \omega, \alpha^0)$ which is an interior point of V defined by (3.8), a sufficiently small constant $\epsilon > 0$, and an upper bound Q which is the maximum number of unconstrained minimizations to be performed.

Step 2. Starting from $(x^k, y^k(\omega), \omega, \alpha^k)$, apply a standard quasi-Newton line search of MATLAB software to solve the unconstrained smooth convex program (3.6), (3.7) and the following unconstrained smooth convex program:

$$\begin{aligned}&-\min_{x, y(\omega), \omega, \alpha} \mathcal{L}_\gamma((x, y(\omega), \omega, \alpha), W_k) \\ &\quad = \tfrac{1}{\gamma} \ln \Big\{ \exp[\gamma(\alpha^k - \alpha)] + \exp[\gamma(-\alpha)] + \exp[\gamma(\alpha - 1)] \\ &\qquad + \exp[\gamma(\tfrac{1}{L} \sum_{\omega \in \Omega_L} f(x, y(\omega), \omega)\zeta(\omega) - \mu_{\tilde{S}_0}^{-1}(\alpha))] \\ &\qquad + \exp[\gamma(\langle F(x, y(\omega), \omega), y(\omega)\rangle - \mu_{\tilde{S}_1}^{-1}(\alpha))] \\ &\qquad + \exp[\gamma(-\langle F(x, y(\omega), \omega), y(\omega)\rangle - \mu_{\tilde{S}_2}^{-1}(\alpha))] \\ &\qquad + \sum_{j=3}^{l+2} \exp[\gamma(-d'_{j-2} F(x, y(\omega), \omega) + \mu_{\tilde{S}_j}^{-1}(\alpha))] \\ &\qquad + \sum_{i=1}^{l} \exp[\gamma(-d_i(x) y)] \Big\}\end{aligned} \tag{3.10}$$

with a sufficiently large γ. Denote its solution by $(x^{k+1}, y^{k+1}(\omega), \omega, \alpha^{k+1})$ in the light of Algorithm 3.1.

Step 3. If $k > 1$ and $\|(x^{k+1}, y^{k+1}(\omega), \omega, \alpha^{k+1}) - (x^k, y^k(\omega), \omega, \alpha^k)\|_2 \le \epsilon$, then the computation terminates with $(x^{k+1}, y^{k+1}(\omega), \omega, \alpha^{k+1})$ as the solution. If $k > Q$, then the computation terminates with a failure.

Step 4. $k \leftarrow k+1$ and go to *Step* 2.

From Algorithm 3.2, it follows that $\min_{x,y(\omega),\omega,\alpha} \mathcal{L}_\gamma((x, y(\omega), \omega, \alpha), W_k)$ provides a centre of W_k, as $\gamma \to \infty$. By using a moderately large γ, we can obtain an accurate approximation. Also because of the special "log-exponential" form of $\mathcal{L}_\gamma((x, y(\omega), \omega, \alpha), W_k)$, we can avoid most overflow problems in computation. Moreover, since problem (3.10) is an unconstrained, smooth, and convex optimization program, the commonly used solution methods, such as the quasi-Newton line search of MATLAB software, can be readily applied.

Remark 3.1. We note that Algorithm 3.1 appears in *Step* 2 of Algorithm 3.2. It is the fuzzy constraints in (1.1) that yields a fuzzy objective. Hence, a class of new and interesting regularization smoothing approximation approaches must be chosen to define two parameters in (3.6) and (3.7), and to employ for solving problem (3.10) which is equivalent to the stochastic parametric optimization problem (3.8).

STEP V Comprehensive convergence theorems based on Algorithm 3.2 should be proved.

In the sequel, we first give the following lemmas and results.

Lemma 3.1. Let the function $\varphi : \Omega \to \mathbb{R}$ be continuous. Then we have

$$\lim_{L\to\infty} \frac{1}{L} \sum_{\omega\in\Omega_L} \varphi(\omega)\zeta(\omega) = \int_\Omega \varphi(\omega)\zeta(\omega)d\omega.$$

Proof. Taking $N = L$, $\bar{I}^s = \Omega$, $J = \Omega_L$, $x_i = \omega_i$ $(i = 1, 2, \cdots)$ and $f = \varphi\zeta$, then from the results (2.2) and (2.3) given in Chapter 2 of [24, pp. 13-14], the result holds. This completes the proof. □

Remark 3.2. By Lemma 3.1, we know immediately that

$$\lim_{L\to\infty} \frac{1}{L} \sum_{\omega\in\Omega_L} f(x, y(\omega), \omega)\zeta(\omega) = \int_\Omega f(x, y(\omega), \omega)\zeta(\omega)d\omega \tag{3.11}$$

and particularly,

$$\lim_{L\to\infty} \frac{1}{L} \sum_{\omega\in\Omega_L} \zeta(\omega) = \int_\Omega \zeta(\omega)d\omega = 1. \tag{3.12}$$

Lemma 3.2. If $\psi(x)$ is continuous, strictly increasing and linear over a convex set U in $\mathbb{R}^n$, then its inverse ψ^{-1} is linear.

Theorem 3.1. Suppose that condition $(\mathbf{H}_C)$ holds, the set $U \subset \mathbb{R}^n$ is nonempty and bounded, $F : \mathbb{R}^{n+m} \times \Omega \to \mathbb{R}^m$ is continuously differentiable, and $f : \mathbb{R}^{n+m} \times \Omega \to \mathbb{R}$ is Hölder continuous in $(x, y(\cdot))$ on $U \times \mathbb{R}^m$ with order $\sigma > 0$ and Hölder constant $\varsigma(\omega) > 0$ for all $\omega \in \Omega_L$ satisfying

$$\int_\Omega \varsigma(\omega)d\zeta(\omega) < +\infty.$$

Then

(i) problem (3.6) has at least one optimal solution when L is large enough;

(ii) $(x^*, y^*(\cdot))$ is an optimal solution of problem (2.4) when x^* is an accumulation point of the sequence $\{x^L\}$ and $y^*(\cdot)$ is defined by

$$y^*(\omega) := \max_{i=1,2,\cdots,l} \{-F(x^*, 0, \omega), -d_i'(x^*)F(x^*, 0, \omega), 0\}, \quad \omega \in \Omega. \tag{3.13}$$

Proof. (i) Let $\mathcal{F}_L$ be the feasible region of problem (3.6). It is not difficult to see that $\mathcal{F}_L$ is a nonempty and closed set and the objective function of problem (3.6) is bounded below on $\mathcal{F}_L$. Thus,

there exists a sequence $\{(x^k, y^k(\omega))_{\omega\in\Omega_L}\} \subset \mathcal{F}_L$ such that

$$\lim_{k\to\infty} \frac{1}{L} \sum_{\omega\in\Omega_L} f(x^k, y^k(\omega), \omega)\zeta(\omega)$$
$$= \inf_{(x,y(\omega),\omega)_{\omega\in\Omega_L}} \frac{1}{L} \sum_{\omega\in\Omega_L} f(x, y(\omega), \omega)\zeta(\omega). \tag{3.14}$$

It follows from the boundedness of U and the Hölder continuity of f that the sequence $\{x^k\}$ and the function f are bounded.

On the other hand, noting that $(x^k, y^k(\omega))_{\omega\in\Omega_L} \in \mathcal{F}_L$ for every k, we have

$$0 \le y^k(\omega) \perp \frac{1}{L} \sum_{\omega\in\Omega_L} F(x^k, y^k(\omega), \omega)\zeta(\omega) \ge 0, \tag{3.15}$$

where the symbol $\perp$ means the two vectors are perpendicular to each other. Assume that the sequence $\{y^k(\omega)\}_{\omega\in\Omega_L}$ is unbounded. Taking a subsequence if necessary, let

$$\lim_{\substack{k\to\infty \\ \omega\in\Omega_L}} \|y^k(\omega)\| = +\infty, \lim_{\substack{k\to\infty \\ \omega\in\Omega_L}} \frac{y^k(\omega)}{\|y^k(\omega)\|} = \bar{y}(\omega), \|\bar{y}(\omega)\| = 1. \tag{3.16}$$

Then, for all $\omega \in \Omega_L$, dividing (3.15) by $\|y^k(\omega)\|$ and letting $k \to +\infty$, we have for any $x \in U$,

$$0 \le \bar{y}(\omega) \perp \frac{1}{L} \sum_{\omega\in\Omega_L} F(x, \bar{y}(\omega), \omega)\zeta(\omega) \ge 0.$$

This contradicts (3.16) by the continuous differentiability of F, and so $\{y^k(\omega)\}$ is bounded for each $\omega \in \Omega_L$ with $\zeta(\omega) > 0$. For any $\omega \in \Omega_L$ with $\zeta(\omega) = 0$, we redefine $y^k(\omega)$ by

$$y^k(\omega) := \max_{i=1,2,\cdots,l} \{-F(x^k, 0, \omega), -d_i'(x^k)F(x^k, 0, \omega), 0\}.$$

Hence, the sequence $\{(x^k, y^k(\omega))_{\omega\in\Omega_L}\}$ is bounded and (3.14) remains valid. Therefore, the closeness of $\mathcal{F}_L$ implies that any accumulation point of $\{(x^k, y^k(\omega))_{\omega\in\Omega_L}\}$ must be an optimal solution of problem (3.6).

(ii) By the assumptions, the sequence $\{x^L\}$ contains a subsequence converging to x^*. Without loss of generality, we suppose $\lim_{L\to\infty} x^L = x^*$.

Firstly, we prove that $(x^*, y^*(\cdot))$ is feasible to problem (3.6). To this end, we define

$$\hat{y}^L(\omega) := \max_{i=1,2,\cdots,l} \left\{-F(x^L, 0, \omega), -d_i'(x^L)F(x^L, 0, \omega), 0\right\}, \quad \omega \in \Omega. \tag{3.17}$$

It is obvious that $(x^*, \hat{y}^L(\omega))_{\omega\in\Omega_L}$ is feasible to problem (3.6) for every L. Since $F(x^*, y^*(\omega), \omega) \ge 0$ by the definition (3.13), it is sufficient to show that

$$(y^*(\omega))^T F(x^*, y^*(\omega), \omega) = 0, \quad \omega \in \Omega. \tag{3.18}$$

Let $\bar{\omega} \in \Omega$ be fixed. Since the sample set Ω_L is chosen to be asymptotically dense in Ω, there exists a sequence $\{\bar{\omega}_L\}$ of samples such that $\bar{\omega}_L \in \Omega_L$ for each L and $\lim_{L\to\infty} \bar{\omega}_L = \bar{\omega}$. Thus, we obtain

$$(\hat{y}^L(\bar{\omega}_L))^T F(x^L, \hat{y}^L(\bar{\omega}_L), \bar{\omega}_L) = 0, \quad L = 1, 2, \cdots.$$

Letting $L \to +\infty$ and taking the continuity of the functions $F(x, y(\cdot), \cdot)$ on the compact set Ω into account, we have

$$(y^*(\bar{\omega}))^T F(x^*, y^*(\bar{\omega}), \bar{\omega}) = 0.$$

By the arbitrariness of $\bar{\omega} \in \Omega$, now we know that (3.18) immediately holds. This completes the proof of the feasibility of $(x^*, y^*(\cdot))$ in (3.6).

Next, let $(x, y(\cdot))$ be an arbitrary feasible solution of (3.6). It follows from the results of (i) and obvious that $(x, y(\omega), \omega)_{\omega\in\Omega_L}$ is feasible to problem (3.6) for any L. Moreover, from the Höcontinuity of f, we have

$$\begin{aligned}
&\frac{1}{L}\sum_{\omega\in\Omega_L}\left[f(x^L, y^L(\omega), \omega) - f(x^L, \hat{y}^L(\omega), \omega)\right]\zeta(\omega)\\
&= \frac{1}{L}\sum_{\omega\in\Omega_L}\left[f(x^L, y^L(\omega), \omega) - f(x^L, \hat{y}^L(\omega), \omega)\right]\zeta(\omega)\\
&\leq \frac{1}{L}\sum_{\omega\in\Omega_L}\left[\|y^L(\omega) - \hat{y}^L(\omega)\|\varsigma(\omega)\right]\cdot\zeta(\omega) \to 0 \quad \text{as} \quad k\to\infty, \quad \text{w.p.1.}
\end{aligned}$$

which along with Lemma 3.1 yields

$$\lim_{L\to\infty}\frac{1}{L}\sum_{\omega\in\Omega_L} f(x^k, y^k(\omega), \omega) = \lim_{L\to\infty}\frac{1}{L}\sum_{\omega\in\Omega_L} f(x^*, y^*(\omega), \omega) = E_\omega[f(x^*, y^*(\omega), \omega)] \quad \text{w.p.1,}$$

which indicates that $(x^*, y^*(\cdot))$ is an optimal solution of problem (1.1) with probability one and the feasibility of $(x^L, y^L(\omega), \omega)_{\omega\in\Omega_L}$ in (3.6) that $(x^L, \hat{y}^L(\omega), \omega)_{\omega\in\Omega_L}$ is also an optimal solution of problem (3.6). Thus, since f is Hölder continuous in $(x, y(\cdot))$ on $U\times\mathbb{R}^m$, we obtain

$$\begin{aligned}
&\frac{1}{L}\sum_{\omega\in\Omega_L}\Big[f(x^*, y^*(\omega), \omega) - f(x, y(\omega), \omega)\Big]\zeta(\omega)\\
&\leq \frac{1}{L}\sum_{\omega\in\Omega_L}\left[f(x^*, y^*(\omega), \omega) - f(x^L, \hat{y}^L(\omega), \omega)\right]\zeta(\omega)\\
&\leq \frac{1}{L}\sum_{\omega\in\Omega_L}\left|f(x^*, y^*(\omega), \omega) - f(x^L, \hat{y}^L(\omega), \omega)\right|\zeta(\omega)\\
&\leq \frac{1}{L}\sum_{\omega\in\Omega_L}\zeta(\omega)\cdot\left[\|x^L - x^*\| + \|\hat{y}^L(\omega) - y^*(\omega)\|\right]\varsigma(\omega). \qquad (3.19)
\end{aligned}$$

It follows from (3.12) that the sequence $\left\{\frac{1}{L}\sum_{\omega\in\Omega_L}\zeta(\omega)\right\}$ is bounded. This yields

$$\lim_{L\to\infty}\frac{1}{L}\sum_{\omega\in\Omega_L}\Big|f(x^*, y^*(\omega), \omega) - f(x^L, \hat{y}^L(\omega), \omega)\Big|\zeta(\omega) = 0. \qquad (3.20)$$

Thus, by letting $L\to+\infty$ in (3.19) and taking (3.11) and (3.20) into account, we have

$$\int_\Omega f(x^*, y^*(\omega), \omega)\zeta(\omega)d\omega \leq \int_\Omega f(x, y(\omega), \omega)\zeta(\omega)d\omega,$$

which implies that x^* together with $y^*(\cdot)$ constitutes an optimal solution of problem (3.6). This completes the proof. □

Similarly, by Lemma 3.1, (3.11), (3.12) and proof of Theorem 3.1, we have the following result.

Theorem 3.2. Assume that condition $(\mathbf{H}_C)$ holds, and f, F and U are the same as in Theorem 3.1. Then

(i) problem (3.7) has at least one optimal solution when L is large enough;

(ii) $(x^*, y^*(\cdot))$ is an optimal solution of problem (2.5) when x^* is an accumulation point of the sequence $\{x^L\}$ and $y^*(\cdot)$ is defined by

$$y^*(\omega) := \max_{i=1,2,\cdots,l}\{-t_1 - F(x^*, 0, \omega), -t_2 + F(x^*, 0, \omega), -t_{i+2} - d_i'(x^*)F(x^*, 0, \omega), 0\}, \quad \omega\in\Omega.$$

Now, consider the case that the membership function of each fuzzy stochastic inequality and the objective function $E_\omega[f(x, y(\omega), \omega)]$ in (3.3) is continuous, strictly increasing, and linear over the

corresponding tolerance interval. A commonly used example in fuzzy set theory is that $\psi(x) = 1 - bx^{\beta}$ with $b > 0$ and $\beta > 1$. In this case, from the theory of convex analysis [27], Lemma 3.2, and Theorems 3.1 and 3.2, we have the following simple result.

Theorem 3.3. Suppose that condition $(\mathbf{H}_C)$ holds. If $F : \mathbb{R}^{n+m} \times \Omega \to \mathbb{R}^m$ is monotone in the second variable, and $\mu_{\tilde{\Omega}_z}(x, y(\omega), \omega)$ is continuous, strictly increasing and linear for all $z \in C(x, \omega)$ and any $(x, y(\omega), \omega) \in \mathbb{R}^{n+m} \times \Omega$, then we can find an optimal solution $(x^*, y^*(\cdot))$ of the stochastic optimization problem (1.1) by solving the following stochastic parametric optimization problem: (3.8), which can be readily approximated by the iterative sequence $\{(x^{k+1}, y^{k+1}(\omega), \omega, \alpha^{k+1})\}$ generated by Algorithm 3.2.

4 Simulation analysis

In this section, we shall give an example to illustrate the validity of our approaches.

Taking $n = 2, m = 3, l = 2$, $U = [1, 14] \times [1, 14]$, $d_1(x) = (-1, -1, 3)$, $d_2(x) = (-n2, 1, -1)$,

$$f(x, y(\omega), \omega) = (x_1 - y_1)^2 + x_2 y_2 + 2\omega,\ F(x, y(\omega), \omega) = \begin{pmatrix} -2x_1 + y_1 - 3y_2 + \omega \\ x_1 + x_2 + 3y_1 - y_3 - 2\omega \\ -2x_2 + y_2 + 2y_3 - \omega \end{pmatrix},$$

$$C(x, \omega) = \left\{ y(\omega) = (y_1, y_2, y_3)^T \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} d_1(x) \\ d_2(x) \end{pmatrix} y(\omega) \geq 0 \right\},$$

and Letting $d_1'(x) = (0, 3, 1)$ and $d_2'(x) = (1, 2, 0)$ in (3.3), and $\zeta(\omega_\ell) = p_\ell\ (\ell = 1, 2, \cdots, L)$ in (3.5), then we have

$$\begin{aligned}
&\varphi(x, y(\omega), \omega) := E_\omega[(x_1 - y_1)^2 + x_2 y_2 + 2\omega] = \tfrac{1}{L} \textstyle\sum_{\ell=1}^{L} [(x_1 - y_1)^2 + x_2 y_2 + 2\omega_\ell] p_\ell, \\
&f_1(x, y(\omega), \omega) = \langle F(x, y(\omega), \omega), y(\omega) \rangle = -2x_1 y_1 + x_1 y_2 + x_2 y_2 - 2x_2 y_3 + y_1^2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad + \omega y_1 - 2\omega y_2 + 2y_3^2 - \omega y_3, \\
&f_2(x, y(\omega), \omega) = \langle -F(x, y(\omega), \omega), y(\omega) \rangle = 2x_1 y_1 - x_1 y_2 - x_2 y_2 + 2x_2 y_3 - y_1^2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad - \omega y_1 + 2\omega y_2 - 2y_3^2 + \omega y_3, \\
&f_3(x, y(\omega), \omega) = d_1'(x) F(x, y(\omega), \omega) = 3x_1 + x_2 + 9y_1 + y_2 - y_3 - 7\omega, \\
&f_4(x, y(\omega), \omega) = d_2'(x) F(x, y(\omega), \omega) = 2x_2 + 7y_1 - 3y_2 - 2y_3 - 3\omega.
\end{aligned}$$

Thus, problem (3.5) is equivalent to the following generalized fuzzy stochastic inequality constrained optimization program:

$$\begin{array}{ll}
\min & \varphi(x, y(\omega), \omega) \\
\text{s.t.} & 1 \leq x_1, x_2 \leq 14, \quad y_1, y_2, y_3 \geq 0, \\
& -y_1 - y_2 + y_3 \geq 0, \quad -2y_1 + y_2 - y_3 \geq 0, \\
& f_\iota(x, y(\omega), \omega) \gtrsim 0, \quad \iota = 1, 2, 3, 4,
\end{array} \tag{4.1}$$

with the membership function $\mu_{\tilde{S}_\tau}(x, y(\omega), \omega)$ $(\tau = 1, 2, 3, 4)$, being specified as $t_1 = 9, t_2 = 2, t_3 = 6, t_4 = 10$,

$$\mu_{\tilde{S}_1}(x, y(\omega), \omega) = \begin{cases} 1, & \text{if } f_1(x) \geq 0, \\ 1 - \frac{f_1(x, y(\omega), \omega)}{9}, & \text{if } f_1(x, y(\omega), \omega) \in [-9, 0), \\ 0, & \text{if } f_1(x, y(\omega), \omega) < -9, \end{cases}$$

$$\mu_{\tilde{S}_2}(x, y(\omega), \omega) = \begin{cases} 1, & \text{if } f_2(x, y(\omega), \omega) \geq 0, \\ 1 - \frac{f_2(x, y(\omega), \omega)}{2}, & \text{if } f_2(x, y(\omega), \omega) \in [-2, 0), \\ 0, & \text{if } f_2(x) < -2, \end{cases}$$

$$\mu_{\tilde{S}_3}(x, y(\omega), \omega) = \begin{cases} 1, & \text{if } f_3(x, y(\omega), \omega) \geq 0, \\ 1 - \frac{f_3(x, y(\omega), \omega)}{6}, & \text{if } f_3(x, y(\omega), \omega) \in [-6, 0), \\ 0, & \text{if } f_3(x, y(\omega), \omega) < -6, \end{cases}$$

$$\mu_{\tilde{S}_4}(x, y(\omega), \omega) = \begin{cases} 1, & \text{if } f_4(x) \geq 0, \\ 1 - \frac{f_4(x, y(\omega), \omega)}{10}, & \text{if } f_4(x, y(\omega), \omega) \in [-10, 0), \\ 0, & \text{if } f_4(x) < -10, \end{cases}$$

and

$$\mu_{\tilde{S}_0}(x,y(\omega),\omega)=\begin{cases}1, & \text{if} \quad \varphi(x,y(\omega),\omega)<\underline{f},\\ \frac{\bar{f}-\varphi(x,y(\omega),\omega)}{\bar{f}-\underline{f}}, & \text{if} \quad \varphi(x,y(\omega),\omega)\in[\underline{f},\bar{f}),\\ 0, & \text{if} \quad \langle\varphi(x,y(\omega),\omega)\geq\bar{f},\end{cases}$$

where

$$\begin{aligned}\bar{f}=\min \quad & \varphi(x,y(\omega),\omega)\\ \text{s.t.} \quad & 1\leq x_1,x_2\leq 14, \quad y_1,y_2,y_3\geq 0,\\ & -y_1-y_2+3y_3\geq 0, \quad -2y_1+y_2-y_3\geq 0,\\ & f_\iota(x,y(\omega),\omega)\geq 0, \quad \iota=1,2,3,4,\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}\underline{f}=\min \quad & \varphi(x,y(\omega),\omega)\\ \text{s.t.} \quad & 1\leq x_1,x_2\leq 14, \quad y_1,y_2,y_3\geq 0,,\\ & -y_1-y_2+3y_3\geq 0, \quad -2y_1+y_2-y_3\geq 0,\\ & f_1(x,y(\omega),\omega)\geq -9, \quad f_2(x,y(\omega),\omega)\geq -2,\\ & f_3(x,y(\omega),\omega)\geq -6, \quad f_4(x,y(\omega),\omega)\geq -10.\end{aligned} \tag{4.3}$$

By Bellman and Zadeh's method of fuzzy decision making [15] and Algorithm 3.2, now we know that the conditions of Theorem 3.3 hold, and so an optimal solution of the problem (4.1) can be obtained by solving the following unconstrained and smooth nonlinear parametric optimization problem:

$$\begin{aligned}\min_{x,y(\omega),\omega,\alpha} \tfrac{1}{\gamma}\ln\Big\{ & \exp[\gamma(\alpha^k-\alpha)]+\exp[\gamma(-\alpha)]+\exp[\gamma(\alpha-1)]\\ & +\exp[\gamma(\varphi(x,y(\omega),\omega)-(\bar{f}-\alpha(\bar{f}-\underline{f})))]\\ & +\exp[\gamma(f_1(x,y(\omega),\omega)-9(1-\alpha))]\\ & +\exp[\gamma(f_2(x,y(\omega),\omega)-2(1-\alpha))]\\ & +\exp[\gamma(-f_3(x,y(\omega),\omega)+6(1-\alpha))]\\ & +\exp[\gamma(-f_4(x,y(\omega),\omega)+10(1-\alpha))]\\ & +\exp[\gamma(y_1+y_2-3y_3)]+\exp[\gamma(2y_1-y_2+y_3)]\\ & +\exp[\gamma(1-x_1)]+\exp[\gamma(1-x_2)]+\exp[\gamma(x_1-14)]\\ & +\exp[\gamma(x_2-14)]+\exp[\gamma(-y_1)]+\exp[\gamma(-y_2)]+\exp[\gamma(-y_3)]\Big\}\end{aligned} \tag{4.4}$$

with γ being sufficiently large, where the optimal values of $\bar{f}$ and $\underline{f}$ are obtained by computing (4.2) and (4.3), respectively.

Choosing $x^0=(4.0000,2.0000)$, $y^0(\omega)=(1.0547,1.0564,0.1574)$ and $\alpha^0=0.2$ and setting $L=3$ with the probability $p_1=0.1590, p_2=0.6821, p_3=0.1589$, and $\epsilon=10^{-5}$, $Q=10^6$ and fixed $\gamma=12$, then for each iteration of Algorithm 3.2, we first generate the random variable ω by using normrnd function (that is, normal distribution function) of MATLAB 7.0 software. Secondly, we solve from problems (4.2) and (4.3) to problem (4.4) in turn by the commonly used quasi-Newton line search of MATLAB software 7.0.

Here, the first layer iteration searching optimization is to solve problems (4.2) and (4.3), respectively. And the second optimizing process is to find the optimal solution of problem (4.1) via solving the unconstrained and smooth nonlinear parametric optimization problem (4.4). We only present four optimal solution $(x^*,y^*(\cdot))$ with respect to the random variable ω and the corresponding membership degree α^* for whole stochastic optimization problem, which is listed in Table 1. Further, Table 2 show that each iteration calculation results including iterative solutions with the random variable $\omega=0.128808$ for the second optimizing process to this problem. The results for every step in Table 2 (i.e., $k=0,1,2,\cdots,11$) come from the first layer iteration process, which are too much and so they are omitted.

Table 1: The optimal solution with the random variable and membership degree

k	$(x^*,\ y^*(\omega),\ \omega)$	α^*
1	(0.984674, 1.158197, 0.036645, 0.228989, 0.128690, 0.128808)	0.954363
2	(0.987781, 1.219403, 0.024577, 0.187961, 0.097964, 0.345629)	0.949652
3	(0.984337, 1.156955, 0.036586, 0.232004, 0.127346, 0.191080)	0.953185
4	(0.988056, 1.226168, 0.023580, 0.184104, 0.095786, 0.353731)	0.949391

Table 2: Data for Computational results with the random variable $\omega = 0.128808$

k	$(x^k,\ y^k)$	α^k	Iterations No.
0	(4.0000, 2.0000, 1.0547, 1.0564, 0.1574)	0.2	46
1	(0.989779,1.252773,0.027679,0.173089,0.106607)	0.871896	16
2	(0.985831,1.166544,0.034618,0.217474,0.125437)	0.924577	13
3	(0.984928,1.159731,0.036190,0.226463,0.128032)	0.945141	15
4	(0.984726,1.158504,0.036552,0.228473,0.128555)	0.951749	14
5	(0.984687,1.158268,0.036623,0.228867,0.128658)	0.953643	9
6	(0.984677,1.158241,0.036639,0.228955,0.128681)	0.954166	4
7	(0.984676,1.158237,0.036638,0.228959,0.128678)	0.954309	3
8	(0.984676,1.158233,0.036638,0.228962,0.128678)	0.954348	5
9	(0.984674,1.158198,0.036645,0.228989,0.128690)	0.954359	1
10	(0.984674,1.158197,0.036645,0.228989,0.128690)	0.954362	1
11	(0.984674,1.158197,0.036645,0.228989,0.128690)	0.954363	

5 Concluding remarks

In this paper, by developing a class of new regularization smoothing approximation approaches, we investigated approximation solvability of the following fuzzy parametric variational inequality constrained stochastic optimization problems in n-dimension real numeral set $\mathbb{R}^n$:

$$\begin{array}{ll} \min\limits_{x,y(\cdot)} & E_\omega[f(x,y(\omega),\omega)] \\ \text{s.t.} & x \in U, \\ & y(\omega) \in C(x,\omega), \\ & \langle F(x,y(\omega),\omega), z(\omega)-y(\omega)\rangle \gtrapprox 0, \quad \forall z(\omega) \in C(x,\omega), \end{array} \tag{5.1}$$

which has been very little studied by right of the known theories and approaches in the literature. It is because the existence of the random variable and the fuzzified version mean that (5.1) involves multiple complementarity-type constraints, and solving problem (5.1) is more difficult than solving an ordinary mathematical program with (fuzzy) equilibrium constraints or stochastic mathematical program with equilibrium constraints.

Based on the notion of tolerance approach with entropic regularization and fuzzy set theory, we first showed that solving the stochastic optimization problem with fuzzy parametric variational inequality constraints is equivalent to solving a fuzzy complementarity constrained stochastic optimization problem, which can be converted to a regular nonlinear parametric optimization problem with continuous random variables. Then, we constructed a centre iterative algorithm and developed a class of new regularization smoothing approximation approaches for solving a problem with continuous random variables based on quasi-Monte Carlo estimate and entropic regularization technique, and discussed a comprehensive convergence theory for approximating the resulting optimization problem. Finally, numerical example was provided to illustrate our main results applying quasi-Newton line search of MATLAB software.

We remark that in the paper, based on the concept that fuzzy constraints should yield a fuzzy

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objective, we must choose a class of new regularization smoothing approximation approaches to define the objective function value of two optimization problems as the parameters in an equivalent stochastic parametric optimization problem. Hence, the problem presented in this paper is brand new and the method is also new and interesting.

Whether the corresponding results of Theorem 3.3 hold when the objective function is a fuzzy stochastic function, the constraints are fuzzy implicit variational inequalities (such as in [14, 17]) or elliptic inequalities subject to physical phenomenon, and the numerical testing is some large-scale applications, which are still **open questions** to be solved in further research.

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The Split Common Fixed Point Problem for Demicontractive Mappings in Banach Spaces

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Abstract. In this paper, based on the work by Moudafi and inspired by Takahashi and Xu, we try to investigate the split common fixed point problems for the class of demicontractive mappings in the setting of two Banach spaces, and obtain the strong and weak convergence theorems. The results presented in the paper improve and extend some recent well-known corresponding results.

Keywords: split common fixed point problem; demicontractive mapping; Demiclosed principle, weak and strong convergence theorems.

2010 AMS Subject Classification: 47H09, 49J25.

1 Introduction and Preliminaries

The split common fixed point problem was introduced by Moudafi [1] in 2010. Moudafi proposed an iteration scheme and obtained a weak convergence theorem of the split common fixed point problem for demicontractive mappings in the setting of two Hilbert spaces. Since then, many authors investigated the split common fixed point problems of other nonlinear mappings in the setting of two Hilbert spaces (see [2-7]). At the beginning of 2015, Takahashi [8] first attempted to introduce and consider the split feasibility problem and split common null point problem in the setting of one Hilbert space and one Banach space. By using hybrid methods and Halperns type methods under suitable conditions, some strong and weak convergence theorems for such problems are obtained. The results presented in [8] seem to be the first outside Hilbert spaces. This naturally brings us to solve the split common fixed point problem for demicontractive mappings in the setting of two Banach space.

Let E_1 and E_2 be two real Banach spaces, and $A : E_1 \to E_2$ be a bounded linear operator such that $A \neq 0$. The split common fixed point problem (SCFP) for nonlinear mappings S and T is to find a point $x \in E_1$ such thst

$$x \in F(S) \text{ and } Ax \in F(T), \tag{1.1}$$

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where $F(S)$ and $F(T)$ denote the sets of fixed points of S and T, respectively. We use Γ to denote the set of solutions of SCFP for mappings S and T, that is,

$$\Gamma = \{x \in F(S) |\ Ax \in F(T)\}.$$

In this paper, we use the following algorithm to approximate a split common fixed point of demicontractive mappings in the setting of two Banach spaces.

Algorithm: Let E_1 and E_2 be two real Banach spaces, $A : E_1 \to E_2$ be a bounded linear operator, A^* be the adjoint operator of A and J_i be the normalized duality mapping from E_i to $2^{E_i^*}$, $i = 1, 2$. Now, we define the iterative scheme $\{x_n\}$:

Let $x_1 \in E_1$ be arbitrary, for all $n \geq 1$, set

$$y_n = x_n + \gamma J_1^{-1} A^* J_2(T - I)Ax_n, \tag{1.2}$$

$$x_{n+1} = (1 - \alpha_n)y_n + \alpha_n S(y_n), \tag{1.3}$$

where $S : E_1 \to E_1$ and $T : E_2 \to E_2$ are two demicontractive mappings.

Under some suitable conditions, the iterative scheme $\{x_n\}$ is shown to converge weakly and strongly to a split common fixed point of demicontractive mappings T and S. Our result extends the split common fixed point problem from Hilbert spaces to Banach spaces.

In order to solve this problem mentioned above, we recall the following concepts and results.

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E, We denote the value of $y^* \in E$ at $x \in E$ by $\langle x, y^*\rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$.

We recall that $T : E \to E$ is demicontractive (see for example [9]) if there exists a constant $\eta \in [0, 1)$ such that

$$\|Tx - q\|^2 \leq \|x - q\|^2 + \eta\|x - Tx\|^2, \ \ \forall (x, q) \in E \times F(T). \tag{1.4}$$

An operator satisfying (1.4) will be referred to as a η-demicontractive mapping.

It is worth noting that the class of demicontractive maps contains important operators such as the quasi-nonexpansive maps and the strictly pseudocontractive maps with fixed points.

A mapping $T : E \to E$ is called quasi-nonexpansive, if

$$\|Tx - q\| \leq \|x - q\|$$

for all $(x, q) \in E \times F(T)$. A mapping $T : E \to E$ is strictly pseudocontractive, if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \beta\|x - y - (Tx - Ty)\|^2$$

for all $(x, y) \in E \times E$ and for some $\beta \in [0, 1)$.

A mapping $T : E \to E$ is called demiclosed at zero, if for any sequence $\{x_n\} \subset E$ and $x \in E$, we have

$$x_n \rightharpoonup x, \ \ (I - T)(x_n) \to 0 \Rightarrow x \in F(T).$$

A mapping $S : E \to E$ is said to be semi-compact, if for any sequence $\{x_n\}$ in E such that $\|x_n - Sx_n\| \to 0, (n \to \infty)$, there exists subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $x^* \in E$.

The normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \ \ \forall x \in E.$$

Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \ \text{ exists.}$$

In the case, E is called smooth. E is smooth if and only if J is single-valued. We denote the single-valued normalized duality mapping by J.

The modulus of convexity of E is defined by

$$\delta_E(\epsilon) = \inf\Big\{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon\Big\},$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. E is said to be p-uniformly convex, if there exists a constant $a > 0$ such that $\delta_E(\epsilon) \geq a\epsilon^p$ for all $0 < \epsilon \leq 2$.

Let $\rho_E : [0, \infty) \to [0, \infty)$ be the modulus of smoothness of E defined by

$$\rho_E(t) = \sup\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x \in U, \|y\| \leq t\}.$$

A Banach space E is said to be uniformly smooth if $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$. Let q be a fixed real number with $q > 1$. Then a Banach space E is said to be q-uniformly smooth if there exists a constant $b > 0$ such that $\rho_E(t) \leq bt^q$ for all $t > 0$. It is well known that every q-uniformly smooth Banach space is uniformly smooth.

A Banach space E is said to satisfy the Opial's condition [10] if for any sequence $\{x_n\} \subset E$, $x_n \rightharpoonup x$ implies

$$\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|,$$

for all $y \in E$ with $y \neq x$.

Lemma 1.1. [11] *Let E be a 2-uniformly convex Banach space. Then the following inequality holds:*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)c\|x - y\|^2, \ \ \forall x, y \in E, \tag{1.5}$$

where $0 \leq \lambda \leq 1$, $c = \mu(1) > 0$,

$$\begin{aligned} \mu(t) : \ &= \ \inf\Big\{\frac{\lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \|\lambda x + (1 - \lambda)y\|^2}{\lambda(1 - \lambda)} : \\ &\quad\ 0 < \lambda < 1, x, y \in E \ \ and \ \ \|x - y\| = t\Big\} \\ &> \ 0. \end{aligned}$$

Lemma 1.2. [11] *Let E be a 2-uniformly smooth Banach space with the best smoothness constants $\kappa > 0$. Then the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, Jy \rangle + 2\|\kappa y\|^2,$$

for all $x, y \in E$.

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2 Main Results

Lemma 2.1. *Let E_1 be a real 2-uniformly convex and 2-uniformly smooth Banach spaces with the best smoothness constant κ satisfying $0 < \kappa < \frac{1}{\sqrt{2}}$, E_2 be a real Banach space, and $A : E_1 \to E_2$ be a bounded linear operator. Let $S : E_1 \to E_1$ be β-demicontractive and $T : E_2 \to E_2$ be η-demicontractive with $F(S) \neq \emptyset$ and $F(T) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by algorithm (1.2)-(1.3) is Féjer-monotone with respect to $\Gamma = \{x \in F(S) | Ax \in F(T)\}$, that is, for every $z \in \Gamma$,*

$$\|x_{n+1} - z\| \leq \|x_n - z\|, \ \ \forall n \in N,$$

where $0 < \gamma < \min\left\{\frac{1-\eta}{\|A\|^2}, \frac{1-2\kappa}{\|A\|^2}\right\}$ *and* $\alpha_n \in (0, 1 - \frac{\beta}{c}], \ \ \beta < c = \mu(1)$.

Proof. Let $z \in \Gamma$. Then $z \in F(S)$ and $Az \in F(T)$. It follows from Lemma 1.1 and (1.3) that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|(1-\alpha_n)y_n + \alpha_n S(y_n) - z\|^2 \\
&= \|(1-\alpha_n)(y_n - z) + \alpha_n(S(y_n) - z)\|^2 \\
&\leq (1-\alpha_n)\|y_n - z\|^2 + \alpha_n\|S(y_n) - z\|^2 \\
&\quad -\alpha_n(1-\alpha_n)c\|S(y_n) - y_n\|^2 \\
&\leq (1-\alpha_n)\|y_n - z\|^2 + \alpha_n\|y_n - z\|^2 + \alpha_n\beta\|S(y_n) - y_n\|^2 \\
&\quad -\alpha_n(1-\alpha_n)c\|S(y_n) - y_n\|^2 \\
&\leq \|y_n - z\|^2 - \alpha_n(c - \beta - c\alpha_n)\|S(y_n) - y_n\|^2,
\end{aligned} \tag{2.1}$$

where $c = \mu(1)$.

On the other hand, It follows from (1.2) and Lemma 1.2 that

$$\begin{aligned}
\|y_n - z\|^2 &= \|x_n + \gamma J_1^{-1}A^*J_2(T-I)Ax_n - z\|^2 \\
&= \|x_n - z + \gamma J_1^{-1}A^*J_2(T-I)Ax_n\|^2 \\
&\leq \|\gamma J_1^{-1}A^*J_2(T-I)Ax_n\|^2 + 2\gamma\langle x_n - z, A^*J_2(T-I)Ax_n\rangle \\
&\quad +2\kappa^2\|x_n - z\|^2 \\
&\leq \gamma^2\|A\|^2\|(T-I)Ax_n\|^2 + 2\gamma\langle Ax_n - Az, J_2(T-I)Ax_n\rangle \\
&\quad +2\kappa^2\|x_n - z\|^2 \\
&= \gamma^2\|A\|^2\|(T-I)Ax_n\|^2 + 2\kappa^2\|x_n - z\|^2 \\
&\quad 2\gamma\langle Ax_n - TAx_n + TAx_n - Az, J_2(T-I)Ax_n\rangle \\
&\leq \gamma^2\|A\|^2\|(T-I)Ax_n\|^2 + 2\kappa^2\|x_n - z\|^2 \\
&\quad -2\gamma\|(T-I)Ax_n\|^2 + 2\gamma\langle TAx_n - Az, J_2(T-I)Ax_n\rangle \\
&\leq (\gamma^2\|A\|^2 - 2\gamma)\|(T-I)Ax_n\|^2 + 2\kappa^2\|x_n - z\|^2 \\
&\quad +\gamma(\|TAx_n - Az\|^2 + \|(T-I)Ax_n\|^2) \\
&\leq 7(\gamma^2\|A\|^2 - 2\gamma)\|(T-I)Ax_n\|^2 + 2\kappa^2\|x_n - z\|^2 \\
&\quad +\gamma(\|Ax_n - Az\|^2 + \eta|(Ax_n - TAx_n\|^2 + \|(T-I)Ax_n\|^2) \\
&\leq (2\kappa^2 + \gamma\|A\|^2)\|x_n - z\|^2 - \gamma(1 - \eta - \gamma\|A\|^2)\|(T-I)Ax_n\|^2,
\end{aligned}$$

where A^* is the adjoint operator of A and J_i is the normalized duality mapping from E_i to $2^{E_i^*}, i = 1, 2$.

In addition, since $0 < \kappa < \frac{1}{\sqrt{2}}$ and $0 < \gamma < \frac{1-2\kappa^2}{\|A\|^2}$, $0 < \gamma\|A\|^2 + 2\kappa^2 < 1$, so we have

$$\|y_n - z\|^2 \leq \|x_n - z\|^2 - \gamma(1 - \eta - \gamma\|A\|^2)\|(T-I)Ax_n\|^2. \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$\begin{aligned} \|x_{n+1}-z\|^2 &\leq \|x_n-z\|^2-\gamma(1-\eta-\gamma\|A\|^2)\|(T-I)Ax_n\|^2 \\ &\quad -\alpha_n(c-\beta-c\alpha_n)\|S(y_n)-y_n\|^2. \end{aligned} \tag{2.3}$$

Finally, by the assumptions on γ and α_n, we obtain the desired result.

Theorem 2.2. *Let E_1 be a real 2-uniformly convex and 2-uniformly smooth Banach space satisfying Opial's condition with the best smoothness constant κ satisfying $0<\kappa<\frac{1}{\sqrt{2}}$, and E_2 be a real Banach space. Let $A: E_1 \to E_2$ be a bounded linear operator, $S: E_1 \to E_1$ and $T: E_2 \to E_2$ be two demicontractive mappings with constants β and η with $F(S) \neq \emptyset$ and $F(T) \neq \emptyset$, respectively. Assume that $I-S$ and $I-T$ are demiclosed at zero. If $\Gamma \neq \emptyset$, then the sequence $\{x_n\}$ generated by algorithm (1.2)-(1.3) converges weakly to a split common fixed point $x \in \Gamma$, for $0<\gamma<\min\left\{\frac{1-\eta}{\|A\|^2}, \frac{1-2\kappa}{\|A\|^2}\right\}$, $\alpha_n \in (\delta, 1-\frac{\beta}{c}-\delta)$, $\beta < c = \mu(1)$, and for small enough $\delta > 0$.*

Proof. From (2.3) and the fact that $0<\gamma<\min\left\{\frac{1-\eta}{\|A\|^2}, \frac{1-2\kappa}{\|A\|^2}\right\}$ and $\alpha_n \in (\delta, 1-\frac{\beta}{c}-\delta)$, we obtain that the sequence $\{\|x_n-z\|\}$ is monotonically decreasing and thus converges to some positive real limit $l(z)$. From (2.3), we have

$$\gamma(1-\eta-\gamma\|A\|^2)\|(I-T)Ax_n\|^2 \leq \|x_n-z\|^2-\|x_{n+1}-z\|^2.$$

Therefore,

$$\lim_{n\to\infty}\|(I-T)Ax_n\|=0. \tag{2.4}$$

From the *Féjer*-monotonicity of $\{x_n\}$, it follows that the sequence is bounded. Denoting by x a weak-cluster point of $\{x_n\}$. Let $k=0,1,2,...$ be the sequence of indices, such that $x_{n_k} \rightharpoonup x$, as $k\to\infty$. Then from (2.4) and demiclosedness of $I-T$ at zero, we obtain $T(Ax)=Ax$, that is, $Ax \in F(T)$.

Now, by setting $y_n = x_n+\gamma J_1^{-1}A^*J_2(I-T)Ax_n$, it follows that $y_{n_k} \rightharpoonup x$. Again from (2.3), we obtain

$$\alpha_n(c-\beta-c\alpha_n)\|y_n-S(y_n)\|^2 \leq \|x_n-z\|^2-\|x_{n+1}-z\|^2.$$

Using the convergence of the sequence $\{\|x_n-z\|\}$, we get

$$\lim_{n\to\infty}\|y_n-S(y_n)\|=0, \tag{2.5}$$

which combined with the demiclosedness of $I-S$ at zero and the weak convergence of $\{y_{n_k}\}$ to y yields $S(x)=x$. Hence, $x\in F(S)$ and therefore $x \in \Gamma$. Since E_1 satises Opial's condition, we know that $\{x_n\}$ converges weakly to $x \in \Gamma$.

Theorem 2.3. *Let E_1 be a real 2-uniformly convex and 2-uniformly smooth Banach space satisfying Opial's condition with the best smoothness constant κ satisfying $0<\kappa<\frac{1}{\sqrt{2}}$, and E_2 be a real Banach space. Let $A: E_1 \to E_2$ be a bounded linear operator, $S: E_1 \to E_1$ and $T: E_2 \to E_2$ be two demicontractive mappings with constants β and η with $F(S) \neq \emptyset$ and $F(T) \neq \emptyset$, respectively. Assume that $I-S$ and $I-T$ are demiclosed at zero. If $\Gamma \neq \emptyset$ and S is semi-compact, then the sequence $\{x_n\}$ generated by algorithm (1.2)-(1.3) converges strongly to a split common fixed point $x \in \Gamma$, for $0<\gamma< min\{\frac{1-\eta}{\|A\|^2}, \frac{1-2\kappa}{\|A\|^2}\}$, $\alpha_n \in (\delta, 1-\frac{\beta}{c}-\delta)$, $\beta < c = \mu(1)$, and for a small enough $\delta > 0$.*

Proof. It follows from (1.2) that

$$\|x_n-y_n\| = \|J_1(x_n-y_n)\| = \|\gamma A^*J_2(I-T)Ax_n\|,$$

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and so, from (2.4) we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{2.6}$$

Since S is semi-compact, from (2.5), there exist subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that $\{y_{n_j}\}$ converges strongly to $x^* \in E_1$. Using (2.6), we know that $\{x_{n_j}\}$ converges strongly to x^*. By Theorem 2.2, we know that $\{x_n\}$ converges weakly to x, so we have $x^* = x$. Since $\lim_{n\to\infty} \|x_n - x\|$ exists and $\lim_{j\to\infty} \|x_{n_j} - x\| = 0$, we know that $\{x_n\}$ converges strongly to $x \in \Gamma$.

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Iterated binomial transform of the k-Lucas sequence

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Abstract

In this study, we apply "r" times the binomial transform to k-Lucas sequence. Also, the Binet formula, summation, generating function of this transform are found using recurrence relation. Finally, we give the properties of iterated binomial transform with classical Lucas sequence.

Keywords: k-Lucas sequence, iterated binomial transform, Pell sequence.

Ams Classification: 11B65, 11B83.

1 Introduction and Preliminaries

There are so many studies in the literature that concern about the special number sequences such as Fibonacci, Lucas and generalized Fibonacci anad Lucas numbers (see, for example [1]-[3], and the references cited therein). In Fibonacci and Lucas numbers, there clearly exists the term Golden ratio which is defined as the ratio of two consecutive of these numbers that converges to $\alpha = \frac{1+\sqrt{5}}{2}$. It is also clear that the ratio has so many applications in, specially, Physics, Engineering, Architecture, etc.[4]. Also, many generalizations of the Fibonacci sequence have been introduced and studied matrix applications of this sequence in [13]-[16].

For $n \geq 1$, k-Lucas sequence is defined by the recursive equation:

$$L_{k,n+1} = kL_{k,n} + L_{k,n-1}, \qquad L_{k,0} = 2 \text{ and } L_{k,1} = k. \tag{1.1}$$

In addition, some matrix-based transforms can be introduced for a given sequence. Binomial transform is one of these transforms and there are also other ones such as rising and falling binomial transforms(see [5]-[12]). Given

an integer sequence $X = \{x_0, x_1, x_2, \ldots\}$, the binomial transform B of the sequence X, $B(X) = \{b_n\}$, is given by

$$b_n = \sum_{i=0}^{n} \binom{n}{i} x_i.$$

In [10], authors gave the application of the several class of transforms to the k-Lucas sequence. For example, for $n \geq 1$, authors obtained recurrence relation of the binomial transform for k-Lucas sequence

$$b_{k,n+1} = (2+k)\, b_{k,n} - k b_{k,n-1}, \quad b_{k,0} = 2 \text{ and } b_{k,1} = k+2.$$

Falcon [11] studied the iterated application of some Binomial transforms to the k-Fibonacci sequence. For example, author obtained recurrence relation of the iterated binomial transform for k-Fibonacci sequence

$$c_{k,n+1}^{(r)} = (2r+k)\, c_{k,n}^{(r)} - \left(r^2 + kr - 1\right) c_{k,n-1}^{(r)}, \quad c_{k,0}^{(r)} = 0 \text{ and } c_{k,1}^{(r)} = 1.$$

Motivated by [11, 12], the goal of this paper is to apply iteratively the binomial transform to the k-Lucas sequence. Also, the properties of this transform are found by recurrence relation. Finally, the relation of between the transform and the iterated binomial transform of k-Fibonacci sequence by deriving new formulas are illustrated.

2 Iterated Binomial Transform of k-Lucas Sequences

In this section, we will mainly focus on iterated binomial transforms of k - Lucas sequences to get some important results. In fact, we will also present the recurrence relation, Binet formula, summation, generating function of the transform and relationships betweeen of the transform and iterated binomial transform of k-Fibonacci sequence.

The iterated binomial transform of the k-Lucas sequences is demonstrated by $B_k^{(r)} = \left\{ b_{k,n}^{(r)} \right\}$, where $b_{k,n}^{(r)}$ is obtained by applying "r" times the binomial transform to k-Lucas sequence. It is obvious that $b_{k,0}^{(r)} = 2$ and $b_{k,1}^{(r)} = 2r + k$.

The following lemma will be the key proof of the next theorems.

Lemma 2.1 *For $n \geq 0$ and $r \geq 1$, the following equality hold:*

$$b_{k,n+1}^{(r)} = b_{k,n}^{(r)} + \sum_{j=0}^{n} \binom{n}{j} b_{k,j+1}^{(r-1)}.$$

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Proof. By using definition of binomial transform and the well known binomial equality

$$\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1},$$

we obtain

$$\begin{aligned} b_{k,n+1}^{(r)} &= \sum_{j=0}^{n+1} \binom{n+1}{j} b_{k,j}^{(r-1)} \\ &= \sum_{j=1}^{n+1} \binom{n+1}{j} b_{k,j}^{(r-1)} + b_{k,0}^{(r-1)} \end{aligned}$$

$$\begin{aligned} b_{k,n+1}^{(r)} &= \sum_{j=1}^{n+1} \binom{n}{j} b_{k,j}^{(r-1)} + \sum_{j=1}^{n+1} \binom{n}{j-1} b_{k,j}^{(r-1)} + b_{k,0}^{(r-1)} \\ &= \sum_{j=0}^{n+1} \binom{n}{j} b_{k,j}^{(r-1)} + \sum_{j=0}^{n+1} \binom{n}{j-1} b_{k,j}^{(r-1)} \\ &= \sum_{j=0}^{n} \binom{n}{j} b_{k,j}^{(r-1)} + \sum_{j=-1}^{n} \binom{n}{j} b_{k,j+1}^{(r-1)} \\ &= b_{k,n}^{(r)} + \sum_{j=0}^{n} \binom{n}{j} b_{k,j+1}^{(r-1)} \end{aligned}$$

which is desired result. ■

In [10], the authors obtained the following equality for binomial transform of k-Lucas sequences. However, in here, we obtain the equality in terms of iterated binomial transform of the k-Lucas sequences as a consequence of Lemma 2.1. To do that we take $r = 1$ in Lemma 2.1:

$$b_{k,n+1} = b_{k,n} + \sum_{j=0}^{n} \binom{n}{j} L_{k,j+1}.$$

Theorem 2.1 *For $n \geq 0$ and $r \geq 1$, the recurrence relation of sequence $\left\{ b_{k,n}^{(r)} \right\}$ is*

$$b_{k,n+1}^{(r)} = (2r+k)\, b_{k,n}^{(r)} - \left(r^2 + kr - 1\right) b_{k,n-1}^{(r)}, \tag{2.1}$$

with initial conditions $b_{k,0}^{(r)} = 2$ and $b_{k,1}^{(r)} = 2r + k$.

Proof. The proof will be done by induction steps on r and n.

First of all, for $r=1$, from the equality 2.2 in [10], it is true $b_{k,n+1} = (2+k)\,b_{k,n} - kb_{k,n-1}$.

Let us consider definition of iterated binomial transform, then we have

$$b_{k,2}^{(r)} = k^2 + 2rk + 2r^2 + 2.$$

The initial conditions are

$$b_{k,0}^{(r)} = 2 \text{ and } b_{k,1}^{(r)} = 2r + k.$$

Hence, for $n=1$, the Eq. (2.1) is true, that is $b_{k,2}^{(r)} = (2r+k)\,b_{k,1}^{(r)} - \left(r^2 + kr - 1\right) b_{k,0}^{(r)}$.

Actually, by assuming the Eq. (2.1) holds for all $(r-1,n)$ and $(r,n-1)$, that is,

$$b_{k,n+1}^{(r-1)} = (2r-2+k)\,b_{k,n}^{(r-1)} - \left((r-1)^2 + k(r-1) - 1\right) b_{k,n-1}^{(r-1)},$$

and

$$b_{k,n}^{(r)} = (2r+k)\,b_{k,n-1}^{(r)} - \left(r^2 + kr - 1\right) b_{k,n-2}^{(r)}.$$

Now, by taking into account Lemma 2.1, we obtain

$$\begin{aligned} b_{k,n+1}^{(r)} &= b_{k,n}^{(r)} + \sum_{j=0}^{n} \binom{n}{j} b_{k,j+1}^{(r-1)} \\ &= \sum_{j=0}^{n} \binom{n}{j} b_{k,j}^{(r-1)} + \sum_{j=0}^{n} \binom{n}{j} b_{k,j+1}^{(r-1)} \\ &= \sum_{j=1}^{n} \binom{n}{j} \left(b_{k,j}^{(r-1)} + b_{k,j+1}^{(r-1)}\right) + b_{k,0}^{(r-1)} + b_{k,1}^{(r-1)}. \end{aligned}$$

By reconsidering our assumption, we write

$$\begin{aligned} b_{k,n+1}^{(r)} &= \sum_{j=1}^{n} \binom{n}{j} \left(b_{k,j}^{(r-1)} + (2r-2+k)\,b_{k,j}^{(r-1)} - \left(r^2 - 2r + kr - k\right) b_{k,j-1}^{(r-1)}\right) + b_{k,0}^{(r-1)} + b_{k,1}^{(r-1)} \\ &= (2r+k-1) \sum_{j=1}^{n} \binom{n}{j} b_{k,j}^{(r-1)} - \left(r^2 - 2r + kr - k\right) \sum_{j=1}^{n} \binom{n}{j} b_{k,j-1}^{(r-1)} + b_{k,0}^{(r-1)} + b_{k,1}^{(r-1)} \\ &= (2r+k-1) \sum_{j=0}^{n} \binom{n}{j} b_{k,j}^{(r-1)} - \left(r^2 - 2r + kr - k\right) \sum_{j=1}^{n} \binom{n}{j} b_{k,j-1}^{(r-1)} + b_{k,0}^{(r-1)} + b_{k,1}^{(r-1)} \\ &\quad - (2r+k-1)\,b_{k,0}^{(r-1)} \\ &= (2r+k-1)\,b_{k,n}^{(r)} - \left(r^2 - 2r + kr - k\right) \sum_{j=1}^{n} \binom{n}{j} b_{k,j-1}^{(r-1)} + (2-2r-k)\,b_{k,0}^{(r-1)} + b_{k,1}^{(r-1)}. \end{aligned}$$

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Then we have

$$b_{k,n+1}^{(r)}-(2r+k-1)\,b_{k,n}^{(r)}=-\left(r^2-2r+kr-k\right)\sum_{j=1}^{n}\binom{n}{j}b_{k,j-1}^{(r-1)}+4-2r-k. \tag{2.2}$$

By taking $n\to n-1$, it is

$$\begin{aligned}
b_{k,n}^{(r)} &= (2r+k-1)\,b_{k,n-1}^{(r)}-\left(r^2-2r+kr-k\right)\sum_{j=1}^{n-1}\binom{n-1}{j}b_{k,j-1}^{(r-1)}+4-2r-k\\
&= (2r+k-1)\,b_{k,n-1}^{(r)}-\left(r^2-2r+kr-k\right)\sum_{j=1}^{n}\left[\binom{n}{j}-\binom{n-1}{j-1}\right]b_{k,j-1}^{(r-1)}+4-2r-k\\
&= (2r+k-1)\,b_{k,n-1}^{(r)}-\left(r^2-2r+kr-k\right)\sum_{j=1}^{n}\binom{n}{j}b_{k,j-1}^{(r-1)}\\
&\quad+\left(r^2-2r+kr-k\right)\sum_{j=1}^{n}\binom{n-1}{j-1}b_{k,j-1}^{(r-1)}+4-2r-k
\end{aligned}$$

$$\begin{aligned}
b_{k,n}^{(r)} &= (2r+k-1)\,b_{k,n-1}^{(r)}-\left(r^2-2r+kr-k\right)\sum_{j=1}^{n}\binom{n}{j}b_{k,j-1}^{(r-1)}\\
&\quad+\left(r^2-2r+kr-k\right)\sum_{j=0}^{n-1}\binom{n-1}{j}b_{k,j}^{(r-1)}+4-2r-k\\
&= (2r+k-1)\,b_{k,n-1}^{(r)}-\left(r^2-2r+kr-k\right)\sum_{j=1}^{n}\binom{n}{j}b_{k,j-1}^{(r-1)}\\
&\quad+\left(r^2-2r+kr-k\right)b_{k,n-1}^{(r)}+4-2r-k\\
&= \left(r^2+kr-1\right)b_{k,n-1}^{(r)}-\left(r^2-2r+kr-k\right)\sum_{j=1}^{n}\binom{n}{j}b_{k,j-1}^{(r-1)}+4-2r-k.
\end{aligned}$$

Hence, we have

$$b_{k,n}^{(r)}-\left(r^2+kr-1\right)b_{k,n-1}^{(r)}=-\left(r^2-2r+kr-k\right)\sum_{j=1}^{n}\binom{n}{j}b_{k,j-1}^{(r-1)}+4-2r-k.$$

If last expression put in place in the equation (2.2), then we get

$$\begin{aligned}
b_{k,n+1}^{(r)} &= (2r+k-1)\,b_{k,n}^{(r)}+b_{k,n}^{(r)}-\left(r^2+kr-1\right)b_{k,n-1}^{(r)}\\
&= (2r+k)\,b_{k,n}^{(r)}-\left(r^2+kr-1\right)b_{k,n-1}^{(r)}
\end{aligned}$$

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which completed the proof of this theorem. ■

The characteristic equation of sequence $\left\{b_{k,n}^{(r)}\right\}$ in (2.1) is $\lambda^2-(2r+k)\lambda+r^2+kr-1=0$. Let λ_1 and λ_2 be the roots of this equation. Then, Binet's formulas of sequence $\left\{b_{k,n}^{(r)}\right\}$ can be expressed as

$$b_{k,n}^{(r)}=\left(\frac{k+\sqrt{k^2+4}}{2}+r\right)^n+\left(\frac{k-\sqrt{k^2+4}}{2}+r\right)^n. \tag{2.3}$$

In here, we obtain the equalities given in [10] in terms of iterated binomial transform of the k-Lucas sequences as a consequence of Theorem 2.1. To do that we take $r=1$ in Theorem 2.1 and the Eq. (2.3):

$$b_{k,n+1}=(2+k)\,b_{k,n}-kb_{k,n-1},$$

and

$$b_{k,n}=\left(\frac{k+2+\sqrt{k^2+4}}{2}\right)^n+\left(\frac{k+2-\sqrt{k^2+4}}{2}\right)^n.$$

Now, we give the sum of iterated binomial transform for k-Lucas sequences.

Theorem 2.2 *Sum of sequence* $\left\{b_{k,n}^{(r)}\right\}$ *is*

$$\sum_{i=0}^{n-1}b_{k,i}^{(r)}=\frac{\left(r^2+kr-1\right)b_{k,n-1}^{(r)}-b_{k,n}^{(r)}-k-2r+2}{r^2+kr-k-2r}.$$

Proof. By considering Eq. (2.3), we have

$$\sum_{i=0}^{n-1}b_{k,i}^{(r)}=\sum_{i=0}^{n-1}\left(\lambda_1^i+\lambda_2^i\right).$$

Then we obtain

$$\sum_{i=0}^{n-1}b_{k,i}^{(r)}=\left(\frac{\lambda_1^n-1}{\lambda_1-1}\right)+\left(\frac{\lambda_2^n-1}{\lambda_2-1}\right).$$

Afterward, by taking into account equations $\lambda_1.\lambda_2=r^2+kr-1$ and $\lambda_1+\lambda_2=k+2r$, we conclude

$$\sum_{i=0}^{n-1}b_{k,i}^{(r)}=\frac{\left(r^2+kr-1\right)b_{k,n-1}^{(r)}-b_{k,n}^{(r)}-k-2r+2}{r^2+kr-k-2r}.$$

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■

Note that, if we take $r=1$ in Theorem 2.2, we obtain the summation of binomial transform for k-Lucas sequence:

$$\sum_{i=0}^{n-1} b_{k,i} = b_{k,n} - k b_{k,n-1} + k$$

Theorem 2.3 *The generating function of the iterated binomial transform for* $\{L_{k,n}\}$ *is*

$$\sum_{i=0}^{\infty} b_{k,i}^{(r)} x^i = \frac{2-(2r+k)\,x}{1-(2r+k)\,x+(r^2+kr-1)\,x^2}.$$

Proof. Assume that $b(k,x,r) = \sum\limits_{i=0}^{\infty} b_{k,i}^{(r)} x^i$ is the generating function of the iterated binomial transform for $\{L_{k,n}\}$. From Theorem 2.1, we obtain

$$\begin{aligned}
b(k,x,r) &= b_{k,0}^{(r)} + b_{k,1}^{(r)} x + \sum_{i=2}^{\infty} \left((2r+k)\, b_{k,i-1}^{(r)} - \left(r^2+kr-1\right) b_{k,i-2}^{(r)} \right) x^i \\
&= b_{k,0}^{(r)} + b_{k,1}^{(r)} x - (2r+k)\, b_{k,0}^{(r)} x + (2r+k)\, x \sum_{i=0}^{\infty} b_{k,i}^{(r)} x^i \\
&\quad - \left(r^2+kr-1\right) x^2 \sum_{i=0}^{\infty} b_{k,i}^{(r)} x^i \\
&= b_{k,0}^{(r)} + \left(b_{k,1}^{(r)} - (2r+k)\, b_{k,0}^{(r)} \right) x + (2r+k)\, x b(k,x,r) \\
&\quad - \left(r^2+kr-1\right) x^2\, b(k,x,r).
\end{aligned}$$

Now rearrangement of the equation implies that

$$b(k,x,r) = \frac{b_{k,0}^{(r)} + \left(b_{k,1}^{(r)} - (2r+k)\, b_{k,0}^{(r)} \right) x}{1-(2r+k)\,x+(r^2+kr-1)\,x^2},$$

which equals to the $\sum\limits_{i=0}^{\infty} b_{k,i}^{(r)} x^i$ in theorem. Hence, the result. ■

In here, we obtain the generating function given in [10] in terms of iterated binomial transform of the k-Lucas sequences as a consequence of Theorem 2.3. To do that we take $r=1$ in Theorem 2.3:

$$\sum_{i=0}^{\infty} b_{k,i} x^i = \frac{2-(2+k)\,x}{1-(2+k)\,x+kx^2}.$$

7

In the following theorem, we present the relationship between the iterated binomial transform of k-Lucas sequence and iterated binomial transform of k-Fibonacci sequence.

Theorem 2.4 *For $n > 0$, the relationship of between the transforms $\left\{b_{k,n}^{(r)}\right\}$ and $\left\{c_{k,n}^{(r)}\right\}$ is illustrated by following way:*

$$b_{k,n}^{(r)} = c_{k,n+1}^{(r)} - \left(r^2 + kr - 1\right) c_{k,n-1}^{(r)}, \tag{2.4}$$

where $b_{k,n}^{(r)}$ is the iterated binomial transform of k-Lucas sequence and $c_{k,n}^{(r)}$ is the iterated binomial transform of k-Fibonacci sequence.

Proof. By using the Eq.(2.4), let be

$$b_{k,n}^{(r)} = Xc_{k,n+1}^{(r)} + Yc_{k,n-1}^{(r)}.$$

If we take $n = 1$ and 2, we have the system

$$\begin{cases} b_{k,1}^{(r)} = Xc_{k,2}^{(r)} + Yc_{k,0}^{(r)}, \\ b_{k,2}^{(r)} = Xc_{k,3}^{(r)} + Yc_{k,1}^{(r)}. \end{cases}$$

By considering definition of the iterated binomial transforms for k-Lucas, k-Fibonacci sequence and Cramer rule for the system, we obtain

$$\begin{cases} 2r + k = (2r + k) X, \\ k^2 + 2rk + 2r^2 + 2 = \left(3r^2 + 3rk + k^2 + 1\right) X + Y \end{cases}$$

and

$$X = 1 \text{ and } Y = -\left(r^2 + kr - 1\right)$$

which is completed the proof of this theorem. ■

Note that, if we take $r = 1$ in Theorem 2.4, we obtain the relationship of between the binomial transform for k-Lucas sequence and the binomial transform for k-Fibonacci sequence:

$$b_{k,n} = c_{k,n+1} - kc_{k,n-1}.$$

Corollary 2.1 *We should note that choosing $k = 1$ in the all results of section 2, it is actually obtained some properties of the iterated binomial transform for classical Lucas sequence such that the recurrence relation, Binet formula, summation, generating function and relationship of between binomial transforms for Fibonacci and Lucas sequences.*

Corollary 2.2 *We should note that choosing $k = 2$ in the all results of section 2, it is actually obtained some properties of the iterated binomial transform for classical Pell-Lucas sequence such that the recurrence relation, Binet formula, summation, generating function and relationship of between binomial transforms for Pell and Pell-Lucas sequences.*

Conclusion 2.1 *In this paper, we define the iterated binomial transform for k-Lucas sequence and present some properties of this transform. By the results in Sections 2 of this paper, we have a great opportunity to compare and obtain some new properties over this transform. This is the main aim of this paper. Thus, we extend some recent result in the literature.*

In the future studies on the iterated binomial transform for number sequences, we expect that the following topics will bring a new insight.

(1) *It would be interesting to study the iterated binomial transform for Fibonacci and Lucas matrix sequences,*

(2) *Also, it would be interesting to study the iterated binomial transform for Pell and Pell-Lucas matrix sequences.*

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Nielsen fixed point theory for digital images

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Abstract

In this paper, we introduce the Nielsen fixed point theory in digital images. We also deal with some important properties of the Nielsen number and calculate the Nielsen number of some digital images. We get some new results using digital covering maps and Nielsen number.

Keywords: Fixed point, Nielsen number, digital homotopy.
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1 Introduction

Digital topology is often used in computer graphics, pattern recognition and image processing. This topic has been studied by important researchers such as Rosenfeld, Kong, Kopperman, Boxer, Karaca, Han, etc. Their goal is to determine not only similarities but also differences between digital images and topology.

Fixed point theory with applications is an important area in topology. This theory continues to develop with new computations and come out of new invariants. Nielsen fixed point theorem is a notable theorem in this theory because it gives a way to count fixed points. One of the main goals in digital topology is to classify digital images. For this reason, we use the Nielsen number which is a powerful invariant for digital images.

In 1920s, Jakop Nielsen introduced the Nielsen theory and the Nielsen number. He focused on both the existence problem of fixed points and the problem of determining the minimal number of fixed points in the homotopy classes. He did this by introducing the Nielsen number of a self map. This number is a homotopy invariant lower bound for the number of fixed points of the map. In this area, there are significant works [10, 11, 12, 16, 17, 18, 19].

Boxer [6] introduces the digital covering space and showed that the existence of digital universal covering spaces. Boxer and Karaca [7] classify digital covering spaces using the conjugacy class corresponding to a digital covering space. Boxer and Karaca [8] study digital versions of some properties of covering spaces from algebraic topology. Karaca and Ege [20] get some results related to the simplicial homology groups of $2D$ digital images. Ege and Karaca [13] give characteristic properties of the simplicial homology groups.

This paper is organized as follows. The second section provides the general notions of digital images, digital homotopy, digital covering spaces and digital homology groups. In Section 3 we present the Nielsen fixed point theorem for digital images, give some examples and properties. In Section 4 we discuss about the relation between Nielsen theory and digital universal covering space. We finally make some conclusions about this topic.

2 Preliminaries

A digital image consists of a pair (X, κ), where $\mathbb{Z}$ is the set of integers, $X \subset \mathbb{Z}^n$ for some positive integer n, and κ indicates an adjacency relation for the members of X.

Definition 2.1. [3]. For a positive integer l with $1 \leq l \leq n$ and two distinct points $p = (p_1, p_2, \ldots, p_n)$, $q = (q_1, q_2, \ldots, q_n) \in \mathbb{Z}^n$, p and q are *c_l-adjacent*, if

(1) there are at most l indices i such that $|p_i - q_i| = 1$, and

(2) for all other indices j such that $|p_j - q_j| \neq 1$, $p_j = q_j$.

The notation c_l represents the number of points $q \in \mathbb{Z}^n$ that are adjacent to a given point $p \in \mathbb{Z}^n$. Thus, in $\mathbb{Z}$, we have $c_1 = 2$-adjacency; in $\mathbb{Z}^2$, we have $c_1 = 4$-adjacency and $c_2 = 8$-adjacency; in $\mathbb{Z}^3$, we have $c_1 = 6$-adjacency, $c_2 = 18$-adjacency, and $c_3 = 26$-adjacency [5]. *A κ-neighbor* of $p \in \mathbb{Z}^n$ [3] is a point of $\mathbb{Z}^n$ that is κ-adjacent to p.

A digital interval [2] is defined by $[a,b]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid a \leq z \leq b\}$ where $a, b \in \mathbb{Z}$ and $a < b$. A digital image $X \subset \mathbb{Z}^n$ is *κ-connected* [15] if and only if for every pair of different points $x, y \in X$, there is a set $\{x_0, x_1, \ldots, x_r\}$ of points of a digital image X such that $x = x_0$, $y = x_r$ and x_i and x_{i+1} are κ-neighbors where $i = 0, 1, \ldots, r-1$.

Definition 2.2. [3]. Let $X \subset \mathbb{Z}^{n_0}$ and $Y \subset \mathbb{Z}^{n_1}$ be digital images with κ_0-adjacency and κ_1-adjacency, respectively. A function $f : X \longrightarrow Y$ is said to be *(κ_0, κ_1)-continuous* if for every κ_0-connected subset U of X, $f(U)$ is a κ_1-connected subset of Y. We say that such a function is digitally continuous.

Proposition 2.3. *[3]. Let $X \subset \mathbb{Z}^{n_0}$ and $Y \subset \mathbb{Z}^{n_1}$ be digital images with κ_0-adjacency and κ_1-adjacency, respectively. The function $f : X \to Y$ is (κ_0, κ_1)-continuous if and only if for every κ_0-adjacent points $\{x_0, x_1\}$ of X, either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are κ_1-adjacent in Y.*

In a digital image X, if there is a $(2, \kappa)$-continuous function $f : [0, m]_{\mathbb{Z}} \to X$ such that $f(0) = x$ and $f(m) = y$, then there exists *a digital κ-path* [6] from x to y. If $f(0) = f(m)$, then we say that f is *digital κ-loop* and the point $f(0)$ is the base point of the loop f. When a digital loop f is a constant function, it is said to be *a trivial loop*.

Definition 2.4. Let $(X, \kappa_0) \subset \mathbb{Z}^{n_0}$ and $(Y, \kappa_1) \subset \mathbb{Z}^{n_1}$ be digital images. A function $f : X \to Y$ is called *a (κ_0, κ_1)-isomorphism* [2] if f is (κ_0, κ_1)-continuous and bijective and $f^{-1} : Y \to X$ is (κ_1, κ_0)-continuous.

Definition 2.5. [3]. Let $(X, \kappa_0) \subset \mathbb{Z}^{n_0}$ and $(Y, \kappa_1) \subset \mathbb{Z}^{n_1}$ be digital images. We say that two (κ_0, κ_1)-continuous functions $f, g : X \to Y$ are digitally (κ_0, κ_1)-homotopic in Y if there is a positive integer m and a function $H : X \times [0, m]_{\mathbb{Z}} \to Y$ such that

- for all $x \in X$, $H(x, 0) = f(x)$ and $H(x, m) = g(x)$;
- for all $x \in X$, the induced function $H_x : [0, m]_{\mathbb{Z}} \to Y$ defined by
$$H_x(t) = H(x, t) \quad \text{for all } t \in [0, m]_{\mathbb{Z}},$$
is $(2, \kappa_1)$-continuous; and
- for all $t \in [0, m]_{\mathbb{Z}}$, the induced function $H_t : X \to Y$ defined by
$$H_t(x) = H(x, t) \quad \text{for all } x \in X,$$
is (κ_0, κ_1)-continuous.

The function H is called a digital (κ_0, κ_1)-homotopy between f and g. If these functions are digitally (κ_0, κ_1)-homotopic, it is denoted $f \simeq_{(\kappa_0, \kappa_1)} g$. The digital (κ_0, κ_1)-homotopy relation [3] is equivalence among digitally continuous functions $f : (X, \kappa_0) \to (Y, \kappa_1)$.

If $f : [0, m_1]_{\mathbb{Z}} \to X$ and $g : [0, m_2]_{\mathbb{Z}} \to X$ are digital κ-paths with $f(m_1) = g(0)$, then define the product $(f * g) : [0, m_1 + m_2]_{\mathbb{Z}} \to X$ [3] by
$$(f * g)(t) = \begin{cases} f(t), & t \in [0, m_1]_{\mathbb{Z}} \\ g(t - m_1), & t \in [m_1, m_1 + m_2]_{\mathbb{Z}}. \end{cases}$$
Let f and f' be κ-loops in a digital image (X, x_0). We say f' is a trivial extension of f [3] if there are sets of κ-paths $\{f_1, \ldots, f_r\}$ and $\{F_1, \ldots, F_p\}$ in X such that:
(1) $r \leq p$,
(2) $f = f_1 * \ldots * f_r$,
(3) $f' = F_1 * \ldots * F_p$,
(4) There are indices $1 \leq i_1 < i_2 < \ldots < i_r \leq p$ such that $F_{i_j} = f_j$, $1 \leq j \leq r$ and $i \neq \{i_1, \ldots, i_r\}$ implies F_i is a trivial loop.

If $f, g : [0, m]_{\mathbb{Z}} \to X$ are κ-paths such that $f(0) = g(0)$ and $f(m) = g(m)$, then a homotopy
$$H : [0, m]_{\mathbb{Z}} \times [0, M]_{\mathbb{Z}} \to X$$
between f and g such that for all $t \in [0, M]_{\mathbb{Z}}$, $H(0, t) = f(0)$ and $H(m, t) = f(m)$, holds the endpoints fixed.

Two loops f, f_0 with the same base point $x_0 \in X$ belong to the same loop class $[f]_X$ if they have trivial extensions that can be joined by a homotopy that holds the endpoints fixed (see [4]).

Let (E, κ) be a digital image and let ε be a positive integer. The κ-neighborhood of $e_0 \in E$ with radius ε is the set
$$N_\kappa(e_0, \varepsilon) = \{e \in E \mid l_\kappa(e_0, e) \leq \varepsilon\} \cup \{e_0\},$$
where $l_\kappa(e_0, e)$ is the length of a shortest κ-path from e_0 to e in E (see [14]).

Definition 2.6. [6]. Let (E, κ_0) and (B, κ_1) be digital images. A map $p : E \to B$ is called a (κ_0, κ_1)-covering map if the followings are true:
1. p is a (κ_0, κ_1)-continuous surjection.
2. for each $b \in B$, there exists an indexing set M such that $p^{-1}(b)$ can be indexed as $p^{-1}(b) = \{e_i | i \in M\}$ and the following conditions hold:
- $p^{-1}(N_{\kappa_1}(b, 1)) = \bigcup_{i \in M} N_{\kappa_0}(e_i, 1)$,
- if $i, j \in M$, $i \neq j$, then $N_{\kappa_0}(e_i, 1) \cap N_{\kappa_0}(e_j, 1) = \emptyset$,
- the restriction map $p|_{N_{\kappa_0}(e_i,1)} : N_{\kappa_0}(e_i, 1) \to N_{\kappa_1}(b, 1)$ is a (κ_0, κ_1)-isomorphism for all $i \in M$.

Let (E, κ_0), (B, κ_1) and (X, κ_2) be digital images, let $p : E \to B$ be a (κ_0, κ_1)-covering map, and $f : X \to B$ be (κ_2, κ_1)-continuous. A lifting of f with respect to p is a (κ_2, κ_0)-continuous function $\tilde{f} : X \to E$ such that $p \circ \tilde{f} = f$ (see [14]).

Definition 2.7. [6]. Let (E_0, p_0, B) be a (κ_0, κ_B)-covering. Suppose $\mathcal{C}$ is a set of (κ_E, κ_B)-coverings of B such that for every $(E, p, B) \in \mathcal{C}$, there is a (κ_0, κ_E)-covering (E_0, p_E, E). Then the pair (E_0, p_0) is a universal covering space of B for the set $\mathcal{C}$.

Definition 2.8. [21]. Let S be a set of nonempty subset of a digital image (X, κ). Then the members of S are called *simplexes* of (X, κ), if the followings hold:
- If p and q are distinct points of $s \in S$, then p and q are κ-adjacent,
- If $s \in S$ and $\emptyset \neq t \subset s$, then $t \in S$.

An m-simplex is a simplex S such that $|S| = m + 1$. For a digital m-simplex P, if P' is a nonempty proper subset of P, then P' is called *a face* of P.

Definition 2.9. [1]. Let (X, κ) be a finite collection of digital m-simplices, $0 \leq m \leq d$ for some non-negative integer d. If the followings hold, then (X, κ) is called *a finite digital simplicial complex*:
- If P belongs to X, then every face of P also belongs to X.
- If $P, Q \in X$, then $P \cap Q$ is either empty or a common face of P and Q.

Definition 2.10. [1]. Let $(X, \kappa) \subset \mathbb{Z}^n$ be a digital oriented simplicial complex with m-dimension. $C_q^\kappa(X)$ is a free abelian group with basis all digital (κ, q)-simplices in X. A homomorphism $\partial_q : C_q^\kappa(X) \longrightarrow C_{q-1}^\kappa(X)$ called *the boundary operator*. If $\sigma = [v_0, \ldots, v_q]$ is an oriented simplex with $0 < q \leq m$, ∂_q is defined by

$$\partial_q \sigma = \partial_q [v_0, \ldots, v_q] = \sum_{i=0}^{q} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_q]$$

where $\hat{v}_i$ means the vertex v_i is to be deleted from the array.

We remark that for $q < 0, m < q$, since $C_q^\kappa(X)$ is the trivial group, the operator ∂_q is the trivial homomorphism for $q \leq 0$, $m < q$. We notice that $\partial_{q-1} \circ \partial_q = 0$ [1] for $q \geq 0$.

Definition 2.11. [1]. Let $(X, \kappa) \subset \mathbb{Z}^n$ be a digital oriented simplicial complex with m-dimension.
- $Z_q^\kappa(X) = \text{Ker } \partial_q$ is called the group of digital simplicial q-cycles.
- $B_q^\kappa(X) = \text{Im } \partial_{q+1}$ is called the group of digital simplicial q-boundaries.
- $H_q^\kappa(X) = Z_q^\kappa(X) / B_q^\kappa(X)$ is called the qth digital simplicial homology group.

3 Nielsen Theory for Digital Images

Let (X, κ) be a digital image and let $f : X \to X$ be a digital map. The fixed point set of f is $Fix(f) = \{x \in X : f(x) = x\}$. The main object of study in topological fixed point theory is the minimum number of fixed points which is denoted by $M[f]$ among all digital maps (κ, κ)-homotopic to f. For example, $M[f] = 0$ means that there is a digital map g which is (κ, κ)-homotopic to f such that $g(x) \neq x$ for all $x \in X$.

To calculate $M[f]$ we have to examine the fixed point sets of every map homotopic to f. In the fixed point theory, it is made use of a homotopy invariant, called the Nielsen number of f. Its computation requires only a knowledge of the map f itself.

Definition 3.1. Let $f : (X, \kappa_1) \longrightarrow (Y, \kappa_2)$ be a (κ_1, κ_2)-continuous map where (X, κ_1) and (Y, κ_2) are digital images. Then f induces homomorphisms $f_* : H_*^{\kappa_1}(X) \longrightarrow H_*^{\kappa_2}(Y)$ and f_* can be thought of as a homomorphisms of the integers. The integer $deg(f)$ to which the number 1 gets sent is called the degree of the map f.

Definition 3.2. Let (X, κ) be a digital image, $A \subset X$ and $f : A \to X$ a digital map. We define the *fixed point index* of f as $ind(f) = deg(F)$ where $F(x) = x - f(x)$ and $x \in X$.

Some properties of fixed point index can be given as follows. We don't prove these because they are proved similarly in Algebraic Topology.

1. (Homotopy invariance) Let $A \subset X \times [0,m]_{\mathbb{Z}}$ be digital image with κ-adjacency and $F : A \to X$ be a digital map such that
$$Fix(F) = \{(x,t) \in A : F(x,t) = x\}.$$
Then $ind(f_0) = ind(f_m)$, where $f_t = F(-,t)$ for $0 \leq t < m$ and a positive integer m.
2. (Commutativity) Let (X, κ_1) and (Y, κ_2) be digital images and let $f : A \to Y$ and $g : B \to X$ be digital (κ_1, κ_2)-continuous maps, respectively, where $A \subset X$, $B \subset Y$. Then $Fix(gf) = Fix(fg)$ and $ind(fg) = ind(gf)$.

Now we define Nielsen number for digital images.

Definition 3.3. Let (X, κ) be a digital image and $f : (X, \kappa) \to (X, \kappa)$ a self-map. Two fixed points $x, y \in Fix(f)$ are Nielsen related if and only if there is a κ-path $c : [0,m]_{\mathbb{Z}} \to X$ satisfying $c(0) = x$, $c(m) = y$ and the κ-paths c, $f \circ c$ are fixed end point homotopic, i.e. there is a digital map $H : [0,n]_{\mathbb{Z}} \times [0,m]_{\mathbb{Z}} \to X$ satisfying

$$H(t,0) = c(t), \quad H(t,m) = f \circ c(t), \quad H(0,s) = x, \quad H(n,s) = y.$$

This is an equivalence relation, hence $Fix(f)$ splits into disjoint Nielsen classes. A fixed point class F is essential if its index is nonzero. The number of essential fixed point classes is called the *Nielsen number* of f, denoted $N(f)$.

We give some characteristic examples about the Nielsen number.

Example 3.4. Let (X, κ) be a digital image. If $f : X \to X$ is a constant digital map, then $N(f) = 1$.

Since the boundary $Bd(I^{n+1})$ of an $(n+1)$-cube I^{n+1} is homeomorphic to n-sphere S^n, we can represent a digital sphere by using the boundary of a digital cube. Boxer [5] defines sphere-like digital image as $S_n = [-1,1]_{\mathbb{Z}}^{n+1} \setminus \{0_{n+1}\} \subset \mathbb{Z}^{n+1}$, where 0_n denotes the origin of $\mathbb{Z}^n$.

Example 3.5. $S_1 = \{c_0 = (1,0), c_1 = (1,1), c_2 = (0,1), c_3 = (-1,1), c_4 = (-1,0), c_5 = (-1,-1), c_6 = (0,-1), c_7 = (1,-1)\}$ is digital 1-sphere with 4-adjacency in $\mathbb{Z}^2$. Let $f : (S_1, 4) \to (S_1, 4)$ be a digital map of degree 1. Then f can be considered as identity map and is $(4,4)$-homotopic to a fixed point free map. Thus we have $N(f) = 0$.

Let's give some important properties of Nielsen number for digital images.

Theorem 3.6. *Let (X, κ) be any digital image. If $f \simeq_{(\kappa,\kappa)} g : X \to X$, then $N(f) = N(g)$.*

Proof. We must show that there is a bijection between sets of essential classes of f and g. Let $H(t,s)$ be a digital (κ,κ)-homotopy from f and g. For every Nielsen class $A \subset Fix(f)$, there is one $A' \subset Fix(H)$ containing A. Let

$$B = \{x \in X \mid (x,m) \in A'\}.$$

So B is a Nielsen class of g or is empty. From homotopy invariance index property, we have $ind(f,A) = ind(g,B)$. If A is essential, then B is essential. As a result, we find a map from the set of essential classes of f to the set of essential classes of g.

On the other hand, $H(x, m-t)$ gives the inverse map. Consequently, we get $N(f) = N(g)$. □

Theorem 3.7. *Let (X, κ) be a digital image and $f : X \to X$ be a digital map. Any digital map g digital (κ,κ)-homotopic to f has at least $N(f)$ fixed points.*

Proof. Using Theorem 3.6, we have $N(f) = N(g)$. Since each essential Nielsen class of g is nonempty, we get $M[g] \geq N(g)$ where
$$M[g] = \min\{\#Fixg \mid g \simeq_{(\kappa,\kappa)} f : X \to X\}.$$
□

Theorem 3.8. *Let (X, κ) and (Y, κ') be any digital images, $f : X \to Y$ and $g : Y \to X$ be digital maps. Then $N(g \circ f) = N(f \circ g)$.*

Proof. For digital maps f and g, if we use commutativity property of fixed point index, i.e. $Fix(f \circ g) = Fix(g \circ f)$ and $ind(f \circ g) = ind(g \circ f)$, we have a bijection which preserves index between the sets of essential Nielsen classes. As a result, we have $N(g \circ f) = N(f \circ g)$. □

Lemma 3.9. *Let $A \subset X$ be digital image with κ-adjacency and $f : X \to X$ be digital map such that $f(X) \subset A$, where X is any digital image with κ-adjacency. If $f_A : A \to A$ is the restriction of f, then $N(f_A) = N(f)$.*

Proof. Let $i : A \to X$ be inclusion map. Assume that $g : X \to A$ is given by $g(x) = f(x)$. Using Theorem 3.8, we conclude

$$N(f) = N(i \circ g) = N(g \circ i) = N(f_A)$$

because $i \circ g = f$ and $g \circ i = f_A$. □

Theorem 3.10. *Let (X, κ) and $(Y, \kappa^{'})$ be any two digital images. Suppose that $h : X \to Y$ is a digital $(\kappa, \kappa^{'})$-homotopy equivalence and let the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & X \arrow[d, "h"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

be digital $(\kappa, \kappa^{'})$-homotopy commutative, i.e. $h \circ f \simeq_{(\kappa,\kappa')} g \circ h$. Then $N(f) = N(g)$.

Proof. Assume that the digital $(\kappa^{'}, \kappa)$-homotopy inverse of h is $m : Y \to X$. Then $m \circ h \simeq_{(\kappa,\kappa)} 1_X$ and $h \circ m \simeq_{(\kappa',\kappa')} 1_Y$. By Theorem 3.6 and Theorem 3.8, we have

$$\begin{aligned} N(f) = N(f(mh)) = N((fm)h) &= N(h(fm)) \\ &= N((hf)m) \\ &= N((gh)m) \\ &= N(g(hm)) \\ &= N(g). \end{aligned}$$

□

Theorem 3.11. *Let (X, κ) be a digital image and $f : (X, \kappa) \to (X, \kappa)$ be a digital map. $N(f)$ is a lower bound for the number of fixed points in the homotopy class of f, i.e.*

$$0 \leq N(f) \leq M[f] := \min\{\#Fixg \mid g \simeq_{(\kappa,\kappa)} f : X \to X\}.$$

Proof. By the definition of Nielsen number, we have $N(f) \geq 0$. On the other hand, since we know that each Nielsen class contains at least one fixed point of f, we conclude that $0 \leq N(f) \leq M[f]$. □

4 Nielsen Theory and Digital Universal Covering Spaces

Since there is a connection between the digital fundamental group and the digital universal covering of a space, same results can be also obtained by the lifts of the considered maps to the digital universal coverings. Let (X, κ) be a digital image and $p : \tilde{X} \to \tilde{X}$ be the digital universal covering of X, with group π of covering transformations. Let $\tilde{f} : \tilde{X} \to \tilde{X}$ be a lifting of f, i.e., have a commutative diagram

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "p"'] & \tilde{X} \arrow[d, "p"] \\ X \arrow[r, "f"'] & X \end{tikzcd}$$

If $\tilde{f}^{'}$ is another lifting of $\tilde{f}$, then $\tilde{f}^{'} = \alpha \circ \tilde{f}$ for some $\alpha \in \pi$. The set of all liftings of f is $\{\alpha \circ \tilde{f} \mid \alpha \in \pi\}$.

For any $\alpha \in \pi$, $\tilde{f} \circ \alpha$ is a lifting of f and so we have $\alpha^{'} \circ \tilde{f} = \tilde{f} \circ \alpha$ for some $\alpha^{'} \in \pi$. This defines a homomorphism $\varphi : \pi \to \pi$ given by $\varphi(\alpha) = \alpha^{'}$. Define the Reidemeister action of π on π as follows:

$$\begin{aligned} &\pi \times \pi \to \pi \\ &(\gamma, \alpha) \mapsto \gamma\alpha\varphi(\gamma)^{-1}, \end{aligned}$$

where $\gamma, \alpha \in \pi$. This defines an equivalence relation. We say that the Reidemeister classes of its equivalence classes. The set of the Reidemeister classes determined by φ is denoted by $\mathcal{R}[\varphi] = \{[\alpha] \mid \alpha \in \pi\}$.

Theorem 4.1. *Let (X, κ) be a digital image, $f : X \to X$ be a digital map and $\tilde{f} : \tilde{X} \to \tilde{X}$ be a lifting of f. Then $[\alpha] = [\alpha^{'}]$ if and only if $p(Fix(\alpha \circ \tilde{f})) = p(Fix(\alpha^{'} \circ \tilde{f}))$, where $p : \tilde{X} \to X$ is a digital covering map of f and $\alpha, \alpha^{'} \in \pi$.*

Proof. For necessary condition, since the fixed point sets of any two digital homotopic maps are same, we conclude that

$$\begin{aligned}[\alpha] = [\alpha^{'}] &\Rightarrow \alpha \simeq_{(\tilde{\kappa},\tilde{\kappa})} \alpha^{'} \\ &\Rightarrow \alpha \circ \tilde{f} \simeq_{(\tilde{\kappa},\tilde{\kappa})} \alpha^{'} \circ \tilde{f} \\ &\Rightarrow Fix(\alpha \circ \tilde{f}) = Fix(\alpha^{'} \circ \tilde{f}) \\ &\Rightarrow p(Fix(\alpha \circ \tilde{f})) = p(Fix(\alpha^{'} \circ \tilde{f})).\end{aligned}$$

For sufficient condition, let $a \in p(Fix(\alpha \circ \tilde{f})) = p(Fix(\alpha^{'} \circ \tilde{f}))$. Then we have $p(\tilde{a}) = a$ and $p(\tilde{a^{'}}) = a$. Moreover, we get

$$\alpha \circ \tilde{f}(\tilde{a}) = \tilde{a} = 1_{\tilde{X}}(\tilde{a}) \quad \text{and} \quad \alpha^{'} \circ \tilde{f}(\tilde{a}) = \tilde{a^{'}} = 1_{\tilde{X}}(\tilde{a^{'}})$$

Finally, we can say

$$\alpha(\tilde{f}(\tilde{a})) = \alpha^{'}(\tilde{f}(\tilde{a})) \quad \Rightarrow \quad \alpha = \alpha^{'} \quad \Rightarrow \quad \alpha \simeq_{(\tilde{\kappa},\tilde{\kappa})} \alpha^{'},$$

where $\tilde{a}$ is any point of $\tilde{X}$. As a result, we have $[\alpha] = [\alpha^{'}]$. □

Corollary 4.2. *If $p(Fix(\alpha\tilde{f}))$ is any fixed point class, then*

$$Fix(f) = \coprod_{[\alpha]\in\mathcal{R}[\varphi]} p(Fix(\alpha\tilde{f})),$$

where $[\alpha]$ is a Reidemeister class.

Lemma 4.3. *Let $(X,\kappa), (\tilde{X},\tilde{\kappa})$ be digital images, $f : X \to X$ be a digital map and $\tilde{f} : \tilde{X} \to \tilde{X}$ be any lifting of f. Then any two points in $p(Fix(\tilde{f})) \subset Fix(f)$ are Nielsen related. We have also*

$$Fix(f) = \bigcup_{\tilde{f'}} p(Fix(\tilde{f'}))$$

where $\tilde{f'}$ is a lifting of f.

Proof. Let a and b be any two points in $p(Fix(\tilde{f}))$. We say that

$$p(\tilde{a}) = \tilde{a}, \quad p(\tilde{b}) = \tilde{b}, \quad \tilde{f}(\tilde{a}) = \tilde{a}, \quad \tilde{f}(\tilde{b}) = \tilde{b},$$

for some $\tilde{a}, \tilde{b} \in Fix(\tilde{f})$. We denote a $\tilde{\kappa}$-path from $\tilde{a}$ to $\tilde{b}$ in $\tilde{X}$ by $\tilde{\theta}$. There is a digital homotopy $\tilde{H}$ such that

$$\tilde{H} : \tilde{X} \times [0,m]_{\mathbb{Z}} \to \tilde{X}$$

$$\tilde{H}(\tilde{x},0) = \tilde{\theta} \quad \text{and} \quad \tilde{H}(\tilde{x},m) = \tilde{f} \circ \tilde{\theta}$$

because $\tilde{X}$ is $\tilde{\kappa}$-connected digital image. Then we have $p \circ \tilde{H} = H : X \times [0,m']_{\mathbb{Z}} \to X$ is a digital homotopy between two κ-paths

$$\theta = p \circ \tilde{\theta} \quad \text{and} \quad f \circ \theta = p \circ (\tilde{f} \circ \tilde{\theta})$$

which join two points $a, b \in Fix(f)$. As a result, a and b are Nielsen related points.

Now we prove the latter statement. Let $u \in Fix(f)$ and $\tilde{u} \in p^{-1}(u)$. Then $f(u) = u$ and $p^{-1}(u) = \tilde{u}$. Moreover, $\tilde{f'}(\tilde{u}) = \tilde{u}$ because $\tilde{f'}$ is a lifting of f. We can say the following result.

$$\tilde{f'}(\tilde{u}) = \tilde{u} \quad \Rightarrow \quad p \circ \tilde{f'}(\tilde{u}) = p(\tilde{u}) = u \quad \Rightarrow \quad u \in p(Fix(\tilde{f'})).$$

Consequently, we have $Fix(f) = \bigcup_{\tilde{f'}} p(Fix(\tilde{f'}))$. □

Let $p : \tilde{X} \to X$ be a digital universal covering map. Let

$$\mathcal{O}_X = \{\alpha \in \tilde{X} \to \tilde{X} : p \circ \alpha = p\}$$

denote the group of deck transformations of this digital covering map.

Lemma 4.4. *Let $\mathcal{C}$ be the set of liftings of f and $\tilde{f}, \tilde{f'} \in \mathcal{C}$. If $p(Fix(\tilde{f})) = p(Fix(\tilde{f'})) \neq \emptyset$, then there is an $\alpha \in \mathcal{O}_X$ such that $\alpha \circ \tilde{f} = \tilde{f'} \circ \alpha$.*

Proof. If $p(Fix(\tilde{f})) = p(Fix(\tilde{f}')) \neq \emptyset$, then there are two points $\tilde{x}, \tilde{x'}$ such that $p(\tilde{x}) = p(\tilde{x'})$ where

$$\tilde{x} \in Fix(\tilde{f}) \quad \Rightarrow \quad \tilde{f}(\tilde{x}) = \tilde{x} \quad \text{and} \quad \tilde{x'} \in Fix(\tilde{f}') \quad \Rightarrow \quad \tilde{f}'(\tilde{x'}) = \tilde{x'}.$$

Since $\alpha \in \mathcal{O}_X$, i.e. $p \circ \alpha = p$, we have $\alpha(\tilde{x}) = \tilde{x'}$. We conclude that

$$\tilde{f}' \circ \alpha(\tilde{x}) = \tilde{f}'(\tilde{x'}) = \tilde{x'} = \alpha(\tilde{x}) = \alpha(\tilde{f}(\tilde{x})) = \alpha \circ \tilde{f}(\tilde{x}).$$

As a result, we have $\alpha\tilde{f} = \tilde{f}'\alpha$. □

Lemma 4.5. *Let $\tilde{f}' = \alpha\tilde{f}\alpha^{-1}$ for an $\alpha \in \mathcal{O}_X$. Then $p(Fix(\tilde{f})) = p(Fix(\tilde{f}'))$.*

Proof. By assumption, we have $p(\alpha(\tilde{x})) = p(\tilde{x})$. If $u \in p(Fix(\tilde{f}))$, then $p(\tilde{x}) = u$ and $\tilde{f}(\tilde{x}) = \tilde{x}$. Since

$$p \circ \tilde{f}(\tilde{x}) = u \quad \Rightarrow \quad p \circ \alpha^{-1} \circ \tilde{f}' \circ \alpha(\tilde{x}) = p \circ \tilde{f}'(\tilde{x'}) = u,$$

we have $u \in p(Fix(\tilde{f}'))$. As a result, $p(Fix(\tilde{f})) = p(Fix(\tilde{f}'))$. □

5 Conclusion

The essential aim of this paper is to determine fixed point properties for a digital image. This work can play an important role in digital images because Nielsen theory gives an information about the number of fixed points of a map. Since the Nielsen number is a powerful invariant in digital images, we think that this work will be useful for fixed point theory, especially Nielsen theory.

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A FIXED POINT THEOREM AND STABILITY OF ADDITIVE-CUBIC FUNCTIONAL EQUATIONS IN MODULAR SPACES

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Abstract. In this paper, we investigate a fixed point theorem for a mapping without the condition of bounded orbit in a modular space, whose induced modular is lower semi-continunous. Using this fixed point theorem, we prove the generalized Hyers-Ulam stability for an additive-cubic functional equation in modular spaces without $\triangle_2$-conditions and the convexity.

1. Introduction and preliminaries

The question of stability for a generic functional equation was originated in 1940 by Ulam [14]. Concerning a group homomorphism, Ulam posted the question asking how likely to an automorphism a function should behave in order to guarantee the existence of an automorphism near such functions. Hyers [3] gave the first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [1] for additive mappings and by Rassias [12] for linear mappings by considering an unbounded Cauchy difference, the latter of which has influenced many developments in the stability theory. This area is then referred to as the generalized Hyers-Ulam stability. In 1994, P. Găvruta [2] generalized these theorems for approximate additive mappings controlled by the unbounded Cauchy difference with regular conditions.

A problem that mathematicians has dealt with is "how to generalize the classical function space L^p". A first attempt was made by Birnhaum and Orlicz in 1931. This generalization found many applications in differential and intergral equations with kernls of nonpower types. The more abstract generalization was given by Nakano [10] in 1950 based on replacing the particular integral form of the functional by an abstract one that satisfies some good properties. This functional was called *modular*. This idea was refined, generalized by Musielak and Orlicz [8] in 1959 and studied by many authors ([4], [7], [11], [19]).

Recently, Sadeghi [13] presented a fixed point method to prove the generalized Hyers-Ulam stability of functional equations in modular spaces with the $\triangle_2$-condition and Wongkum, Chaipunya, and Kumam [15] proved the fixed point theorem and the generalized Hyers-Ulam stability for quadratic mappings in a modular space whose modular is convex, lower semi-continuous but do not satisfy the $\triangle_2$-condition.

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In this paper, we investigate a fixed point theorem in modular spaces, whose induced modular is lower semi-continuous, for a mapping with some conditions in place of the condition of bounded orbit and using this fixed point theorem, we will prove the generalized Hyers-Ulam stability for the following additive-cubic functional equation

$$f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-2f(2x)+4f(x)=0 \tag{1.1}$$

in modular spaces without $\triangle_2$-conditions and the convexity.

In fact, the equation (1.1) has been studied in various spaces. For example, in quasi-Banach spaces ([9]), in F-spaces ([16]), in non-Archimedean fuzzy normed spaces ([17]), and in intuitionistic fuzzy normed spaces ([18]), etc. Unlike Banach spaces and F-spaces, due to the absence of the triangle inequality in modular spaces, we need subtle caculations in the proofs of Lemma 1.4 and Theorem 2.2.

Definition 1.1. Let X be a vector space over a field $\mathbb{K}(\mathbb{R}$ or $\mathbb{C})$.
(1) A generalized functional $\rho : X \longrightarrow [0,\infty]$ is called *a modular* if
(M1) $\rho(x) = 0$ if and only if $x = 0$,
(M2) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$, and
(M3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for all $x, y \in X$ and for all nonnegative real numbers α, β with $\alpha + \beta = 1$.
(2) If (M3) is replaced by
(M4) $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$
for all $x, y \in X$ and for all nonnegative real numbers α, β with $\alpha + \beta = 1$, then we say that ρ is a *convex modular*.

Remark 1.2. Let ρ be a modular on a vector space X. Then by (M1) and (M3), we can easily show that for any positive real number δ with $\delta < 1$,

$$\rho(\delta x) \leq \rho(x)$$

for all $x \in X$.

For any modular ρ on X, the modular space X_ρ is defined by

$$X_\rho := \{x \in X \mid \rho(\lambda x) \to 0 \ as \ \lambda \to 0\}.$$

Let X_ρ be a modular space and let $\{x_n\}$ be a sequence in X_ρ. Then (i) $\{x_n\}$ is called *ρ-convergent* to a point $x \in X_\rho$ if $\rho(x_n - x) \to 0$ as $n \to \infty$, (ii) $\{x_n\}$ is called *ρ-Cauchy* if for any $\epsilon > 0$, there is a $k \in \mathbb{N}$ such that $\rho(x_n - x_m) < \epsilon$ for all $m, n \in \mathbb{N}$ with $n, m \geq k$, and (iii) a subset K of X_ρ is called *ρ-complete* if each ρ-Cauchy sequence is ρ-convergent to an element of K.

Another unnatural behavior one usually encounter is that the convergence of a sequence $\{x_n\}$ to x does not imply that $\{cx_n\}$ converges to cx for some $c \in \mathbb{K}$. Thus, many mathematicians imposed some additional conditions for a modular to meet in order to make the multiples of $\{x_n\}$ converge naturally. Such preferences are referred to mostly under the term related to the $\triangle_2$-conditions.

A modular space X_ρ is said to *satisfy the $\triangle_2$-condition* if there exists $k \geq 2$ such that $\rho(2x) \leq k\rho(x)$ for all $x \in X_\rho$. Some authors varied the notion so that only $k > 0$ is required and called it *the $\triangle_2$-type condition.* In fact, one may see that these two notions coincide. There are still a number of equivalent notions related to the $\triangle_2$-conditions.

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In [5], Khamsi proved a series of fixed point theorems in modular spaces where the modulars do not satisfy $\triangle_2$-conditions. His results exploit one unifying hypothesis in which the boundedness of an orbit is assumed.

Lemma 1.3. *(see [5]) Let X_ρ be a modular space whose induced modular is lower semi-continuous and let $C \subseteq X_\rho$ be a ρ-complete subset. If $T : C \longrightarrow C$ is a ρ-contraction, that is, there is a constant $L \in [0, 1)$ such that*

$$\rho(Tx - Ty) \leq L\rho(x - y), \ \ \forall x, y \in C$$

and T has a bounded orbit at a point $x_0 \in C$, that is,

$$\sup\{\rho(T^n x_0 - T^m x_0) \mid n, m \in \mathbb{N} \cup \{0\}\} < \infty$$

then the sequence $\{T^n x_0\}$ is ρ-convergent to a point $w \in C$.

Now, we will prove a fixed point theorem in modular spaces where the map T do not assume to be the boundedness of an orbit. Our results exploit one unifying hypothesis in which some conditions are assumed.

Lemma 1.4. *Let X_ρ be a modular space whose induced modular is lower semi-continuous and let $C \subseteq X_\rho$ be a ρ-complete subset. Let $T : C \longrightarrow C$ be a mapping such that*

$$2Tx = T2x, \ \ \forall x \in C. \tag{1.2}$$

Suppose that there is a constant $L \in [0, 1)$ with

$$\rho(2Tx - 2Ty) \leq L\rho(x - y), \ \ \forall x, y \in C \tag{1.3}$$

and $\rho(Tx_o - x_o) < \infty$ at $x_o \in C$. Then the sequence $\{T^n \frac{x_0}{4}\}$ is ρ-convergent to some point $w \in C$ and

$$\rho(\frac{x_0}{4} - w) \leq \frac{2}{1 - L}\rho(Tx_0 - x_0). \tag{1.4}$$

Proof. By (M1) and (M3), we have $\rho(Tx - Ty) \leq \rho(2Tx - 2Ty)$ and so, by (1.3), T is a ρ-contration. Hence we have

$$\begin{aligned}\rho(\frac{1}{2}T^2x_0 - \frac{1}{2}x_0) &\leq \rho(T^2x_0 - Tx_0) + \rho(Tx_0 - x_0)\\ &\leq (L + 1)\rho(Tx_0 - x_0).\end{aligned}$$

Let $Gx = 2Tx$ for all $x \in C$. By (1.3), we have

$$\begin{aligned}\rho(\frac{1}{2}T^nx_0 - \frac{1}{2}x_0) &\leq \rho(T^nx_0 - Tx_0) + \rho(Tx_0 - x_0)\\ &= \rho\Big(\frac{1}{2}G(T^{n-1}x_0) - \frac{1}{2}Gx_0\Big) + \rho(Tx_0 - x_0)\\ &\leq L\rho\Big(\frac{1}{2}T^{n-1}x_0 - \frac{1}{2}x_0\Big) + \rho(Tx_0 - x_0)\end{aligned}$$

for all $n \in \mathbb{N}$ with $n \geq 2$ and by induction, we have

$$\rho(\frac{1}{2}T^nx_0 - \frac{1}{2}x_0) \leq \Sigma_{k=0}^{n-1}L^k\rho(Tx_0 - x_0) \leq \frac{1}{1 - L}\rho(Tx_0 - x_0)$$

for all $n \in \mathbb{N}$. For any non-negative integers m, n with $m > n$,

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$$\begin{aligned}\rho(\frac{1}{4}T^n x_0 - \frac{1}{4}T^m x_0) &\leq \rho(\frac{1}{2}T^n x_0 - \frac{1}{2}x_0) + \rho(\frac{1}{2}T^m x_0 - \frac{1}{2}x_0) \\ &\leq \frac{2}{1-L}\rho(Tx_0 - x_0).\end{aligned} \tag{1.5}$$

By (1.2), T has a bounded orbit at a point $\frac{x_0}{4} \in C$ and thus by Lemma 1.3, $\{T^n \frac{x_0}{4}\}$ is ρ-convergent to a point $\omega \in C$. Since ρ is lower semi-continuous, by taking $n = 0$ and $m \to \infty$ in (1.5), we have (1.4). □

If ρ is convex, then Lemma 1.4 can be replaced by the following lemma.

Lemma 1.5. *All conditions in Lemma 1.4 are assumed. Suppose that ρ is convex and $0 \leq L < 2$. Then the sequence $\{T^n \frac{x_0}{4}\}$ is ρ-convergent to some point $w \in C$ and*

$$\rho(\frac{x_0}{4} - w) \leq \frac{1}{2-L}\rho(Tx_0 - x_0). \tag{1.6}$$

Proof. By (M1) and (M4), we have $\rho(Tx - Ty) \leq \frac{1}{2}\rho(2Tx - 2Ty)$ and since $0 \leq L < 2$, by (1.3), T is a ρ-contration. Hence by (M4), we have

$$\begin{aligned}\rho(\frac{1}{2}T^2 x_0 - \frac{1}{2}x_0) &\leq \frac{1}{2}\rho(T^2 x_0 - Tx_0) + \frac{1}{2}\rho(Tx_0 - x_0) \\ &\leq (\frac{1}{4}L + \frac{1}{2})\rho(Tx_0 - x_0).\end{aligned}$$

Let $Gx = 2Tx$ for all $x \in C$. By (1.3), we have

$$\begin{aligned}\rho(\frac{1}{2}T^n x_0 - \frac{1}{2}x_0) &\leq \frac{1}{2}\rho(T^n x_0 - Tx_0) + \frac{1}{2}\rho(Tx_0 - x_0) \\ &= \rho\Big(\frac{1}{2}G(T^{n-1}x_0) - \frac{1}{2}Gx_0\Big) + \frac{1}{2}\rho(Tx_0 - x_0) \\ &\leq \frac{1}{2}L\rho\Big(\frac{1}{2}T^{n-1}x_0 - \frac{1}{2}x_0\Big) + \frac{1}{2}\rho(Tx_0 - x_0)\end{aligned}$$

for all $n \in \mathbb{N}$ with $n \geq 2$ and by induction, we have

$$\rho(\frac{1}{2}T^n x_0 - \frac{1}{2}x_0) \leq \Sigma_{k=0}^{n-1}\frac{L^k}{2^{k+1}}\rho(Tx_0 - x_0) \leq \frac{1}{2-L}\rho(Tx_0 - x_0)$$

for all $n \in \mathbb{N}$. For any non-negative integers m, n with $m > n$,

$$\begin{aligned}\rho(\frac{1}{4}T^n x_0 - \frac{1}{4}T^m x_0) &\leq \frac{1}{2}\rho(\frac{1}{2}T^n x_0 - \frac{1}{2}x_0) + \frac{1}{2}\rho(\frac{1}{2}T^m x_0 - \frac{1}{2}x_0) \\ &\leq \frac{1}{2-L}\rho(Tx_0 - x_0).\end{aligned}$$

The rest of the proof is similar to Lemma 1.4. □

Let ρ be a modular on X, V a linear space. Define a set $\mathbb{M}$ by

$$\mathbb{M} := \{g : V \longrightarrow X_\rho \mid g(0) = 0\}$$

and a generalized function $\widetilde{\rho}$ on $\mathbb{M}$ by

$$\widetilde{\rho}(g) := \inf\{c > 0 \mid \rho(g(x)) \leq c\psi(x, x),\ \forall x \in V\},$$

for each $g \in \mathbb{M}$, where $\psi : V^2 \longrightarrow [0, \infty)$ a mapping. Then $\mathbb{M}$ is a linear space, $\widetilde{\rho}$ is a modular on $\mathbb{M}$. Furthermore, if ρ is convex, then $\widetilde{\rho}$ is also convex([15]).

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Lemma 1.6. *Let V be a linear space, X_ρ a ρ-complete modular space, where ρ is lower semi-continuous and $f : V \longrightarrow X_\rho$ a mapping with $f(0) = 0$. Let $\psi : V^2 \longrightarrow [0, \infty)$ be a mapping. Then we have the following :*
(1) *$\mathbb{M}_{\widetilde{\rho}} = \mathbb{M}$ and $\mathbb{M}_{\widetilde{\rho}}$ is $\widetilde{\rho}$-complete.*
(2) *$\widetilde{\rho}$ is lower semi-continuous.*

Proof. (1) By the definition of $\mathbb{M}_{\widetilde{\rho}}$, $\mathbb{M}_{\widetilde{\rho}} = \mathbb{M}$. Let $\epsilon > 0$ be given. Take any $\widetilde{\rho}$-Cauchy sequence $\{g_n\}$ in $\mathbb{M}_{\widetilde{\rho}}$. Then there is an $l \in \mathbb{N}$ such that for $n, m \in \mathbb{N}$ with $n, m \geq l$,

$$\rho(g_n(x) - g_m(x)) \leq \epsilon\psi(x, x) \tag{1.7}$$

for all $x \in V$. Hence $\{g_n(x)\}$ is a ρ-Cauchy sequence in X_ρ for all $x \in X$. Since X_ρ is ρ-complete, there is a mapping $g : V \longrightarrow X_\rho$ such that $\rho(g_n(x) - g(x)) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $x \in X$. Then there is an $m \in \mathbb{N}$ such that

$$\rho(g_m(0) - g(0)) = \rho(g(0)) \leq \epsilon$$

and hence $g \in \mathbb{M}_{\widetilde{\rho}}$. Since ρ is a lower semi-continuous, by (1.7), we have

$$\rho(g_n(x) - g(x)) \leq \liminf_{m \longrightarrow \infty} \rho(g_n(x) - g_m(x)) \leq \epsilon\psi(x, x)$$

for all $x \in X$. Hence $\mathbb{M}_{\widetilde{\rho}}$ is $\widetilde{\rho}$-complete.

(2) Suppose that $\{g_n\}$ is a sequence in $\mathbb{M}_{\widetilde{\rho}}$ which is $\widetilde{\rho}$-convergent to $g \in \mathbb{M}_{\widetilde{\rho}}$. Let $\epsilon > 0$. Then for any $n \in \mathbb{N}$, there is a positive real number c_n such that

$$\widetilde{\rho}(g_n) \leq c_n \leq \widetilde{\rho}(g_n) + \epsilon$$

and so

$$\rho(g(x)) \leq \liminf_{n \to \infty} \rho(g_n(x)) \leq \liminf_{n \to \infty} c_n\psi(x, x) \leq \Big(\liminf_{n \to \infty} \rho(g_n(x)) + \epsilon\Big)\psi(x, x)$$

for all $x \in X$. Hence $\widetilde{\rho}$ is lower semi-continuous. □

2. The generalized Hyers-Ulam stability for (1.1) in modular spaces

Throughout this section, we assume that every modular is lower semi-continuous. In this section, we will prove the generalized Hyers-Ulam stability for (1.1) by using our fixed point theorem. We can easily show the following lemma.

Lemma 2.1. *Let X and Y be vector spaces. Let $f : X \longrightarrow Y$ satisfies (1.1) and $f(0) = 0$. Then we have :*
(1) *f is additive if and only if $f(2x) = 2f(x)$ for all $x \in X$.*
(2) *f is cubic if and only if $f(2x) = 8f(x)$ for all $x \in X$.*

For any mapping $g : X \longrightarrow Y$, let

$$g_a(x) = \frac{1}{4}(g(2x) - 8g(x)), \ \ g_c(x) = g(2x) - 2g(x)$$

and

$$Dg(x, y) = g(2x + y) + g(2x - y) - 2g(x + y) - 2g(x - y) - 2g(2x) + 4g(x).$$

Theorem 2.2. *Let V be a linear space, X_ρ a ρ-complete modular space. Suppose that $f : V \longrightarrow X_\rho$ satisfies $f(0) = 0$ and*

$$\rho(Df(x, y)) \le \phi(x, y) \tag{2.1}$$

for all $x, y \in V$, where $\phi : V^2 \longrightarrow [0, \infty)$ is a mapping such that

$$\phi(2x, 2y) \le L\phi(x, y) \tag{2.2}$$

for some L with $0 \le L < 1$ and for all $x, y \in V$. Then there exists a unique additive-cubic mapping $F : V \longrightarrow X_\rho$ such that

$$\rho(F(x) - \frac{3}{16}f(x)) \le \frac{4}{1-L}\psi(x, x) \tag{2.3}$$

for all $x \in V$, where $\psi(x, y) = \phi(x, 2y) + \phi(x, y) + \phi(0, y)$.

Proof. Let $\psi(x, y) = \phi(x, 2y) + \phi(x, y) + \phi(0, y)$. Then by Lemma 1.6, $\mathbb{M}_{\widetilde{\rho}} = \mathbb{M}$ is $\widetilde{\rho}$-complete and $\widetilde{\rho}$ is lower semi-continuous.

Define $T_a : \mathbb{M}_{\widetilde{\rho}} \longrightarrow \mathbb{M}_{\widetilde{\rho}}$ by $T_a g(x) = \frac{1}{2}g(2x)$ for all $g \in \mathbb{M}_{\widetilde{\rho}}$ and all $x \in V$. Let $g, h \in \mathbb{M}_{\widetilde{\rho}}$. Suppose that $\widetilde{\rho}(g - h) \le c$ for some positive real number c. Then by (2.3) and Remark 1.2, we have

$$\rho(T_a g(x) - T_a h(x)) \le \rho(g(2x) - h(2x)) \le Lc\psi(x, x)$$

for all $x \in V$ and so $\widetilde{\rho}(T_a g - T_a h) \le L\widetilde{\rho}(g - h)$. Hence T_a is a $\widetilde{\rho}$-contraction. By (2.1), we get

$$\rho(f(x) + f(-x)) \le \phi(0, x), \tag{2.4}$$

$$\rho(f(3x) - 4f(2x) + 5f(x)) \le \phi(x, x), \tag{2.5}$$

and

$$\rho(f(4x) - 2f(3x) - 2f(2x) - 2f(-x) + 4f(x)) \le \phi(x, 2x), \tag{2.6}$$

for all $x \in V$. By (2.4), (2.5), and (2.6), we obtain

$$\rho(T_a f_a(x) - f_a(x)) = \rho(\frac{1}{2}f_a(2x) - f_a(x)) \le \phi(x, 2x) + \phi(x, x) + \phi(0, x) = \psi(x, x)$$

for all $x \in V$ and hence we have

$$\widetilde{\rho}(T_a f_a - f_a) \le 1. \tag{2.7}$$

Let $Gg = 2T_a g$ for all $g \in \mathbb{M}_{\widetilde{\rho}}$. Then

$$Gg(x) = g(2x)$$

for all $g \in \mathbb{M}_{\widetilde{\rho}}$ and for all $x \in V$. Suppose that $\widetilde{\rho}(g - h) \le c$ for some positive real number c, where $g, h \in \mathbb{M}_{\widetilde{\rho}}$. Then $\rho(g(x) - h(x)) \le c\psi(x, x)$ for all $x \in V$ and by (2.2), we have

$$\rho(Gg(x) - Gh(x)) = \rho(g(2x) - h(2x)) \le c\psi(2x, 2x) \le cL\psi(x, x)$$

for all $x \in V$. Hence $\widetilde{\rho}(Gg - Gh) \le cL$ and so

$$\widetilde{\rho}(Gg - Gh) \le L\widetilde{\rho}(g - h).$$

Since T_a is linear, by Lemma 1.4, there is an $A \in \mathbb{M}_{\widetilde{\rho}}$ such that $\{T_a^n \frac{f_a}{4}\}$ is $\widetilde{\rho}$-convergent to A. In fact, we get

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$$\lim_{n\to\infty} \rho(\frac{1}{2^{n+2}} f_a(2^n x) - A(x)) = 0 \tag{2.8}$$

for all $x \in V$. Since $\widetilde{\rho}$ is lower semi-continuous, we get

$$\widetilde{\rho}(T_a A - A) \le \liminf_{n\to\infty} \widetilde{\rho}(T_a A - T_a^{n+1}\frac{f_a}{4}) \le \liminf_{n\to\infty} L\widetilde{\rho}(A - T_a^n \frac{f_a}{4}) = 0$$

and hence A is a fixed point of T_a in $\mathbb{M}_{\widetilde{\rho}}$. Replacing x and y by $2^n x$ and $2^n y$ in (2.1), respectively, by (2.2), we have

$$\begin{aligned}
&\rho(\frac{1}{2^{n+2}} Df_a(2^n x, 2^n y)) \\
&\le \rho(\frac{1}{2^{n+3}} Df(2^{n+1}x, 2^{n+1}y)) + \rho(\frac{1}{2^n} Df(2^n x, 2^n y)) \\
&\le L^{n+1}\phi(x,y) + L^n\phi(x,y)
\end{aligned}$$

for all $x, y \in V$ and for all $n \in \mathbb{N}$. Hence we get

$$\lim_{n\to\infty} \rho\Big(\frac{1}{2^{n+2}} Df_a(2^n x, 2^n y)\Big) = 0 \tag{2.9}$$

for all $x, y \in V$. Note that

$$\begin{aligned}
&\rho\Big(\frac{1}{2^{n+10}} Df_a(2^n x, 2^n y) - \frac{1}{2^8} DA(x,y)\Big) \\
&\le \rho\Big(\frac{1}{2^{n+9}} f_a(2^n(2x+y)) - \frac{1}{2^7} A(2x+y)\Big) + \rho\Big(\frac{1}{2^{n+8}} f_a(2^n(2x-y)) - \frac{1}{2^6} A(2x-y)\Big) \\
&+ \rho\Big(\frac{1}{2^{n+7}} 2f_a(2^n(x+y)) - \frac{1}{2^5} 2A(x+y)\Big) + \rho\Big(\frac{1}{2^{n+6}} 2f_a(2^n(x-y)) - \frac{1}{2^4} 2A(x-y)\Big) \\
&+ \rho\Big(\frac{1}{2^{n+5}} 2f_a(2^n 2x) - \frac{1}{2^3} 2A(2x)\Big) + \rho\Big(\frac{1}{2^{n+5}} 2f_a(2^n(x-y)) - \frac{1}{2^3} 4A(x-y)\Big)
\end{aligned}$$

for all $x, y \in V$ and for all $n \in \mathbb{N}$. Hence we have

$$\lim_{n\to\infty} \rho\Big(\frac{1}{2^{n+10}} Df_a(2^n x, 2^n y) - \frac{1}{2^8} DA(x,y)\Big) = 0 \tag{2.10}$$

for all $x, y \in V$. Since

$$\rho\Big(\frac{1}{2^9} DA(x,y)\Big) \le \rho\Big(\frac{1}{2^{n+10}} Df_a(2^n x, 2^n y) - \frac{1}{2^8} DA(x,y)\Big) + \rho\Big(\frac{1}{2^{n+10}} Df_a(2^n x, 2^n y)\Big)$$

for all $x, y \in V$ and for all $n \in \mathbb{N}$, by (2.9) and (2.10), we get

$$DA(x,y) = 0 \tag{2.11}$$

for all $x, y \in V$. By (1.4) in Lemma 1.4, we get

$$\widetilde{\rho}(A - \frac{1}{4} f_a) \le \frac{2}{1-L}. \tag{2.12}$$

Define $T_c : \mathbb{M}_{\widetilde{\rho}} \longrightarrow \mathbb{M}_{\widetilde{\rho}}$ by $T_c g(x) = \frac{1}{8} g(2x)$ for all $g \in \mathbb{M}_{\widetilde{\rho}}$ and all $x \in V$. By (2.4), (2.5), and (2.6), we obtain

$$\rho(\frac{1}{2^3} f_c(2x) - f_c(x)) \le \psi(x,x)$$

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for all $x \in V$ and hence

$$\widetilde{\rho}(T_c f_c - f_c) \leq 1. \tag{2.13}$$

Let $Hg = 2T_c g$ for all $g \in \mathbb{M}_{\widetilde{\rho}}$. Then

$$Hg(x) = \frac{1}{4}g(2x).$$

for all $g \in \mathbb{M}_{\widetilde{\rho}}$ and for all $x \in V$. Let $g, h \in \mathbb{M}_{\widetilde{\rho}}$. Suppose that $\widetilde{\rho}(g - h) \leq c$ for some positive real number c. Then $\rho(g(x) - h(x)) \leq c\psi(x, x)$ for all $x \in V$ and by (2.3), we get

$$\rho(Hg(x) - Hh(x)) = \rho(\frac{1}{4}g(2x) - \frac{1}{4}h(2x)) \leq c\psi(2x, 2x) \leq cL\psi(x, x)$$

for all $x \in V$. Hence $\widetilde{\rho}(Hg - Hh) \leq cL$ and so

$$\widetilde{\rho}(Hg - Hh) \leq L\widetilde{\rho}(g - h).$$

Since T_c is linear, by Lemma 1.4, there is a $C \in \mathbb{M}_{\widetilde{\rho}}$ such that $\{T_c^n \frac{1}{4} f_c\}$ is $\widetilde{\rho}$-convergent to C. Since $\widetilde{\rho}$ is lower semi-continuous, we get

$$\widetilde{\rho}(T_c C - C) \leq \liminf_{n\to\infty} \widetilde{\rho}(T_c C - T_c^{n+1}\frac{1}{4}f_c) \leq \liminf_{n\to\infty} L\widetilde{\rho}(C - T_c^n \frac{1}{4}f_c) = 0$$

and hence C is a fixed point of T_c in $\mathbb{M}_{\widetilde{\rho}}$. Replacing x and y by $2^n x$ and $2^n y$ in (2.1), respectively, by (2.2), we have

$$\begin{aligned}
&\rho(\frac{1}{2^{3n+2}}Df_c(2^n x, 2^n y)) \\
&\leq \rho(\frac{1}{2^{3n+1}}Df(2^{n+1}x, 2^{n+1}y)) + \rho(\frac{1}{2^{3n}}Df(2^n x, 2^n y)) \\
&\leq L^{n+1}\phi(x, y) + L^n\phi(x, y)
\end{aligned}$$

for all $x, y \in V$. Hence we get

$$\lim_{n\to\infty} \rho\Big(\frac{1}{2^{3n+2}}Df_c(2^n x, 2^n y)\Big) = 0$$

for all $x, y \in V$. Similar to A, we have

$$DC(x, y) = 0 \tag{2.14}$$

for all $x, y \in V$ and by (1.4) in Lemma 1.4, we get

$$\rho(C(x) - \frac{1}{4}f_c(x)) \leq \frac{2}{1 - L}\psi(x, x)$$

for all $x \in X$. Hence we have

$$\widetilde{\rho}(C - \frac{1}{4}f_c) \leq \frac{2}{1 - L}. \tag{2.15}$$

Let $F = \frac{1}{8}C - \frac{1}{2}A$. Since A is a fixed point of T_a, $A(2x) = 2A(x)$ for all $x \in X$ and similarly, $C(2x) = 8C(x)$ for all $x \in X$. By Lemma 2.1, A is additive and C is cubic. Hence F is an additive-cubic mapping. Since $f(x) = \frac{1}{6}f_c(x) - \frac{2}{3}f_a(x)$, we have

$$\widetilde{\rho}(F - \frac{3}{16}f) \leq \widetilde{\rho}(A - \frac{1}{4}f_a) + \widetilde{\rho}(\frac{1}{4}C - \frac{1}{16}f_c) \leq \widetilde{\rho}(A - \frac{1}{4}f_a) + \widetilde{\rho}(C - \frac{1}{4}f_c),$$

and hence by (2.12) and (2.15), we have (2.3).

To prove the uniquness of F, let $K : V \longrightarrow X_\rho$ be another additive-cubic mapping with (2.3). By (2.3), we get

$$\begin{aligned}\rho(\frac{1}{4}K(x) - \frac{1}{4}F(x)) &\leq \rho(K(x) - \frac{3}{16}f(x)) + \rho(F(x) - \frac{3}{16}f(x)))\\ &\leq \frac{8}{1-L}\psi(x,x)\end{aligned}$$

for all $x \in V$ and so

$$\begin{aligned}\rho(\frac{1}{16}K_a(x) - \frac{1}{16}F_a(x)) &\leq \rho(\frac{1}{32}K(2x) - \frac{1}{32}F(2x)) + \rho(\frac{1}{4}K(x) - \frac{1}{4}F(x)))\\ &\leq \frac{8(1+L)}{1-L}\psi(x,x)\end{aligned}$$

for all $x \in V$. Since F_a and K_a are fixed points of T_a, we have

$$\begin{aligned}&\rho(\frac{1}{16}K_a(x) - \frac{1}{16}F_a(x)) = \rho(\frac{1}{16}T_a^n K_a(x) - \frac{1}{16}T_a^n F_a(x))\\ &\leq \frac{8(1+L)}{1-L}L^n\psi(x,x)\end{aligned}$$

for all $x \in V$ and for all $n \in \mathbb{N}$. Letting $n \to \infty$ in the last inequality, we have $F_a = K_a$ and similarly, we have $F_c = K_c$. Thus $F = K$. □

Comparing the results in a modular and a convex modular, we may see that the coefficient in the case of convex modular is smaller.

Theorem 2.3. *Suppose that every assumption of Theorem 2.2 holds, ρ is convex and $0 \leq L < 2$. Then there exists a unique additive-cubic mapping $F : V \longrightarrow X_\rho$ such that*

$$\rho(F(x) - \frac{3}{16}f(x)) \leq \frac{5}{32(2-L)}\psi(x,x) \tag{2.16}$$

for all $x \in V$, where $\psi(x,y) = \phi(x,2y) + \phi(x,y) + \phi(0,y)$.

Proof. Define $T_a : \mathbb{M}_{\widetilde{\rho}} \longrightarrow \mathbb{M}_{\widetilde{\rho}}$ by $T_a g(x) = \frac{1}{2}g(2x)$ for all $g \in \mathbb{M}_{\widetilde{\rho}}$ and all $x \in V$. Let $g, h \in \mathbb{M}_{\widetilde{\rho}}$. Suppose that $\widetilde{\rho}(g-h) \leq c$ for some positive real number c. Then by (2.3) and (M4), we have

$$\rho(T_a g(x) - T_a h(x)) \leq \frac{1}{2}\rho(g(2x) - h(2x)) \leq \frac{1}{2}Lc\psi(x,x)$$

for all $x \in V$ and so $\widetilde{\rho}(T_a g - T_a h) \leq \frac{1}{2}L\widetilde{\rho}(g-h)$. Hence T_a is a $\widetilde{\rho}$-contraction. By (2.1), we get

$$\rho(f(x) + f(-x)) \leq \phi(0,x), \tag{2.17}$$

$$\rho(f(3x) - 4f(2x) + 5f(x)) \leq \phi(x,x), \tag{2.18}$$

and

$$\rho(f(4x) - 2f(3x) - 2f(2x) - 2f(-x) + 4f(x)) \leq \phi(x,2x), \tag{2.19}$$

for all $x \in V$. By (2.17), (2.18), and (2.19), we obtain

$$\rho(\frac{1}{2}f_a(2x) - f_a(x)) \leq \frac{1}{8}\phi(x,2x) + \frac{1}{4}\phi(x,x) + \frac{1}{4}\phi(0,x) \leq \frac{1}{4}\psi(x,x)$$

for all $x \in V$ and hence

$$\rho(T_a f_a(x) - f_a(x)) \leq \frac{1}{4}\psi(x,x)$$

for all $g \in \mathbb{M}_{\widetilde{\rho}}$ and all $x \in V$. Hence we have

$$\widetilde{\rho}(T_a f_a - f_a) \leq \frac{1}{4}. \tag{2.20}$$

Let $Gg = 2T_a g$ for all $g \in \mathbb{M}_{\widetilde{\rho}}$. Then

$$Gg(x) = g(2x)$$

for all $x \in V$. Suppose that $\widetilde{\rho}(g-h) \leq c$ for some positive real number c. Then $\rho(g(x)-h(x)) \leq c\psi(x,x)$ for all $x \in V$ and by (2.2), we have

$$\rho(Gg(x) - Gh(x)) = \rho(g(2x) - h(2x)) \leq c\psi(2x,2x) \leq cL\psi(x,x)$$

for all $x \in V$. Hence $\widetilde{\rho}(Gg - Gh) \leq cL$ and so

$$\widetilde{\rho}(Gg - Gh) \leq L\widetilde{\rho}(g-h).$$

Since T_a is linear, by Lemma 1.5, there is an $A \in \mathbb{M}_{\widetilde{\rho}}$ such that $\{T_a^n \frac{f_a}{4}\}$ is $\widetilde{\rho}$-convergent to A. In fact, we get

$$\lim_{n\to\infty} \rho(\frac{1}{2^{n+2}} f_a(2^n x) - A(x)) = 0$$

for all $x \in V$. Since $\widetilde{\rho}$ is lower semi-continuous, we get

$$\widetilde{\rho}(T_a A - A) \leq \liminf_{n\to\infty} \widetilde{\rho}(T_a A - T_a^{n+1}\frac{f_a}{4}) \leq \liminf_{n\to\infty} L\widetilde{\rho}(A - T_a^n \frac{f_a}{4}) = 0$$

and hence A is a fixed point of T_a in $\mathbb{M}_{\widetilde{\rho}}$. Similar to Theorem 2.2, we have

$$DA(x,y) = 0 \tag{2.21}$$

for all $x, y \in V$ and by (1.6) in Lemma 1.5 and (2.20), we get

$$\widetilde{\rho}(A - \frac{1}{4}f_a) \leq \frac{1}{4(2-L)}. \tag{2.22}$$

Define $T_c : \mathbb{M}_{\widetilde{\rho}} \longrightarrow \mathbb{M}_{\widetilde{\rho}}$ by $T_c g(x) = \frac{1}{8}g(2x)$ for all $g \in \mathbb{M}_{\widetilde{\rho}}$ and all $x \in V$. By (2.17), (2.18), and (2.19), we obtain

$$\rho(\frac{1}{2^3} f_c(2x) - f_c(x)) \leq \frac{1}{4}\psi(x,x)$$

for all $x \in V$ and hence

$$\widetilde{\rho}(T_c f_c - f_c) \leq \frac{1}{4}. \tag{2.23}$$

Let $Hg = 2T_c g$ for all $g \in \mathbb{M}_{\widetilde{\rho}}$. Then

$$Hg(x) = \frac{1}{4}g(2x).$$

for all $g \in \mathbb{M}_{\widetilde{\rho}}$ and for all $x \in V$. Let $g, h \in \mathbb{M}_{\widetilde{\rho}}$. Suppose that $\widetilde{\rho}(g-h) \leq c$ for some positive real number c. Then $\rho(g(x)-h(x)) \leq c\psi(x,x)$ for all $x \in V$ and by (2.3), we get

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$$\rho(Hg(x) - Hh(x)) = \rho(\frac{1}{4}g(2x) - \frac{1}{4}h(2x)) \leq c\psi(2x, 2x) \leq cL\psi(x, x)$$

for all $x \in V$. Hence $\widetilde{\rho}(Gg - Gh) \leq cL$ and so

$$\widetilde{\rho}(Hg - Hh) \leq L\widetilde{\rho}(g - h).$$

Since T_c is linear, by Lemma 1.5, there is a $C \in \mathbb{M}_{\widetilde{\rho}}$ such that $\{T_c^n \frac{1}{4} f_c\}$ is $\widetilde{\rho}$-convergent to C. Since $\widetilde{\rho}$ is lower semi-continuous, we get

$$\widetilde{\rho}(T_cC - C) \leq \liminf_{n\to\infty} \widetilde{\rho}(T_cC - T_c^{n+1}\frac{1}{4}f_c) \leq \liminf_{n\to\infty} L\widetilde{\rho}(C - T_c^n\frac{1}{4}f_c) = 0$$

and hence C is a fixed point of T_c in $\mathbb{M}_{\widetilde{\rho}}$. Similar to Theorem 2.2, we get

$$DC(x, y) = 0 \tag{2.24}$$

for all $x, y \in V$ and by (1.6) in Lemma 1.5, we get

$$\rho(C(x) - \frac{1}{4}f_c(x)) \leq \frac{1}{4(2 - L)}\psi(x, x)$$

for all $x \in X$. Hence we have

$$\widetilde{\rho}(C - \frac{1}{4}f_c) \leq \frac{1}{4(2 - L)}. \tag{2.25}$$

Let $F = \frac{1}{8}C - \frac{1}{2}A$. Since A is a fixed point of T_a, $A(2x) = 2A(x)$ for all $x \in X$ and similarly, $C(2x) = 8C(x)$ for all $x \in X$. By Lemma 2.1, A is additive and C is cubic. Hence F is an additive-cubic mapping. Since $f(x) = \frac{1}{6}f_c(x) - \frac{2}{3}f_a(x)$, by (2.22) and (2.25), we have

$$\widetilde{\rho}(F - \frac{3}{16}f) \leq \frac{1}{2}\widetilde{\rho}(A - \frac{1}{4}f_a) + \frac{1}{2}\widetilde{\rho}(\frac{1}{4}C - \frac{1}{16}f_c) \leq \frac{1}{2}\widetilde{\rho}(A - \frac{1}{4}f_a) + \frac{1}{8}\widetilde{\rho}(C - \frac{1}{4}f_c).$$

and hence we have (2.16). The rest of the proof is similar to Theorem 2.2. □

Remark 2.4. Sadeghi [13] proved the generalized Hyers-Ulam stability of functional equations in modular spaces with the $\triangle_2$-condition and in [15], authors proved the stability of mappings $f : V \longrightarrow X_\rho$ and $\phi : V^2 \longrightarrow [0, \infty)$ satisfying $f(0) = 0$,

$$\lim_{n\to\infty} \frac{\phi(2^n x, 2^n y)}{4^n} = 0,\ \phi(2x, 2x) \leq 4L\phi(x, x),\ \forall x, y \in V, \tag{2.26}$$

and

$$\rho(4f(x + y) + 4f(x - y) - 8f(x) - 8f(y)) \leq \phi(x, y),\ \forall x, y \in V$$

for some real number L with $0 \leq L < \frac{1}{2}$ whose codomain is equipped with a convex and lower semi-continuous modular without $\triangle_2$-conditions. Our results guarantee the stability of an additive-cubic mapping, whose induced modular is lower semi-continuous without the convexity and $\triangle_2$-conditions if $0 \leq L < \frac{1}{4}$. Further, in [15], authors left whether the multiple of 4 on the left side of the inequality (6) can be dropped as a problem. We can solve the problem by using Lemma 1.4 and its proof is similar to the proof in Theorem 2.2.

In fact, suppose that $\phi : V^2 \longrightarrow [0, \infty)$ is a mapping with (2.26) and that $0 \leq L < \frac{1}{4}$. Let $f : V \longrightarrow X_\rho$ be a mapping such that $f(0) = 0$ and

$$\rho(f(x + y) + f(x - y) - 2f(x) - 2f(y)) \leq \phi(x, y) \tag{2.27}$$

for all $x, y \in X$. Let $\psi(x,y) = \phi(x,y)$ for all $x, y \in V$, $f_0(x) = 4f(x)$, and $Tg(x) = \frac{1}{4}g(2x)$. Then by (2.27), we have

$$\rho(Tf_0(x) - f_0(x)) = \rho(\frac{1}{4}f_0(2x) - f_0(x)) \le \phi(x,x)$$

for all $x \in V$ and so

$$\widetilde{\rho}(Tf_0 - f_0) \le 1. \tag{2.28}$$

Moreover, by (2.26), we have

$$\widetilde{\rho}(2Tg - 2Th) \le 4L\widetilde{\rho}(g - h). \tag{2.29}$$

Since $0 \le L < \frac{1}{4}$, by Lemma 1.4, there is a fixed point $Q \in \mathbb{M}_{\widetilde{\rho}}$ such that $\{T^n \frac{f_0}{4}\} = \{T^n f\}$ converges to Q in X_ρ and

$$\rho(Q(x) - f(x)) \le \frac{2}{1-4L}\phi(x,x) \tag{2.30}$$

for all $x \in V$. We can show that Q is a quadratic mapping ([15]).

Further, suppose that ρ is convex and $0 \le L < 1$. Then (2.28) and (2.29) can be replaced by

$$\widetilde{\rho}(Tf_0 - f_0) \le \frac{1}{4}$$

and

$$\widetilde{\rho}(2Tg - 2Th) \le 2L\widetilde{\rho}(g - h),$$

respectively. By Lemma 1.5 and (1.6),

$$\rho(Q(x) - f(x)) \le \frac{1}{4(2-2L)}\phi(x,x) = \frac{1}{8(1-L)}\phi(x,x)$$

for all $x \in V$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Results on value-shared of admissible function and non-admissible function in the unit disc *

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Abstract

In this paper, we consider the uniqueness problem of admissible functions and non-admissible functions sharing some values in the unit disc. We obtain: If f_1 is admissible and f_2 is inadmissible satisfying $\lim\limits_{r\to 1^-} T(r, f_2) = \infty$, $a_j(j = 1, 2, \dots; q)$ be q distinct complex numbers. Then

(i) $f_1(z), f_2(z)$ can share at most three values a_1, a_2, a_3 IM;

(ii) $f_1(z), f_2(z)$ can share at most five values $a_j(j = 1, 2, \dots; 5)$ with reduced weight 1. Our results of this paper are improvement of the uniqueness theorems of meromorphic functions sharing some values on the whole complex plane which given by Yi and Cao.

Key words: uniqueness; meromorphic function; admissible; non-admissible.
Mathematical Subject Classification (2010): 30D 35.

1 Introduction and Main Results

In what follows, we shall assume that reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions such as the proximity function $m(r, f)$, counting function $N(r, f)$, characteristic function $T(r, f)$, the first and second main theorems, lemma on the logarithmic derivatives etc. of Nevalinna theory, (see Hayman [7] , Yang [16] and Yi and Yang [19]). For a meromorphic function f, $S(r, f)$ denotes any quantity satisfying $S(r, f) = o(T(r, f))$ for all r outside a possible exceptional set of finite logarithmic measure.

We use $\mathbb{C}$ to denote the open complex plane, $\widehat{\mathbb{C}} := \mathbb{C} \bigcup \{\infty\}$ to denote the extended complex plane, and $\mathbb{D} = \{z : |z| < 1\}$ to denote the unit disc.

Let f, g be two non-constant meromorphic functions in $\mathbb{D}$ and $a \in \widehat{\mathbb{C}}$. If $E(a, \mathbb{D}, f) = E(a, \mathbb{D}, g)$, we say f and g share a CM(counting multiplicities) in $\mathbb{D}$. If $\overline{E}(a, \mathbb{D}, f) =$

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$\overline{E}(a,\mathbb{D},g)$, we say f and g share a IM(ignoring multiplicities) in $\mathbb{D}$. If $\mathbb{D}$ is replaced by $\mathbb{C}$, we give the simple notation as before, $E(a,f),\overline{E}(a,f)$ and so on(see [16]).

R.Nevanlinna [12] proved the following well-known theorems.

Theorem 1.1 (see [12]) *If f and g are two non-constant meromorphic functions that share five distinct values a_1,a_2,a_3,a_4,a_5 IM in $\mathbb{C}$, then $f(z)\equiv g(z)$.*

Theorem 1.2 (see [12]) *If f and g are two distinct non-constant meromorphic functions that share four distinct values a_1,a_2,a_3,a_4 CM in $\mathbb{C}$, then f is a Möbius transformation of g , two of the shared values, say a_1 and a_2 are Picard values, and the cross ratio $(a_1,a_2,a_3,a_4)=-1$.*

After their very work, the uniqueness of meromorphic functions with shared values in the whole complex plane attracted many investigations (see [16]). In 1987 and 1988, Yi [17, 18] dealt with the problems of multiple values and uniquness of meromorphic functions sharing some values in the whole complex plane by adopting L.Yang's method and obtained some results which improved the concerning theorems due to Gopalakrishna and Bhoosnurmath's [6], Ueda [14]. To state the theorems, we will explain some notations as follows.

Let $f(z)$ be a non-constant meromorphic function, an arbitrary complex number $a\in\hat{\mathbb{C}}$, and k be a positive integer. We use $\overline{E}_{k)}(a,f)$ to denote the set of zeros of $f-a$, with multiplicities no greater than k, in which each zero counted only once. We say that $f(z)$ and $g(z)$ share the value a with reduced weight k, if $\overline{E}_{k)}(a,f)=\overline{E}_{k)}(a,g)$.

In 1987, Yi [17] obtained the uniqueness theorems concerning multiple values of meromorphic functions as follows.

Theorem 1.3 *(see [17, 19, Theorem 3.15]). Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, $a_j(j=1,2,\dots,q)$ be q distinct complex numbers, and let $k_j(j=1,2,\dots,q)$ be positive integers or ∞ satisfying*

$$k_1\geq k_2\geq\cdots\geq k_q\geq 1. \tag{1}$$

If

$$\overline{E}_{k_j)}(a_j,f)=\overline{E}_{k_j)}(a_j,g)\quad(j=1,2,\dots,q)$$

and

$$\sum_{j=3}^{q}\frac{k_j}{k_j+1}>2, \tag{2}$$

then $f(z)\equiv g(z)$.

In recent, it is an interesting topic to investigate the uniqueness with shared values in the subregion of the complex plane such as the unit disc, an angular domain, see [1, 2, 9, 10, 11, 15, 20, 21, 22]. In 1999, Fang [5] studied the uniqueness problem of admissible meromorphic functions in the unit disc $\mathbb{D}$ sharing two sets and three sets. Later, there were some results of uniqueness of meromorphic function in the unit disc concerning admissible functions. To state some uniqueness theorems of meromorphic functions in the unit disc $\mathbb{D}$, we need the following basic notations and definitions of meromorphic functions in $\mathbb{D}$(see [3], [4], [8]).

Definition 1.1 *Let f be a meromorphic function in $\mathbb{D}$ and $\lim_{r\to 1^-} T(r,f)=\infty$. Then*

$$\alpha(f) := \limsup_{r\to 1^-} \frac{T(r,f)}{-\log(1-r)}$$

is called the index of inadmissibility of f. If $\alpha(f)=\infty$, f is called admissible.

Definition 1.2 *Let f be a meromorphic function in $\mathbb{D}$ and $\lim_{r\to 1^-} T(r,f)=\infty$. Then*

$$\rho(f) := \limsup_{r\to 1^-} \frac{\log^+ T(r,f)}{-\log(1-r)}$$

is called the order (of growth) of f.

For admissible functions, the following theorem plays a very important role in studies the uniqueness problems of meromorphic functions in the unit disc.

Theorem 1.4 (see [13, Theorem 3]). *Let f be an admissible meromorphic function in $\mathbb{D}$, q be a positive integer and $a_1, a_2, \ldots, a_q$ be pairwise distinct complex numbers. Then, for $r \to 1^-$, $r \notin E$,*

$$(q-2)T(r,f) \leq \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f-a_j}\right) + S(r,f),$$

where $E \subset (0,1)$ is a possibly occurring exceptional set with $\int_E \frac{dr}{1-r} < \infty$. If the order of f is finite, the remainder $S(r,f)$ is a $O\left(\log \frac{1}{1-r}\right)$ without any exceptional set.

In 2005, Titzhoff [13] investigated the uniqueness of two admissible functions in the unit disc $\mathbb{D}$ by using the Second Main Theorem for admissible functions (Theorem 1.4) and obtained the five values theorem in the unit disc $\mathbb{D}$ as follows.

Theorem 1.5 (see [13]). *If two admissible function f, g share five distinct values, then $f \equiv g$.*

In 2009, Mao and Liu [11] gave a different method to investigate the uniqueness problem of meromorphic functions in unit disc and obtained the following results.

Theorem 1.6 (see [11]). *Let f, g be two meromorphic functions in $\mathbb{D}$, $a_j \in \widehat{\mathbb{C}}(j = 1, 2, \ldots, 5)$ be five distinct values, and $\Delta(\theta_0, \delta)(0 < \delta < \pi)$ be an angular domain such that for some $a \in \widehat{\mathbb{C}}$,*

$$\limsup_{r\to 1^-} \frac{\log n(r, \Delta(\theta_0, \delta/2), f(z)=a)}{\log \frac{1}{1-r}} = \tau > 1. \tag{3}$$

If f and g share $a_j(j=1,2,\ldots,5)$ IM in $\Delta(\theta_0,\delta)$, then $f(z) \equiv g(z)$.

Remark 1.1 *In fact, the condition (3) implies that f is admissible in the unit disc. Therefore, Theorem 1.6 is one result of uniqueness of admissible functions in the unit disc.*

3

For admissible functions in the unit disc $\mathbb{D}$, from Theorem 1.4, using the same argument as in the proofs of Theorem 1.3, we can easily get the following results.

Theorem 1.7 *Let $f_1(z)$ and $f_2(z)$ be two admissible meromorphic functions in $\mathbb{D}$, $a_j (j = 1, 2, \ldots, q)$ be q distinct complex numbers, and let $k_j (j = 1, 2, \ldots, q)$ be positive integers or ∞ satisfying (1). If $f_1(z)$ and $f_2(z)$ satisfy*

$$\overline{E}_{k_j)}(a_j, \mathbb{D}, f_1) = \overline{E}_{k_j)}(a_j, \mathbb{D}, f_2) \quad (j = 1, 2, \ldots, q) \tag{4}$$

and (2), then $f_1(z) \equiv f_2(z)$, where $\overline{E}_{k)}(a, \mathbb{D}, f)$ to denote the set of zeros of $f - a$ in $\mathbb{D}$, with multiplicities no greater than k, in which each zero counted only once.

Remark 1.2 *For $a \in \widehat{\mathbb{C}}$ and a positive integer k, we can say that $f_1(z), f_2(z)$ share the value a in $\mathbb{D}$ with reduced weight k, if $\overline{E}_{k)}(a, \mathbb{D}, f_1) = \overline{E}_{k)}(a, \mathbb{D}, f_2)$.*

Similar to the corollary of Theorem 1.3 (see [19, Corollary,pp.181.]), we can get the following corollary.

Corollary 1.1 *Let $f_1(z)$ and $f_2(z)$ be two admissible meromorphic functions in $\mathbb{D}$, $a_j (j = 1, 2, \ldots, q)$ be q distinct complex numbers, and let $k_j (j = 1, 2, \ldots, q)$ be positive integers or ∞ satisfying (1) and (4).*

(i) if $q = 7$, then $f_1(z) \equiv f_2(z)$;
(ii) if $q = 6$ and $k_3 \geq 2$, then $f_1(z) \equiv f_2(z)$;
(iii) if $q = 5$, $k_3 \geq 3$ and $k_5 \geq 2$, then $f_1(z) \equiv f_2(z)$;
(iv) if $q = 5$ and $k_4 \geq 4$, then $f_1(z) \equiv f_2(z)$;
(v) if $q = 5$, $k_3 \geq 5$ and $k_4 \geq 3$, then $f_1(z) \equiv f_2(z)$;
(vi) if $q = 5$, $k_3 \geq 6$ and $k_4 \geq 2$, then $f_1(z) \equiv f_2(z)$.

Remark 1.3 *In Theorem 1.5, the conclusion $f(z) \equiv g(z)$ holds when $q = 5$ and $k_j = \infty$ $(j = 1, 2, \ldots, 5)$. From Corollary 1.1, we can get that $f_1(z) \equiv f_2(z)$ when $q = 5$ and k_j $(j = 1, 2, \ldots, 5)$ satisfy any of the four conditions (i)-(iv). Hence, Corollary 1.1 is an improvement of Theorem 1.5.*

For non-admissible functions, the following theorem also plays a very important role in studies theirs uniqueness problems.

Theorem 1.8 (see [13, Theorem 2]). *Let f be a meromorphic function in $\mathbb{D}$ and $\lim_{r \to 1^-} T(r, f) = \infty$, q be a positive integer and $a_1, a_2, \ldots, a_q$ be pairwise distinct complex numbers. Then, for $r \to 1^-$, $r \notin E$,*

$$(q - 2)T(r, f) \leq \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f - a_j}\right) + \log \frac{1}{1 - r} + S(r, f).$$

Remark 1.4 *(i) In contrast to admissible functions, the term $\log \frac{1}{1-r}$ in Theorem 1.8 does not necessarily enter the remainder $S(r, f)$ because the non-admissible function f may have $T(r, f) = O\left(\log \frac{1}{1-r}\right)$.*

(ii) If $0 < \alpha(f) < \infty$, we can see that $S(r, f) = o\left(\log \frac{1}{1-r}\right)$ holds in Theorem 1.8 without a possible exception set.

From Theorem 1.8 and Remark 1.4, we can see that the uniqueness of non-admissible functions is more intricate than the case of admissible functions.

In this paper, we will deal with the uniqueness problem of non-admissible functions in $\mathbb{D}$. We use Υ_α to denote the class of non-admissible functions satisfying the condition: $\alpha(f) = \alpha (0 < \alpha < \infty)$ for $f \in \Upsilon_\alpha$. For the class Υ_α, we get the following results

Theorem 1.9 *Let $f(z) \in \Upsilon_\alpha$, $a_j (j = 1, 2, \ldots, q)$ are q distinct complex numbers. If $q = 5 + [\frac{2}{k} + \frac{k+1}{k\alpha}]$, then there does not exist $g(z) (\not\equiv f(z)) \in \Upsilon_\alpha$ satisfying*

$$\overline{E}_{k)}(a_j, \mathbb{D}, f) = \overline{E}_{k)}(a_j, \mathbb{D}, g), \quad (j = 1, 2, \ldots, q), \tag{5}$$

where $[x]$ denotes the largest integer less than or equal to x.

Corollary 1.2 *Let $f(z) \in \Upsilon_\alpha$. Then $f(z)$ is uniquely determined in Υ_α by one of the following cases:*

(i) if f have seven point-sets $\overline{E}_{1)}(a_j, \mathbb{D}, f) (j = 1, 2, \ldots, 7)$ and $\alpha > 1$;

(ii) if f have six point-sets $\overline{E}_{2)}(a_j, \mathbb{D}, f) (j = 1, 2, \ldots, 6)$ and $\alpha > \frac{3}{2}$;

(iii) if f have five point-sets $\overline{E}_{3)}(a_j, \mathbb{D}, f) (j = 1, 2, \ldots, 5)$ and $\alpha > 4$.

Remark 1.5 *For Corollary 1.1, we can see that the conclusion (iii) in Corollary 1.1 is an improvement of Theorem 1.6. In fact, the conclusion of Theorem 1.6 is that non-constant meromorphic function f is uniquely determined in $\mathbb{D}$ by five point-sets $\overline{E}_{\infty)}(a_j, \mathbb{D}, f) (j = 1, 2, \ldots, 5)$ and $\alpha(f) = \infty$.*

Theorem 1.10 *Let $\alpha > 12$ and $f(z) \in \Upsilon_\alpha$, $a_j (j = 1, 2, \ldots, 5)$ be five distinct complex numbers. Then $f(z)$ is uniquely determined in Υ by three point-sets $\overline{E}_{3)}(a_j, \mathbb{D}, f) (j = 1, 2, 3)$ and two point-sets $\overline{E}_{2)}(a_j, \mathbb{D}, f) (j = 4, 5)$.*

Furthermore, for the uniqueness of regular inadmissibility functions we obtain the following theorems

Theorem 1.11 *Let $a_j (j = 1, 2, \ldots, q)$ be q distinct complex numbers, and let $k_j (j = 1, 2, \ldots, q)$ be positive integers or ∞ satisfying (1). If $f_1(z), f_2(z)$ be non-constant regular inadmissibility functions satisfying $0 < \alpha(f_1), \alpha(f_2) < \infty$, (4) and*

$$\sum_{j=3}^{q} \frac{k_j}{k_j + 1} - 2 > \frac{2}{\alpha(f_1) + \alpha(f_2)}, \tag{6}$$

then $f_1(z) \equiv f_2(z)$.

From Theorem 1.11, similar to Corollary 1.1, we can get the following results easily.

Corollary 1.3 *Let $a_j (j = 1, 2, \ldots, q)$ be q distinct complex numbers, and let $k_j (j = 1, 2, \ldots, q)$ be positive integers or ∞ satisfying (1), $\alpha := \min\{\alpha(f_1), \alpha(f_2)\}$. And let $f_1(z), f_2(z)$ be non-constant regular inadmissibility functions satisfying $0 < \alpha(f_1), \alpha(f_2) < \infty$ and (4),*

(i) if $\alpha > 1$, $q = 7$ and $k_7 \geq 2$, then $f_1(z) \equiv f_2(z)$;

(ii) if $\alpha > 1$, $q = 6$ and $k_6 \geq 4$, then $f_1(z) \equiv f_2(z)$;

(iii) if $\alpha > 2$ *and* $q = 7$*, then* $f_1(z) \equiv f_2(z)$*;*
(iv) if $\alpha > 3$*,* $q = 6$ *and* $k_3 \geq 2$*, then* $f_1(z) \equiv f_2(z)$*;*
(v) if $\alpha > 6$*,* $q = 5$*,* $k_3 \geq 3$ *and* $k_5 \geq 2$*, then* $f_1(z) \equiv f_2(z)$*;*
(vi) if $\alpha > 10$*,* $q = 5$ *and* $k_4 \geq 4$*, then* $f_1(z) \equiv f_2(z)$*;*
(vii) if $\alpha > 12$*,* $q = 5$*,* $k_3 \geq 5$ *and* $k_4 \geq 3$*, then* $f_1(z) \equiv f_2(z)$*;*
(viii) if $\alpha > 42$*,* $q = 5$*,* $k_3 \geq 6$ *and* $k_4 \geq 2$*, then* $f_1(z) \equiv f_2(z)$*.*

Remark 1.6 *In Corollary 1.1,* $f_1(z), f_2(z)$ *are all admissible functions, that is,* $\alpha(f_1) = \infty$ *and* $\alpha(f_2) = \infty$*. From the conclusions of Corollary 1.3, we see that* $f_1(z) \equiv f_2(z)$ *holds when* $f_1(z), f_2(z)$ *are non-constant regular inadmissibility functions with* $\min\{\alpha(f_1), \alpha(f_2)\} > \zeta$ *and* ζ *a positive constant. Hence, Corollary 1.3 is an improvement of Corollary 1.1.*

The following theorem will show that an admissible function can share sufficiently many values concerning multiple values with another inadmissible function.

Theorem 1.12 *If* f_1 *is admissible and* f_2 *is inadmissible satisfying* $\lim_{r\to 1^-} T(r, f_2) = \infty$*,* $a_j (j = 1, 2, \ldots, q)$ *be* q *distinct complex numbers, and let* $k_j (j = 1, 2, \ldots, q)$ *be positive integers or* ∞ *satisfying (1). Then*

$$\sum_{j=2}^{q} \frac{k_j}{k_j + 1} - 2 > 0 \tag{7}$$

and (4) do not hold at same time.

Corollary 1.4 *If* f_1 *is admissible and* f_2 *is inadmissible satisfying* $\lim_{r\to 1^-} T(r, f_2) = \infty$*,* $a_j (j = 1, 2, \ldots, q)$ *be* q *distinct complex numbers. Then*

(i) $f_1(z), f_2(z)$ *can share at most three values* a_1, a_2, a_3 *IM;*

(ii) $f_1(z), f_2(z)$ *can share at most five values* $a_j (j = 1, 2, \ldots, 5)$ *with reduced weight 1;*

And any one of the following cases can not hold

(iii) $q = 4$ *and* $\overline{E}_{k_1)}(a_1, \mathbb{D}, f_1) = \overline{E}_{k_1)}(a_1, \mathbb{D}, f_2)$ $(k_1 \geq 6)$*,* $\overline{E}_{6)}(a_2, \mathbb{D}, f_1) = \overline{E}_{6)}(a_2, \mathbb{D}, f_2)$*,* $\overline{E}_{2)}(a_3, \mathbb{D}, f_1) = \overline{E}_{2)}(a_3, \mathbb{D}, f_2)$ *and* $\overline{E}_{1)}(a_4, \mathbb{D}, f_1) = \overline{E}_{1)}(a_4, \mathbb{D}, f_2)$*;*

(iv) $q = 4$ *and* $\overline{E}_{k_1)}(a_1, \mathbb{D}, f_1) = \overline{E}_{k_1)}(a_1, \mathbb{D}, f_2)$ $(k_1 \geq 3)$*,* $\overline{E}_{3)}(a_2, \mathbb{D}, f_1) = \overline{E}_{3)}(a_2, \mathbb{D}, f_2)$*,* $\overline{E}_{2)}(a_3, \mathbb{D}, f_1) = \overline{E}_{2)}(a_3, \mathbb{D}, f_2)$ *and* $\overline{E}_{2)}(a_4, \mathbb{D}, f_1) = \overline{E}_{2)}(a_4, \mathbb{D}, f_2)$*;*

(v) $q = 5$ *and* $\overline{E}_{k)}(a_i, \mathbb{D}, f_1) = \overline{E}_{k)}(a, \mathbb{D}, f_2)$ $(k \geq 2, i = 1, 2)$*,* $\overline{E}_{1)}(a_j, \mathbb{D}, f_1) = \overline{E}_{1)}(a_j, \mathbb{D}, f_2)$ $(j = 3, 4, 5)$*.*

2 Some Lemmas

To prove our results, we will require the following lemmas.

Lemma 2.1 (see [13, Lemma 1]). *Let* $f(z), g(z)$ *satisfy* $\lim_{r\to 1^-} T(r, f) = \infty$ *and* $\lim_{r\to 1^-} T(r, g) = \infty$*. If there is a* $K \in (0, \infty)$ *with*

$$T(r, f) \leq KT(r, g) + S(r, f) + S(r, g),$$

then each $S(r, f)$ *is also an* $S(r, g)$*.*

Lemma 2.2 (see [19, Lemma 3.4]). *Let $f(z)$ be a non-constant meromorphic function, a be an arbitrary complex number, and k be a positive integer. Then*

$$\overline{N}\left(r,\frac{1}{f-a}\right)\leq \frac{k}{k+1}\overline{N}_{k)}\left(r,\frac{1}{f-a}\right)+\frac{1}{k+1}N\left(r,\frac{1}{f-a}\right),$$

and

$$\overline{N}\left(r,\frac{1}{f-a}\right)\leq \frac{k}{k+1}\overline{N}_{k)}\left(r,\frac{1}{f-a}\right)+\frac{1}{k+1}T(r,f)+O(1),$$

where $\overline{N}_{k)}\left(r,\frac{1}{f-a}\right)$ are denoted by the zeros of $f-a$ in $|z|\leq r$, whose multiplicities are not greater than k and are counted only once.

From Lemma 2.2 and Theorems 1.4 and 1.8, we can get the following Lemma

Lemma 2.3 *Let $f(z)$ be a meromorphic function in $\mathbb{D}$ and $\lim_{r\to 1^-}T(r,f)=\infty$, $a_j(j=1,2,\ldots,q)$ be q distinct complex numbers, and $k_j(j=1,2,\ldots,q)$ be positive integers or ∞. If f is an admissible function, then*

$$\left(q-2-\sum_{j=1}^{q}\frac{1}{k_j+1}\right)T(r,f)\leq \sum_{j=1}^{q}\frac{k_j}{k_j+1}\overline{N}_{k_j)}\left(r,\frac{1}{f-a_j}\right)+S(r,f);$$

If f is a non-admissible function, then

$$\left(q-2-\sum_{j=1}^{q}\frac{1}{k_j+1}\right)T(r,f)\leq \sum_{j=1}^{q}\frac{k_j}{k_j+1}\overline{N}_{k_j)}\left(r,\frac{1}{f-a_j}\right)+\log\frac{1}{1-r}+S(r,f),$$

where $S(r,f)$ is stated as in Theorem 1.4.

3 Proofs of Theorems 1.9 and 1.10

3.1 The Proof of Theorem 1.9

Suppose that there exists $g(z)\in\Upsilon_\alpha$ satisfying (5) and $f(z)\not\equiv g(z)$. Without loss of generality, we can assume that $a_j(j=1,2,\ldots,q)$ are all finite numbers, otherwise, a suitable linear transformation will be done. Since $f(z),g(z)\in\Upsilon_\alpha$, from Lemma 2.3, we have

$$\left(q-2-\frac{q}{k+1}\right)T(r,f)\leq \frac{k}{k+1}\sum_{j=1}^{q}\overline{N}_{k)}\left(r,\frac{1}{f-a_j}\right)+\log\frac{1}{1-r}+S(r,f). \tag{8}$$

If follows form (5) that

$$\sum_{j=1}^{q}\overline{N}_{k)}\left(r,\frac{1}{f-a_j}\right)\leq N\left(r,\frac{1}{f-g}\right)\leq T(r,f)+T(r,g)+O(1). \tag{9}$$

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From (8) and (9), we have

$$\left(\frac{qk}{k+1}-\frac{3k+2}{k+1}\right)T(r,f)\leq\frac{k}{k+1}T(r,g)+\log\frac{1}{1-r}+S(r,f).$$

Similarly, we have

$$\left(\frac{qk}{k+1}-\frac{3k+2}{k+1}\right)T(r,g)\leq\frac{k}{k+1}T(r,f)+\log\frac{1}{1-r}+S(r,g).$$

Combining the above two inequalities, we get

$$\left(\frac{qk}{k+1}-\frac{4k+2}{k+1}\right)\{T(r,f)+T(r,g)\}\leq 2\log\frac{1}{1-r}+S(r,f)+S(r,g). \tag{10}$$

Since $f(z),g(z)\in\Upsilon_\alpha$ and $0<\alpha<\infty$, from the definition of index and $q=5+[\frac{2}{k}+\frac{k+1}{k\alpha}]$, we have for $0<\varepsilon<\alpha-\frac{k+1}{qk-4k-2}$, there exists a sequence $\{r_m\}\to 1^-$ such that

$$T(r_m,f)>(\alpha-\varepsilon)\log\frac{1}{1-r_m},\quad T(r_m,g)>(\alpha-\varepsilon)\log\frac{1}{1-r_m}, \tag{11}$$

for all $m\to\infty$. From $f(z),g(z)\in\Upsilon_\alpha$ and the assumptions of Theorem 1.9, we can see that $S(r,f)=o\left(\log\frac{1}{1-r}\right)$ and $S(r,g)=o\left(\log\frac{1}{1-r}\right)$. From this fact and (10)-(11), we have

$$\left\{2\left(\frac{qk}{k+1}-\frac{4k+2}{k+1}\right)(\alpha-\varepsilon)-2\right\}\log\frac{1}{1-r_m}<o\left(\log\frac{1}{1-r_m}\right). \tag{12}$$

From (12) and $2\left(\frac{qk}{k+1}-\frac{4k+2}{k+1}\right)(\alpha-\varepsilon)-2>0$, we can get a contradiction. Hence, we have $f(z)\equiv g(z)$.

Thus, this completes the proof of Theorem 1.9.

3.2 The Proof of Theorem 1.10

Suppose that there exists $g(z)\in\Upsilon_\alpha$ satisfying $f(z)\not\equiv g(z)$ and

$$\begin{aligned}\overline{E}_{3)}(a_j,\mathbb{D},f)&=\overline{E}_{3)}(a_j,\mathbb{D},g),\quad (j=1,2,3)\\ \overline{E}_{2)}(a_j,\mathbb{D},f)&=\overline{E}_{2)}(a_j,\mathbb{D},g),\quad (j=4,5).\end{aligned} \tag{13}$$

Without loss of generality, we can assume that $a_j(j=1,2,\ldots,5)$ are all finite numbers, otherwise, a suitable linear transformation will be done. Since $f(z),g(z)\in\Upsilon_\alpha$, from Lemma 2.3, we have

$$\begin{aligned}&\left(5-2-\frac{3}{4}-\frac{2}{3}\right)T(r,f)\\ \leq&\frac{3}{4}\sum_{j=1}^{3}\overline{N}_{3)}\left(r,\frac{1}{f-a_j}\right)+\frac{2}{3}\sum_{j=4}^{5}\overline{N}_{2)}\left(r,\frac{1}{f-a_j}\right)+\log\frac{1}{1-r}+S(r,f)\\ \leq&\frac{3}{4}\left(\sum_{j=1}^{3}\overline{N}_{3)}\left(r,\frac{1}{f-a_j}\right)+\sum_{j=4}^{5}\overline{N}_{2)}\left(r,\frac{1}{f-a_j}\right)\right)+\log\frac{1}{1-r}+S(r,f).\end{aligned} \tag{14}$$

From (13), we have

$$\sum_{j=1}^{3}\overline{N}_{3)}\left(r,\frac{1}{f-a_j}\right)+\sum_{j=4}^{5}\overline{N}_{2)}\left(r,\frac{1}{f-a_j}\right)\leq N\left(r,\frac{1}{f-g}\right)\leq T(r,f)+T(r,g)+O(1).$$

From this inequality and (14), we have

$$\frac{5}{6}T(r,f)\leq\frac{3}{4}T(r,g)+\log\frac{1}{1-r}+S(r,f). \tag{15}$$

Similarly, we have

$$\frac{5}{6}T(r,g)\leq\frac{3}{4}T(r,f)+\log\frac{1}{1-r}+S(r,g). \tag{16}$$

Since $f(z), g(z) \in \Upsilon_\alpha$ and $\alpha > 12$, from the definition of index, we have for any $\varepsilon(0 < \varepsilon < \alpha - 12)$, there exists a sequence $\{r_m\} \to 1^-$ satisfying (11) for all $m \to \infty$. From this fact and (15)-(16), we have

$$\left(\frac{1}{6}(\alpha-\varepsilon)-2\right)\log\frac{1}{1-r_m}<o\left(\log\frac{1}{1-r_m}\right). \tag{17}$$

Since $\alpha > 12$ and $0 < \varepsilon < \alpha - 12$, we have $\frac{1}{6}(\alpha-\varepsilon)-2\frac{1}{6}(\alpha-\varepsilon)-2 > 0$, a contradiction. Hence, we have $f(z) \equiv g(z)$.

Thus, this completes the proof of Theorem 1.10.

4 Proof of Theorem 1.11

Without loss of generality, we may assume that all $a_j (j = 1, 2, \dots, q)$ are finite, otherwise, a suitable Möbius transformation will be done. From Lemma 2.3, we have

$$\begin{aligned}\left(q-2-\sum_{j=1}^{q}\frac{1}{k_j+1}\right)T(r,f_1)\leq&\sum_{j=1}^{q}\frac{k_j}{k_j+1}\overline{N}_{k_j)}\left(r,\frac{1}{f_1-a_j}\right)+\log\frac{1}{1-r}\\&+S(r,f_1).\end{aligned} \tag{18}$$

From (1), we have

$$\frac{1}{2}\leq\frac{k_q}{k_q+1}\leq\cdots\leq\frac{k_2}{k_2+1}\leq\frac{k_1}{k_1+1}\leq 1. \tag{19}$$

From (18) and (19), we have

$$\begin{aligned}(20)\qquad &\left(\sum_{j=1}^{q}\frac{k_j}{k_j+1}-2\right)T(r,f_1)\\ \leq&\frac{k_3}{k_3+1}\sum_{j=1}^{q}\overline{N}_{k_j)}\left(r,\frac{1}{f_1-a_j}\right)+\sum_{j=1}^{2}\left(\frac{k_j}{k_j+1}-\frac{k_3}{k_3+1}\right)\overline{N}_{k_j)}\left(r,\frac{1}{f_1-a_j}\right)\\ &+\log\frac{1}{1-r}+S(r,f_1)\\ \leq&\frac{k_3}{k_3+1}\sum_{j=1}^{q}\overline{N}_{k_j)}\left(r,\frac{1}{f_1-a_j}\right)+\sum_{j=1}^{2}\left(\frac{k_j}{k_j+1}-\frac{k_3}{k_3+1}\right)T(r,f_1)\\ &+\log\frac{1}{1-r}+S(r,f_1).\end{aligned}$$

If $f_1(z)\not\equiv f_2(z)$, from the assumptions of Theorem 1.11, we have

$$(21)\qquad \sum_{j=1}^{q}\overline{N}_{k_j)}\left(r,\frac{1}{f_1-a_j}\right)\leq N\left(r,\frac{1}{f_1-f_2}\right)\leq T(r,f_1)+T(r,f_2)+O(1).$$

From this inequality, we have

$$(22)\qquad \left(\sum_{j=3}^{q}\frac{k_j}{k_j+1}+\frac{k_3}{k_3+1}-2\right)T(r,f_1)\leq\frac{k_3}{k_3+1}T(r,f_2)+\log\frac{1}{1-r}+S(r,f_1).$$

Similarly, we have

$$(23)\qquad \left(\sum_{j=3}^{q}\frac{k_j}{k_j+1}+\frac{k_3}{k_3+1}-2\right)T(r,f_2)\leq\frac{k_3}{k_3+1}T(r,f_1)+\log\frac{1}{1-r}+S(r,f_2).$$

Since $0<\alpha(f_1),\alpha(f_2)<\infty$, we have $S(r,f_1)=o\left(\log\frac{1}{1-r}\right),S(r,f_2)=o\left(\log\frac{1}{1-r}\right)$. And from the definition of index, for any ε satisfying

$$0<2\varepsilon<\min\left\{\alpha(f_1),\alpha(f_2),\alpha(f_1)+\alpha(f_2)-\frac{2}{\sum_{j=3}^{q}\frac{k_j}{k_j+1}}\right\},$$

there exists a sequence $\{r_m\}\to1^-$ such that

$$(24)\qquad T(r_m,f_1)>(\alpha(f_1)-\varepsilon)\log\frac{1}{1-r_m},\quad T(r_m,f_2)>(\alpha(f_2)-\varepsilon)\log\frac{1}{1-r_m},$$

for all $m\to\infty$. From (22)-(24), we have

$$(25)\qquad \left((\alpha(f_1)+\alpha(f_2)-2\varepsilon)\sum_{j=3}^{q}\frac{k_j}{k_j+1}-2\right)\log\frac{1}{1-r_m}<o\left(\log\frac{1}{1-r_m}\right).$$

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Since $0 < 2\varepsilon < \alpha(f_1)+\alpha(f_2)-\frac{2}{\Sigma_{j=3}^{q}\frac{k_j}{k_j+1}}$, we have $(\alpha(f_1)+\alpha(f_2)-2\varepsilon)\sum_{j=3}^{q}\frac{k_j}{k_j+1}-2>0$, a contradiction. Hence, we have $f_1(z)\equiv f_2(z)$.

Thus, this completes the proof of Theorem 1.11.

5 Proofs of Theorem 1.12 and Corollary 1.4

5.1 The Proof of Theorem 1.12

We will employ the proof by contradiction, that is, suppose that (4) and (7) hold at the same time. Since $f_1(z)$ is admissible, from Lemma 2.3, and by using the same argument as in Theorem 1.11, we can easily get

$$\left(\sum_{j=3}^{q}\frac{k_j}{k_j+1}+\frac{2k_2}{k_2+1}-2\right)T(r,f_1)\le\frac{k_2}{k_2+1}(T(r,f_1)+T(r,f_2))+S(r,f_1),$$

that is,

$$\left(\sum_{j=2}^{q}\frac{k_j}{k_j+1}-2\right)T(r,f_1)\le\frac{k_2}{k_2+1}T(r,f_2)+S(r,f_1). \tag{26}$$

Set $K=\sum_{j=2}^{q}\frac{k_j}{k_j+1}-2$. If $K>0$, from (26), we have

$$T(r,f_1)\le K'T(r,f_2)+S(r,f_1), \tag{27}$$

where $K'=\frac{1}{K}\frac{k_2}{k_2+1}$. Since $k_j>0(j=1,2,\ldots,q)$, we have $K'>0$ as $K>0$. From this and Lemma 2.1, we can get that each $S(r,f_1)$ is also an $S(r,f_2)$. Since $f_1(z)$ is admissible and $f_2(z)$ is non-admissible, we can get $T(r,f_2)=S(r,f_1)$. Thus, we have

$$T(r,f_2)=S(r,f_1)=S(r,f_2)=o(T(r,f_2)). \tag{28}$$

Since $\lim_{r\to 1^-}T(r,f_2)=\infty$ and (28), we can get a contradiction.

Hence, we prove that (4) and (7) do not hold at the same time.

6 The Proof of Corollary 1.4

(i) Suppose that $f_1(z),f_2(z)$ share four values $a_j(j=1,2,3,4)$ IM, that is, $k_j=\infty(j=1,2,3,4)$. Since $f_1(z)$ is admissible, from Theorem 1.4, we have

$$2T(r,f_1)\le\sum_{j=1}^{4}N\left(r,\frac{1}{f_1-a_j}\right)+S(r,f_1). \tag{29}$$

Since $f_1(z),f_2(z)$ share four values $a_j(j=1,2,3,4)$ IM, we have

$$\sum_{j=1}^{4}N\left(r,\frac{1}{f_1-a_j}\right)\le N\left(r,\frac{1}{f_1-f_2}\right)\le T(r,f_1)+T(r,f_2)+O(1). \tag{30}$$

From (29) and (30), we have

$$T(r, f_1) \leq T(r, f_2) + S(r, f_1). \tag{31}$$

By Lemma 2.1, similar to the proof of Theorem 1.12, we have

$$T(r, f_2) = S(r, f_1) = S(r, f_2) = o(T(r, f_2)).$$

From this and $\lim_{r\to 1^-} T(r, f_2) = \infty$, we can get a contradiction.

Thus, this completes (i) of Corollary 1.4.

Similar to the proof of Corollary 1.4 (i), we can prove (iii),(iv) and (v) of Corollary 1.4 easily. Here we omit the detail.

(ii) Suppose that f_1, f_2 share six values $a_j (j = 1, 2, \ldots, 6)$ with reduced weight 1, that is,

$$\overline{E}_{1)}(a_j, \mathbb{D}, f_1) = \overline{E}_{1)}(a_j, \mathbb{D}, f_2), \quad (j = 1, 2, \ldots, 6), \tag{32}$$

and $k_1 = k_2 = \cdots = k_6 = 1$. Then, we can deduce that

$$\sum_{j=2}^{6} \frac{k_j}{k_j + 1} - 2 = 5 \times \frac{1}{2} - 2 = \frac{1}{2} > 0.$$

From this and the conclusion of Theorem 1.12, we get a contradiction.

Thus, this completes the proof of Corollary 1.4.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

HYX, LZY and CFY completed the main part of this article, HYX corrected the main theorems. All authors read and approved the final manuscript.

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COMPOSITIONS INVOLVING SCHUR HARMONICALLY CONVEX FUNCTIONS

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Abstract. The decision theorem of the Schur harmonic convexity for the compositions involving Schur harmonically convex functions is established and used to determine the Schur harmonic convexity of some symmetric functions.

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Keywords: Schur harmonically convex function; harmonically convex function; composite function; symmetric function

1. Introduction

Throughout the article, $\mathbb{R}$ denotes the set of real numbers, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ denotes n-tuple (n-dimensional real vectors), the set of vectors can be written as

$$\mathbb{R}^n = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) : x_i, \in \mathbb{R}, i = 1, 2, \ldots, n\},$$

$$\mathbb{R}^n_{++} = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) : x_i > 0, i = 1, 2, \ldots, n\},$$

$$\mathbb{R}^n_{+} = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) : x_i \geq 0, i = 1, 2, \ldots, n\}.$$

In particular, the notations $\mathbb{R}$, $\mathbb{R}_{++}$ and $\mathbb{R}_+$ denote $\mathbb{R}^1$, $\mathbb{R}^1_{++}$ and $\mathbb{R}^1_+$, respectively.

The following conclusion is proved in reference [1, p. 91], [2, p. 64-65].

Theorem A. *Let the interval* $[a, b] \subset \mathbb{R}$, $\varphi : \mathbb{R}^n \to \mathbb{R}$, $f : [a, b] \to \mathbb{R}$ *and* $\psi(x_1, x_2, \ldots, x_n) = \varphi(f(x_1), f(x_2), \ldots, f(x_n)) : [a, b]^n \to \mathbb{R}$.

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(*i*) *If φ is increasing and Schur-convex and f is convex, then ψ is Schur-convex.*

(*ii*) *If φ is increasing and Schur-concave and f is concave, then ψ is Schur-concave.*

(*iii*) *If φ is decreasing and Schur-convex and f is concave, then ψ is Schur-convex.*

(*iv*) *If φ is increasing and Schur-convex and f is increasing and convex, then ψ is increasing and Schur-convex.*

(*v*) *If φ is decreasing and Schur-convex and f is decreasing and concave, then ψ is increasing and Schur-convex.*

(*vi*) *If φ is increasing and Schur-convex and f is decreasing and convex, then ψ is decreasing and Schur-convex.*

(*vii*) *If φ is decreasing and Schur-convex and f is increasing and concave, then ψ is decreasing and Schur-convex.*

(*viii*) *If φ is decreasing and Schur-concave and f is decreasing and convex, then ψ is increasing and Schur-concave.*

Theorem A is very effective for determine of the Schur-convexity of the composite functions.

The Schur harmonically convex functions were proposed by Chu et al. [3, 4, 5] in 2009. The theory of majorization was enriched and expanded by using this concepts. Regarding the Schur harmonically convex functions, the aim of this paper is to establish the following theorem which is similar to Theorem A.

Theorem 1. *Let the interval* $[a,b] \subset \mathbb{R}_{++}$, $\varphi : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$, $f : [a,b] \to \mathbb{R}_{++}$ *and* $\psi(x_1, x_2, \ldots, x_n) = \varphi(f(x_1), f(x_2), \ldots, f(x_n)) : [a,b]^n \to \mathbb{R}_{++}$.

(*i*) *If φ is increasing and Schur harmonically convex and f is harmonically convex, then ψ is Schur harmonically convex.*

(*ii*) *If φ is increasing and Schur harmonically concave and f is harmonically concave, then ψ is Schur harmonically concave.*

(*iii*) *If φ is decreasing and Schur harmonically convex and f is harmonically concave, then ψ is Schur harmonically convex.*

(*iv*) *If φ is increasing and Schur harmonically convex and f is increasing and harmonically convex, then ψ is increasing and Schur harmonically convex.*

(*v*) *If φ is decreasing and Schur harmonically convex and f is decreasing and harmonically concave, then ψ is increasing and Schur harmonically convex.*

(*vi*) *If φ is increasing and Schur harmonically convex and f is decreasing and harmonically convex, then ψ is decreasing and Schur harmonically convex.*

(*vii*) *If φ is decreasing and Schur harmonically convex and f is increasing and harmonically concave, then ψ is decreasing and Schur harmonically convex.*

(*viii*) *If φ is decreasing and Schur harmonically concave and f is decreasing and harmonically convex, then ψ is increasing and Schur harmonically concave.*

2. Definitions and lemmas

In order to prove our results, in this section we will recall useful definitions and lemmas.

Definition 1. [1, 2] Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

(*i*) $\boldsymbol{x} \geq \boldsymbol{y}$ means $x_i \geq y_i$ for all $i = 1, 2, \ldots, n$.

(*ii*) Let $\Omega \subset \mathbb{R}^n$, φ: $\Omega \to \mathbb{R}$ is said to be increasing if $\boldsymbol{x} \geq \boldsymbol{y}$ implies $\varphi(\boldsymbol{x}) \geq \varphi(\boldsymbol{y})$. φ is said to be decreasing if and only if $-\varphi$ is increasing.

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Definition 2. [1, 2] Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

We say $\boldsymbol{y}$ majorizes $\boldsymbol{x}$ ($\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$), denoted by $\boldsymbol{x} \prec \boldsymbol{y}$, if $\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$ for $k = 1, 2, \ldots, n-1$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, where $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.

Definition 3. [1, 2] Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

(i) A set $\Omega \subset \mathbb{R}^n$ is said to be a convex set if

$$\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} = (\alpha x_1 + (1-\alpha)y_1, \alpha x_2 + (1-\alpha)y_2, \ldots, \alpha x_n + (1-\alpha)y_n) \in \Omega$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, and $\alpha \in [0, 1]$.

(ii) Let $\Omega \subset \mathbb{R}^n$ be convex set. A function φ: $\Omega \to \mathbb{R}$ is said to be a convex function on Ω if

$$\varphi\left(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}\right) \leq \alpha\varphi(\boldsymbol{x}) + (1-\alpha)\varphi(\boldsymbol{y})$$

holds for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, and $\alpha \in [0, 1]$. φ is said to be a concave function on Ω if and only if $-\varphi$ is convex function on Ω.

(iii) Let $\Omega \subset \mathbb{R}^n$. A function φ: $\Omega \to \mathbb{R}$ is said to be a Schur-convex function on Ω if $\boldsymbol{x} \prec \boldsymbol{y}$ on Ω implies $\varphi\left(\boldsymbol{x}\right) \leq \varphi\left(\boldsymbol{y}\right)$. A function φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex function on Ω.

Lemma 1. (*Schur-convex function decision theorem*)[1, 2] *: Let $\Omega \subset \mathbb{R}^n$ be symmetric and have a nonempty interior convex set. Ω^0 is the interior of Ω. φ : $\Omega \to \mathbb{R}$ is continuous on Ω and differentiable in Ω^0. Then φ is the Schur $-$*

convex (*or Schur* − *concave, respectively*) *function if and only if* φ *is symmetric on* Ω *and*

$$(x_1 - x_2)\left(\frac{\partial\varphi}{\partial x_1} - \frac{\partial\varphi}{\partial x_2}\right) \geq 0(or\ \ \leq 0,\ respectively) \tag{1}$$

holds for any $\boldsymbol{x} \in \Omega^0$.

Definition 4. [6] Let $\Omega \subset \mathbb{R}^n_{++}$.

(i) A set Ω is said to be a harmonically convex set if $\dfrac{\boldsymbol{xy}}{\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}} \in \Omega$ for every $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ and $\lambda \in [0,1]$, where $\boldsymbol{xy} = \sum_{i=1}^n x_i y_i$ and $\dfrac{1}{\boldsymbol{x}} = (\dfrac{1}{x_1}, \dfrac{1}{x_2}, \ldots, \dfrac{1}{x_n})$.

(ii) Let $\Omega \subset \mathbb{R}^n_{++}$ be a harmonically convex set. A function $\varphi : \Omega \to \mathbb{R}_{++}$ be a continuous function, then φ is called a harmonically convex (or concave, respectively) function, if

$$\varphi\left(\frac{1}{\frac{\alpha}{\boldsymbol{x}} + \frac{1-\alpha}{\boldsymbol{y}}}\right) \leq (or \geq, respectively) \frac{1}{\frac{\alpha}{\varphi(\boldsymbol{x})} + \frac{1-\alpha}{\varphi(\boldsymbol{y})}}$$

holds for any $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, and $\alpha \in [0,1]$.

(iii) A function $\varphi : \Omega \to \mathbb{R}_{++}$ is said to be a Schur harmonically convex (or concave, respectively) function on Ω if $\dfrac{1}{\boldsymbol{x}} \prec \dfrac{1}{\boldsymbol{y}}$ implies $\varphi(\boldsymbol{x}) \leq$ (or $\geq$, respectively) $\varphi(\boldsymbol{y})$.

By Definition 4, it is not difficult to prove the following propositions.

Proposition 1. Let $\Omega \subset \mathbb{R}^n_{++}$ be a set, and let $\dfrac{1}{\Omega} = \{(\dfrac{1}{x_1}, \dfrac{1}{x_2}, \ldots, \dfrac{1}{x_n}) : (x_1, x_2, \ldots, x_n) \in \Omega\}$. Then $\varphi : \Omega \to \mathbb{R}_{++}$ is a Schur harmonically convex (or concave, respectively) function on Ω if and only if $\varphi(\dfrac{1}{\boldsymbol{x}})$ is a Schur-convex (or concave, respectively) function on $\dfrac{1}{\Omega}$.

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In fact, for any $\boldsymbol{u},\boldsymbol{v}\in\frac{1}{\Omega}$, there exist $\boldsymbol{x},\boldsymbol{y}\in\Omega$ such that $\boldsymbol{u}=\frac{1}{\boldsymbol{x}},\boldsymbol{v}=\frac{1}{\boldsymbol{y}}$. Let $\boldsymbol{u}\prec\boldsymbol{v}$, that is $\frac{1}{\boldsymbol{x}}\prec\frac{1}{\boldsymbol{y}}$, if $\varphi:\Omega\to\mathbb{R}_{++}$ is a Schur harmonically convex (or concave, respectively) function on Ω, then $\varphi(\boldsymbol{x})\leq$ (or $\geq$, respectively)$\varphi(\boldsymbol{y})$, namely, $\varphi(\frac{1}{\boldsymbol{u}})\leq$ (or $\geq$, respectively)$\varphi(\frac{1}{\boldsymbol{v}})$, this means that $\varphi(\frac{1}{\boldsymbol{x}})$ is a Schur-convex (or concave, respectively) function on $\frac{1}{\Omega}$. The necessity is proved. The sufficiency can be similar to proof.

Proposition 2. $f:[a,b](\subset\mathbb{R}_{++})\to\mathbb{R}_{++}$ is harmonically convex (or concave, respectively) if and only if $g(x)=\frac{1}{f(\frac{1}{x})}$ is concave (or convex, respectively) on $\left[\frac{1}{b},\frac{1}{a}\right]$.

In fact, for any $x,y\in\left[\frac{1}{b},\frac{1}{a}\right]$, then $\frac{1}{x},\frac{1}{y}\in[a,b]$. If $f:[a,b](\subset\mathbb{R}_{++})\to\mathbb{R}_{++}$ is harmonically convex (or concave, respectively), then

$$f\left(\frac{1}{\alpha x+(1-\alpha)y}\right)\leq(or\ \geq,respectively)\frac{1}{\frac{\alpha}{f(\frac{1}{x})}+\frac{1-\alpha}{f(\frac{1}{y})}},$$

this is

$$\frac{1}{f(\frac{1}{\alpha x+(1-\alpha)y})}\geq(or\ \leq,respectively)\frac{\alpha}{f(\frac{1}{x})}+\frac{1-\alpha}{f(\frac{1}{y})},$$

this means that $g(x)=\frac{1}{f(\frac{1}{x})}$ is concave (or convex, respectively) on $\left[\frac{1}{b},\frac{1}{a}\right]$. The necessity is proved. The sufficiency can be similar to proof.

Lemma 2. (*Schur harmonically convex function decision theorem*)[5] *Let $\Omega\subset\mathbb{R}^n_{++}$ be a symmetric and harmonically convex set with inner points, and let $\varphi:\Omega\to\mathbb{R}_{++}$ be a continuously symmetric function which is differentiable on interior Ω^0. Then φ is Schur harmonically convex (or Schur harmonically concave, respectively) on Ω if and only if*

$$(x_1-x_2)\left(x_1^2\frac{\partial\varphi(\boldsymbol{x})}{\partial x_1}-x_2^2\frac{\partial\varphi(\boldsymbol{x})}{\partial x_2}\right)\geq0\quad(or\leq0,\ respectively),\quad\boldsymbol{x}\in\Omega^0.\qquad(2)$$

3. Proof of main results

Proof of Theorem 1. We only give the proof of Theorem 1 (vi) in detail. Similar argument leads to the proof of the rest part.

If φ is increasing and Schur harmonically convex and f is decreasing and harmonically convex, then by Proposition 1, it follows that $\varphi(\frac{1}{x_1},\frac{1}{x_2},\dots,\frac{1}{x_n})$ is decreasing and Schur convex, and by Proposition 2, it follows that $g(x)=\dfrac{1}{f(\frac{1}{x})}$ is decreasing and concave on $\left[\frac{1}{b},\frac{1}{a}\right]$. And then from Theorem A (iii), it follows that

$$\varphi\left(\frac{1}{g(x_1)},\frac{1}{g(x_2)},\dots,\frac{1}{g(x_n)}\right)=\varphi\left(f(\frac{1}{x_1}),f(\frac{1}{x_2}),\dots,f(\frac{1}{x_n})\right)$$

is increasing and Schur-convex. Again by Proposition 1, it follows that

$$\psi(x_1,x_2,\dots,x_n)=\varphi(f(x_1),f(x_2),\dots,f(x_n))$$

is decreasing and Schur harmonically convex.

4. Applications

Let $\boldsymbol{x}=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$. Its elementary symmetric functions are

$$E_r(\boldsymbol{x})=E_r(x_1,x_2,\dots,x_n)=\sum_{1\le i_1<i_2<\cdots<i_r\le n}\prod_{j=1}^{r}x_{i_j},\ \ r=1,2,\dots,n,$$

and defined $E_0(\boldsymbol{x})=1$, and $E_r(\boldsymbol{x})=0$ for $r<0$ or $r>n$. The dual forms of the elementary symmetric functions are

$$E_r^*(\boldsymbol{x})=E_r^*(x_1,x_2,\dots,x_n)=\prod_{1\le i_1<i_2<\cdots<i_r\le n}\sum_{j=1}^{r}x_{i_j},\ \ r=1,2,\dots,n,$$

and defined $E_0^*(\boldsymbol{x})=1$, and $E_r^*(\boldsymbol{x})=0$ for $r<0$ or $r>n$.

It is well-known that $E_r(\boldsymbol{x})$ is a increasing and Schur-concave function on $\mathbb{R}^n_+$[1].

In [7, 6], Shi proved that $E_r^*(\boldsymbol{x})$ is a increasing and Schur-concave function on $\mathbb{R}^n_+$.

Theorem 2. *For $r = 1, 2, \dots, n, n \geq 2$, $E_r(\boldsymbol{x})$ and $E_r^*(\boldsymbol{x})$ are Schur harmonically convex function on $\mathbb{R}^n_{++}$.*

Proof. Noting that

$$E_r(\boldsymbol{x}) = x_1x_2E_{r-2}(x_3, x_4, \dots, x_n) + (x_1 + x_2)E_{r-1}(x_3, x_4, \dots, x_n)$$
$$+ E_r(x_3, x_4, \dots, x_n), \qquad r = 1, 2, \dots, n,$$

then

$$\begin{aligned}
&(x_1 - x_2)\left(x_1^2\frac{\partial E_r(\boldsymbol{x})}{\partial x_1} - x_2^2\frac{\partial E_r(\boldsymbol{x})}{\partial x_2}\right)\\
=&(x_1 - x_2)[x_1^2(x_2E_{r-2}(x_3, x_4, \dots, x_n) + E_{r-1}(x_3, x_4, \dots, x_n)) -\\
&x_2^2(x_1E_{r-2}(x_3, x_4, \dots, x_n) + E_{r-1}(x_3, x_4, \dots, x_n))]\\
=&(x_1 - x_2)^2[x_1x_2E_{r-2}(x_3, x_4, \dots, x_n) + (x_1 + x_2)E_{r-1}(x_3, x_4, \dots, x_n)] \geq 0,
\end{aligned}$$

by Lemma 2, it follows that $E_r(\boldsymbol{x})$ is Schur harmonically convex on $\mathbb{R}^n_{++}$.

By a direct, though tedious, calculation, and according to Lemma 2, $E_1^*(\boldsymbol{x})$, $E_2^*(\boldsymbol{x})$ is Schur harmonically convex on $\mathbb{R}^n_{++}$. When $r > 2$, it is easy to see that

$$E_r^*(\boldsymbol{x}) = E_r^*(x_1, x_2, \dots, x_n) = E_r^*(x_2, x_3, \dots, x_n) \times \prod_{2 \leq i_1 < i_2 < \dots < i_{r-1} \leq n} (x_1 + \sum_{j=1}^{r-1} x_{i_j}),$$

then

$$\log E_r^*(\boldsymbol{x}) = \log E_r^*(x_2, x_3, \dots, x_n) + \sum_{2 \leq i_1 < i_2 < \dots < i_{r-1} \leq n} \log(x_1 + \sum_{j=1}^{r-1} x_{i_j}).$$

Now, it leads to

$$\frac{1}{E_r^*(\boldsymbol{x})}\frac{\partial E_r^*(\boldsymbol{x})}{\partial x_1} = \sum_{2 \leq i_1 < i_2 < \dots < i_{r-1} \leq n} \frac{1}{x_1 + \sum_{j=1}^{r-1} x_{i_j}},$$

and then

$$\frac{\partial E_r^*(\boldsymbol{x})}{\partial x_1} = E_r^*(\boldsymbol{x})\times$$
$$\left[\sum_{3\le i_1<i_2<\cdots<i_{r-1}\le n}\frac{1}{x_1+\sum_{j=1}^{r-1}x_{i_j}}+\sum_{3\le i_1<i_2<\cdots<i_{r-2}\le n}\frac{1}{x_1+x_2+\sum_{j=1}^{r-2}x_{i_j}}\right].$$

By the same arguments,

$$\frac{\partial E_r^*(\boldsymbol{x})}{\partial x_2} = E_r^*(\boldsymbol{x})\times$$
$$\left[\sum_{3\le i_1<i_2<\cdots<i_{r-1}\le n}\frac{1}{x_2+\sum_{j=1}^{r-1}x_{i_j}}+\sum_{3\le i_1<i_2<\cdots<i_{r-2}\le n}\frac{1}{x_1+x_2+\sum_{j=1}^{r-2}x_{i_j}}\right].$$

Thus,

$$\begin{aligned}
&(x_1-x_2)\left(x_1^2\frac{\partial E_r^*(\boldsymbol{x})}{\partial x_1}-x_2^2\frac{\partial E_r^*(\boldsymbol{x})}{\partial x_2}\right)\\
=&(x_1-x_2)E_r^*(\boldsymbol{x})\times\left[\sum_{3\le i_1<i_2<\cdots<i_{r-1}\le n}\left(\frac{x_1^2}{x_1+\sum_{j=1}^{r-1}x_{i_j}}-\frac{x_2^2}{x_2+\sum_{j=1}^{r-1}x_{i_j}}\right)+\right.\\
&\qquad\left.(x_1^2-x_2^2)\cdot\sum_{3\le i_1<i_2<\cdots<i_{r-2}\le n}\frac{1}{x_1+x_2+\sum_{j=1}^{r-2}x_{i_j}}\right]\\
=&(x_1-x_2)^2E_r^*(\boldsymbol{x})\times\left[\sum_{3\le i_1<i_2<\cdots<i_{r-1}\le n}\frac{x_1x_2+(x_1+x_2)\sum_{j=1}^{r-1}x_{i_j}}{(x_1+\sum_{j=1}^{r-1}x_{i_j})(x_2+\sum_{j=1}^{r-1}x_{i_j})}+\right.\\
&\qquad\left.(x_1+x_2)\cdot\sum_{3\le i_1<i_2<\cdots<i_{r-2}\le n}\frac{1}{x_1+x_2+\sum_{j=1}^{r-2}x_{i_j}}\right]\ge 0,
\end{aligned}$$

by Lemma 2, it follows that $E_r^*(\boldsymbol{x})$ is Schur harmonically convex on $\mathbb{R}_{++}^n$. □

For $\boldsymbol{x}=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$, the complete symmetric functions $c_n(\boldsymbol{x},r)$ are defined as

$$c_n(\boldsymbol{x},r)=\sum_{i_1+i_2+\cdots+i_n=r}\prod_{j=1}^{n}x_j^{i_j},\ r=1,2,\ldots,n,$$

where $c_0(\boldsymbol{x},r)=1,\ r\in\{1,2,\ldots,n\},\ i_1,i_2,\ldots,i_n$ are non-negative integers.

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The dual forms of the complete symmetric functions $c_n^*(\boldsymbol{x}, r)$ are

$$c_n^*(\boldsymbol{x}, r) = \prod_{i_1+i_2+\cdots+i_n=r} \sum_{j=1}^{n} i_j x_j,\ \ r = 1, 2, \ldots, n,$$

where $i_j (j = 1, 2, \ldots, n)$ are non-negative integers.

Guan [8] discussed the Schur-convexity of $c_n(\boldsymbol{x}, r)$ and proved that $c_n(\boldsymbol{x}, r)$ is increasing and Schur-convex on $\mathbb{R}^n_{++}$. Subsequently, Chu et al. [5] proved that $c_n(\boldsymbol{x}, r)$ is Schur harmonically convex on $\mathbb{R}^n_{++}$.

Zhang and Shi [9] proved that $c_n^*(\boldsymbol{x}, r)$ is increasing, Schur-concave and Schur harmonically convex on $\mathbb{R}^n_{++}$.

In the following, we prove that the Schur harmonic convexity of the composite functions involving the above symmetric functions and their dual form by using Theorem 1.

Theorem 3. *The following symmetric functions are increasing and Schur harmonically convex on* $(0, 1)^n, r = 1, 2, \ldots, n,$

$$E_r\left(\frac{1+\boldsymbol{x}}{1-\boldsymbol{x}}\right) = \sum_{1\le i_1<i_2<\cdots<i_r\le n} \prod_{j=1}^{r} \frac{1+x_{i_j}}{1-x_{i_j}}, \tag{3}$$

$$E_r^*\left(\frac{1+\boldsymbol{x}}{1-\boldsymbol{x}}\right) = \prod_{1\le i_1<i_2<\cdots<i_r\le n} \sum_{j=1}^{r} \frac{1+x_{i_j}}{1-x_{i_j}}, \tag{4}$$

$$c_n\left(\frac{1+\boldsymbol{x}}{1-\boldsymbol{x}}, r\right) = \sum_{i_1+i_2+\cdots+i_n=r} \prod_{j=1}^{n} \left(\frac{1+x_j}{1-x_j}\right)^{i_j} \tag{5}$$

and

$$c_n^*\left(\frac{1+\boldsymbol{x}}{1-\boldsymbol{x}}, r\right) = \prod_{i_1+i_2+\cdots+i_n=r} \sum_{j=1}^{n} i_j \left(\frac{1+x_j}{1-x_j}\right). \tag{6}$$

Proof. Let $f(x) = \dfrac{1+x}{1-x}, x \in (0, 1)$. Then $f(x) > 0$, $f'(x) = \dfrac{2}{(1-x)^2} > 0$, so f is increasing on $(0, 1)$.

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And let $g(x) = \dfrac{1}{f(\frac{1}{x})} = \dfrac{x-1}{x+1}$. Then $g^{''}(x) = -\dfrac{4}{(x+1)^3} < 0$, this means that $\dfrac{1}{f(\frac{1}{x})}$ is concave on $(1, \infty)$, by Proposition 2, it follows that f is harmonically convex on $(0, 1)$. Since $E_r(\boldsymbol{x})$, $E_r^*(\boldsymbol{x})$, $c_n(\boldsymbol{x}, r)$ and $c_n^*(\boldsymbol{x}, r)$ are all increasing and Schur harmonically convex function on $\mathbb{R}_{++}^n$, by Theorem 1 (iv), it follows that Theorem 3 holds. □

Remark 1. By Lemma 2, Xia and Chu [10] proved that $E_r\left(\dfrac{1+\boldsymbol{x}}{1-\boldsymbol{x}}\right)$ is Schur harmonically convex on $(0,1)^n$. By the properties of Schur harmonically convex function, Shi and Zhang [11] proved that $E_r^*\left(\dfrac{1+\boldsymbol{x}}{1-\boldsymbol{x}}\right)$ is Schur harmonically convex on $(0,1)^n$. By Theorem 1, we give a new proof.

Theorem 4. *The following symmetric functions are increasing and Schur harmonically convex on* $\mathbb{R}_{++}^n, r = 1, 2, \ldots, n$,

$$E_r\left(\boldsymbol{x}^{\frac{1}{r}}\right) = \sum_{1 \le i_1 < i_2 < \cdots < i_r \le n} \prod_{j=1}^{r} x_{i_j}^{\frac{1}{r}}, \tag{7}$$

$$E_r^*\left(\boldsymbol{x}^{\frac{1}{r}}\right) = \prod_{1 \le i_1 < i_2 < \cdots < i_r \le n} \sum_{j=1}^{r} x_{i_j}^{\frac{1}{r}}, \tag{8}$$

$$c_n\left(\boldsymbol{x}^{\frac{1}{r}}, r\right) = \sum_{i_1 + i_2 + \cdots + i_n = r} \prod_{j=1}^{n} x_j^{\frac{i_j}{r}} \tag{9}$$

and

$$c_n^*\left(\boldsymbol{x}^{\frac{1}{r}}, r\right) = \prod_{i_1 + i_2 + \cdots + i_n = r} \sum_{j=1}^{n} i_j x_j^{\frac{1}{r}}. \tag{10}$$

Proof. For $r \ge 1$, let $p(x) = x^{\frac{1}{r}}, x \in \mathbb{R}_{++}$. Then $p^{'}(x) = \frac{1}{r}x^{\frac{1}{r}-1} > 0$, so p is increasing on $\mathbb{R}_{++}$.

And let $q(x) = \dfrac{1}{p(\frac{1}{x})} = x^{\frac{1}{r}} = p(x)$. Then $q^{''}(x) = \frac{1}{r}(\frac{1}{r} - 1)x^{\frac{1}{r}-2} \le 0$, this means that $\dfrac{1}{p(\frac{1}{x})}$ is concave on $\mathbb{R}_{++}$, by Proposition 2, it follows that p is harmonically convex on $\mathbb{R}_{++}$. Since $E_r(\boldsymbol{x})$, $E_r^*(\boldsymbol{x})$, $c_n(\boldsymbol{x}, r)$ and $c_n^*(\boldsymbol{x}, r)$ are all increasing and

Schur harmonically convex function on $\mathbb{R}^n_{++}$, by Theorem 1 (iv), it follows that Theorem 4 holds. $\square$

Remark 2. By Lemma 2, Chu and Lv [3] proved that the Hamy's symmetric function $E_r\left(\boldsymbol{x}^{\frac{1}{r}}\right)$ is Schur harmonically convex on $\mathbb{R}^n_{++}$. Later, K. Z. Guan and R. K. Guan [12] further studied the harmonic convexity of the generalized Hamy symmetric function.

By Lemma 2, Meng et al. [13] proved that the dual form of the Hamy's symmetric function $E_r^*\left(\boldsymbol{x}^{\frac{1}{r}}\right)$ is Schur harmonically convex on $\mathbb{R}^n_{++}$.

By Lemma 2, Chu and Sun [4] proved that $c_n\left(\boldsymbol{x}^{\frac{1}{r}}, r\right)$ is Schur harmonically convex on $\mathbb{R}^n_{++}$.

By Theorem 1, we give a new proof.

Since $f(x) = \dfrac{1+x}{1-x}$ is increasing and harmonically convex on $(0,1)$, from Theorem 1 (iv) and Theorem 4, it follows

Theorem 5. *The following symmetric functions are increasing and Schur harmonically convex on* $(0,1)^n, r = 1, 2, \ldots, n$,

$$E_r\left(\left(\frac{1+\boldsymbol{x}}{1-\boldsymbol{x}}\right)^{\frac{1}{r}}\right) = \sum_{1\le i_1<i_2<\cdots<i_r\le n} \prod_{j=1}^{r} \left(\frac{1+x_{i_j}}{1-x_{i_j}}\right)^{\frac{1}{r}}, \tag{11}$$

$$E_r^*\left(\left(\frac{1+\boldsymbol{x}}{1-\boldsymbol{x}}\right)^{\frac{1}{r}}\right) = \prod_{1\le i_1<i_2<\cdots<i_r\le n} \sum_{j=1}^{r} \left(\frac{1+x_{i_j}}{1-x_{i_j}}\right)^{\frac{1}{r}}, \tag{12}$$

$$c_n\left(\left(\frac{1+\boldsymbol{x}}{1-\boldsymbol{x}}\right)^{\frac{1}{r}}, r\right) = \sum_{i_1+i_2+\cdots+i_n=r} \prod_{j=1}^{n} \left(\frac{1+x_j}{1-x_j}\right)^{\frac{i_j}{r}} \tag{13}$$

and

$$c_n^*\left(\left(\frac{1+\boldsymbol{x}}{1-\boldsymbol{x}}\right)^{\frac{1}{r}}, r\right) = \prod_{i_1+i_2+\cdots+i_n=r} \sum_{j=1}^{n} i_j \left(\frac{1+x_j}{1-x_j}\right)^{\frac{1}{r}}. \tag{14}$$

Remark 3. By Lemma 2, Long and Chu [14] proved that $E_r^*\left(\left(\frac{1+\boldsymbol{x}}{1-\boldsymbol{x}}\right)^{\frac{1}{r}}\right)$ is Schur harmonically convex on $(0,1)^n$. By Theorem 1, we give a new proof.

Theorem 6. *The following symmetric functions are increasing and Schur harmonically convex on* $(0,1)^n, r = 1, 2, \ldots, n$,

$$E_r\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}\right) = \sum_{1\le i_1<i_2<\cdots<i_r\le n} \prod_{j=1}^{r} \frac{x_{i_j}}{1-x_{i_j}}, \tag{15}$$

$$E_r^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}\right) = \prod_{1\le i_1<i_2<\cdots<i_r\le n} \sum_{j=1}^{r} \frac{x_{i_j}}{1-x_{i_j}}, \tag{16}$$

$$c_n\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right) = \sum_{i_1+i_2+\cdots+i_n=r} \prod_{j=1}^{n} \left(\frac{x_j}{1-x_j}\right)^{i_j} \tag{17}$$

and

$$c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right) = \prod_{i_1+i_2+\cdots+i_n=r} \sum_{j=1}^{n} i_j \left(\frac{x_j}{1-x_j}\right). \tag{18}$$

Proof. Let $h(x) = \dfrac{x}{1-x}, x \in (0,1)$. Then $h'(x) = \dfrac{1}{(1-x)^2} > 0$, so h is increasing on $(0,1)$.

And let $k(x) = \dfrac{1}{h(\frac{1}{x})} = x - 1$. Then $k''(x) = 0$, this means that $\dfrac{1}{h(\frac{1}{x})}$ is concave on $(1,\infty)$, by Proposition 2, it follows that h is harmonically convex on $(0,1)$. Since $E_r(\boldsymbol{x})$, $E_r^*(\boldsymbol{x})$, $c_n(\boldsymbol{x}, r)$ and $c_n^*(\boldsymbol{x}, r)$ are all increasing and Schur harmonically convex function on $\mathbb{R}_{++}^n$, by Theorem 1 (iv), it follows that Theorem 5 holds. □

Remark 4. By the judgement theorem of Schur harmonic convexity for a class of symmetric functions, Shi and Zhang [15] proved that $E_r\left(\dfrac{\boldsymbol{x}}{1-\boldsymbol{x}}\right)$ is Schur harmonically convex on $(0,1)^n$. Here by Theorem 1, we give a new proof.

By the properties of Schur harmonically convex function, Shi and Zhang [11] proved that $E_r^*\left(\dfrac{\boldsymbol{x}}{1-\boldsymbol{x}}\right)$ is Schur harmonically convex on $\left[\dfrac{1}{2}, 1\right)^n$. By Theorem 1, this conclusion is extended to the collection $(0,1)^n$.

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By Lemma 2, Sun et al. [16] proved that $c_n\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$ is Schur harmonically convex on $[0,1)^n$, here by Theorem 1, we give a new proof.

Since $f(x)=\frac{x}{1-x}$ is increasing and harmonically convex on $(0,1)$, from Theorem 1 (iv) and Theorem 4, it follows

Theorem 7. *The following symmetric functions are increasing and Schur harmonically convex on* $(0,1)^n, r=1,2,\ldots,n$,

$$E_r\left(\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}\right)^{\frac{1}{r}}\right)=\sum_{1\le i_1<i_2<\cdots<i_r\le n}\prod_{j=1}^{r}\left(\frac{x_{i_j}}{1-x_{i_j}}\right)^{\frac{1}{r}}, \tag{19}$$

$$E_r^*\left(\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}\right)^{\frac{1}{r}}\right)=\prod_{1\le i_1<i_2<\cdots<i_r\le n}\sum_{j=1}^{r}\left(\frac{x_{i_j}}{1-x_{i_j}}\right)^{\frac{1}{r}}, \tag{20}$$

$$c_n\left(\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}\right)^{\frac{1}{r}}, r\right)=\sum_{i_1+i_2+\cdots+i_n=r}\prod_{j=1}^{n}\left(\frac{x_j}{1-x_j}\right)^{\frac{i_j}{r}} \tag{21}$$

and

$$c_n^*\left(\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}\right)^{\frac{1}{r}}, r\right)=\prod_{i_1+i_2+\cdots+i_n=r}\sum_{j=1}^{n} i_j\left(\frac{x_j}{1-x_j}\right)^{\frac{1}{r}}. \tag{22}$$

Remark 5. By Lemma 2, Sun [17] proved that $E_r\left(\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}\right)^{\frac{1}{r}}\right)$ is Schur harmonically convex on $[0,1)^n$. Here by Theorem 1, we give a new proof.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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A NOTE ON DEGENERATE GENERALIZED q-GENOCCHI POLYNOMIALS

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Abstract. In this paper,we consider degenerate generalized q-Genocchi polynomials arising from p-adic fermionic q-integral on $\mathbb{Z}_p$. We found some interesting identities of these polynomials.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$. Let q be an indeterminate in $\mathbb{C}_p$ such that $|q-1|_p < p^{-\frac{1}{p-1}}$.

Let $f(x)$ be a continuous function on $\mathbb{Z}_p$. Then the *p-adic fermionic q-integral on $\mathbb{Z}_p$* is defined by Kim to be

$$\begin{aligned} \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_{-q}\left(x+p^N\mathbb{Z}_p\right) \\ &= \frac{[2]_q}{2} \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x q^x, \text{ (see [9]).} \end{aligned} \tag{1.1}$$

Thus, by (1.1), we get

$$q\int_{\mathbb{Z}_p} f(x+1)d\mu_{-q}(x) + \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = [2]_q f(0), \tag{1.2}$$

and

$$q^n\int_{\mathbb{Z}_p} f(x+n)d\mu_{-q}(x) + (-1)^{n-1}\int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = [2]_q \sum_{l=0}^{n-1} f(l)q^l(-1)^{n-1-l}, \tag{1.3}$$

where $n \in \mathbb{N}$ (see [5-10, 12]).

It is known that the *q-Euler polynomials* are given by the generating function as follows:

$$\int_{\mathbb{Z}_p} e^{(x+y)t}d\mu_{-q}(y) = \frac{[2]_q}{qe^t+1}e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}. \tag{1.4}$$

When $x = 0$, $E_{n,q} = E_{n,q}(0)$ are called *q-Euler numbers* (see [5, 9, 12]).

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Recently, degenerate q-Euler polynomials are introduced by he generating function as follows:

$$\frac{[2]_q}{q(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\mathcal{E}_{n,q,\lambda}(x)\frac{t^n}{n!},\ \text{(see [9])}. \tag{1.5}$$

It is known that the *q-Genocchi polynomials* are given by the generating function as follows:

$$\int_{\mathbb{Z}_p} te^{(x+y)t}d\mu_{-q}(y)=\frac{t[2]_q}{qe^t+1}e^{xt}=\sum_{n=0}^{\infty}G_{n,q}(x)\frac{t^n}{n!}. \tag{1.6}$$

When $x=0$, $G_{n,q}=E_{n,q}(0)$ are called *q-Genocchi numbers* (see [1, 2, 4, 6-8]).

Now, the degenerate q-Genocchi polynomials are introduced by the generating function as follows:

$$\frac{t[2]_q}{q(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\mathcal{G}_{n,q,\lambda}(x)\frac{t^n}{n!}. \tag{1.7}$$

Note that $\lim_{\lambda\to 0}\mathcal{G}_{n,q,\lambda}(x)=G_{n,q}(x)$, $(n\geq 0)$, (see [3]).

For $d\in\mathbb{N}$ with $d\equiv$ (mod 2) and $(d,p)=1$, we set

$$X=\varprojlim_{N}\mathbb{Z}/dp^N\mathbb{Z},\ X^*=\bigcup_{0<a<dp,\ a\nmid p}\left(a+dp\mathbb{Z}_p\right),$$

and

$$a+dp^N\mathbb{Z}_p=\left\{x\in X\mid x\equiv a\ (\bmod\ dp^N)\right\},$$

where $a\in\mathbb{Z}$ with $0\leq a<dp^N-1$.

For $d\in\mathbb{N}$ with $d\equiv 1$ (mod 2), let us assume that χ is a Dirichlet character with conductor d. Now, we consider the generalized q-Genocchi polynomials attached to χ which are given by the generating function to be

$$\begin{aligned}\int_X \chi(y)te^{(x+y)t}d\mu_{-q}(y)&=\left(\frac{t[2]_q}{q^de^{dt}+1}\sum_{a=0}^{d-1}q^q(-1)^a\chi(a)e^{at}\right)e^{xt}\\&=\sum_{n=0}^{\infty}G_{n,q,\chi}(x)\frac{t^n}{n!},\ \text{(see [4-6, 8])}.\end{aligned} \tag{1.8}$$

When $x=0$, $G_{n,q,\chi}=G_{n,q,\chi}(0)$ are called *generalized q-Genocchi numbers attached to χ*.

One of the most recent papers on the theory of Genocchi polynomials and numbers is the paper T. Kim(see [6-8]), which deals mainly with the theory of Genocchi polynomials and numbers. Facts on Bernoulli polynomials and Euler polynomials, to which Genocchi polynomials may be related, has been derived in Volkenborn integral (see [3]). While a lot of the properties of Genocchi polynomials bear a striking resemblance to the properties of Beroulli and Euler polynomials, some properties are rather different. Note that Genocchi polynomials occur naturally in the areas of elementary number theory, complex analytic number theory, homotopy theory, differential topology, theory of modular forms, p-adic analytic number theory, quantum physics (see [1-13]).

In the viewpoint of (1.8), we consider degenerate generalized q-Genocchi polynomials which are derived from the fermionic q-integral on $\mathbb{Z}_p$. The purpose of this paper is to investigate some properties and identities of degenerate generalized q-Genocchi polynomials.

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2. Degenerate generalized q-Genocchi polynomials

In this section, we assume that $\lambda, t \in \mathbb{C}_p$ with $|\lambda t|_p < p^{-\frac{1}{p-1}}$. For $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, let χ be a Dirichlet's character with conductor d.

In the viewpoint of (1.8), we consider degenerate generalized q-Genocchi polynomials which are given by the generating function to be

$$\begin{aligned}
&\int_X \chi(y)t(1+\lambda t)^{\frac{x+y}{\lambda}}d\mu_{-q}(y)\\
&=\left(\frac{t[2]_q}{q^d(1+\lambda t)^{\frac{d}{\lambda}}+1}\sum_{a=0}^{d-1}q^a(-1)^a\chi(a)(1+\lambda t)^{\frac{a}{\lambda}}\right)(1+\lambda t)^{\frac{x}{\lambda}} \qquad (2.1)\\
&=\sum_{n=0}^{\infty}\mathcal{G}_{n,\chi,q,\lambda}(x)\frac{t^n}{n!},
\end{aligned}$$

where $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$.

From (1.5) and (2.1), we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{G}_{n,\chi,q,\lambda}(x)\frac{t^n}{n!}&=\left(\frac{t[2]_q}{q^d(1+\lambda t)^{\frac{d}{\lambda}}+1}\sum_{a=0}^{d-1}q^a(-1)^a\chi(a)(1+\lambda t)^{\frac{a+x}{\lambda}}\right)\\
&=\frac{[2]_q}{[2]_{q^d}}\sum_{a=0}^{d-1}q^a(-1)^a\chi(a)\left(\frac{t[2]_{q^d}}{q^d(1+\lambda t)^{\frac{d}{\lambda}}+1}(1+\lambda t)^{\frac{d}{\lambda}\frac{a+x}{d}}\right) \qquad (2.2)\\
&=\sum_{n=0}^{\infty}\left(\frac{[2]_q}{[2]_{q^d}}\sum_{a=0}^{d-1}q^a(-1)^a\chi(a)\mathcal{G}_{n,q^d,\frac{\lambda}{d}}\left(\frac{a+x}{d}\right)\frac{d^nt^{n+1}}{n!}\right).
\end{aligned}$$

Thus, by (2.2), we get

$$\mathcal{G}_{n,\chi,q,\lambda}(x)=\frac{nd^{n-1}[2]_q}{[2]_{q^d}}\sum_{a=0}^{d-1}q^a(-1)^a\chi(a)\mathcal{G}_{n-1,q^d,\frac{\lambda}{d}}\left(\frac{a+x}{d}\right),\ (n\geq 0). \qquad (2.3)$$

Therefore, by (2.3), we obtain the following theorem.

Theorem 2.1. *For $d \in \mathbb{N}$ with $d \equiv 1$ (mod 2), $n \geq 0$, we have*

$$\begin{aligned}
\mathcal{G}_{n,\chi,q,\lambda}(x)&=\int_X \chi(y)t(x+y|\lambda)_n d\mu_{-q}(y)\\
&=\frac{n[2]_q}{[2]_{q^d}}d^{n-1}\sum_{a=0}^{d-1}\chi(a)q^a(-1)^a\mathcal{G}_{n-1,q^d,\frac{\lambda}{d}}\left(\frac{a+x}{d}\right),
\end{aligned}$$

where

$$\begin{aligned}
(x|\lambda)_n&=x(x-\lambda)\cdots(x-\lambda(n-1))\\
&=\lambda^n\left(\frac{x}{\lambda}\right)_n.
\end{aligned}$$

For $n \geq 0$, we observe that

$$\begin{aligned}
(x+y|\lambda)_n&=\lambda^n\left(\frac{x+y}{\lambda}\right)_n=\lambda^n\sum_{l=0}^{n}S_1(n,l)\left(\frac{x+y}{\lambda}\right)^l\\
&=\sum_{l=0}^{n}S_1(n,l)\lambda^{n-l}(x+y)^l. \qquad (2.4)
\end{aligned}$$

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By (2.4), we get

$$\begin{aligned}\int_X \chi(y)t(x+y|\lambda)_n d\mu_{-q}(y) &= \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l}\int_X \chi(y)t(x+y)^l d\mu_{-q}(y)\\ &= \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l}G_{l,q,\chi}(x),\ (n\geq 0),\end{aligned} \tag{2.5}$$

where $S_1(n,l)$ is the Stirling number of the first kind. Therefore, by (2.5), we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$\mathcal{G}_{n,\chi,q,\lambda}(x) = \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l}G_{l,q,\chi}(x).$$

By replacing t by $\frac{1}{\lambda}(e^{\lambda t}-1)$ in (2.1), we get

$$\begin{aligned}\int_X \chi(y)\frac{1}{\lambda}(e^{\lambda t}-1)e^{(x+y)t}d\mu_{-q}(y) &= \sum_{m=0}^{\infty}\mathcal{G}_{m,\chi,q,\lambda}(x)\frac{1}{m!}\left(\frac{1}{\lambda}(e^{\lambda t}-1)\right)^m\\ =\sum_{m=0}^{\infty}\mathcal{G}_{m,\chi,q,\lambda}(x)\lambda^{-m}\sum_{n=m}^{\infty}S_2(n,m)\frac{\lambda^n}{n!}t^n\\ =\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\lambda^{n-m}S_2(n,m)\mathcal{G}_{m,\chi,q,\lambda}(x)\right)\frac{t^n}{n!},\end{aligned} \tag{2.6}$$

where $S_2(n,m)$ is the Stirling number of the second kind.

From (1.8), we note that

$$\begin{aligned}\int_X \chi(y)te^{(x+y)t}d\mu_{-q}(y) &= \left(\frac{t[2]_q}{q^d e^{dt}+1}\right)\sum_{a=0}^{d-1}q^a(-1)^a\chi(a)e^{(a+x)t}\\ &= \sum_{n=0}^{\infty}G_{n,q,\chi}(x)\frac{t^n}{n!}.\end{aligned} \tag{2.7}$$

By multiplying t on the both side (2.6), we get

$$\int_X \chi(y)te^{(x+y)t}d\mu_{-q}(y) = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{1}{e^{\lambda t}-1}\lambda^{n-m+1}S_2(n,m)\mathcal{G}_{m,\chi,q,\lambda}(x)\right)\frac{t^{n+1}}{n!}. \tag{2.8}$$

Therefore, by (2.6), (2.7) and (2.8), we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, we have*

$$G_{n,q,\chi}(x) = \sum_{m=0}^{n-1}\frac{n}{e^{\lambda t}-1}\lambda^{n-m}S_2(n-1,m)\mathcal{G}_{m,\chi,q,\lambda}(x).$$

Let $d\in\mathbb{N}$ with $d\equiv 1 \pmod 2$. From (1.3), we have

$$\begin{aligned}&q^d\int_X (x+d|\lambda)_n\chi(x)d\mu_{-q}(x)+\int_X (x|\lambda)_n\chi(x)d\mu_{-q}(x)\\ =&[2]_q\sum_{a=0}^{d-1}\chi(a)q^q(-1)^a(a|\lambda)_n,\ (n\geq 0).\end{aligned} \tag{2.9}$$

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Therefore, by Theorem 2.1 and (2.8), we obtain the following theorem.

Theorem 2.4. *For $d \in \mathbb{N}$ with $d \equiv 1$ (mod 2), $n \geq 0$, we have*

$$q^d \mathcal{G}_{n,\chi,q,\lambda}(d) + \mathcal{G}_{n,\chi,q,\lambda} = t[2]_q \sum_{a=0}^{d-1} \chi(a) q^q (-1)^a (a|\lambda)_n,$$

where $\mathcal{G}_{n,\chi,q,\lambda} = \mathcal{G}_{n,\chi,q,\lambda}(0)$ are called degenerate generalized q-Genocchi numbers attached to χ.

Now, we observe that

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{G}_{n,\chi,q,\lambda}(x)\frac{t^n}{n!} &= \left(\frac{t[2]_q \sum_{a=0}^{d-1} q^a(-1)^a \chi(a)(1+\lambda t)^{\frac{a}{\lambda}}}{q^d(1+\lambda t)^{\frac{d}{\lambda}}+1}\right)(1+\lambda t)^{\frac{x}{\lambda}} \\ &= \left(\sum_{m=0}^{\infty} \mathcal{G}_{n,\chi,q,\lambda}\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}(x|\lambda)_l \frac{t^l}{l!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\mathcal{G}_{m,\chi,q,\lambda}(x|\lambda)_{n-m}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.10}$$

Thus, by comparing the coefficients on the both sides, we obtain the following theorem.

Theorem 2.5. *For $n \geq 0$, we have*

$$\mathcal{G}_{n,\chi,q,\lambda}(x) = \sum_{m=0}^{n}\binom{n}{m}\mathcal{G}_{m,\chi,q,\lambda}(x|\lambda)_{n-m}.$$

Now, we observe that

$$\begin{aligned}\frac{(-1)^n}{n!}\mathcal{G}_{n,\chi,q,\lambda} =&\frac{(-1)^n}{n!}\int_X \chi(x)t(x|\lambda)_n d\mu_{-q}(x) \\ =&\lambda^n \int_X \binom{-\frac{x}{\lambda}+n-1}{n}\chi(x)t d\mu_{-q}(x) \\ =&\lambda^n \sum_{l=0}^{n}\binom{n-1}{l-1}\frac{(-1)^l}{\lambda^l l!}\int_X \chi(x)t(x|-\lambda)_l d\mu_{-q}(x) \\ =&\sum_{l=0}^{n}\binom{n-1}{l-1}\lambda^{n-l}(-1)^l\frac{1}{l!}\mathcal{G}_{l,\chi,q,-\lambda} \\ =&\sum_{l=1}^{n}\binom{n-1}{l-1}\lambda^{n-l}(-1)^l\frac{\mathcal{G}_{l,\chi,q,-\lambda}}{l!}.\end{aligned} \tag{2.11}$$

Therefore, by (2.11), we obtain the following theorem.

Theorem 2.6. *For $n \geq 0$, we have*

$$\frac{(-1)^n}{n!}\mathcal{G}_{n,\chi,q,\lambda} = \sum_{l=1}^{n}\binom{n-1}{l-1}\lambda^{n-l}(-1)^l\frac{\mathcal{G}_{l,\chi,q,-\lambda}}{l!}.$$

Note that

$$\lim_{\lambda\to 0}\mathcal{G}_{n,\chi,q,\lambda}(x) = G_{n,q,\chi}(x), \ (n \geq 0).$$

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Cubic soft ideals in BCK/BCI-algebras

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Abstract. The notions of cubic soft $\circ$-subalgebras and (closed) cubic soft ideals in BCK/BCI-algebras are introduced, and related properties are investigated. Relations between cubic soft subalgebras, cubic soft $\circ$-subalgebras and (closed) cubic soft ideals are discussed. Conditions for a cubic soft subalgebras to be a (closed) cubic soft ideals are provided. Characterizations of cubic soft ideals are considered. R-union and R-intersection of cubic soft ideals are discussed.

1. Introduction

To solve complicated problems in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [8]. Maji et al. [5] and Molodtsov [8] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [8] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At present, works on the soft set theory are progressing rapidly. Maji et al. [5] described the application of soft set theory to a decision making problem. Maji et al. [6] also studied several operations on the theory of soft sets. Jun et al. [2, 4] applied the notion of soft sets to BCK/BCI-algebras and d-algebras.

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Combining cubic sets and soft sets, Muhiuddin and Al-roqi [10] introduced the notions of (internal, external) cubic soft sets, P-cubic (resp. R-cubic) soft subsets, R-union (resp. R-intersection, P-union, P-intersection) of cubic soft sets, and the complement of a cubic soft set. They investigated several related properties, and applied the notion of cubic soft sets to BCK/BCI-algebras. In [9], Muhiuddin et al. considered several basic operations of cubic soft sets, namely, "AND" operation and "OR" operation based on the P-order and the R-order. They provided an example to illustrate that the R-union of two internal cubic soft sets might not be internal. They also discussed conditions for the R-union of two internal cubic soft sets to be an internal cubic soft set, and investigated several properties of cubic soft subalgebras in BCK/BCI-algebras based on a parameter.

In this paper, we introduce the notions of cubic soft $\circ$-subalgebras and (closed) cubic soft ideals in BCK/BCI-algebras, and investigate related properties. We consider relations between cubic soft subalgebras, cubic soft $\circ$-subalgebras and (closed) cubic soft ideals, and provide conditions for a cubic soft subalgebras to be a (closed) cubic soft ideals. We discuss characterizations of cubic soft ideals. We show that the R-intersection of cubic soft ideals is a cubic soft ideal. We also show that if parameter sets are mutually disjoint then the R-union of cubic soft ideals is a cubic soft ideal. We provide an example to show that the R-union of cubic soft ideals is not a cubic soft ideal when parameter sets are not disjoint.

2. Preliminaries

An algebra $(X;*,0)$ of type $(2,0)$ is called a *BCI-algebra* if it satisfies the following axioms:

(I) $(\forall x,y,z\in X)\ (((x*y)*(x*z))*(z*y)=0)$,

(II) $(\forall x,y\in X)\ ((x*(x*y))*y=0)$,

(III) $(\forall x\in X)\ (x*x=0)$,

(IV) $(\forall x,y\in X)\ (x*y=0, y*x=0 \Rightarrow x=y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x\in X)\ (0*x=0)$,

then X is called a *BCK-algebra*. Any BCK/BCI-algebra X satisfies the following conditions:

(a1) $(\forall x\in X)\ (x*0=x)$,

(a2) $(\forall x,y,z\in X)\ (x*y=0 \Rightarrow (x*z)*(y*z)=0,\ (z*y)*(z*x)=0)$,

(a3) $(\forall x,y,z\in X)\ ((x*y)*z=(x*z)*y)$,

(a4) $(\forall x,y,z\in X)\ (((x*z)*(y*z))*(x*y)=0)$.

We can define a partial ordering $\leq$ by $x\leq y$ if and only if $x*y=0$. A BCK-algebra X is said to be with *condition* (S) if, for all $x,y\in X$, the set $\{z\in X \mid z*x\leq y\}$ has a greatest element, written $x\circ y$. A BCI-algebra X is said to be *p-semisimple* if its BCK-part is equal to $\{0\}$. In a p-semisimple BCI-algebra, the following conditions are valid:

(a5) $(\forall x,y\in X)\ (0*(x*y)=y*x)$.

(a6) $(\forall x,y\in X)\ (x*(x*y)=y)$.

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies, for all $x, y \in X$, the following conditions:

(b1) $0 \in I$,

(b2) $x * y \in I,\ y \in I \ \Rightarrow\ x \in I$.

An ideal I of a BCI-algebra X is said to be *closed* if $0 * x \in I$ for all $x \in I$. We refer the reader to the books [1, 7] for further information regarding BCK/BCI-algebras.

By an *interval number* we mean a closed subinterval $\tilde{a} = [a^-, a^+]$ of I, where $0 \le a^- \le a^+ \le 1$. Denote by $[I]$ the set of all interval numbers. Let us define what is known as *refined minimum* and *refined maximun* (briefly, rmin and rmax) of two elements in $[I]$. We also define the symbols "$\succeq$", "$\preceq$", "$=$" in case of two elements in $[I]$. Consider two interval numbers $\tilde{a}_1 := \left[a_1^-, a_1^+\right]$ and $\tilde{a}_2 := \left[a_2^-, a_2^+\right]$. Then

$$\begin{aligned}
&\operatorname{rmin}\{\tilde{a}_1, \tilde{a}_2\} = \left[\min\left\{a_1^-, a_2^-\right\}, \min\left\{a_1^+, a_2^+\right\}\right],\\
&\operatorname{rmax}\{\tilde{a}_1, \tilde{a}_2\} = \left[\max\left\{a_1^-, a_2^-\right\}, \max\left\{a_1^+, a_2^+\right\}\right],\\
&\tilde{a}_1 \succeq \tilde{a}_2 \text{ if and only if } a_1^- \ge a_2^- \text{ and } a_1^+ \ge a_2^+,
\end{aligned}$$

and similarly we may have $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 = \tilde{a}_2$. To say $\tilde{a}_1 \succ \tilde{a}_2$ (resp. $\tilde{a}_1 \prec \tilde{a}_2$) we mean $\tilde{a}_1 \succeq \tilde{a}_2$ and $\tilde{a}_1 \ne \tilde{a}_2$ (resp. $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 \ne \tilde{a}_2$).

Let X be a nonempty set. A function $A : X \to [I]$ is called an *interval-valued fuzzy set* (briefly, an *IVF set*) in X. Let $[I]^X$ stand for the set of all IVF sets in X. For every $A \in [I]^X$ and $x \in X$, $A(x) = [A^-(x), A^+(x)]$ is called the *degree* of membership of an element x to A, where $A^- : X \to I$ and $A^+ : X \to I$ are fuzzy sets in X which are called a *lower fuzzy set* and an *upper fuzzy set* in X, respectively. For simplicity, we denote $A = [A^-, A^+]$.

Molodtsov [8] defined the soft set in the following way: Let U be an initial universe set and E be a set of parameters. Let $\mathcal{P}(U)$ denotes the power set of U and $A \subset E$.

Definition 2.1 ([8])**.** A pair (F, A) is called a *soft set* over U, where F is a mapping given by

$$F : A \to \mathcal{P}(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set (F, A). Clearly, a soft set is not a set. For illustration, Molodtsov considered several examples in [8].

Definition 2.2 ([3])**.** Let U be a universe. By a *cubic set* in U we mean a structure

$$\mathscr{A} = \{\langle x, A(x), \lambda(x)\rangle \mid x \in U\}$$

in which A is an IVF set in U and λ is a fuzzy set in U.

In what follows, a cubic set $\mathscr{A} = \{\langle x, \bar{\mu}_A(x), \lambda_A(x)\rangle \mid x \in U\}$ is simply denoted by $\mathscr{A} = \langle \bar{\mu}_A, \lambda_A\rangle$, and denote by $\mathcal{C}^U$ the collection of all cubic sets in U.

Definition 2.3 ([10]). Let U be an initial universe set and let E be a set of parameters. A *cubic soft set* over U is defined to be a pair $(\mathscr{F}, A)$ where $\mathscr{F}$ is a mapping from A to $\mathcal{C}^U$ and $A \subset E$. Note that the pair $(\mathscr{F}, A)$ can be represented as the following set:

$$(\mathscr{F}, A) := \{\mathscr{F}(\varepsilon) \mid \varepsilon \in A\} \text{ where } \mathscr{F}(\varepsilon) = \langle \bar{\mu}_{\mathscr{F}(\varepsilon)}, \lambda_{\mathscr{F}(\varepsilon)} \rangle.$$

3. Cubic soft ideals

In what follows, let U be an initial universe set which is a BCK/BCI-algebra unless otherwise specified.

Definition 3.1 ([10]). A cubic soft set $(\mathscr{F}, A)$ over U is said to be a *cubic soft BCK/BCI-algebra* over U based on a parameter ε (briefly, *ε-cubic soft subalgebra* over U) if there exists a parameter $\varepsilon \in A$ such that

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(x * y) \succeq \text{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y)\} \tag{3.1}$$

$$\lambda_{\mathscr{F}(\varepsilon)}(x * y) \leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(x), \lambda_{\mathscr{F}(\varepsilon)}(y)\} \tag{3.2}$$

for all $x, y \in U$. If $(\mathscr{F}, A)$ is an ε-cubic soft subalgebra over U for all $\varepsilon \in A$, we say that $(\mathscr{F}, A)$ is a *cubic soft subalgebra* over U.

Definition 3.2. Let U be a BCK-algebra with the condition (S). Given a parameter $\varepsilon \in A$, a cubic soft set $(\mathscr{F}, A)$ over U is said to be a *cubic soft $\circ$-subalgebra* over U based on ε (briefly, *ε-cubic soft $\circ$-subalgebra* over U) if it satisfies the following conditions:

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(x \circ y) \succeq \text{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y)\} \tag{3.3}$$

$$\lambda_{\mathscr{F}(\varepsilon)}(x \circ y) \leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(x), \lambda_{\mathscr{F}(\varepsilon)}(y)\} \tag{3.4}$$

for all $x, y \in U$. If $(\mathscr{F}, A)$ is an ε-cubic soft $\circ$-subalgebra over U for all $\varepsilon \in A$, we say that $(\mathscr{F}, A)$ is a *cubic soft $\circ$-subalgebra* over U.

Definition 3.3. Given a parameter $\varepsilon \in A$, a cubic soft set $(\mathscr{F}, A)$ over U is said to be a *cubic soft ideal* over U based on ε (briefly, *ε-cubic soft ideal* over U) if it satisfies the following conditions:

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(0) \succeq \bar{\mu}_{\mathscr{F}(\varepsilon)}(x),\ \lambda_{\mathscr{F}(\varepsilon)}(0) \leq \lambda_{\mathscr{F}(\varepsilon)}(x), \tag{3.5}$$

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(x) \succeq \text{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x * y), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y)\} \tag{3.6}$$

$$\lambda_{\mathscr{F}(\varepsilon)}(x) \leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(x * y), \lambda_{\mathscr{F}(\varepsilon)}(y)\} \tag{3.7}$$

for all $x, y \in U$. If $(\mathscr{F}, A)$ is an ε-cubic soft ideal over U for all $\varepsilon \in A$, we say that $(\mathscr{F}, A)$ is a *cubic soft ideal* over U.

Example 3.4. Let $(Y, *, 0)$ be a BCI-algebra and consider the adjoint BCI-algebra $(\mathbb{Z}, -, 0)$ of the additive group $(\mathbb{Z}, +, 0)$ of integers. Then the direct product $U := Y \times \mathbb{Z}$ of Y and $\mathbb{Z}$ is a BCI-algebra (see [1]). For any $\varepsilon \in A$, let $(\mathscr{F}, A)$ be a soft set over U defined by

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(x) = \begin{cases} \tilde{a} = [a^-, a^+] (\neq [0, 0]) & \text{if } x \in Y \times \mathbb{N}_0, \\ [0,0] & \text{otherwise,} \end{cases}$$

$$\lambda_{\mathscr{F}(\varepsilon)}(x) = \begin{cases} s & \text{if } x \in Y \times \mathbb{N}_0, \\ t & \text{otherwise,} \end{cases}$$

where $\mathbb{N}$ is the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $s, t \in [0, 1]$ with $s < t$. Then $(\mathscr{F}, A)$ is an ε-cubic soft ideal over U.

Example 3.5. Let $U = \{0, a, b, c\}$ be a BCI-algebra with the following Cayley table 1:

Table 1. Cayley table of the operation $*$

$*$	0	a	b	c
0	0	0	0	c
a	a	0	0	c
b	b	b	0	c
c	c	c	c	0

Let $(\mathscr{F}, A)$ be a cubic soft set over U, where $A = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, with the tabular representation in Table 2.

Table 2. Tabular representation of the cubic soft set $(\mathscr{F}, A)$

	ε_1	ε_2	ε_3
0	$\langle[0.5, 0.8],\ 0.6\rangle$	$\langle[0.8, 1.0],\ 0.1\rangle$	$\langle[0.4, 0.6],\ 0.7\rangle$
a	$\langle[0.8, 0.9],\ 0.7\rangle$	$\langle[0.3, 0.7],\ 0.8\rangle$	$\langle[0.1, 0.2],\ 0.7\rangle$
b	$\langle[0.1, 0.7],\ 0.5\rangle$	$\langle[0.3, 0.7],\ 0.8\rangle$	$\langle[0.1, 0.7],\ 0.3\rangle$
c	$\langle[0.2, 0.6],\ 0.9\rangle$	$\langle[0.3, 0.7],\ 0.8\rangle$	$\langle[0.3, 0.6],\ 0.2\rangle$

Then $(\mathscr{F}, A)$ is not an ε_1-cubic soft ideal over U since

$$\bar{\mu}_{\mathscr{F}(\varepsilon_1)}(0) = [0.5, 0.8] \not\succeq [0.8, 0.9] = \bar{\mu}_{\mathscr{F}(\varepsilon_1)}(a).$$

We know that $(\mathscr{F}, A)$ is an ε_2-cubic soft ideal over U. $(\mathscr{F}, A)$ is not an ε_3-cubic soft ideal over U since

$$\bar{\mu}_{\mathscr{F}(\varepsilon_3)}(a) = [0.1, 0.2] \not\succeq [0.3, 0.6] = \operatorname{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon_3)}(a * c), \bar{\mu}_{\mathscr{F}(\varepsilon_3)}(c)\}.$$

Proposition 3.6. *If $(\mathscr{F}, A)$ is an ε-cubic soft ideal over U, then*

$$(\forall x, y \in U)\left(x \leq y \ \Rightarrow\ \bar{\mu}_{\mathscr{F}(\varepsilon)}(y) \preceq \bar{\mu}_{\mathscr{F}(\varepsilon)}(x),\ \lambda_{\mathscr{F}(\varepsilon)}(y) \geq \lambda_{\mathscr{F}(\varepsilon)}(x)\right).$$

Proof. Let $x, y \in U$ be such that $x \leq y$. Then $x * y = 0$, and so

$$\begin{aligned} \bar{\mu}_{\mathscr{F}(\varepsilon)}(x) &\succeq \operatorname{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x * y), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y)\} \\ &= \operatorname{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(0), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y)\} = \bar{\mu}_{\mathscr{F}(\varepsilon)}(y) \end{aligned}$$

and

$$\begin{aligned} \lambda_{\mathscr{F}(\varepsilon)}(x) &\leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(x * y), \lambda_{\mathscr{F}(\varepsilon)}(y)\} \\ &= \max\{\lambda_{\mathscr{F}(\varepsilon)}(0), \lambda_{\mathscr{F}(\varepsilon)}(y)\} = \lambda_{\mathscr{F}(\varepsilon)}(y). \end{aligned}$$

This completes the proof. $\square$

Proposition 3.7. *Let $(\mathscr{F}, A)$ be an ε-cubic soft ideal over U for a parameter $\varepsilon \in A$. If the inequality $x * y \leq z$ holds in U, then*

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(x) \succeq \operatorname{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(y), \bar{\mu}_{\mathscr{F}(\varepsilon)}(z)\}$$

and $\lambda_{\mathscr{F}(\varepsilon)}(x) \leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(y), \lambda_{\mathscr{F}(\varepsilon)}(z)\}$.

Proof. Assume that $x * y \leq z$ for all $x, y, z \in U$. Then

$$\begin{aligned} \bar{\mu}_{\mathscr{F}(\varepsilon)}(x * y) &\succeq \operatorname{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}((x * y) * z), \bar{\mu}_{\mathscr{F}(\varepsilon)}(z)\} \\ &= \operatorname{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(0), \bar{\mu}_{\mathscr{F}(\varepsilon)}(z)\} = \bar{\mu}_{\mathscr{F}(\varepsilon)}(z) \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \lambda_{\mathscr{F}(\varepsilon)}(x * y) &\leq \max\{\lambda_{\mathscr{F}(\varepsilon)}((x * y) * z), \lambda_{\mathscr{F}(\varepsilon)}(z)\} \\ &= \max\{\lambda_{\mathscr{F}(\varepsilon)}(0), \lambda_{\mathscr{F}(\varepsilon)}(z)\} = \lambda_{\mathscr{F}(\varepsilon)}(z), \end{aligned} \tag{3.9}$$

which implies from (3.6) and (3.7) that

$$\begin{aligned} \bar{\mu}_{\mathscr{F}(\varepsilon)}(x) &\succeq \operatorname{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x * y), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y)\} \\ &\succeq \operatorname{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(y), \bar{\mu}_{\mathscr{F}(\varepsilon)}(z)\} \end{aligned}$$

and

$$\begin{aligned} \lambda_{\mathscr{F}(\varepsilon)}(x) &\leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(x * y), \lambda_{\mathscr{F}(\varepsilon)}(y)\} \\ &\leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(y), \lambda_{\mathscr{F}(\varepsilon)}(z)\}. \end{aligned}$$

This completes the proof. □

Theorem 3.8. *In a BCK-algebra U with the condition (S), every ε-cubic soft ideal $(\mathscr{F}, A)$ over U is an ε-cubic soft $\circ$-subalgebra over U for all $\varepsilon \in A$.*

Proof. Let $\varepsilon \in A$. Since U has the condition (S), we have $(x \circ y) * x \leq y$ for all $x, y \in U$. Hence (3.6), (3.7) and Proposition 3.6 imply that

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(x \circ y) \succeq \operatorname{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}((x \circ y) * x), \bar{\mu}_{\mathscr{F}(\varepsilon)}(x)\} \succeq \operatorname{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y)\}$$

and

$$\lambda_{\mathscr{F}(\varepsilon)}(x \circ y) \leq \max\{\lambda_{\mathscr{F}(\varepsilon)}((x \circ y) * x), \lambda_{\mathscr{F}(\varepsilon)}(x)\} \leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(x), \lambda_{\mathscr{F}(\varepsilon)}(y)\}$$

for all $x, y \in U$. Therefore $(\mathscr{F}, A)$ is an ε-cubic soft $\circ$-subalgebra over U for all $\varepsilon \in A$. □

Theorem 3.9. *In a BCK-algebra U, if $(\mathscr{F}, A)$ is an ε-cubic soft ideal over U, then it is an ε-cubic soft subalgebra over U for all $\varepsilon \in A$.*

Proof. Let $(\mathscr{F}, A)$ be an ε-cubic soft ideal over U where $\varepsilon \in A$. For any $x, y \in U$, we have

$$\begin{aligned} \bar{\mu}_{\mathscr{F}(\varepsilon)}(x * y) &\succeq \operatorname{rmin}\left\{\bar{\mu}_{\mathscr{F}(\varepsilon)}((x * y) * x), \bar{\mu}_{\mathscr{F}(\varepsilon)}(x)\right\} \\ &= \operatorname{rmin}\left\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(0), \bar{\mu}_{\mathscr{F}(\varepsilon)}(x)\right\} \\ &= \bar{\mu}_{\mathscr{F}(\varepsilon)}(x) \end{aligned}$$

and $\lambda_{\mathscr{F}(\varepsilon)}(x * y) \leq \max\{\lambda_{\mathscr{F}(\varepsilon)}((x * y) * x), \lambda_{\mathscr{F}(\varepsilon)}(x)\} = \max\{\lambda_{\mathscr{F}(\varepsilon)}(0), \lambda_{\mathscr{F}(\varepsilon)}(x)\} = \lambda_{\mathscr{F}(\varepsilon)}(x)$. Therefore $(\mathscr{F}, A)$ is an ε-cubic soft subalgebra over U. □

Theorem 3.9 is not true in a BCI-algebra. In fact, the ε-cubic soft ideal $(\mathscr{F}, A)$ in Example 3.4 is not an ε-cubic soft subalgebra over U since

$$\begin{aligned}\bar{\mu}_{\mathscr{F}(\varepsilon)}\left((0,0)*(0,1)\right) &= \bar{\mu}_{\mathscr{F}(\varepsilon)}(0,-1) = [0,0] \not\succeq \tilde{a} = [a^-, a^+] \\ &= \operatorname{rmin}\left\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(0,0), \bar{\mu}_{\mathscr{F}(\varepsilon)}(0,1)\right\}\end{aligned}$$

and/or

$$\lambda_{\mathscr{F}(\varepsilon)}\left((0,0)*(0,1)\right) = \lambda_{\mathscr{F}(\varepsilon)}(0,-1) = t \not\leq s = \max\left\{\lambda_{\mathscr{F}(\varepsilon)}(0,0), \lambda_{\mathscr{F}(\varepsilon)}(0,1)\right\}.$$

Definition 3.10. Let U be a BCI-algebra and $\varepsilon \in A$. An ε-cubic soft ideal $(\mathscr{F}, A)$ over U is said to be *closed* if $\bar{\mu}_{\mathscr{F}(\varepsilon)}(0*x) \succeq \bar{\mu}_{\mathscr{F}(\varepsilon)}(x)$ and $\lambda_{\mathscr{F}(\varepsilon)}(0*x) \leq \lambda_{\mathscr{F}(\varepsilon)}(x)$ for all $x \in U$.

Example 3.11. The ε_2-cubic soft ideal $(\mathscr{F}, A)$ in Example 3.5 is closed.

Theorem 3.12. *In a BCI-algebra, every closed cubic soft ideal is a cubic soft subalgebra.*

Proof. Let $(\mathscr{F}, A)$ be a closed cubic soft ideal over U. Then $\bar{\mu}_{\mathscr{F}(\varepsilon)}(0*x) \succeq \bar{\mu}_{\mathscr{F}(\varepsilon)}(x)$ and $\lambda_{\mathscr{F}(\varepsilon)}(0*x) \leq \lambda_{\mathscr{F}(\varepsilon)}(x)$ for all $x \in U$. It follows from (a3), (3.6) and (3.7) that

$$\begin{aligned}\bar{\mu}_{\mathscr{F}(\varepsilon)}(x*y) &\succeq \operatorname{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}((x*y)*x), \bar{\mu}_{\mathscr{F}(\varepsilon)}(x)\} \\ &= \operatorname{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(0*y), \bar{\mu}_{\mathscr{F}(\varepsilon)}(x)\} \\ &\succeq \operatorname{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(y), \bar{\mu}_{\mathscr{F}(\varepsilon)}(x)\}\end{aligned}$$

and

$$\begin{aligned}\lambda_{\mathscr{F}(\varepsilon)}(x*y) &\succeq \operatorname{rmin}\{\lambda_{\mathscr{F}(\varepsilon)}((x*y)*x), \lambda_{\mathscr{F}(\varepsilon)}(x)\} \\ &= \max\{\lambda_{\mathscr{F}(\varepsilon)}(0*y), \lambda_{\mathscr{F}(\varepsilon)}(x)\} \\ &\leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(y), \lambda_{\mathscr{F}(\varepsilon)}(x)\}\end{aligned}$$

for all $x, y \in U$. Therefore $(\mathscr{F}, A)$ is a cubic soft ideal over U. $\square$

We provide a condition for a cubic soft subalgebra over U to be a (closed) cubic soft ideal over U.

Theorem 3.13. *In a p-semisimple BCI-algebra U, every cubic soft subalgebra over U is a closed cubic soft ideal over U.*

Proof. Let $(\mathscr{F}, A)$ be a cubic soft subalgebra over a p-semisimple BCI-algebra U and let $\varepsilon \in A$ be a parameter. For every $x \in U$, we have

$$\begin{aligned}&\bar{\mu}_{\mathscr{F}(\varepsilon)}(0) = \bar{\mu}_{\mathscr{F}(\varepsilon)}(x*x) \succeq \operatorname{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x), \bar{\mu}_{\mathscr{F}(\varepsilon)}(x)\} = \bar{\mu}_{\mathscr{F}(\varepsilon)}(x), \\ &\lambda_{\mathscr{F}(\varepsilon)}(0) = \lambda_{\mathscr{F}(\varepsilon)}(x*x) \leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(x), \lambda_{\mathscr{F}(\varepsilon)}(x)\} = \lambda_{\mathscr{F}(\varepsilon)}(x).\end{aligned} \tag{3.10}$$

Using (3.1), (3.2) and (3.10), we get

$$\begin{aligned}&\bar{\mu}_{\mathscr{F}(\varepsilon)}(0*x) \succeq \operatorname{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(0), \bar{\mu}_{\mathscr{F}(\varepsilon)}(x)\} = \bar{\mu}_{\mathscr{F}(\varepsilon)}(x), \\ &\lambda_{\mathscr{F}(\varepsilon)}(0*x) \leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(0), \lambda_{\mathscr{F}(\varepsilon)}(x)\} = \lambda_{\mathscr{F}(\varepsilon)}(x).\end{aligned} \tag{3.11}$$

For any $x, y \in U$, we have

$$\begin{aligned}\bar{\mu}_{\mathscr{F}(\varepsilon)}(x) &= \bar{\mu}_{\mathscr{F}(\varepsilon)}(y*(y*x)) \succeq \operatorname{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(y), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y*x)\}\\ &= \operatorname{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(y), \bar{\mu}_{\mathscr{F}(\varepsilon)}(0*(x*y))\}\\ &\succeq \operatorname{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x*y), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y)\}\end{aligned}$$

and

$$\begin{aligned}\lambda_{\mathscr{F}(\varepsilon)}(x) &= \lambda_{\mathscr{F}(\varepsilon)}(y*(y*x)) \leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(y), \lambda_{\mathscr{F}(\varepsilon)}(y*x)\}\\ &= \max\{\lambda_{\mathscr{F}(\varepsilon)}(y), \lambda_{\mathscr{F}(\varepsilon)}(0*(x*y))\}\\ &\leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(x*y), \lambda_{\mathscr{F}(\varepsilon)}(y)\}\end{aligned}$$

by using (a6), (3.1), (3.2), (a5) and (3.11). Therefore $(\mathscr{F}, A)$ is a closed cubic soft ideal over U. □

Corollary 3.14. *If a BCI-algebra U satisfies any one of the following conditions:*

- $U = \{0 * x \mid x \in U\}$,
- *every element of U is minimal,*
- $(\forall x, y \in U)\ (x * (0 * y) = y * (0 * x))$,
- $(\forall x \in U)\ (0 * x = 0 \ \Rightarrow\ x = 0)$,
- $(\forall x, y \in U)\ ((x * y) * z = x * (y * z))$,
- $(\forall x, y \in U)\ (x * y = y * x)$,
- $(\forall x \in U)\ (0 * x = x)$,
- $(\forall x, y, z \in U)\ ((x * y) * (x * z) = z * y)$,

then every cubic soft subalgebra over U is a closed cubic soft ideal over U.

Theorem 3.15. *For a cubic soft set $(\mathscr{F}, A)$ over a BCK-algebra U with condition (S) and a parameter $\varepsilon \in A$, the following are equivalent.*

(i) *$(\mathscr{F}, A)$ is an ε-cubic soft ideal over U.*

(ii) *For every $x, y, z \in U$, if $x \leq y \circ z$, then $\bar{\mu}_{\mathscr{F}(\varepsilon)}(x) \succeq \operatorname{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(y), \bar{\mu}_{\mathscr{F}(\varepsilon)}(z)\}$ and $\lambda_{\mathscr{F}(\varepsilon)}(x) \leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(y), \lambda_{\mathscr{F}(\varepsilon)}(z)\}$.*

Proof. Assume that $(\mathscr{F}, A)$ is an ε-cubic soft ideal over U and $x \leq y \circ z$ for all $x, y, z \in U$. Then

$$\begin{aligned}\bar{\mu}_{\mathscr{F}(\varepsilon)}(x) &\succeq \operatorname{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x*(y\circ z)), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y\circ z)\}\\ &= \operatorname{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(0), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y\circ z)\}\\ &= \bar{\mu}_{\mathscr{F}(\varepsilon)}(y\circ z)\\ &\succeq \operatorname{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(y), \bar{\mu}_{\mathscr{F}(\varepsilon)}(z)\}\end{aligned}$$

and

$$\begin{aligned}\lambda_{\mathscr{F}(\varepsilon)}(x) &\leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(x*(y\circ z)), \lambda_{\mathscr{F}(\varepsilon)}(y\circ z)\}\\ &= \max\{\lambda_{\mathscr{F}(\varepsilon)}(0), \lambda_{\mathscr{F}(\varepsilon)}(y\circ z)\}\\ &= \lambda_{\mathscr{F}(\varepsilon)}(y\circ z)\\ &\leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(y), \lambda_{\mathscr{F}(\varepsilon)}(z)\}.\end{aligned}$$

Conversely suppose that (ii) is valid. Since $0 \leq x \circ x$ for all $x \in U$, it follows from (ii) that

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(0) \succeq \operatorname{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x), \bar{\mu}_{\mathscr{F}(\varepsilon)}(x)\} = \bar{\mu}_{\mathscr{F}(\varepsilon)}(x)$$

and

$$\lambda_{\mathscr{F}(\varepsilon)}(0) \leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(x), \lambda_{\mathscr{F}(\varepsilon)}(x)\} = \lambda_{\mathscr{F}(\varepsilon)}(x)$$

for all $x \in U$. Since $x \leq (x * y) \circ y$ for all $x, y \in U$, we have

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(x) \succeq \operatorname{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x * y), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y)\}$$

and

$$\lambda_{\mathscr{F}(\varepsilon)}(x) \leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(x * y), \lambda_{\mathscr{F}(\varepsilon)}(y)\}$$

for all $x, y \in U$. Therefore $(\mathscr{F}, A)$ is an ε-cubic soft ideal over U. □

Theorem 3.16. *Given a parameter $\varepsilon \in A$, a cubic soft set $(\mathscr{F}, A)$ over U is an ε-cubic soft ideal over U if and only if the nonempty sets*

$$\bar{\mu}^{\Leftarrow}_{\mathscr{F}(\varepsilon)}[\delta_1, \delta_2] := \left\{x \in U \mid \bar{\mu}_{\mathscr{F}(\varepsilon)}(x) \succeq [\delta_1, \delta_2]\right\}$$

and

$$\lambda^{\rightarrow}_{\mathscr{F}(\varepsilon)}(t) := \left\{x \in U \mid \lambda_{\mathscr{F}(\varepsilon)}(x) \leq t\right\}$$

are ideals of U for all $[\delta_1, \delta_2] \in [I]$ and $t \in [0, 1]$.

Proof. Assume that a cubic soft set $(\mathscr{F}, A)$ over U is an ε-cubic soft ideal over U. Suppose that $\bar{\mu}^{\Leftarrow}_{\mathscr{F}(\varepsilon)}[\delta_1, \delta_2] \cap \lambda^{\rightarrow}_{\mathscr{F}(\varepsilon)}(t) \neq \emptyset$ for all $[\delta_1, \delta_2] \in [I]$ and $t \in [0, 1]$. Obviously, $0 \in \bar{\mu}^{\Leftarrow}_{\mathscr{F}(\varepsilon)}[\delta_1, \delta_2] \cap \lambda^{\rightarrow}_{\mathscr{F}(\varepsilon)}(t)$. Let x and y be elements of U such that $x*y \in \bar{\mu}^{\Leftarrow}_{\mathscr{F}(\varepsilon)}[\delta_1, \delta_2]$ and $y \in \bar{\mu}^{\Leftarrow}_{\mathscr{F}(\varepsilon)}[\delta_1, \delta_2]$. Then $\bar{\mu}_{\mathscr{F}(\varepsilon)}(x*y) \succeq [\delta_1, \delta_2]$ and $\bar{\mu}_{\mathscr{F}(\varepsilon)}(y) \succeq [\delta_1, \delta_2]$. It follows from (3.6) that

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(x) \succeq \operatorname{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x * y), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y)\} \succeq \operatorname{rmin}\{[\delta_1, \delta_2], [\delta_1, \delta_2]\} = [\delta_1, \delta_2].$$

Hence $x \in \bar{\mu}^{\Leftarrow}_{\mathscr{F}(\varepsilon)}[\delta_1, \delta_2]$. Now if $x * y,\ y \in \lambda^{\rightarrow}_{\mathscr{F}(\varepsilon)}(t)$, then $\lambda_{\mathscr{F}(\varepsilon)}(x * y) \leq t$ and $\lambda_{\mathscr{F}(\varepsilon)}(y) \leq t$. Using (3.7), we have $\lambda_{\mathscr{F}(\varepsilon)}(x) \leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(x * y), \lambda_{\mathscr{F}(\varepsilon)}(y)\} \leq t$, and so $x \in \lambda^{\rightarrow}_{\mathscr{F}(\varepsilon)}(t)$. Therefore $\bar{\mu}^{\Leftarrow}_{\mathscr{F}(\varepsilon)}[\delta_1, \delta_2]$ and $\lambda^{\rightarrow}_{\mathscr{F}(\varepsilon)}(t)$ are ideals of U.

Conversely, suppose that $\bar{\mu}^{\Leftarrow}_{\mathscr{F}(\varepsilon)}[\delta_1, \delta_2]$ and $\lambda^{\rightarrow}_{\mathscr{F}(\varepsilon)}(t)$ are ideals of U for all $[\delta_1, \delta_2] \in [I]$ and $t \in [0, 1]$. Assume that there exists $a \in U$ such that $\bar{\mu}_{\mathscr{F}(\varepsilon)}(0) \not\succeq \bar{\mu}_{\mathscr{F}(\varepsilon)}(a)$ or $\lambda_{\mathscr{F}(\varepsilon)}(0) > \lambda_{\mathscr{F}(\varepsilon)}(a)$. Let $\bar{\mu}_{\mathscr{F}(\varepsilon)}(0) = [0^-, 0^+]$ and $\bar{\mu}_{\mathscr{F}(\varepsilon)}(a) = [a^-, a^+]$. Then $0^- < a^-$ and $0^+ < a^+$ which imply that $0^- < \delta_1 < a^-$ and $0^+ < \delta_2 < a^+$, that is,

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(0) = [0^-, 0^+] < [\delta_1, \delta_2] < [a^-, a^+]$$

by taking $[\delta_1, \delta_2] := \left[\frac{1}{2}(0^- + a^-), \frac{1}{2}(0^+ + a^+)\right]$. Hence $0 \notin \bar{\mu}^{\Leftarrow}_{\mathscr{F}(\varepsilon)}[\delta_1, \delta_2]$. Also $0 \notin \lambda^{\rightarrow}_{\mathscr{F}(\varepsilon)}(a_t)$ where $a_t = \lambda_{\mathscr{F}(\varepsilon)}(a)$. This is a contradiction, and so (3.5) is valid. Assume that there exist $a, b \in U$ such that

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(a) \not\succeq \operatorname{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(a * b),\ \bar{\mu}_{\mathscr{F}(\varepsilon)}(b)\} \tag{3.12}$$

or

$$\bar{\lambda}_{\mathscr{F}(\varepsilon)}(a) > \max\{\bar{\lambda}_{\mathscr{F}(\varepsilon)}(a*b),\ \bar{\lambda}_{\mathscr{F}(\varepsilon)}(b)\}. \tag{3.13}$$

For the case (3.12), let $\bar{\mu}_{\mathscr{F}(\varepsilon)}(a) = [\delta_1, \delta_2]$, $\bar{\mu}_{\mathscr{F}(\varepsilon)}(a*b) = [\gamma_1, \gamma_2]$ and $\bar{\mu}_{\mathscr{F}(\varepsilon)}(b) = [\gamma_3, \gamma_4]$. Then

$$[\delta_1, \delta_2] \prec \mathrm{rmin}\{[\gamma_1, \gamma_2], [\gamma_3, \gamma_4]\} = [\min\{\gamma_1, \gamma_3\}, \min\{\gamma_2, \gamma_4\}].$$

Hence $\delta_1 < \min\{\gamma_1, \gamma_3\}$ and $\delta_2 < \min\{\gamma_2, \gamma_4\}$. Taking

$$[\tau_1, \tau_2] = \tfrac{1}{2}\left(\bar{\mu}_{\mathscr{F}(\varepsilon)}(a) + \mathrm{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(a*b), \bar{\mu}_{\mathscr{F}(\varepsilon)}(b)\}\right)$$

implies that

$$\begin{aligned}[\tau_1, \tau_2] &= \tfrac{1}{2}\left([\delta_1, \delta_2] + [\min\{\gamma_1, \gamma_3\}, \min\{\gamma_2, \gamma_4\}]\right) \\ &= \left[\tfrac{1}{2}\left(\delta_1 + \min\{\gamma_1, \gamma_3\}\right), \tfrac{1}{2}\left(\delta_2 + \min\{\gamma_2, \gamma_4\}\right)\right].\end{aligned}$$

It follows that

$$\begin{aligned}\min\{\gamma_1, \gamma_3\} > \tau_1 = \tfrac{1}{2}\left(\delta_1 + \min\{\gamma_1, \gamma_3\}\right) > \delta_1, \\ \min\{\gamma_2, \gamma_4\} > \tau_2 = \tfrac{1}{2}\left(\delta_2 + \min\{\gamma_2, \gamma_4\}\right) > \delta_2,\end{aligned}$$

and so that

$$[\min\{\gamma_1, \gamma_3\}, \min\{\gamma_2, \gamma_4\}] \succ [\tau_1, \tau_2] \succ [\delta_1, \delta_2] = \bar{\mu}_{\mathscr{F}(\varepsilon)}(a).$$

Therefore $a \notin \bar{\mu}^{\Leftarrow}_{\mathscr{F}(\varepsilon)}[\tau_1, \tau_2]$. On the other hand, we know that

$$\begin{aligned}\bar{\mu}_{\mathscr{F}(\varepsilon)}(a*b) = [\gamma_1, \gamma_2] \succeq [\min\{\gamma_1, \gamma_3\}, \min\{\gamma_2, \gamma_4\}] \succ [\tau_1, \tau_2], \\ \bar{\mu}_{\mathscr{F}(\varepsilon)}(b) = [\gamma_3, \gamma_4] \succeq [\min\{\gamma_1, \gamma_3\}, \min\{\gamma_2, \gamma_4\}] \succ [\tau_1, \tau_2],\end{aligned}$$

which imply that $a*b$, $b \in \bar{\mu}^{\Leftarrow}_{\mathscr{F}(\varepsilon)}[\tau_1, \tau_2]$. This is a contradiction, and so

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(x) \succeq \mathrm{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x*y), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y)\}$$

for all $x, y \in U$. Now, (3.13) implies that there exists $t_0 \in (0, 1)$ such that

$$\lambda_{\mathscr{F}(\varepsilon)}(a) \geq t_0 > \max\{\lambda_{\mathscr{F}(\varepsilon)}(a*b), \lambda_{\mathscr{F}(\varepsilon)}(b)\}.$$

Hence $a*b$, $b \in \lambda^{\rightarrow}_{\mathscr{F}(\varepsilon)}(t_0)$ but $a \notin \lambda^{\rightarrow}_{\mathscr{F}(\varepsilon)}(t_0)$. This is a contradiction, and therefore

$$\lambda_{\mathscr{F}(\varepsilon)}(x) \leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(x*y), \lambda_{\mathscr{F}(\varepsilon)}(y)\}$$

for all $x, y \in U$. Consequently, $(\mathscr{F}, A)$ is an ε-cubic soft ideal over U. □

Definition 3.17 ([10])**.** The *R-union* of cubic soft sets $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ over U is a cubic soft set $(\mathscr{H}, C)$ where $C = A \cup B$ and

$$\mathscr{H}(\varepsilon) = \begin{cases} \mathscr{F}(\varepsilon) & \text{if } \varepsilon \in A \setminus B, \\ \mathscr{G}(\varepsilon) & \text{if } \varepsilon \in B \setminus A, \\ \mathscr{F}(\varepsilon) \cup_R \mathscr{G}(\varepsilon) & \text{if } \varepsilon \in A \cap B \end{cases}$$

for all $\varepsilon \in C$. This is denoted by $(\mathscr{H}, C) = (\mathscr{F}, A) \Cup_R (\mathscr{G}, B)$. Also the *R-intersection* of cubic soft sets $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ over U is a cubic soft set $(\mathscr{H}, C)$ where $C = A \cup B$ and

$$\mathscr{H}(\varepsilon) = \begin{cases} \mathscr{F}(\varepsilon) & \text{if } \varepsilon \in A \setminus B, \\ \mathscr{G}(\varepsilon) & \text{if } \varepsilon \in B \setminus A, \\ \mathscr{F}(\varepsilon) \cap_R \mathscr{G}(\varepsilon) & \text{if } \varepsilon \in A \cap B \end{cases}$$

for all $\varepsilon \in C$. This is denoted by $(\mathscr{H}, C) = (\mathscr{F}, A) \Cap_R (\mathscr{G}, B)$.

Theorem 3.18. *If $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ are cubic soft ideals over U, then so is the R-intersection $(\mathscr{H}, C) = (\mathscr{F}, A) \Cap_R (\mathscr{G}, B)$ of $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$.*

Proof. Straightforward. □

Theorem 3.19. *Let $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ be cubic soft ideals over U. If A and B are disjoint, then the R-union of $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ is a cubic soft ideal over U.*

Proof. By means of Definition 3.17, we can write $(\mathscr{F}, A) \Cup_R (\mathscr{G}, B) = (\mathscr{H}, C)$, where $C = A \cup B$ and for all $\varepsilon \in C$,

$$\mathscr{H}(\varepsilon) = \begin{cases} \mathscr{F}(\varepsilon) & \text{if } \varepsilon \in A \setminus B, \\ \mathscr{G}(\varepsilon) & \text{if } \varepsilon \in B \setminus A, \\ \mathscr{F}(\varepsilon) \cup_R \mathscr{G}(\varepsilon) & \text{if } \varepsilon \in A \cap B \end{cases}$$

Since $A \cap B = \emptyset$, either $\varepsilon \in A \setminus B$ or $\varepsilon \in B \setminus A$ for all $\varepsilon \in C$. If $\varepsilon \in A \setminus B$, then $\mathscr{H}(\varepsilon) = \mathscr{F}(\varepsilon)$ is a cubic soft ideal over U. If $\varepsilon \in B \setminus A$, then $\mathscr{H}(\varepsilon) = \mathscr{G}(\varepsilon)$ is a cubic soft ideal over U. Hence $(\mathscr{H}, C) = (\mathscr{F}, A) \Cup_R (\mathscr{G}, B)$ is a cubic soft ideal over U. □

The following example shows that Theorem 3.19 is not valid if A and B are not disjoint.

Example 3.20. Let $U = \{0, a, b, c\}$ be a BCI-algebra with the Cayley table in Table 3.

Table 3. Cayley table of the operation $*$

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Consider sets of parameters $A := \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ and $B := \{\varepsilon_3, \varepsilon_4\}$. Then A and B are not disjoint. Let $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ be cubic soft sets over U with the tabular representations in Table 4 and Table 5, respectively.

Then $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ are cubic soft ideals over U, and the R-union $(\mathscr{H}, C) = (\mathscr{F}, A) \Cup_R (\mathscr{G}, B)$ of $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ is represented by Table 6.

Note that

$$\bar{\mu}_{\mathscr{H}(\varepsilon_3)}(c) = [0.4, 0.7] \not\succeq [0.6, 0.8] = \text{rmin}\{\bar{\mu}_{\mathscr{H}(\varepsilon_3)}(c * a), \bar{\mu}_{\mathscr{H}(\varepsilon_3)}(a)\}$$

Table 4. Tabular representation of the cubic soft set $(\mathscr{F}, A)$

	ε_1	ε_2	ε_3
0	$\langle[0.6, 0.8], 0.3\rangle$	$\langle[0.5, 0.9], 0.4\rangle$	$\langle[0.7, 0.9], 0.3\rangle$
a	$\langle[0.3, 0.7], 0.5\rangle$	$\langle[0.2, 0.5], 0.7\rangle$	$\langle[0.6, 0.8], 0.8\rangle$
b	$\langle[0.3, 0.7], 0.5\rangle$	$\langle[0.3, 0.6], 0.7\rangle$	$\langle[0.4, 0.7], 0.8\rangle$
c	$\langle[0.3, 0.7], 0.5\rangle$	$\langle[0.2, 0.5], 0.6\rangle$	$\langle[0.4, 0.7], 0.5\rangle$

Table 5. Tabular representation of the cubic soft set $(\mathscr{G}, B)$

	ε_3	ε_4
0	$\langle[0.7, 1.0], 0.2\rangle$	$\langle[0.4, 0.8], 0.1\rangle$
a	$\langle[0.3, 0.7], 0.7\rangle$	$\langle[0.2, 0.6], 0.3\rangle$
b	$\langle[0.6, 0.8], 0.4\rangle$	$\langle[0.2, 0.6], 0.6\rangle$
c	$\langle[0.3, 0.7], 0.7\rangle$	$\langle[0.4, 0.8], 0.6\rangle$

Table 6. Tabular representation of the cubic soft set $(\mathscr{H}, C)$

	ε_1	ε_2	ε_3	ε_4
0	$\langle[0.6, 0.8], 0.3\rangle$	$\langle[0.5, 0.9], 0.4\rangle$	$\langle[0.7, 1.0], 0.2\rangle$	$\langle[0.4, 0.8], 0.1\rangle$
a	$\langle[0.3, 0.7], 0.5\rangle$	$\langle[0.2, 0.5], 0.7\rangle$	$\langle[0.6, 0.8], 0.7\rangle$	$\langle[0.2, 0.6], 0.3\rangle$
b	$\langle[0.3, 0.7], 0.5\rangle$	$\langle[0.3, 0.6], 0.7\rangle$	$\langle[0.6, 0.8], 0.4\rangle$	$\langle[0.2, 0.6], 0.6\rangle$
c	$\langle[0.3, 0.7], 0.5\rangle$	$\langle[0.2, 0.5], 0.6\rangle$	$\langle[0.4, 0.7], 0.5\rangle$	$\langle[0.4, 0.8], 0.6\rangle$

and/or $\lambda_{\mathscr{H}(\varepsilon_3)}(a) = 0.7 \nleq 0.5 = \max\{\lambda_{\mathscr{H}(\varepsilon_3)}(a * b), \lambda_{\mathscr{H}(\varepsilon_3)}(b)\}$. Hence the R-union $(\mathscr{H}, C) = (\mathscr{F}, A) \Cup_R (\mathscr{G}, B)$ of $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ is not a cubic soft ideal over U.

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Hyers-Ulam stability of the delayed homogeneous matrix difference equation with constructive method

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Abstract

We prove Hyers-Ulam stability of the first order delayed homogeneous matrix difference equation $\vec{x}_{i+p} = \mathbf{A}(i)\vec{x}_i$ for all integers i.

1 Introduction

Throughout this paper, we denote by $\mathbb{C}$, $\mathbb{N}$, $\mathbb{N}_0$, and $\mathbb{Z}$ the set of all complex numbers, of all positive integers, of all nonnegative integers, and the set of all integers, respectively. Given a fixed positive integer n, let $(\mathbb{C}^n, \|\cdot\|_n)$ be a complex normed space, each of whose elements is a column vector, and let $\mathbb{C}^{n\times n}$ be a vector space consisting of all $(n \times n)$ complex matrices. We choose a norm $\|\cdot\|_{n\times n}$ on $\mathbb{C}^{n\times n}$ which is compatible with $\|\cdot\|_n$, *i.e.*, both norms obey

$$\|\mathbf{AB}\|_{n\times n} \leq \|\mathbf{A}\|_{n\times n}\|\mathbf{B}\|_{n\times n} \quad \text{and} \quad \|\mathbf{A}\vec{x}\|_n \leq \|\mathbf{A}\|_{n\times n}\|\vec{x}\|_n \tag{1.1}$$

for all $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{n\times n}$ and $\vec{x} \in \mathbb{C}^n$.

A matrix difference equation is a difference equation with matrix coefficients in which the value of vector at one point depends on the values of preceding points.

In this paper, we prove Hyers-Ulam stability of the first order delayed homogeneous matrix difference equation

$$\vec{x}_{i+p} = \mathbf{A}(i)\vec{x}_i \tag{1.2}$$

for all integers $i \in \mathbb{Z}$, where each transition matrix $\mathbf{A}(i)$ is nonsingular and p is a fixed integer larger than 1. More precisely, we prove that if a vector sequence $\{\vec{y}_i\}_{i\in\mathbb{Z}}$ of $\mathbb{C}^n$ satisfies the inequality

$$\|\vec{y}_{i+p} - \mathbf{A}(i)\vec{y}_i\|_n \leq \varepsilon$$

for all $i \in \mathbb{Z}$, then there exists a solution $\{\vec{x}_i\}_{i\in\mathbb{Z}}$ to the delayed matrix difference equation (1.2) such that the bound for $\|\vec{y}_i - \vec{x}_i\|_n$ depends on ε and the transition matrices $\mathbf{A}(i)$ only. We refer the reader to [1, 2, 3, 4, 6] for the exact definition of Hyers-Ulam stability.

[0] Key words and phrases: difference equation; matrix difference equation; delayed matrix difference equation; Hyers-Ulam stability; approximation.

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2 Preliminaries

Throughout this paper, the transition matrix $\mathbf{A}(i)$ of $\mathbb{C}^{n\times n}$ is defined by

$$\mathbf{A}(i) = \begin{pmatrix} a_{11}(i) & a_{12}(i) & \cdots & a_{1n}(i) \\ a_{21}(i) & a_{22}(i) & \cdots & a_{2n}(i) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(i) & a_{n2}(i) & \cdots & a_{nn}(i) \end{pmatrix}$$

for any integer i. We moreover assume that every $\mathbf{A}(i)$ is nonsingular. We will use the following abbreviation.

$$\Phi(j,k) := \begin{cases} \prod_{i=k}^{j-1} \mathbf{A}(i) = \mathbf{A}(j-1)\mathbf{A}(j-2)\cdots\mathbf{A}(k) & (\text{for } j > k), \\ \mathbf{I}_{n\times n} & (\text{for } j = k), \end{cases} \tag{2.1}$$

where we set $\Phi(j,k) := \big(\Phi(k,j)\big)^{-1} = \mathbf{A}(j)^{-1}\mathbf{A}(j+1)^{-1}\cdots\mathbf{A}(k-1)^{-1}$ for $j < k$ and $\mathbf{I}_{n\times n}$ denotes the $(n\times n)$ identity matrix. Sometimes, we use $\Phi(j)$ and $\Phi^{-1}(k,j)$ instead of $\Phi(j,0)$ and $\big(\Phi(k,j)\big)^{-1}$, respectively.

In the following lemma, we introduce some properties of $\Phi(j,k)$ without proof.

Lemma 2.1 *Assume that n is a fixed positive integer. If the transition matrix $\mathbf{A}(i)$ of $\mathbb{C}^{n\times n}$ is nonsingular for any integer i, then it holds that*

(*i*) $\Phi(j+1,k) = \mathbf{A}(j)\Phi(j,k)$;

(*ii*) $\Phi^{-1}(j,k+1) = \mathbf{A}(k)\Phi^{-1}(j,k)$;

(*iii*) $\mathbf{A}(k-1)^{-1}\Phi^{-1}(j,k) = \Phi^{-1}(j,k-1)$

for all integers j and k.

3 Hyers-Ulam stability of $\vec{x}_{i+p} = \mathbf{A}(i)\vec{x}_i$

We now prove our main theorem concerning Hyers-Ulam stability of the delayed homogeneous matrix difference equation (1.2). Obviously, our theorem is a generalization and an improvement of [5, Theorem 2.1].

Theorem 3.1 *Assume that $n > 0$ and $p > 1$ are fixed integers and ε is a nonnegative real number. For all integers i, assume that $\mathbf{A}(i)$ is a nonsingular $(n\times n)$ complex-valued matrix for which there exists a constant $K > 0$ such that*

$$\sum_{j=0}^{\infty} \left\| \left(\prod_{k=0}^{j} \mathbf{A}(i+kp) \right)^{-1} \right\|_{n\times n} \le K \tag{3.1}$$

for all integers i. If a sequence $\{\vec{y}_i\}_{i\in\mathbb{Z}}$ of $\mathbb{C}^n$ satisfies the inequality

$$\left\|\vec{y}_{i+p} - \mathbf{A}(i)\vec{y}_i\right\|_n \leq \varepsilon \tag{3.2}$$

for all integers i, then there exists a unique solution $\{\vec{T}_i\}_{i\in\mathbb{Z}}$ to the first order delayed homogeneous matrix difference equation (1.2) *such that*

$$\left\|\vec{T}_i - \vec{y}_i\right\|_n \leq K\varepsilon \tag{3.3}$$

for each integer i.

Proof. In view of (3.2), there exists a sequence $\{\vec{\varepsilon}_i\}_{i\in\mathbb{Z}}$ of $\mathbb{C}^n$ such that

$$\vec{y}_{i+p} - \mathbf{A}(i)\vec{y}_i = \vec{\varepsilon}_i \tag{3.4}$$

for all integers i and

$$\sup_{i\in\mathbb{Z}} \left\|\vec{\varepsilon}_i\right\|_n \leq \varepsilon. \tag{3.5}$$

First, we use the induction on m to prove

$$\vec{y}_{i+mp} = \left(\prod_{k=0}^{m-1} \mathbf{A}(i+kp)\right)\vec{y}_i + \sum_{j=0}^{m-1}\left(\prod_{k=j+1}^{m-1} \mathbf{A}(i+kp)\right)\vec{\varepsilon}_{i+jp} \tag{3.6}$$

for all $i \in \mathbb{Z}$ and $m \in \mathbb{N}_0$. Obviously, the equality (3.6) is true for $m \in \{0,1\}$. Assume that the equality (3.6) is true for some positive integer m. It then follows from (3.4) and (3.6) that

$$\begin{aligned}
\vec{y}_{i+(m+1)p} &= \mathbf{A}(i+mp)\vec{y}_{i+mp} + \vec{\varepsilon}_{i+mp} \\
&= \left(\prod_{k=0}^{m} \mathbf{A}(i+kp)\right)\vec{y}_i + \sum_{j=0}^{m-1}\left(\prod_{k=j+1}^{m} \mathbf{A}(i+kp)\right)\vec{\varepsilon}_{i+jp} + \vec{\varepsilon}_{i+mp} \\
&= \left(\prod_{k=0}^{m} \mathbf{A}(i+kp)\right)\vec{y}_i + \sum_{j=0}^{m}\left(\prod_{k=j+1}^{m} \mathbf{A}(i+kp)\right)\vec{\varepsilon}_{i+jp},
\end{aligned}$$

which follows from (3.6) by replacing m with $m+1$.

If we set

$$\vec{T}_i(m) := \left(\prod_{k=0}^{m} \mathbf{A}(i+kp)\right)^{-1} \vec{y}_{i+(m+1)p}$$

for all $i \in \mathbb{Z}$ and $m \in \mathbb{N}_0$, then it follows from (3.6) that

$$\vec{T}_i(m) = \vec{y}_i + \sum_{j=0}^{m}\left(\prod_{k=0}^{j} \mathbf{A}(i+kp)\right)^{-1} \vec{\varepsilon}_{i+jp}. \tag{3.7}$$

Let m and n be nonnegative integers with $n > m$. Then, by (3.7), we have

$$\vec{T}_i(n) - \vec{T}_i(m) = \sum_{j=m+1}^{n} \left(\prod_{k=0}^{j} \mathbf{A}(i+kp) \right)^{-1} \vec{\varepsilon}_{i+jp}$$

for any fixed integer i. In view of (3.5) and (3.1), we further get

$$\begin{aligned}
\big\| \vec{T}_i(n) - \vec{T}_i(m) \big\|_n &= \left\| \sum_{j=m+1}^{n} \left(\prod_{k=0}^{j} \mathbf{A}(i+kp) \right)^{-1} \vec{\varepsilon}_{i+jp} \right\|_n \\
&\le \sum_{j=m+1}^{n} \left\| \left(\prod_{k=0}^{j} \mathbf{A}(i+kp) \right)^{-1} \vec{\varepsilon}_{i+jp} \right\|_n \\
&\le \sum_{j=m+1}^{n} \left\| \left(\prod_{k=0}^{j} \mathbf{A}(i+kp) \right)^{-1} \right\|_{n\times n} \big\| \vec{\varepsilon}_{i+jp} \big\|_n \\
&\le \varepsilon \sum_{j=m+1}^{n} \left\| \left(\prod_{k=0}^{j} \mathbf{A}(i+kp) \right)^{-1} \right\|_{n\times n} \\
&\to 0, \quad \text{as} \quad m \to \infty,
\end{aligned}$$

for every $i \in \mathbb{Z}$. Hence, $\{\vec{T}_i(m)\}_{m\in\mathbb{N}_0}$ is a Cauchy sequence for each fixed $i \in \mathbb{Z}$, and we can define

$$\vec{T}_i := \lim_{m\to\infty} \vec{T}_i(m) \tag{3.8}$$

for each $i \in \mathbb{Z}$.

By (3.4), (3.7), and (3.8), we obtain

$$\begin{aligned}
\vec{T}_{i+p} - \mathbf{A}(i)\vec{T}_i &= \vec{y}_{i+p} + \sum_{j=0}^{\infty} \left(\prod_{k=0}^{j} \mathbf{A}\big(i+(k+1)p\big) \right)^{-1} \vec{\varepsilon}_{i+(j+1)p} \\
&\quad - \mathbf{A}(i)\vec{y}_i - \sum_{j=0}^{\infty} \left(\prod_{k=1}^{j} \mathbf{A}(i+kp) \right)^{-1} \vec{\varepsilon}_{i+jp} \\
&= \vec{y}_{i+p} + \sum_{j=0}^{\infty} \left(\prod_{k=1}^{j+1} \mathbf{A}(i+kp) \right)^{-1} \vec{\varepsilon}_{i+(j+1)p} \\
&\quad - \mathbf{A}(i)\vec{y}_i - \sum_{j=0}^{\infty} \left(\prod_{k=1}^{j} \mathbf{A}(i+kp) \right)^{-1} \vec{\varepsilon}_{i+jp} \\
&= \vec{y}_{i+p} + \sum_{j=1}^{\infty} \left(\prod_{k=1}^{j} \mathbf{A}(i+kp) \right)^{-1} \vec{\varepsilon}_{i+jp} \\
&\quad - \mathbf{A}(i)\vec{y}_i - \sum_{j=0}^{\infty} \left(\prod_{k=1}^{j} \mathbf{A}(i+kp) \right)^{-1} \vec{\varepsilon}_{i+jp} \\
&= \vec{0}
\end{aligned}$$

for all $i \in \mathbb{Z}$. Moreover, it follows from (3.5), (3.1), (3.7), and (3.8) that

$$\begin{aligned}
\|\vec{T}_i - \vec{y}_i\|_n &= \left\| \sum_{j=0}^{\infty} \left(\prod_{k=0}^{j} \mathbf{A}(i+kp) \right)^{-1} \vec{\varepsilon}_{i+jp} \right\|_n \\
&\le \sum_{j=0}^{\infty} \left\| \left(\prod_{k=0}^{j} \mathbf{A}(i+kp) \right)^{-1} \vec{\varepsilon}_{i+jp} \right\|_n \\
&\le \sum_{j=0}^{\infty} \left\| \left(\prod_{k=0}^{j} \mathbf{A}(i+kp) \right)^{-1} \right\|_{n\times n} \|\vec{\varepsilon}_{i+jp}\|_n \\
&\le K\varepsilon
\end{aligned}$$

for all $i \in \mathbb{Z}$.

Finally, we prove the uniqueness of the sequence $\{\vec{T}_i\}_{i\in\mathbb{Z}}$. Assume that $\{\vec{U}_i\}_{i\in\mathbb{Z}}$ is another solution to the difference equation (1.2). By applying the induction on m, we prove that

$$\vec{U}_i = \left(\prod_{k=0}^{m} \mathbf{A}(i+kp) \right)^{-1} \vec{U}_{i+(m+1)p} \tag{3.9}$$

for any $m \in \mathbb{N}_0$. Obviously, (3.9) is true for $m = 0$. Assume now that (3.9) is true for some integer $m \ge 0$. It then follows from (1.2) and (3.9) that

$$\begin{aligned}
\vec{U}_i &= \left(\prod_{k=0}^{m} \mathbf{A}(i+kp) \right)^{-1} \vec{U}_{i+(m+1)p} \\
&= \left(\prod_{k=0}^{m+1} \mathbf{A}(i+kp) \right)^{-1} \mathbf{A}\big(i+(m+1)p\big)\vec{U}_{i+(m+1)p} \\
&= \left(\prod_{k=0}^{m+1} \mathbf{A}(i+kp) \right)^{-1} \vec{U}_{i+(m+2)p},
\end{aligned}$$

which can be obtained from (3.9) by replacing m with $m+1$. Thus, by (3.1), (3.3), and (3.9), we have

$$\begin{aligned}
\|\vec{T}_i - \vec{U}_i\|_n &= \left\| \left(\prod_{k=0}^{m} \mathbf{A}(i+kp) \right)^{-1} \big(\vec{T}_{i+(m+1)p} - \vec{U}_{i+(m+1)p}\big) \right\|_n \\
&\le \left\| \left(\prod_{k=0}^{m} \mathbf{A}(i+kp) \right)^{-1} \right\|_{n\times n} \|\vec{T}_{i+(m+1)p} - \vec{y}_{i+(m+1)p}\|_n \\
&\quad + \left\| \left(\prod_{k=0}^{m} \mathbf{A}(i+kp) \right)^{-1} \right\|_{n\times n} \|\vec{y}_{i+(m+1)p} - \vec{U}_{i+(m+1)p}\|_n \\
&\le 2K\varepsilon \left\| \left(\prod_{k=0}^{m} \mathbf{A}(i+kp) \right)^{-1} \right\|_{n\times n} \\
&\to 0, \quad \text{as} \quad m \to \infty,
\end{aligned}$$

for all $i \in \mathbb{Z}$, which implies the uniqueness of $\{\vec{T}_i\}_{i\in\mathbb{Z}}$. □

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4 Examples

At a glance, the condition (3.1) would seem too strong so that we could seldom find practical examples. But we get rid of such a misunderstanding through introducing a few examples for the sequence $\{\mathbf{A}(i)\}_{i\in\mathbb{Z}}$ of transition matrices which satisfy the condition (3.1).

Example 4.1 *Let us set $n = 1$ and $p = 3$. If $\mathbf{A}(i) = 2^3$ is a (1×1) matrix for every integer i, then we have*

$$\sum_{j=0}^{\infty} \left| \left(\prod_{k=0}^{j} \mathbf{A}(i+3k) \right)^{-1} \right| = \sum_{j=0}^{\infty} \big(\underbrace{2^3 \cdot 2^3 \cdots 2^3}_{j+1} \big)^{-1} = \sum_{j=0}^{\infty} 2^{-3(j+1)} = \frac{1}{7},$$

i.e., the condition (3.1) is satisfied with $K = \frac{1}{7}$.

Assume that a sequence $\{y_i\}_{i\in\mathbb{Z}}$ of complex numbers satisfies the inequality

$$\left| y_{i+3} - 2^3 y_i \right| \leq \varepsilon$$

for all integers i, where ε is an arbitrarily given nonnegative real number. Then, according to Theorem 3.1, there exists a unique sequence $\{x_i\}_{i\in\mathbb{Z}}$ of complex numbers such that

$$x_{i+3} = 2^3 x_i \tag{4.1}$$

and

$$|x_i - y_i| \leq \frac{1}{7}\varepsilon$$

for all integers i.

Indeed, the delayed difference equation (4.1) is strongly related to the nonlinear difference equation

$$x_{i+1} = 2^{i+1} - \frac{2^{2i+1}}{x_i}.$$

Example 4.2 *We consider the difference equation with two variables given as*

$$\begin{pmatrix} u_{i+1} \\ v_{i+1} \end{pmatrix} = \begin{pmatrix} \dfrac{-2^{2i+1}}{(i^2+2i+2)(i^2+1)v_i - (i^2+2i+2)2^{i+1}} \\ \dfrac{2^{i+2}}{i^2+2i+2} - \dfrac{2^{2i+1}}{(i^2+2i+2)(i^2+1)u_i} \end{pmatrix}$$

for all integers i, where $\{u_i\}_{i\in\mathbb{Z}}$ and $\{v_i\}_{i\in\mathbb{Z}}$ are sequences of complex numbers. By a straightforward calculation, we show that

$$\begin{pmatrix} u_{i+2} \\ v_{i+2} \end{pmatrix} = \begin{pmatrix} \dfrac{4(i^2+1)}{i^2+4i+5} & 0 \\ 0 & \dfrac{4(i^2+1)}{i^2+4i+5} \end{pmatrix} \begin{pmatrix} u_i \\ v_i \end{pmatrix}$$

for all integers i. We now define the (2×2) matrix $\mathbf{A}(i)$ by

$$\mathbf{A}(i) := \begin{pmatrix} \dfrac{4(i^2+1)}{i^2+4i+5} & 0 \\ 0 & \dfrac{4(i^2+1)}{i^2+4i+5} \end{pmatrix} = \frac{4(i^2+1)}{(i+2)^2+1}\mathbf{I}_{2\times 2}$$

for every integer i, where $\mathbf{I}_{2\times 2}$ denotes the (2×2) identity matrix. Then we have

$$\left(\prod_{k=0}^{j} \mathbf{A}(i+2k)\right)^{-1} = \frac{(i+2j+2)^2+1}{(i^2+1)4^{j+1}}\mathbf{I}_{2\times 2}$$

for all nonnegative integers j. Hence, we see that

$$\begin{aligned}
\sum_{j=0}^{\infty}\left\|\left(\prod_{k=0}^{j} \mathbf{A}(i+2k)\right)^{-1}\right\|_{\infty} &= \sum_{j=0}^{\infty}\frac{(i+2j+2)^2+1}{(i^2+1)4^{j+1}} \\
&\le \sum_{j=0}^{\infty}\frac{(|i|+2j+2)^2+1}{(i^2+1)4^{j+1}} \\
&\le \sum_{j=0}^{\infty}\frac{1}{4^{j+1}} + \sum_{j=0}^{\infty}\frac{1}{2}\frac{j+1}{4^j} + \sum_{j=0}^{\infty}\frac{(j+1)^2}{4^j} \\
&\le \frac{1}{3} + \sum_{j=0}^{\infty}\frac{1}{2}\frac{1}{2^j} + \sum_{j=0}^{\infty}\frac{9}{4}\frac{1}{2^j} \\
&= \frac{35}{6},
\end{aligned}$$

i.e., the condition (3.1) is satisfied with $K = \frac{35}{6}$.

Let ε is an arbitrarily given nonnegative real number. Assume that a sequence $\{\vec{y}_i\}_{i\in\mathbb{Z}}$ of $\mathbb{C}^2$ satisfies the inequality

$$\left\|\vec{y}_{i+2} - \mathbf{A}(i)\vec{y}_i\right\|_{\infty} \le \varepsilon$$

for all integers i. Then, due to Theorem 3.1 with $n = 2$, $p = 2$, and $K = \frac{35}{6}$, there exists a unique solution $\{\vec{x}_i\}_{i\in\mathbb{Z}}$ to the delayed homogeneous matrix difference equation (1.2) such that

$$\left\|\vec{x}_i - \vec{y}_i\right\|_{\infty} \le \frac{35}{6}\varepsilon$$

for any integer i.

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Mathematical analysis of $(n+3)$-dimensional virus dynamics model

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Abstract

An $(n+3)$-dimensional nonlinear mathematical model for the virus dynamics with humoral immunity and n-stages of infected cells is proposed and analyzed. Two threshold parameters, the basic reproduction number, R_0^M and the humoral immunity number, R_1^M are derived. Utilizing Lyapunov functions and LaSalle's invariance principle, the global asymptotic stability of all steady states of the model is obtained. An example is presented and some numerical simulations are conducted in order to illustrate the dynamical behavior.

Keywords: Virus dynamics; global stability; humoral immunity; Lyapunov function.

1 Introduction

During the past decades many human viruses have been found such as HIV, HBV, HCV and HTLV-I. To understand the virus dynamics, several mathematical models for virus dynamics have been proposed and analyzed (see e.g. [1]-[16]). One of the most important features of mathematical models is the global stability of steady states which gives us a detailed information and enhances our understanding about the virus dynamics. Therefore several researchers studied the global stability of virus dynamics models (see e.g. [5], [6], [7], [8], [9], [11], [12], [13], [14], [19], [20]). Some of these papers consider a single-infected stage for infected cells (see e.g. [5], [6], [7], [11], [12] and [14]). Other works consider double-infected stages for infected cells, the first stage is the latently infected cells which contain viruses but do not produce it and the second stage is the actively infected cells which produce new viruses (see e.g. [8] [9], [19] and [20]). As reported in [21], [22] and [23], due to ongoing viral replication in the virus dynamics process such as HIV, the time from the contact of viruses and uninfected target cells to the death of the cells modeled by dividing the process into n short stages $y_1 \to y_2 \to .... \to y_n$. Georgescu and Hsieh [20] have proposed a virus dynamics model with multi-staged infected cells. However, the model does not consider the immune response.

It should be pointed out that the immune response plays an important role in controlling the disease progression. There are two main responses for immune system, Cytotoxic T Lymphocyte (CTL) immune response and humoral immune response. The function of the CTL cells is to kill the infected cells. The humoral immunity is based on the B cells which produce antibodies to attack the viruses [1]. It is mentioned in [24] that, in malaria, the antibodies are more effective than CTL cells [24]. Several works incorporate the humoral immune response into the virus dynamics models (see e.g. [25]-[31]). Elaiw and AlShamrani [29], [30] studied the global stability of virus dynamics models with double-infected stages for infected cells.

The aim of this paper is to study a general virus dynamics model with multi-staged infected cells and humoral immunity. Our model is an improvement of the model presented in [20] by taking into account the humoral immune response, and by assuming a more general incidence rate which includes the form given in [20]. We use Lyapunov functions and LaSalle's invariance principle to prove the global stability of all the steady states of the model. We show that there exist two bifurcation parameters, the basic reproduction number R_0^M

and the humoral immunity number R_1^M. We establish a set of sufficient conditions which guarantee the global stability of all steady states of the model.

2 The model

In this section we propose the following model:

$$\dot{x} = \lambda - dx - g(x,v), \tag{1}$$

$$\dot{y}_1 = g(x,v) - a_1\phi_1(y_1), \tag{2}$$

$$\dot{y}_i = \tilde{a}_{i-1}\phi_{i-1}(y_{i-1}) - a_i\phi_i(y_i), \quad i = 2,3,...,n, \tag{3}$$

$$\dot{v} = \tilde{a}_n\phi_n(y_n) - pzv - uv, \tag{4}$$

$$\dot{z} = rzv - bz. \tag{5}$$

All parameters and variables have the same identifications given in Section 1. The model is a generalization of several existing model by considering general functions for: (i) the incidence rate of infection $g(x,v)$; (ii) the production rates of infected cells $g(x,v)$ and $\tilde{a}_{i-1}\phi_{i-1}(y_{i-1})$, $i = 2,...,n$; (iii) the removal rate of infected cells $a_i\phi_i(y_i)$, $i = 1,...,n$; (iii) the production rate of viruses $\tilde{a}_n\phi_n(y_n)$. Functions g and ϕ_i are continuously differentiable and satisfy the following conditions:

Condition C1. (i) $g(x,v) > 0$, $g(0,v) = g(x,0) = 0$ for all $x,v > 0$ and
(ii) $\frac{\partial g(x,v)}{\partial x} > 0$, $\frac{\partial g(x,v)}{\partial v} > 0$, $\frac{\partial g(x,0)}{\partial v} > 0$ for all $x,v > 0$.

Condition C2. (i) $g(x,v) \leq v\frac{\partial g(x,0)}{\partial v}$ for all $x,v > 0$ and
(ii) $\left(\dfrac{\partial g(x,0)}{\partial v}\right)' > 0$ for all $x,v > 0$.

Condition C3. (i) $\phi_i(y_i) > 0$ for all $y_i > 0$, $\phi_i(0) = 0$, $i = 1,2,...,n$,
(ii) $\phi_i'(y_i) > 0$ for all $y_i > 0$, $i = 1,2,...,n$, and
(iii) there is $\alpha_i > 0$, $i = 1,...,n$ such that $\phi_i(y_i) \geq \alpha_i y_i$ for all $y_i > 0$.

3 Properties of solutions

In this section, we study some properties of the solutions of the model such as the non-negativity and boundedness.

Proposition 1. Suppose that Conditions C1 and C3 are hold. Then there exist positive numbers M_j, $j = 1,2,...,n+2$, such that the compact set

$$\Theta = \left\{(x,y_1,...,y_n,v,z) \in \mathbb{R}^{n+3}_{\geq 0} : 0 \leq x \leq M_1, 0 \leq y_i \leq M_i, 0 \leq v \leq M_{n+1}, 0 \leq z \leq M_{n+2},\ i = 1,...,n\right\}$$

is positively invariant.

Proof. Since

$$\begin{aligned}
&\dot{x}\mid_{x=0} = \lambda > 0, && \\
&\dot{y}_1\mid_{y_1=0} = g(x,v) \geq 0 && \text{for all } x,v \in [0,\infty), \\
&\dot{y}_i\mid_{y_i=0} = \tilde{a}_{i-1}\phi_{i-1}(y_{i-1}) \geq 0 && \text{for all } y_{i-1} \in [0,\infty),\ i = 2,3,...,n, \\
&\dot{v}\mid_{v=0} = \tilde{a}_n\phi_n(y_n) \geq 0 && \text{for all } y_n \in [0,\infty), \\
&\dot{z}\mid_{z=0} = 0,
\end{aligned}$$

Then, the orthant $\mathbb{R}^{n+3}_{\geq 0}$ is positively invariant for system (1)-(5).

To show the boundedness of the solutions we let $G_1(t) = x(t) + y_1(t)$, then

$$\dot{G}_1 = \lambda - dx - a_1\phi_1(y_1) \leq \lambda - dx - a_1\alpha_1 y_1 \leq \lambda - \delta_1(x + y_1) \leq \lambda - \delta_1 G_1,$$

where $\delta_1 = \min\{d, a_1\alpha_1\}$. It follows that,

$$G_1(t) \le e^{-\delta_1 t}\left(G_1(0) - \frac{\lambda}{\delta_1}\right) + \frac{\lambda}{\delta_1}.$$

Hence, $0 \le G_1(t) \le M_1$ if $G_1(0) \le M_1$ for $t \ge 0$ where $M_1 = \frac{\lambda}{\delta_1}$. The non-negativity of x and y_1 implies that, $0 \le x(t), y_1(t) \le M_1$ if $x(0) + y_1(0) \le M_1$. From Eq. (3) and Condition C3, we have

$$\dot{y}_2 = \tilde{a}_1\phi_1(y_1) - a_2\phi_2(y_2) \le \tilde{a}_1\phi_1(M_1) - a_2\alpha_2 y_2.$$

It follows that, $0 \le y_2(t) \le M_2$ if $y_2(0) \le M_2$, where $M_2 = \dfrac{\tilde{a}_1\phi_1(M_1)}{a_2\alpha_2}$. Similarly, we can show $0 \le y_i(t) \le M_i$ if $y_i(0) \le M_i$, where $M_i = \dfrac{\tilde{a}_{i-1}\phi_{i-1}(M_{i-1})}{a_i\alpha_i}$ $i = 3, ..., n$. Finally, we let $G_2(t) = v(t) + \frac{p}{r}z(t)$, then

$$\begin{aligned}\dot{G}_2 &= \tilde{a}_n\phi_n(y_n) - uv - \frac{pb}{r}z \\ &\le \tilde{a}_n\phi_n(M_n) - \delta_2\left(v + \frac{p}{r}z\right) = \tilde{a}_n\phi_n(M_n) - \delta_2 G_2,\end{aligned}$$

where $\delta_2 = \min\{u, b\}$. It follows that, $0 \le G_2(t) \le M_{n+1}$ if $G_2(0) \le M_{n+1}$, where $M_{n+1} = \dfrac{\tilde{a}_n\phi_n(M_n)}{\delta_2}$. Since $v(t)$ and $z(t)$ are non-negative, then $0 \le v(t) \le M_{n+1}$ and $0 \le z(t) \le M_{n+2}$ if $v(0) + \frac{p}{r}z(0) \le M_{n+1}$, where $M_{n+2} = \frac{r}{p}M_{n+1}$. Therefore, all the variables of the model are bounded and the region Θ is positively invariant with respect to model (1)-(5). □

4 The steady states and biological bifurcations

In this section, we prove the existence of the steady states of system (1)-(5) and derive two bifurcation parameters.

Lemma 1. Assume that Conditions C1-C3 are satisfied, then there exist two bifurcation parameters $R_0^M > R_1^M > 0$ such that

(i) if $R_0^M \le 1$, then the system has only one positive steady state $Q_0 \in \Theta$.

(ii) if $R_1^M \le 1 < R_0^M$, then then the system has only two positive steady states $Q_0 \in \Theta$ and $Q_1 \in \Theta$, and

(iii) if $R_1^M > 1$, then then the system has three positive steady states $Q_0 \in \Theta$, $Q_1 \in \Theta$ and $Q_2 \in \overset{\circ}{\Theta}$.

Proof. At any steady state $E(x, y_1, ..., y_n, v, z)$, the following equations hold:

$$\lambda - dx - g(x, v) = 0, \tag{6}$$

$$g(x, v) - a_1\phi_1(y_1) = 0, \tag{7}$$

$$\tilde{a}_{i-1}\phi_{i-1}(y_{i-1}) - a_i\phi_i(y_i) = 0, \quad i = 2, ..., n, \tag{8}$$

$$\tilde{a}_n\phi_n(y_n) - uv - pzv = 0, \tag{9}$$

$$(rv - b)\, z = 0. \tag{10}$$

Eq. (10) has two possibilities, $z = 0$ and $v = \dfrac{b}{r}$. When $z = 0$, then from Eqs. (6)-(9) we get

$$\lambda - dx = g(x, v) = \left(\prod_{j=1}^{i}\frac{a_j}{\tilde{a}_j}\right)\tilde{a}_i\phi_i(y_i) = \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)uv, \quad i = 1, ..., n, \tag{11}$$

The continuity and strictly increasing properties of ϕ_i imply that ϕ_i^{-1} exists and it is also continuous and strictly increasing [32]. Define $f_i(v) = \phi_i^{-1}\left(\left(\prod_{j=1}^{i}\frac{\tilde{a}_j}{a_j}\right)\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)\frac{uv}{\tilde{a}_i}\right)$, $i = 1, 2, ..., n$, then $f_i(0) = 0$ and $f_i(v) > 0$ for all $v > 0$. From Eq. (11), we get

$$y_i = f_i(v), \qquad x = x_0 - \frac{1}{d}\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)uv, \tag{12}$$

and

$$g\left(x_0 - \frac{1}{d}\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)uv, v\right) - \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)uv = 0, \tag{13}$$

where $x_0 = \lambda/d$. Condition C1 implies that Eq. (13) has two possible solutions $v = 0$ and $v \neq 0$. If $v = 0$, then from Eq. (12), we get the disease-free steady state $Q_0 = (x_0, \overbrace{0, ..., 0, 0}^{n+2})$. Let us consider the case $v \neq 0$. Define

$$\Psi_1(v) = g\left(x_0 - \frac{1}{d}\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)uv, v\right) - \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)uv = 0.$$

We have, $\Psi_1(0) = 0$, and $\Psi_1(\hat{v}) = -\lambda < 0$.where $\hat{v} = \frac{\lambda}{u}\left(\prod_{j=1}^{n}\frac{\tilde{a}_j}{a_j}\right)$. Moreover,

$$\Psi_1'(0) = -\frac{u}{d}\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)\frac{\partial g(x_0, 0)}{\partial x} + \frac{\partial g(x_0, 0)}{\partial v} - \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)u.$$

From Condition C1 we have $\frac{\partial g(x_0,0)}{\partial x} = 0$, then

$$\Psi_1'(0) = u\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)\left(\frac{1}{u}\left(\prod_{j=1}^{n}\frac{\tilde{a}_j}{a_j}\right)\frac{\partial g(x_0, 0)}{\partial v} - 1\right).$$

Therefore, if $\frac{1}{u}\left(\prod_{j=1}^{n}\frac{\tilde{a}_j}{a_j}\right)\frac{\partial g(x_0,0)}{\partial v} > 1$, then $\Psi_1'(0) > 0$ and there exists a $v_1 \in (0, \hat{v})$ such that $\Psi_1(v_1) = 0$. Substituting $v = v_1$ in Eq. (6) and letting

$$\Psi_2(x) = \lambda - dx - g(x, v_1) = 0.$$

According to Condition C1, Ψ_2 is a strictly decreasing, $\Psi_2(0) = \lambda > 0$ and $\Psi_2(x_0) = -g(x_0, v_1) < 0$. Thus, there exists a unique $x_1 \in (0, x_0)$ such that $\Psi_2(x_1) = 0$. On the other hand, from Eq. (12) we have $y_{i,1} = f_i(v_1) > 0$, $i = 1, ..., n$. It follows that, a endemic steady state without humoral immune response $Q_1 = (x_1, y_{1,1}, ..., y_{n,1}, v_1, 0)$ exists when $\frac{1}{u}\left(\prod_{j=1}^{n}\frac{\tilde{a}_j}{a_j}\right)\frac{\partial g(x_0,0)}{\partial v} > 1$. Let us define the basic reproduction number as:

$$R_0^M = \frac{1}{u}\left(\prod_{j=1}^{n}\frac{\tilde{a}_j}{a_j}\right)\frac{\partial g(x_0, 0)}{\partial v}.$$

The other possibility of Eq. (10) is $v = v_2 = \dfrac{b}{r}$. Let

$$\Psi_3(x) = \lambda - dx - g(x, v_2) = 0.$$

Clearly, Ψ_3 is a strictly decreasing, $\Psi_3(0) = \lambda > 0$ and $\Psi_3(x_0) = -g(x_0, v_2) < 0$. Thus, there exists a unique $x_2 \in (0, x_0)$ such that $\Psi_3(x_2) = 0$. It follows that,

$$y_{i,2} = \phi_i^{-1}\left(\left(\prod_{j=1}^{i}\frac{\tilde{a}_j}{a_j}\right)\frac{g(x_2, v_2)}{\tilde{a}_i}\right) > 0.$$

Further, $z_2 = \dfrac{u}{p}(R_1^M - 1)$, where

$$R_1^M = \frac{1}{u}\left(\prod_{j=1}^{n}\frac{\tilde{a}_j}{a_j}\right)\frac{g(x_2, v_2)}{v_2}$$

represents the humoral immunity number. It follows that, if $R_1^M > 1$, then there exists a endemic steady state with humoral immune response $Q_2 = (x_2, y_{1,2}, ..., y_{n,2}, v_2, z_2)$.

Now we show that $Q_0, Q_1 \in \Theta$ and $Q_2 \in \overset{\circ}{\Theta}$. Clearly, $Q_0 \in \Theta$. We have $x_1 \in (0, x_0)$, then

$$0 < x_1 < \frac{\lambda}{d} \leq \frac{\lambda}{\delta_1} = M_1.$$

From Eq. (11), we get

$$a_1\alpha_1 y_{1,1} \leq a_1\phi_1(y_{1,1}) = \lambda - dx_1 < \lambda \Rightarrow 0 < y_{1,1} < \frac{\lambda}{a_1\alpha_1} \leq M_1.$$

Also, from Eq. (8), we have

$$a_2\alpha_2 y_{2,1} \leq a_2\phi_2(y_{2,1}) = \tilde{a}_1\phi_1(y_{1,1}) < \tilde{a}_1\phi_1(M_1) \Rightarrow 0 < y_{2,1} < \frac{\tilde{a}_1\phi_1(M_1)}{a_2\alpha_2} = M_2.$$

Consequently, for $i = 3, ..., n$, we have

$$a_i\alpha_i y_{i,1} \leq a_i\phi_i(y_{i,1}) = \tilde{a}_{i-1}\phi_{i-1}(y_{i-1,1}) < \tilde{a}_{i-1}\phi_{i-1}(M_{i-1}) \Rightarrow 0 < y_{i,1} < \frac{\tilde{a}_{i-1}\phi_{i-1}(M_{i-1})}{a_i\alpha_i} = M_i.$$

Eq. (9) implies that,

$$uv_1 = \tilde{a}_n\phi_n(y_{n,1}) < \tilde{a}_n\phi_n(M_n) \Rightarrow 0 < v_1 < \frac{\tilde{a}_n\phi_n(M_n)}{u} \leq \frac{\tilde{a}_n\phi_n(M_n)}{\delta_2} = M_{n+1}.$$

We have also $z_1 = 0$, then $Q_1 \in \Theta$. Similarly, one can show that $0 < x_2 < M_1$ and $0 < y_{i,2} < M_i$, $i = 1, ..., n$. Now we show that if $R_1^M > 1$, then $0 < v_2 < M_{n+1}$ and $0 < z_2 < M_{n+2}$. From Eq. (9) we have

$$uv_2 + pv_2z_2 = \tilde{a}_n\phi_n(y_{n,2}).$$

Then

$$uv_2 < \tilde{a}_n\phi_n(y_{n,2}) < \tilde{a}_n\phi_n(M_n) \Rightarrow 0 < v_2 < \frac{\tilde{a}_n\phi_n(M_n)}{u} \leq M_{n+1},$$

$$pv_2z_2 < \tilde{a}_n\phi_n(y_{n,2}) < \tilde{a}_n\phi_n(M_n) \Rightarrow 0 < z_2 < \frac{r\tilde{a}_n\phi_n(M_n)}{pb} \leq M_{n+2}.$$

Then, $Q_2 \in \overset{\circ}{\Theta}$. Clearly from Condition C2, we have

$$R_1^M = \frac{1}{u}\left(\prod_{j=1}^{n}\frac{\tilde{a}_j}{a_j}\right)\frac{g(x_2, v_2)}{v_2} \leq \frac{1}{u}\left(\prod_{j=1}^{n}\frac{\tilde{a}_j}{a_j}\right)\frac{\partial g(x_2, 0)}{\partial v} < \frac{1}{u}\left(\prod_{j=1}^{n}\frac{\tilde{a}_j}{a_j}\right)\frac{\partial g(x_0, 0)}{\partial v} = R_0^M. \square$$

5 Global stability analysis

In this section, we study the global stability of system (1)-(5) by constructing suitable Lyapunov functionals. The stability of the disease-free steady state Q_0 will be given in the following result.

Theorem 1. Let Conditions C1-C3 hold true and $R_0^M \leq 1$, then Q_0 is globally asymptotically stable (GAS) in Θ.

Proof. Define

$$V_0(x, y_1, ..., y_n, v, z) = x - x_0 - \int_{x_0}^{x}\lim_{v\to 0^+}\frac{g(x_0, v)}{g(\eta, v)}d\eta + \sum_{i=1}^{n}\left(\prod_{j=1}^{i-1}\frac{a_j}{\tilde{a}_j}\right)y_i + \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)v + \frac{p}{r}\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)z, \quad (14)$$

where $\prod_{j=1}^{0}\frac{a_j}{\tilde{a}_j} = 1$. It is seen that, $V_0(x, y_1, ..., y_n, v, z) > 0$ for all $x, y_1, ..., y_n, v, z > 0$, while $V_0(x_0, \overbrace{0, ..., 0, 0}^{n+2}) = 0$. We calculate $\frac{dV_0}{dt}$ along the solutions of model (1)-(5) as:

$$\frac{dV_0}{dt} = \left(1 - \lim_{v\to 0^+}\frac{g(x_0, v)}{g(x, v)}\right)\dot{x} + \sum_{i=1}^{n}\left(\prod_{j=1}^{i-1}\frac{a_j}{\tilde{a}_j}\right)\dot{y}_i + \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)\dot{v} + \frac{p}{r}\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)\dot{z}. \quad (15)$$

We have

$$\begin{aligned}\sum_{i=1}^{n}\left(\prod_{j=1}^{i-1}\frac{a_j}{\tilde{a}_j}\right)\dot{y}_i &= g(x,v)-a_1\phi_1(y_1)+\sum_{i=2}^{n}\left(\prod_{j=1}^{i-1}\frac{a_j}{\tilde{a}_j}\right)\left(\tilde{a}_{i-1}\phi_{i-1}(y_{i-1})-a_i\phi_i(y_i)\right)\\ &= g(x,v)-\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)\tilde{a}_n\phi_n(y_n).\end{aligned}$$

Then

$$\begin{aligned}\frac{dV_0}{dt} &= dx_0\left(1-\lim_{v\to 0^+}\frac{g(x_0,v)}{g(x,v)}\right)\left(1-\frac{x}{x_0}\right)+g(x,v)\lim_{v\to 0^+}\frac{g(x_0,v)}{g(x,v)}-u\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)v-\frac{pb}{r}\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)z\\ &= dx_0\left(1-\frac{\partial g(x_0,0)/\partial v}{\partial g(x,0)/\partial v}\right)\left(1-\frac{x}{x_0}\right)+u\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)\left(\frac{1}{u}\left(\prod_{j=1}^{n}\frac{\tilde{a}_j}{a_j}\right)\frac{g(x,v)}{v}\frac{\partial g(x_0,0)/\partial v}{\partial g(x,0)/\partial v}-1\right)v\\ &\quad -\frac{pb}{r}\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)z. \end{aligned}\tag{16}$$

From (i) of Condition C2, we have

$$\frac{dV_0}{dt}\le dx_0\left(1-\frac{\partial g(x_0,0)/\partial v}{\partial g(x,0)/\partial v}\right)\left(1-\frac{x}{x_0}\right)+u\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)\left(R_0^M-1\right)v-\frac{pb}{r}\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)z. \tag{17}$$

From (ii) of Condition C1, we get

$$\left(1-\frac{\partial g(x_0,0)/\partial v}{\partial g(x,0)/\partial v}\right)\left(1-\frac{x}{x_0}\right)\le 0,$$

where the equality occurs at $x=x_0$. Therefore, if $R_0^M\le 1$, then $\frac{dV_0}{dt}\le 0$ for all $x,v,z>0$. One can easily show that $\frac{dV_0}{dt}=0$ occurs at Q_0. Using LaSalle's invariance principle, we derive that Q_0 is GAS. □

To prove the global stability of the two steady states Q_1 and Q_2, we need the following condition on the incidence rate function.

Condition C4.
$$\left(1-\frac{g(x,v_i)}{g(x,v)}\right)\left(\frac{g(x,v)}{g(x,v_i)}-\frac{v}{v_i}\right)\le 0,\quad x,v>0,\ i=1,2$$

Lemma 2. Suppose that Conditions C1-C4 are satisfied and $R_0^M>1$. Then x_1,x_2,v_1,v_2 exist satisfying

$$sgn(x_2-x_1)=sgn(v_1-v_2)=sgn(R_1^M-1).$$

Proof. From Condition C1, for $x_1,x_2,v_1,v_2>0$, we have

$$(g(x_2,v_2)-g(x_1,v_2))(x_2-x_1)>0, \tag{18}$$

$$\left(g(x_1,v_2)-g(x_1,v_1)\right)(v_2-v_1)>0. \tag{19}$$

Using Condition C4 with $i=1$, $x=x_1$ and $v=v_2$ we get

$$\left(g(x_1,v_2)v_1-g(x_1,v_1)v_2\right)\left(g(x_1,v_2)-g(x_1,v_1)\right)<0. \tag{20}$$

It follows from inequality (19) that

$$(g(x_1,v_2)v_1-g(x_1,v_1)v_2)(v_1-v_2)>0. \tag{21}$$

First, we claim $sgn(x_2-x_1)=sgn(v_1-v_2)$. Suppose this is not true, i.e., $sgn(x_2-x_1)=sgn(v_2-v_1)$. Using the conditions of the steady states Q_1 and Q_2 we have

$$\begin{aligned}(\lambda-dx_2)-(\lambda-dx_1) &= g(x_2,v_2)-g(x_1,v_1)\\ &= (g(x_2,v_2)-g(x_1,v_2))+(g(x_1,v_2)-g(x_1,v_1)).\end{aligned}$$

Therefore, from inequalities (18) and (19) we get:

$$sgn\left(x_1 - x_2\right) = sgn\left(x_2 - x_1\right),$$

which leads to a contradiction. Thus, $sgn\left(x_2 - x_1\right) = sgn\left(v_1 - v_2\right)$. Using the steady state conditions for Q_1 we have $\frac{1}{u}\left(\prod_{j=1}^{n}\frac{\tilde{a}_j}{a_j}\right)\dfrac{g(x_1,v_1)}{v_1} = 1$, then

$$\begin{aligned} R_1^M - 1 &= \frac{1}{u}\left(\prod_{j=1}^{n}\frac{\tilde{a}_j}{a_j}\right)\frac{g(x_2,v_2)}{v_2} - \frac{1}{u}\left(\prod_{j=1}^{n}\frac{\tilde{a}_j}{a_j}\right)\frac{g(x_1,v_1)}{v_1} \\ &= \frac{1}{u}\left(\prod_{j=1}^{n}\frac{\tilde{a}_j}{a_j}\right)\left[\frac{1}{v_2}\left(g(x_2,v_2) - g(x_1,v_2)\right) + \frac{1}{v_1 v_2}\left(g(x_1,v_2)v_1 - g(x_1,v_1)v_2\right)\right]. \end{aligned}$$

Thus, from inequalities (18) and (21) we get $sgn(R_1^M - 1) = sgn(v_1 - v_2)$. □

Theorem 2. Assume that Conditions C1-C4 are satisfied. If $R_1^M \le 1 < R_0^M$, then Q_1 is GAS in Θ.

Proof. Define:

$$\begin{aligned} V_1(x, y_1, ..., y_n, v, z) &= x - x_1 - \int_{x_1}^{x}\frac{g(x_1,v_1)}{g(\eta,v_1)}d\eta + \sum_{i=1}^{n}\left(\prod_{j=1}^{i-1}\frac{a_j}{\tilde{a}_j}\right)\left(y_i - y_{i,1} - \int_{y_{i,1}}^{y_i}\frac{\phi_i(y_{i,1})}{\phi_i(\eta)}d\eta\right) \\ &+ \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)v_1 H\left(\frac{v}{v_1}\right) + \frac{p}{r}\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)z. \end{aligned} \tag{22}$$

We note that, V_1 is positive and reaches its global minimum at Q_1. Calculating the time derivative of V_1 along the trajectories of system (1)-(5), we obtain

$$\begin{aligned} \frac{dV_1}{dt} &= \left(1 - \frac{g(x_1,v_1)}{g(x,v_1)}\right)\left(\lambda - dx - g(x,v)\right) + \left(1 - \frac{\phi_1(y_{1,1})}{\phi_1(y_1)}\right)\left(g(x,v) - a_1\phi_1(y_1)\right) \\ &+ \sum_{i=2}^{n}\left(\prod_{j=1}^{i-1}\frac{a_j}{\tilde{a}_j}\right)\left(1 - \frac{\phi_i(y_{i,1})}{\phi_i(y_i)}\right)\left(\tilde{a}_{i-1}\phi_{i-1}(y_{i-1}) - a_i\phi_i(y_i)\right) \\ &+ \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)\left(1 - \frac{v_1}{v}\right)\left(\tilde{a}_n\phi_n(y_n) - uv - pzv\right) + \frac{p}{r}\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)\left(rzv - bz\right). \end{aligned} \tag{23}$$

We have

$$\sum_{i=2}^{n}\left(\prod_{j=1}^{i-1}\frac{a_j}{\tilde{a}_j}\right)\left(\tilde{a}_{i-1}\phi_{i-1}(y_{i-1}) - a_i\phi_i(y_i)\right) = a_1\phi_1(y_1) - \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)\tilde{a}_n\phi_n(y_n). \tag{24}$$

Then,

$$\begin{aligned} \frac{dV_1}{dt} &= \left(1 - \frac{g(x_1,v_1)}{g(x,v_1)}\right)(\lambda - dx) + g(x,v)\frac{g(x_1,v_1)}{g(x,v_1)} - \frac{\phi_1(y_{1,1})g(x,v)}{\phi_1(y_1)} \\ &+ a_1\phi_1(y_{1,1}) - \sum_{i=2}^{n}\left(\prod_{j=1}^{i-1}\frac{a_j}{\tilde{a}_j}\right)\frac{\tilde{a}_{i-1}\phi_i(y_{i,1})\phi_{i-1}(y_{i-1})}{\phi_i(y_i)} \\ &+ \sum_{i=2}^{n}\left(\prod_{j=1}^{i-1}\frac{a_j}{\tilde{a}_j}\right)a_i\phi_i(y_{i,1}) - \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)uv - \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)\tilde{a}_n\frac{v_1\phi_n(y_n)}{v} \\ &+ \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)uv_1 + \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)pv_1 z - \frac{pb}{r}\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)z. \end{aligned} \tag{25}$$

Using the steady state conditions for Q_1:

$$\lambda = dx_1 + g(x_1, v_1),$$
$$g(x_1, v_1) = \left(\prod_{j=1}^{i} \frac{a_j}{\tilde{a}_j}\right) \tilde{a}_i \phi_i(y_{i,1}) = \left(\prod_{j=1}^{n} \frac{a_j}{\tilde{a}_j}\right) u v_1, \ i = 1, ..., n.$$

We obtain

$$\begin{aligned}\frac{dV_1}{dt} &= \left(1 - \frac{g(x_1, v_1)}{g(x, v_1)}\right)(dx_1 - dx) + g(x_1, v_1)\left(1 - \frac{g(x_1, v_1)}{g(x, v_1)}\right) + g(x_1, v_1)\frac{g(x, v)}{g(x, v_1)} \\ &- g(x_1, v_1)\frac{\phi_1(y_{1,1})g(x, v)}{\phi_1(y_1)g(x_1, v_1)} + (n+1)\, g(x_1, v_1) - g(x_1, v_1)\sum_{i=2}^{n} \frac{\phi_i(y_{i,1})\phi_{i-1}(y_{i-1})}{\phi_i(y_i)\phi_{i-1}(y_{i-1,1})} \\ &- g(x_1, v_1)\frac{v}{v_1} - g(x_1, v_1)\frac{v_1\phi_n(y_n)}{v\phi_n(y_{n,1})} + p\left(\prod_{j=1}^{n} \frac{a_j}{\tilde{a}_j}\right)\left(v_1 - \frac{b}{r}\right) z. \end{aligned} \tag{26}$$

We can rewrite Eq. (26) as follows

$$\begin{aligned}\frac{dV_1}{dt} &= dx_1\left(1 - \frac{g(x_1, v_1)}{g(x, v_1)}\right)\left(1 - \frac{x}{x_1}\right) + g(x_1, v_1)\left[\frac{g(x, v)}{g(x, v_1)} - \frac{v}{v_1}\right] \\ &+ g(x_1, v_1)\left[(n+2) - \frac{g(x_1, v_1)}{g(x, v_1)} - \frac{\phi_1(y_{1,1})g(x, v)}{\phi_1(y_1)g(x_1, v_1)}\right. \\ &\left. - \sum_{i=2}^{n} \frac{\phi_i(y_{i,1})\phi_{i-1}(y_{i-1})}{\phi_i(y_i)\phi_{i-1}(y_{i-1,1})} - \frac{v_1\phi_n(y_n)}{v\phi_n(y_{n,1})}\right] + p\left(\prod_{j=1}^{n} \frac{a_j}{\tilde{a}_j}\right)(v_1 - v_2)\, z \\ &= dx_1\left(1 - \frac{g(x_1, v_1)}{g(x, v_1)}\right)\left(1 - \frac{x}{x_1}\right) + g(x_1, v_1)\left(1 - \frac{g(x, v_1)}{g(x, v)}\right)\left(\frac{g(x, v)}{g(x, v_1)} - \frac{v}{v_1}\right) \\ &+ g(x_1, v_1)\left[(n+3) - \frac{g(x_1, v_1)}{g(x, v_1)} - \frac{\phi_1(y_{1,1})g(x, v)}{\phi_1(y_1)g(x_1, v_1)} - \sum_{i=2}^{n} \frac{\phi_i(y_{i,1})\phi_{i-1}(y_{i-1})}{\phi_i(y_i)\phi_{i-1}(y_{i-1,1})}\right. \\ &\left. - \frac{v_1\phi_n(y_n)}{v\phi_n(y_{n,1})} - \frac{vg(x, v_1)}{v_1 g(x, v)}\right] + p\left(\prod_{j=1}^{n} \frac{a_j}{\tilde{a}_j}\right)(v_1 - v_2)\, z. \end{aligned} \tag{27}$$

From Conditions C1 and C4, we get that, the first and second terms of Eq. (27) are less than or equal to zero. Since the geometrical mean is less than or equal to the arithmetical mean, then $(n+3) \leq \frac{g(x_1,v_1)}{g(x,v_1)} + \frac{\phi_1(y_{1,1})g(x,v)}{\phi_1(y_1)g(x_1,v_1)} + \sum_{i=2}^{n} \frac{\phi_i(y_{i,1})\phi_{i-1}(y_{i-1})}{\phi_i(y_i)\phi_{i-1}(y_{i-1,1})} + \frac{v_1\phi_n(y_n)}{v\phi_n(y_{n,1})} + \frac{vg(x,v_1)}{v_1 g(x,v)}$. Lemma 2 implies that, if $R_1^M \leq 1$, then $v_1 \leq v_2$. It follows that, $\frac{dV_1}{dt} \leq 0$ for all $x, y_i, v, z > 0$, $i = 1, ..., n$. The solutions of system (1)-(5) are limited to Ω, the largest invariant subset of $\left\{(x, y_1, ..., y_n, v, z) : \frac{dV_1}{dt} = 0\right\}$. We have $\frac{dV_1}{dt} = 0$ at the singleton $\{Q_1\}$. Thus, the global asymptotic stability of the endemic steady state without humoral immune response Q_1 follows from LaSalle's invariance principle. □

Theorem 3. Let Conditions C1-C4 are satisfied and $R_1^M > 1$, then Q_2 is GAS in $\overset{\circ}{\Theta}$.

Proof. We construct a Lyapunov functional as follows:

$$\begin{aligned}V_2(x, y_1, ..., y_n, v, z) &= x - x_2 - \int_{x_2}^{x} \frac{g(x_2, v_2)}{g(\eta, v_2)} d\eta + \sum_{i=1}^{n}\left(\prod_{j=1}^{i-1} \frac{a_j}{\tilde{a}_j}\right)\left(y_i - y_{i,2} - \int_{y_{i,2}}^{y_i} \frac{\phi_i(y_{i,2})}{\phi_i(\eta)} d\eta\right) \\ &+ \left(\prod_{j=1}^{n} \frac{a_j}{\tilde{a}_j}\right) v_2 H\left(\frac{v}{v_2}\right) + \frac{p}{r}\left(\prod_{j=1}^{n} \frac{a_j}{\tilde{a}_j}\right) z_2 H\left(\frac{z}{z_2}\right). \end{aligned} \tag{28}$$

Note that $V_2 > 0$ for all $x, y_1, ..., y_n, v, z > 0$ and $V_2(x_2, y_{1,2}, ..., y_{n,2}, v_2, z_2) = 0$. Function V_2 satisfies:

$$\frac{dV_2}{dt} = \left(1 - \frac{g(x_2, v_2)}{g(x, v_2)}\right)(\lambda - dx - g(x, v)) + \left(1 - \frac{\phi_1(y_{1,2})}{\phi_1(y_1)}\right)(g(x, v) - a_1\phi_1(y_1))$$
$$+ \sum_{i=2}^{n}\left(\prod_{j=1}^{i-1}\frac{a_j}{\tilde{a}_j}\right)\left(1 - \frac{\phi_i(y_{i,2})}{\phi_i(y_i)}\right)(\tilde{a}_{i-1}\phi_{i-1}(y_{i-1}) - a_i\phi_i(y_i))$$
$$+ \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)\left(1 - \frac{v_2}{v}\right)(\tilde{a}_n\phi_n(y_n) - uv - pzv) + \frac{p}{r}\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)\left(1 - \frac{z_2}{z}\right)(rzv - bz)\,. \tag{29}$$

Using Eq. (24), we get

$$\frac{dV_2}{dt} = \left(1 - \frac{g(x_2, v_2)}{g(x, v_2)}\right)(\lambda - dx) + g(x, v)\frac{g(x_2, v_2)}{g(x, v_2)} - \frac{\phi_1(y_{1,2})g(x, v)}{\phi_1(y_1)} + a_1\phi_1(y_{1,2})$$
$$- \sum_{i=2}^{n}\left(\prod_{j=1}^{i-1}\frac{a_j}{\tilde{a}_j}\right)\tilde{a}_{i-1}\frac{\phi_i(y_{i,2})\phi_{i-1}(y_{i-1})}{\phi_i(y_i)} + \sum_{i=2}^{n}\left(\prod_{j=1}^{i-1}\frac{a_j}{\tilde{a}_j}\right)a_i\phi_i(y_{i,2})$$
$$- \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)uv - \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)\tilde{a}_n\frac{v_2\phi_n(y_n)}{v} + \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)uv_2$$
$$+ \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)pv_2z - \frac{pb}{r}\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)z - p\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)vz_2 + \frac{pb}{r}\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)z_2. \tag{30}$$

Using the steady state conditions for Q_2:

$$\lambda = dx_2 + g(x_2, v_2), \quad v_2 = \frac{b}{r},$$
$$g(x_2, v_2) = \left(\prod_{j=1}^{i}\frac{a_j}{\tilde{a}_j}\right)\tilde{a}_i\phi_i(y_{i,2}) = \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)(uv_2 + pv_2z_2)\,,\ i = 1, ..., n,$$

we get

$$\frac{dV_2}{dt} = \left(1 - \frac{g(x_2, v_2)}{g(x, v_2)}\right)(dx_2 - dx) + g(x_2, v_2)\left(1 - \frac{g(x_2, v_2)}{g(x, v_2)}\right) + g(x_2, v_2)\frac{g(x, v)}{g(x, v_2)}$$
$$- g(x_2, v_2)\frac{\phi_1(y_{1,2})g(x, v)}{\phi_1(y_1)g(x_2, v_2)} + (n+1)\,g(x_2, v_2) - g(x_2, v_2)\sum_{i=2}^{n}\frac{\phi_i(y_{i,2})\phi_{i-1}(y_{i-1})}{\phi_i(y_i)\phi_{i-1}(y_{i-1,2})}$$
$$- g(x_2, v_2)\frac{v}{v_2} - g(x_2, v_2)\frac{v_2\phi_n(y_n)}{v\phi_n(y_{n,2})}. \tag{31}$$

We can rewrite Eq. (31) as follows:

$$\frac{dV_2}{dt} = dx_2\left(1 - \frac{g(x_2, v_2)}{g(x, v_2)}\right)\left(1 - \frac{x}{x_2}\right) + g(x_2, v_2)\left(1 - \frac{g(x, v_2)}{g(x, v)}\right)\left(\frac{g(x, v)}{g(x, v_2)} - \frac{v}{v_2}\right)$$
$$+ g(x_2, v_2)\left[(n+3) - \frac{g(x_2, v_2)}{g(x, v_2)} - \frac{\phi_1(y_{1,2})g(x, v)}{\phi_1(y_1)g(x_2, v_2)} - \sum_{i=2}^{n}\frac{\phi_i(y_{i,2})\phi_{i-1}(y_{i-1})}{\phi_i(y_i)\phi_{i-1}(y_{i-1,2})}\right.$$
$$\left.- \frac{v_2\phi_n(y_n)}{v\phi_n(y_{n,2})} - \frac{vg(x, v_2)}{v_2g(x, v)}\right]. \tag{32}$$

We note from Conditions C1 and C4 and the relationship between the arithmetical and geometrical means that, we obtain $\frac{dV_2}{dt} \leq 0$ for all $x, y_1, ..., y_n, v, z > 0$. The solutions of model (1)-(5) are limited to Λ, the largest invariant subset of $\left\{(x, y_1, ..., y_n, v, z) : \frac{dV_2}{dt} = 0\right\}$. It is easy to see that $\frac{dV_2}{dt} = 0$ occurs at Q_2. The global asymptotic stability of Q_2 follows from LaSalle's invariance principle. $\square$

6 Example and numerical simulations

In this section, we introduce an example and perform some numerical simulations to confirm our theoretical results. By using the Lyapunov direct method, we have established a set of conditions on the functions $g(x,v)$ and $\phi_i(y_i)$ and on the parameters R_0^M and R_1^M ensuring the global asymptotic stability of the steady states of model (1)-(5). We consider the following model with two stages (i.e. $n=2$):

$$\dot{x} = \lambda - dx - \frac{\pi x v}{(1+\gamma x)(1+\delta v)}, \tag{33}$$

$$\dot{y}_1 = \frac{\pi x v}{(1+\gamma x)(1+\delta v)} - a_1 y_1, \tag{34}$$

$$\dot{y}_2 = \tilde{a}_1 y_1 - a_2 y_2, \tag{35}$$

$$\dot{v} = \tilde{a}_2 y_2 - pzv - uv, \tag{36}$$

$$\dot{z} = rzv - bz, \tag{37}$$

where $\pi \in (0,\infty)$ and $\gamma, \delta \in [0,\infty)$. In this example we have

$$\phi_i(y_i) = y_i, \ i = 1, ..., n, \ g(x,v) = \frac{\pi x v}{(1+\gamma x)(1+\delta v)},$$

which guarantee that Condition C3 holds true. Now, we verify Conditions C1, C2 and C4. Clearly, $g(x,v) > 0$, $g(0,v) = g(x,0) = 0$ for all $x, v \in (0,\infty)$, and

$$\frac{\partial g(x,v)}{\partial x} = \frac{\pi v}{(1+\gamma x)^2(1+\delta v)}, \qquad \frac{\partial g(x,v)}{\partial v} = \frac{\pi x}{(1+\gamma x)(1+\delta v)^2}, \qquad \frac{\partial g(x,0)}{\partial v} = \frac{\pi x}{1+\gamma x}.$$

Then, for all $x, v \in (0,\infty)$, we have $\frac{\partial g(x,v)}{\partial x} > 0$, $\frac{\partial g(x,v)}{\partial v} > 0$ and $\frac{\partial g(x,0)}{\partial v} > 0$. Therefore Condition C1 is satisfied. We have also

$$g(x,v) = \frac{\pi x v}{(1+\gamma x)(1+\delta v)} \leq \frac{\pi x v}{1+\gamma x} = v\frac{\partial g(x,0)}{\partial v},$$

$$\left(\frac{\partial g(x,0)}{\partial v}\right)' = \frac{\pi}{(1+\gamma x)^2} > 0 \text{ for all } x > 0.$$

It follows that, C2 is satisfied. Moreover,

$$\left(1 - \frac{g(x,v_i)}{g(x,v)}\right)\left(\frac{g(x,v)}{g(x,v_i)} - \frac{v}{v_i}\right) = -\frac{\delta(v-v_i)^2}{v_i(1+\delta v)(1+\delta v_i)} < 0 \text{ for all } v, v_i \in (0,\infty),\ i = 1,2.$$

Thus, C4 is satisfied and the global stability results demonstrated in Theorems 1-3 are guaranteed. The parameters R_0^M and R_1^M are given by:

$$R_0^M = \frac{\tilde{a}_1\tilde{a}_2\pi}{a_1 a_2 u}\frac{x_0}{1+\gamma x_0}, \qquad R_1^M = \frac{\tilde{a}_1\tilde{a}_2\pi}{a_1 a_2 u}\frac{x_2}{(1+\gamma x_2)(1+\delta v_2)}. \tag{38}$$

Now, we will perform some numerical simulations for the model (33)-(37). The values of some parameters of the example are listed in Table 1. The other parameters π, r and γ will be varied. All computations are carried out by MATLAB.

We are interested to study the following cases:

Case (A): Effect of π and r on the stability of steady states:

In this case, we have chosen three different initial conditions:

IC(1): $x(0) = 400$, $y_1(0) = y_2(0) = 1$, $v(0) = 0.2$ and $z(0) = 0.5$,

IC(2): $x(0) = 600$, $y_1(0) = y_2(0) = 2$, $v(0) = 0.5$ and $z(0) = 1$,

IC(3): $x(0) = 800$, $y_1(0) = 5$, $y_2(0) = 3$, $v(0) = 0.9$ and $z(0) = 1.5$.

The evolution of the dynamics of model (33)-(37) was observed over a time interval $[0, 500]$. We fix the value of $\gamma = 0.5$ and change the values of parameters π and r to get three sets as follows:

Table 1: The values of the parameters of model (33)-(37).

Parameter	Value	Parameter	Value	Parameter	Value
λ	10	a_1	1	p	0.5
d	0.01	a_2	1.5	r	Varied
β	Varied	$\tilde{a}_1$	0.5	b	0.3
γ	Varied	$\tilde{a}_2$	1		
δ	0.1	u	3		

Set (I): We choose, $\pi = 4$ and $r = 0.3$. Using the values of the parameters given in Table 1, we compute $R_0^M = 0.89 < 1$ and $R_1^M = 0.80 < 1$, which means that the system has a disease-free steady state Q_0 and it is GAS based on Theorem 1. Evidently, Figures 1-5 show that, the states of the system eventually approach $Q_0 = (1000, 0, 0, 0, 0)$ for the three initial conditions IC(1)-IC(3). This case corresponds to the healthy state where the viruses are cleared.

Set (II): We take $\pi = 5$ and $r = 0.3$. With such choice we have, $R_1^M = 0.99 < 1 < R_0^M = 1.11$. Consequently, Lemma 1 and Theorem 2 state that, Q_1 exists and it is GAS. Figures 1-5 show that the numerical simulations illustrate our theoretical results given in Theorem 2. We observe that, the trajectory of the system will converge to $Q_1 = (140.43, 8.60, 2.87, 0.96, 0)$ for the three initial conditions IC(1)-IC(3). This case corresponds to a chronic infection but with inactive immune response.

Set (III): We choose, $\pi = 5$ and $r = 1$. Then, we calculate $R_0^M = 1.11 > 1$ and $R_1^M = 1.08 > 1$, this means that, the system has three steady states Q_0, Q_1 and Q_2. Thus, from Theorem 3, Q_2 is GAS. From Figures 1-5, we observe a consistency between the numerical results and theoretical results of Theorem 3. We observe that, the trajectory of the system show oscillating behavior for a period before reaching $Q_2 = (709.56, 2.90, 0.97, 0.3, 0.45)$, in the same time frame for the three initial conditions IC(1)-IC(3).

Case (B): Effect of γ on the stability of the steady states

Let us consider π and r be fixed. In this case, we take the values of $\pi = 5$ and $r = 1$, and consider different values of γ. Here we take the initial condition as given in IC(1), while the evolution of the dynamics of model (33)-(37) was observed over a time interval $[0, 600]$. Table 2 contains the values of the bifurcation parameters R_0^M and R_1^M with different values of γ of model (33)-(37).

Table 2: The values of the threshold parameters R_0^M and R_1^M with different values of γ of model (33)-(37).

Different values of γ	R_0^M	R_1^M	The equilibria
0.30	1.85	1.79	$Q_2 = (517.67, 4.82, 1.61, 0.3, 4.72)$
0.40	1.39	1.34	$Q_2 = (637.35, 3.63, 1.21, 0.3, 2.06)$
0.54	1.03	0.996	$Q_1 = (763.16, 2.37, 0.79, 0.26, 0)$
0.55	1.01	0.98	$Q_1 = (926.89, 0.73, 0.24, 0.08, 0)$
0.60	0.92	0.90	$Q_0 = (1000, 0, 0, 0, 0)$
0.70	0.79	0.77	$Q_0 = (1000, 0, 0, 0, 0)$

Table 2 and Figures 6-10 show that, when γ is increased, the infection rate is decreased which leads to an increase in the concentration of the uninfected cells and a decrease on the concentrations of the (first/second) stage of infected cells, free viruses and B cells.

Case (C): Effect of the multiple stages of infected cells on the dynamics of virus dynamics:

To show the effect of multiple stages of infected cells on the dynamical behavior of the virus, we consider

the following model with single stage of infected cells and compare it with model (33)-(37):

$$\dot{x} = \lambda - dx - \frac{\pi x v}{(1+\gamma x)(1+\delta v)}, \tag{39}$$

$$\dot{y}_1 = \frac{\pi x v}{(1+\gamma x)(1+\delta v)} - a_1 y_1, \tag{40}$$

$$\dot{v} = \tilde{a}_1 y_1 - pzv - uv, \tag{41}$$

$$\dot{z} = rzv - bz. \tag{42}$$

Consequently, the bifurcation parameters for this system are given by:

$$R_0^{single} = \frac{\tilde{a}_1 \pi}{a_1 u} \frac{x_0}{1+\gamma x_0}, \qquad R_1^{single} = \frac{\tilde{a}_1 \pi}{a_1 u} \frac{x_2}{(1+\gamma x_2)(1+\delta v_2)}. \tag{43}$$

Since $\tilde{a}_i < a_i$, then from Eqs. (38) and (43) we have

$$R_0^M = \frac{\tilde{a}_1 \tilde{a}_2 \pi}{a_1 a_2 u} \frac{x_0}{1+\gamma x_0} < \frac{\tilde{a}_1 \pi}{a_1 u} \frac{x_0}{1+\gamma x_0} = R_0^{single},$$
$$R_1^M = \frac{\tilde{a}_1 \tilde{a}_2 \pi}{a_1 a_2 u} \frac{x_2}{(1+\gamma x_2)(1+\delta v_2)} < \frac{\tilde{a}_1 \pi}{a_1 u} \frac{x_2}{(1+\gamma x_2)(1+\delta v_2)} = R_1^{single}.$$

Here we consider the following initial condition: $x(0) = 400$, $y_1(0) = 0.5$, $y_2(0) = 1$, $v(0) = 0.2$ and $z(0) = 0.5$. The evolution of the dynamics of models (33)-(37) and (39)-(42) was observed over a time interval $[0, 600]$. Let us consider the values of parameters listed in Table 1 and choose the values $\pi = 3.5$, $r = 1.5$ and $\gamma = 0.5$. By calculating the bifurcation parameters for systems (33)-(37) and (39)-(42), we obtain

$$R_0^M = 0.78 < 1.16 = R_0^{single}, \qquad R_1^M = 0.76 < 1.14 = R_1^{single}.$$

Therefore, with the same values of the parameters, the steady state Q_0 is stable for system (33)-(37) but unstable for system (39)-(42). The presence of multiple stages of infected cells reduces the infection progress. Figures 11-14 show a comparison between the evolution of the uninfected cells, infected cells, free virus particles and B cells of the two systems (33)-(37) and (39)-(42). We observe that, the concentration of uninfected cells of the model with three stages of infected cells is larger than that of system with only one single stage of infected cells (see Figures 11), while the concentrations of first stage of infected cells, viruses and B cells with three stages are less than that of system with a single stage of infected cells (see Figures 12-14). From a biological point of view, the multiple stages of infected cells plays a similar role as antiviral treatment in eliminating the virus. We observe that, if the number of stages of infected cells is increased, then the viral replication is suppressed and the viruses can be cleared from the body. This give us some suggestions on new drugs to increase the number of stages of infected cells.

7 Conclusion

We have studied a general virus dynamics model with humoral immunity. We have assumed that the infected cells passes through n-stages to produce mature viruses. We have obtained two bifurcation parameters, the basic reproduction number and the humoral immunity number. We have established a set of sufficient conditions which guarantee the global stability of the model. The global asymptotic stability of the three steady states, Q_0, Q_1 and Q_2 has been investigated by constructing Lyapunov functionals and using LaSalle's invariance principle. To support our theoretical results, we have presented an example and conducted some numerical simulations.

8 Acknowledgment

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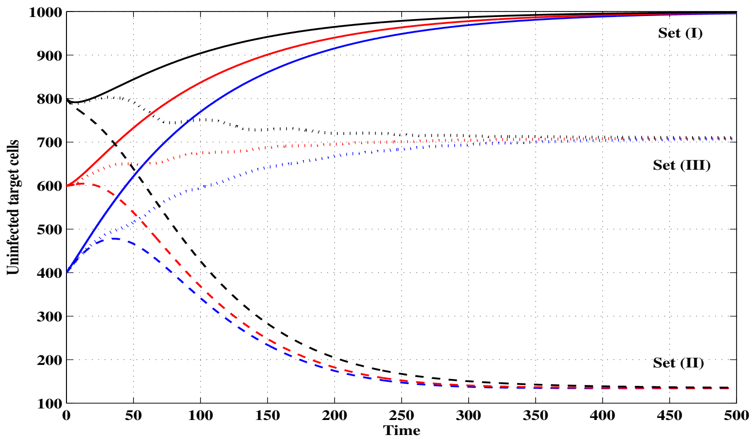

Figure 1: The uninfected cells for model (33)-(37).

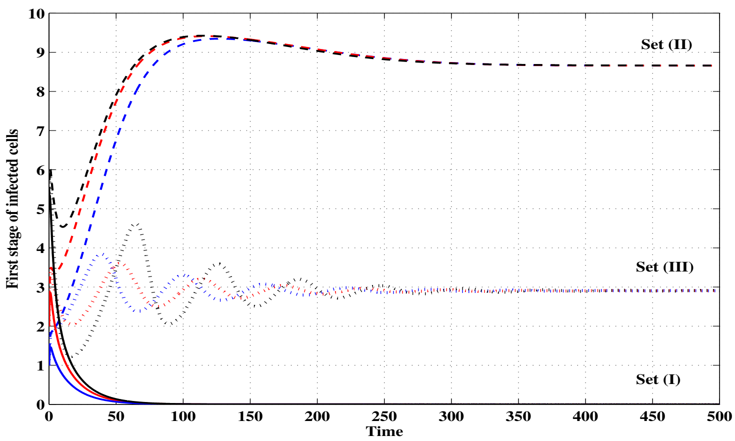

Figure 2: The first stage infected cells for model (33)-(37).

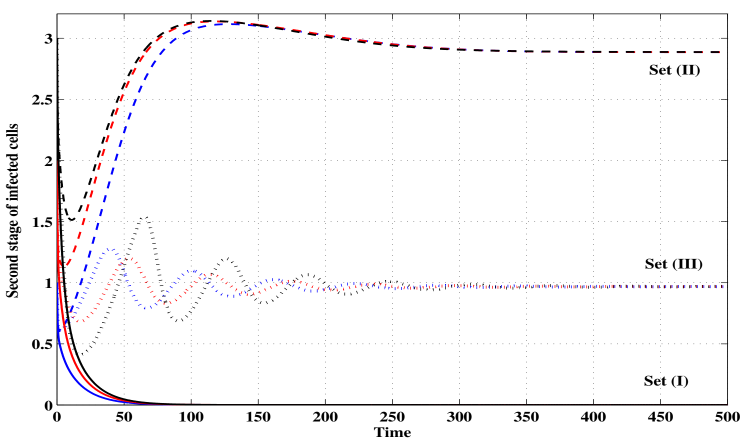

Figure 3: The second stage infected cells for model (33)-(37).

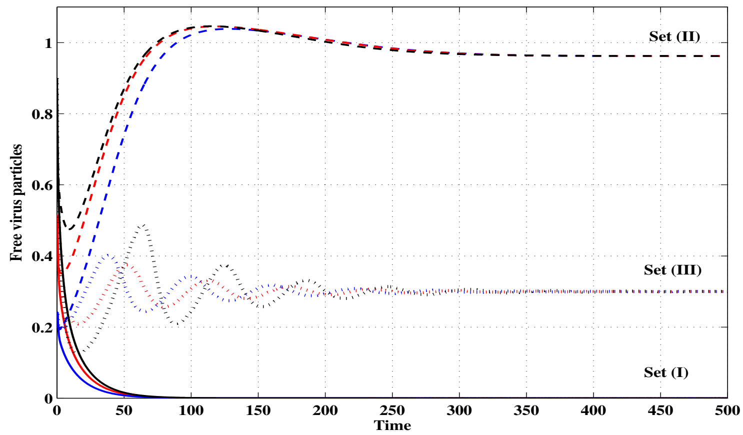

Figure 4: The free virus particles for model (33)-(37).

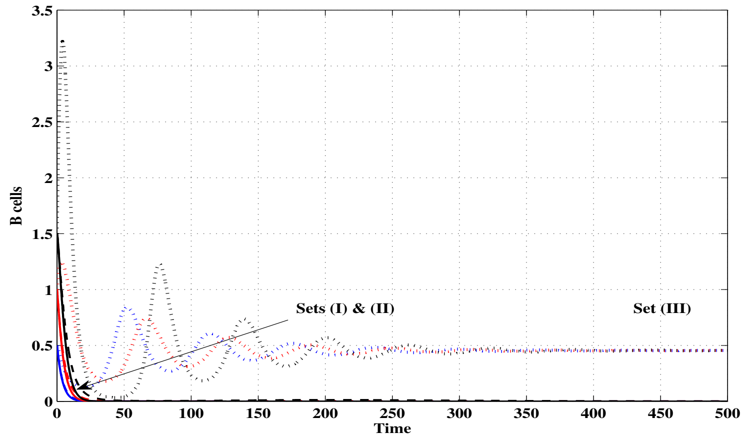

Figure 5: The B cells for model (33)-(37).

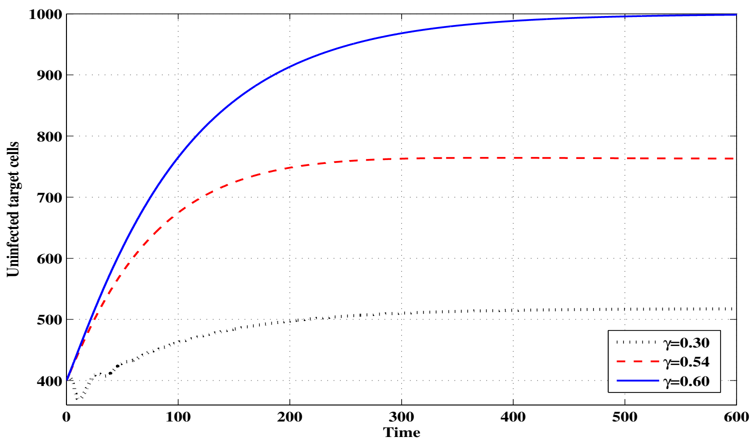

Figure 6: The uninfected target cells for model (33)-(37) under different values of γ.

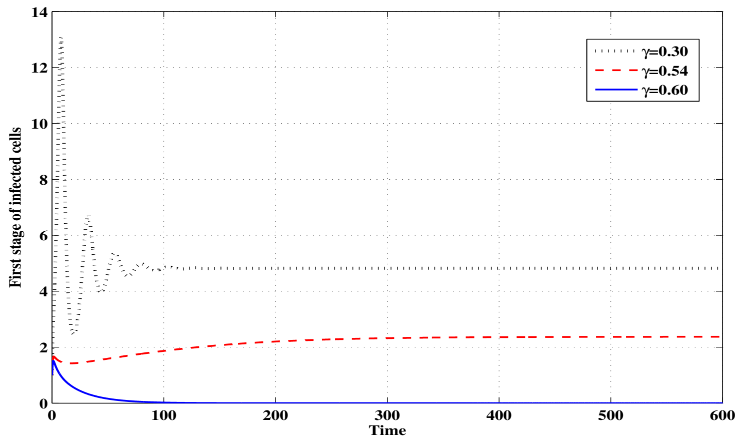

Figure 7: The first stage infected cells for model (33)-(37) under different values of γ.

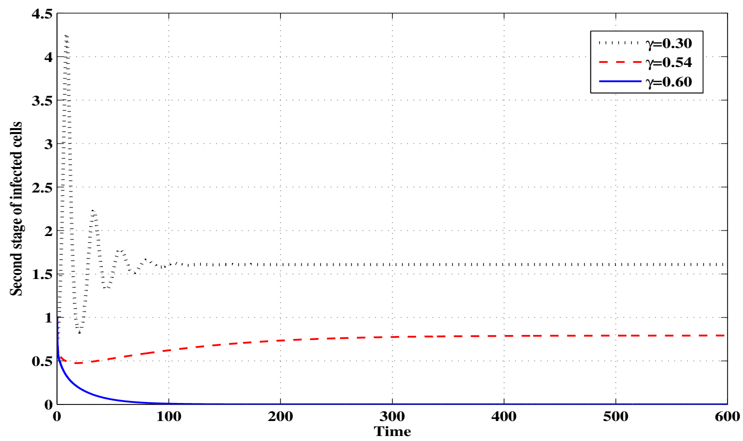

Figure 8: The second stage infected cells for model (33)-(37) under different values of γ.

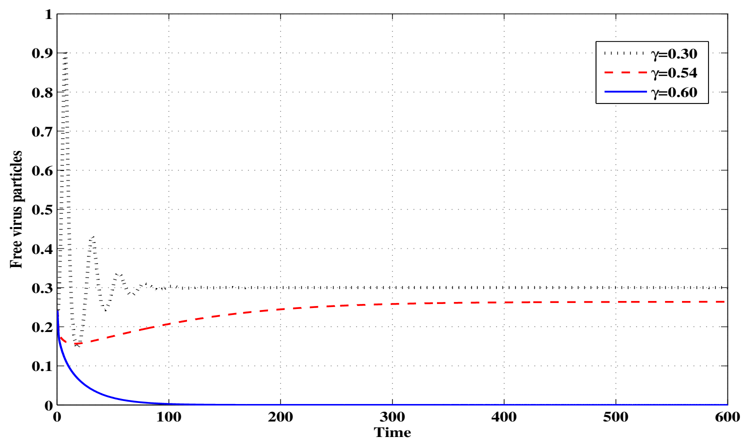

Figure 9: The free virus particles for model (33)-(37) under different values of γ.

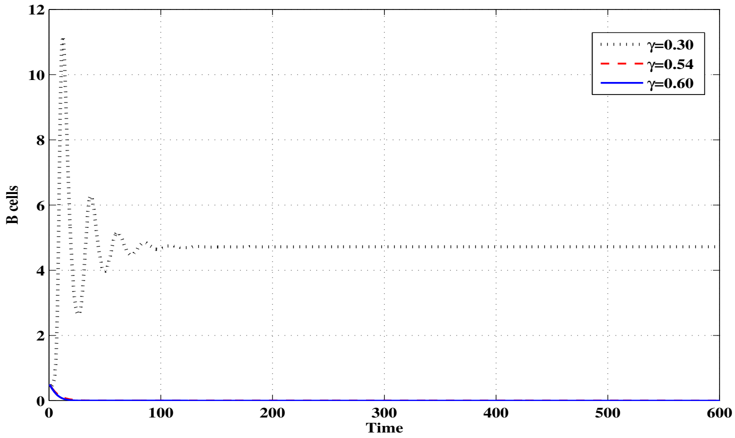

Figure 10: The B cells for model (33)-(37) under different values of γ.

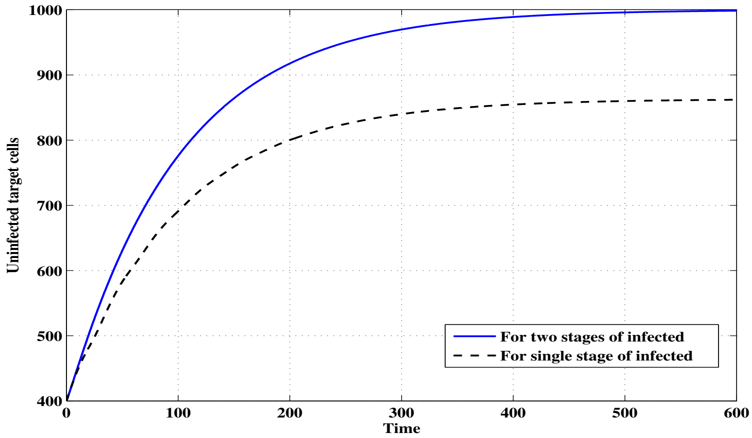

Figure 11: Comparison on the concentration of the uninfected cells for systems (33)-(37) and (39)-(42).

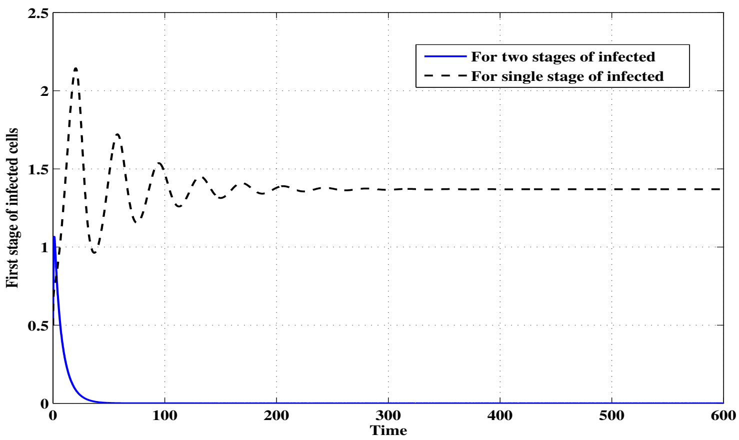

Figure 12: Comparisons on the concentration of the first stage of infected cells for systems (33)-(37) and (39)-(42).

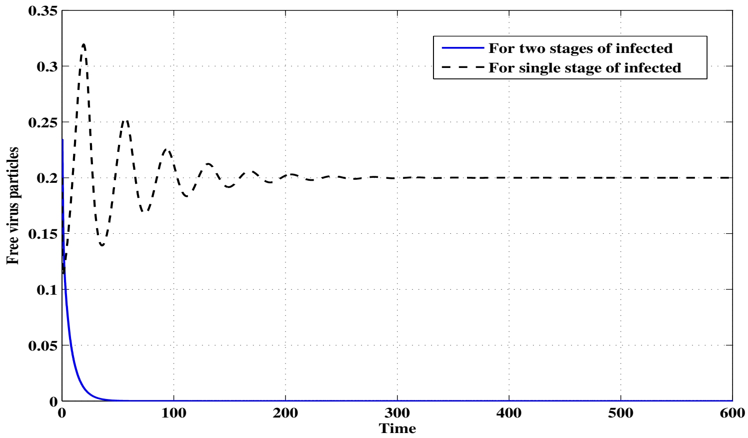

Figure 13: Comparisons on the concentration of the free virus particles for systems (33)-(37) and (39)-(42).

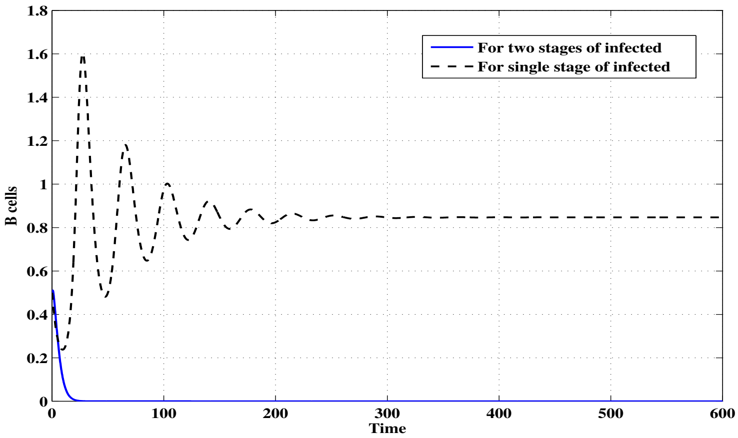

Figure 14: Comparisons on the concentration of the B cells for systems (33)-(37) and (39)-(42).

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On the dynamics of a certain four-order fractional difference equations

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Abstract: This paper is concerned with the following rational recursive sequences

$$x_{n+1}=\frac{x_{n-1}x_{n-2}}{A+By_{n-3}},\ y_{n+1}=\frac{y_{n-1}y_{n-2}}{C+Dx_{n-3}},\ n=0,1,\cdots,$$

where the parameters A, B, C, D are positive constants. The initial condition x_{-3}, x_{-2}, x_{-1}, x_0 and $y_{-3}, y_{-2}, y_{-1}, y_0$ are arbitrary nonnegative real numbers. We give sufficient conditions under which the equilibrium $(0,0)$ of the system is globally asymptotically stable, which extends and includes corresponding results obtained in the cited references [12-17]. Moreover, the asymptotic behavior of others equilibrium points is also studied. Our approach to the problem is based on new variational iteration method for the more general nonlinear difference equations and inequality skills as well as the linearization techniques.

Keywords: recursive sequences; equilibrium point; asymptotical stability; positive solutions.

1. Introduction

Nonlinear Difference equations have been studied because they model numerous real life problems in biology, ecology, physics, economics and so forth [1-5]. Today, with the dramatically development of computer-based computational techniques, difference equations are found to be much appropriate mathematical representations for computer simulation, experiment and computation, which play an important role in realistic applications [6]. Therefore, recently there has been an increasing interest in the study of qualitative analysis of rational difference equations. And the present cardinal problem of

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asymptotic behavior of solutions for a rational difference equation has received extensive attention from researchers (see, e.g., [7-11] and the references therein).

Elabbasy [12] obtained the form of the solutions of the following rational difference system

$$x_{n+1} = \frac{x_{n-1}}{\pm 1 + x_{n-1}y_n}, \quad y_{n+1} = \frac{y_{n-1}}{\mp 1 + y_{n-1}x_n} \tag{1.1}$$

with nonzero real number initial conditions.

In particular, Clark and Kulenovic [13, 14] discussed the global stability properties and asymptotic behavior of solutions for the recursive sequence

$$x_{n+1} = \frac{x_n}{a + cy_n}, \qquad y_{n+1} = \frac{y_n}{b + dx_n}, \qquad n = 0, 1, \cdots, \tag{1.2}$$

where $a, b, c, d \in (0, \infty)$ and the initial conditions x_0 and y_0 are arbitrary nonnegative numbers.

In 2012, Zhang et al. [15] investigated the stability character and asymptotic behavior of the solution for the system of difference equations

$$x_{n+1} = \frac{x_{n-2}}{B + y_{n-2}y_{n-1}y_n}, \quad y_{n+1} = \frac{y_{n-2}}{A + x_{n-2}x_{n-1}x_n}, \quad n = 0, 1, \cdots, \tag{1.3}$$

where $A, B \in (0, \infty)$, and the initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}, y_0 \in (0, \infty)$.

Recently, the following nonlinear two-dimensional difference systems

$$x_{n+1} = \varphi(x_{n-t_1}, y_{n-s_1}), \quad y_{n+1} = \psi(y_{n-s_2}, x_{n-t_2}), \tag{1.4}$$

where t_1, s_1, s_2, t_2 are all positive integers, was studied by Liu et al. [16], in which they gave some sufficient conditions such that every positive solution of this equation converges to the unique equilibrium point.

More recently, in [17] the authors studied analogous results for the system of difference equations

$$x_{n+1} = ax_n + by_{n-1}e^{-x_n}, \quad y_{n+1} = cy_n + dx_{n-1}e^{-y_n}, \tag{1.5}$$

where a, b, c, d are positive constants and the initial values x_1, x_0, y_1, y_0 are positive numbers. For more related work, one can refer to [18-22] and references therein.

Inspired by the above works, the essential problem we consider in this paper is the asymptotic behavior of the solution for the difference equation

$$x_{n+1} = \frac{x_{n-1}x_{n-2}}{A + By_{n-3}}, \quad y_{n+1} = \frac{y_{n-1}y_{n-2}}{C + Dx_{n-3}}, \quad n = 0, 1, \cdots, \tag{1.6}$$

where the initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0 \in (0, \infty)$, $y_{-3}, y_{-2}, y_{-1}, y_0 \in (0, \infty)$ and A, B, C, D are positive constants.

This paper proceeds as follows. In Section 2, we introduce some definitions and preliminary results. The main results and their proofs are given in Section 3.

2. Preliminaries

Let I_x, I_y be some intervals of real numbers and $f: I_x^4 \times I_y^4 \to I_x$, $g: I_x^4 \times I_y^4 \to I_y$ be continuously differentiable functions. Then for every initial conditions $(x_i, y_i) \in I_x \times I_y$, $(i = -3, -2, -1, 0)$, the system of difference equations

$$\begin{cases} x_{n+1} = f(x_n, x_{n-1}, x_{n-2}, x_{n-3}, y_n, y_{n-1}, y_{n-2}, y_{n-3}), \\ y_{n+1} = g(x_n, x_{n-1}, x_{n-2}, x_{n-3}, y_n, y_{n-1}, y_{n-2}, y_{n-3}), \end{cases} \quad n = 0, 1, 2, \cdots, \tag{2.1}$$

has a unique solution $\{(x_n, y_n)\}_{n=-3}^{\infty}$. A point $(\bar{x}, \bar{y}) \in I_x \times I_y$ is called an equilibrium point of (2.1) if $\bar{x} = f(\bar{x}, \bar{x}, \bar{x}, \bar{x}, \bar{y}, \bar{y}, \bar{y}, \bar{y})$, $\bar{y} = g(\bar{x}.\bar{x}, \bar{x}, \bar{x}, \bar{y}, \bar{y}, \bar{y}, \bar{y})$, i. e., $(x_n, y_n) = (\bar{x}, \bar{y})$ for all $n \geq 0$.

Interval $I_x \times I_y$ is called invariant for system (2.1) if, for all $n > 0$, $x_n \in I_x, y_n \in I_y$ when the initial conditions $x_{-3}, x_{-2} x_{-1}, x_0 \in I_x$, $y_{-3}, y_{-2}, y_{-1}, y_0 \in I_y$.

Definition 2.1 Assume that $(\bar{x}, \bar{y})$ is a fixed point of (2.1). Then

(i) $(\bar{x}, \bar{y})$ is said to be stable relative to $I_x \times I_y$ if for every $\varepsilon > 0$, there exits $\delta > 0$ such that for any initial conditions $(x_i, y_i) \in I_x \times I_y$ $(i = -3, -2, -1, 0)$, with $\sum_{i=-3}^{0} |x_i - \bar{x}| < \delta$, $\sum_{i=-3}^{0} |y_i - \bar{y}| < \delta$, implies $|x_n - \bar{x}| < \varepsilon, |y_n - \bar{y}| < \varepsilon$.

(ii) $(\bar{x}, \bar{y})$ is called an attractor relative to $I_x \times I_y$ if for all $(x_i, y_i) \in I_x \times I_y$ $(i = -3, -2, -1, 0)$, $lim_{n\to\infty} x_n = \bar{x}$, $lim_{n\to\infty} y_n = \bar{y}$.

(iii) $(\bar{x}, \bar{y})$ is called asymptotically stable relative to $I_x \times I_y$ if it is stable and an attractor.

(iv) Unstable if it is not stable.

Theorem 2.1 Assume that $X(n+1) = F(X(n))$, $n = 0, 1, \cdots$, is a system of difference equations and $\bar{X}$ is the equilibrium point of this system i.e., $F(\bar{X}) = \bar{X}$.

(i) If all eigenvalues of the Jacobian matrix J_F, evaluated at $\bar{X}$ lie inside the open unit disk $|\lambda| < 1$, then $\bar{X}$ is locally asymptotically stable.

(ii) If all eigenvalues of the Jacobian matrix J_F, evaluated at $\bar{X}$ has modulus greater than one then $\bar{X}$ is unstable.

Definition 2.2 Let p, q, s, t be four nonnegative integers such that $p + q = s + t = n$. Splitting $(x, y) = (x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n)$ into $(x, y) = ([x]_p, [x]_q, [y]_s, [y]_t)$, where $[x]_\sigma$ denotes a vector with σ-components of x, we say that the function $f(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n)$ possesses a mixed monotone property in subsets I^{2n} of R^{2n} if $f([x]_p, [x]_q, [y]_s, [y]_t)$ is monotone nondecreasing in each component of $[x]_p, [y]_s$ and is monotone nonincreasing in each component of $[x]_q, [y]_t$ for $(x, y) \in I^{2n}$. In particular, if $q = t = 0$, then it is said to be monotone nondecreasing

in I^{2n}.

3. The Main Results

In this section, we investigate the asymptotic behavior of the equilibrium points of the systems (1.6). It is easy to know that the systems (1.6) have four equilibrium points $(0,0)$, $(0,C)$, $(A,0)$, and $((A+BC)/(1-BD),(C+AD)/(1-BD))$.

Theorem 3.1 The equilibrium point $(0,0)$ of (1.6) is locally asymptotically stable.

Proof. We can easily obtain that the linearized system of (1.6) about the equilibrium point $(0,0)$ is

$$\varphi_{n+1} = D\varphi_n \tag{3.1}$$

where

$$\varphi_n = \begin{bmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ x_{n-3} \\ y_n \\ y_{n-1} \\ y_{n-2} \\ y_{n-3} \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{3.2}$$

Thus, the characteristic equation of (3.2) is

$$f(\lambda) = \lambda^8 = 0.$$

This shows that all the roots of characteristic equation lie inside unit disk. So the equilibrium（$0,0$）is locally asymptotically stable.

Theorem 3.2 Let

$$\begin{cases} x_{n+1} = f(x_n, x_{n-1}, \cdots, x_{n-k}, y_n, y_{n-1}, \cdots, y_{n-k}), \\ y_{n+1} = g(x_n, x_{n-1}, \cdots, x_{n-k}, y_n, y_{n-1}, \cdots, y_{n-k}), \end{cases} \quad n = 0,1,\cdots, \tag{3.3}$$

$[a,b]$ be an interval of real numbers and assume that $f:[a,b]^{k+1}\times[c,d]^{k+1}\to[a,b]$ and $g:[a,b]^{k+1}\times[c,d]^{k+1}\to[c,d]$ are two continuous functions satisfying the mixed monotone property. If there exit

$$m_0 \le \min\{x_{-k}, x_{-k+1}, \cdots, x_0\} \le \max\{x_{-k}, x_{-k+1}, \cdots, x_0\} \le M_0,$$

and

$$n_0 \le \min\{y_{-k}, y_{-k+1}, \cdots, y_0\} \le \max\{y_{-k}, y_{-k+1}, \cdots, y_0\} \le N_0$$

such that

$$m_0 \le f([m_0]_p,[M_0]_q,[n_0]_s,[N_0]_t) \le f([M_0]_p,[m_0]_q,[N_0]_s,[n_0]_t) \le M_0, \tag{3.4}$$

and

$$n_0 \le g([m_0]_p,[M_0]_q,[n_0]_s,[N_0]_t) \le g([M_0]_p,[m_0]_q,[N_0]_s,[n_0]_t) \le N_0, \tag{3.5}$$

then there exit $(m,M)\in[m_0,M_0]^2$ and $(n,N)\in[n_0,N_0]^2$ satisfying

$$M=f([M]_p,[m]_q,[N]_s,[n]_t),\quad m=f([m]_p,[M]_q,[n]_s,[N]_t),\tag{3.6}$$

and

$$N=g([M]_p,[m]_q,[N]_s,[n]_t),\quad n=g([m]_p,[M]_q,[n]_s,[N]_t).\tag{3.7}$$

Moreover, if $m=M$ and $n=N$, then the system (3.3) has a unique equilibrium point $(\bar{x},\bar{y})\in[m_0,M_0]\times[n_0,N_0]$ and every solution of (3.3) converges to $(\bar{x},\bar{y})$.

Proof. Using m_0,M_0 and n_0,N_0 as two couples of initial iteration, we construct four sequences $\{m_i\},\{M_i\},\{n_i\}$ and $\{N_i\}\ (i=1,2,\cdots)$ from the following equations

$$m_i=f([m_{i-1}]_p,[M_{i-1}]_q,[n_{i-1}]_s,[N_{i-1}]_t),\quad M_i=f([M_{i-1}]_p,[m_{i-1}]_q,[N_{i-1}]_s,[n_{i-1}]_t),$$

and

$$n_i=g([m_{i-1}]_p,[M_{i-1}]_q,[n_{i-1}]_s,[N_{i-1}]_t),\quad N_i=g([M_{i-1}]_p,[m_{i-1}]_q,[N_{i-1}]_s,[n_{i-1}]_t).$$

It is obvious from the mixed monotone property of functions f and g that the sequences $\{m_i\},\{M_i\},\{n_i\}$ and $\{N_i\}\ (i=1,2,\cdots)$ possess the following monotone property

$$m_0\le m_1\le\cdots\le m_i\le\cdots\le M_i\le\cdots\le M_1\le M_0,\tag{3.8}$$

and

$$n_0\le n_1\le\cdots\le n_i\le\cdots\le N_i\le\cdots\le N_1\le N_0,\tag{3.9}$$

where i=0,1,2,$\cdots$.

Moreover, one has

$$m_i\le x_l\le M_i\qquad\text{for }\ l\ge(k+1)i+1,\ i=0,1,2,\cdots.\tag{3.10}$$

and

$$n_i\le y_l\le N_i\qquad\text{for }\ l\ge(k+1)i+1,\ i=0,1,2,\cdots.\tag{3.11}$$

Set

$$m=\lim_{i\to\infty}m_i,\ M=\lim_{i\to\infty}M_i,\ n=\lim_{i\to\infty}n_i,\ N=\lim_{i\to\infty}N_i,\tag{3.12}$$

then

$$m\le\liminf_{i\to\infty}x_i\le\limsup_{i\to\infty}x_i\le M,\quad n\le\liminf_{i\to\infty}y_i\le\limsup_{i\to\infty}y_i\le N.\tag{3.13}$$

By the continuity of f and g, we have

$$M=f([M]_p,[m]_q,[N]_s,[n]_t),\ m=f([m]_p,[M]_q,[n]_s,[N]_t),\tag{3.14}$$

and

$$N=g([M]_p,[m]_q,[N]_s,[n]_t),\ n=g([m]_p,[M]_q,[n]_s,[N]_t).\tag{3.15}$$

Moreover, if $m=M, n=N$, then $m=M=\lim_{i\to\infty}x_i=\bar{x}, n=N=\lim_{i\to\infty}y_i=\bar{y}$, and then the proof is complete.

Theorem 3.3 If $A=C, B=D$, the equilibrium point (0,0) of the systems (1.6) is a global attractor for any initial conditions

$$(x_{-3},x_{-2},x_{-1},x_0,y_{-3},y_{-2},y_{-1},y_0)\in(0,A)^8.$$

5

Proof. Let $(f,g):(0,\infty)^4\times(0,\infty)^4\to(0,\infty)\times(0,\infty)$ be a function defined by

$$f(x_{-3},x_{-2},x_{-1},x_0,y_{-3},y_{-2},y_{-1},y_0)=\frac{x_{n-1}x_{n-2}}{A+By_{n-3}},$$

and

$$g(x_{-3},x_{-2},x_{-1},x_0,y_{-3},y_{-2},y_{-1},y_0)=\frac{y_{n-1}y_{n-2}}{A+Bx_{n-3}}.$$

We can easily see that the functions f and g possess a mixed monotone property in subsets $(0,A)^8$ of R^8.

Let

$$M_0=N_0=\max\{x_{-3},x_{-2},x_{-1},x_0,y_{-3},y_{-2},y_{-1},y_0\},\quad \frac{M_0-A}{B}<n_0=m_0<0.$$

We have

$$m_0\le\frac{m_0^2}{A+BN_0}\le\frac{M_0^2}{A+Bn_0}\le M_0, \tag{3.16}$$

$$n_0\le\frac{n_0^2}{A+BM_0}\le\frac{N_0^2}{A+Bm_0}\le N_0, \tag{3.17}$$

Then from (1.6) and Theorem 3.2, there exit $m,M\in[m_0,M_0]$, $n,N\in[n_0,N_0]$ satisfying

$$m=\frac{m^2}{A+BN},\qquad M=\frac{M^2}{A+Bn}, \tag{3.18}$$

$$n=\frac{n^2}{A+BM},\qquad N=\frac{N^2}{A+Bm}. \tag{3.19}$$

In view of

$$m<M<M_0<A+Bn_0<A+Bn<A+BN,$$

and

$$n<N<N_0<A+Bm_0<A+Bm<A+BM,$$

thus, one has

$$M=m=N=n=0. \tag{3.20}$$

It follows by Theorem 3.2 that the equilibrium point $(0,0)$ of (1.6) is a global attractor. The proof is complete.

Theorem 3.4 The equilibrium point $(0,0)$ of the system (1.6) is global asymptotically stability for any initial conditions

$$(x_{-3},x_{-2},x_{-1},x_0,y_{-3},y_{-2},y_{-1},y_0)\in(0,A)^8.$$

Proof. The result follows from Theorems 3.1 and 3.3.

Theorem 3.5 The equilibrium point $(0,C)$, $(A,0)$ of the system (1.6) is unstable.

Proof. We can easily obtain that the linearized system of the system (1.6) about the equilibrium $(0,C)$ is

$$\varphi_{n+1} = D^{*}\varphi_n , \tag{3.21}$$

where

$$\varphi_n = \begin{bmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ x_{n-3} \\ y_n \\ y_{n-1} \\ y_{n-2} \\ y_{n-3} \end{bmatrix}, \quad D^{*} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -D & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The characteristic equation of the systems (3.20) is

$$P(\lambda)=\lambda^{5}(\lambda^{3} - \lambda - 1). \tag{3.22}$$

It is obvious that $P(1) = -1, P(2) = 160$. It follows by the intermediate value theorem for continuous function that there exists $\lambda > 1$ so that $P(\lambda) = 0$. Therefore, one of the roots of characteristic equation (3.22) lies outside unit disk. According to Theorem 2.1, the equilibrium $(0, C)$ is unstable.

Similarly, we can obtain that the unique equilibrium $(A, 0)$ is unstable.

Theorem 3.6 If $BD < 1$, the equilibrium point $(\bar{x}, \bar{y}) = (\frac{A+BC}{1-BD}, \frac{C+AD}{1-BD})$ is locally asymptotically stable. If $BD > 1$, the equilibrium point $(\bar{x}, \bar{y})$ is unstable.

Proof. We can easily obtain that the linearized system of (1.6) about the equilibrium $(\bar{x}, \bar{y})$ is

$$\varphi_{n+1} = D^{*}\varphi_n , \tag{3.23}$$

where

$$\varphi_n = \begin{bmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ x_{n-3} \\ y_n \\ y_{n-1} \\ y_{n-2} \\ y_{n-3} \end{bmatrix}, \quad D^{*} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -B \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -D & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The characteristic equation of (3.23) is

$$P(\lambda) = \lambda^{8} - BD = 0. \tag{3.24}$$

In view of $BD < 1$, this shows that all the roots of characteristic equation lie inside unit disk, so the unique equilibrium $(\bar{x}, \bar{y})$ is locally asymptotically stable. If $BD > 1$, one of the roots of characteristic equation lie outside unit disk, so the unique equilibrium

$(\bar{x}, \bar{y})$ is unstable.

4. Conclusions

This paper presents the use of a variational iteration method for systems of nonlinear difference equations. This technique is a powerful tool for solving various difference equations and can also be applied to other nonlinear differential equations in mathematical physics. The variational iteration method provides an efficient method to handle the nonlinear structure. We have dealt with the problem of global asymptotic stability analysis for a class of nonlinear difference equations. The general sufficient conditions have been obtained to ensure the existence, uniqueness and global asymptotic stability of the equilibrium point $(0,0)$ for the nonlinear difference equation. These criteria generalize and improve some known results in [12-17]. Moreover, the asymptotic behavior of others equilibrium points is also studied. In addition, the sufficient conditions that we obtained are very simple, which provide flexibility for the application and analysis of nonlinear difference equation.

Remark: Our model and results are different from the existence ones such as those of References [12-17]. In particular, the new variational iteration method can be applied to the models of References [12-17] and the more general nonlinear difference equations. In some sense, we enrich the theoretical results of the difference equations.

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2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

<u>Contribution to a Book</u>

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

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A new result on the almost increasing sequences

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Abstract

In this paper, we have generalized a known theorem on $|\bar{N}, p_n|_k$ summability factors of infinite series to the $\varphi - |A, p_n|_k$ summability by using an almost increasing sequence. This new theorem also includes several new results.

1. INTRODUCTION

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). Let $\sum a_n$ be a given infinite series with partial sums (s_n) and let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ... \tag{1}$$

The series $\sum a_n$ is said to be summable $|A|_k$, $k \geq 1$, if (see [13])

$$\sum_{n=1}^{\infty} n^{k-1} \ \left|\bar{\Delta} A_n(s)\right|^k < \infty, \tag{2}$$

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where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s). \tag{3}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1). \tag{4}$$

The sequence-to-sequence transformation

$$u_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{5}$$

defines the sequence (u_n) of the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [8]). The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} \mid \Delta u_{n-1} \mid^k < \infty, \tag{6}$$

and it is said to be summable $|A, p_n|_k$, $k \geq 1$, if (see [12])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |\bar{\Delta} A_n(s)|^k < \infty, \tag{7}$$

where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s).$$

Let (φ_n) be any sequence of positive real numbers. The series $\sum a_n$ is summable $\varphi - |A, p_n|_k$, $k \geq 1$, if (see [11])

$$\sum_{n=1}^{\infty} \varphi_n^{k-1} |\bar{\Delta} A_n(s)|^k < \infty. \tag{8}$$

If we take $\varphi_n = \frac{P_n}{p_n}$, then $\varphi - |A, p_n|_k$ summability reduces to $|A, p_n|_k$ summability (see [10]). Also, if we take $\varphi_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$, then we get $|\bar{N}, p_n|_k$ summability. If we take $\varphi_n = n$ and $a_{nv} = \frac{p_v}{P_n}$, then we get $|R, p_n|_k$ summability (see [5]). Furthermore, if we take $\varphi_n = n$ and $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then $\varphi - |A, p_n|_k$ summability reduces to $|C, 1|_k$ summability (see [7]).

In [6], Bor has proved the following theorem for $|\bar{N}, p_n|_k$ summability factors of infinite series.

Theorem 1.1. Let (X_n) be an almost increasing sequence and let there be sequences (β_n) and (λ_n) such that

$$| \Delta\lambda_n | \leq \beta_n, \tag{9}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{10}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\beta_n \mid X_n < \infty, \tag{11}$$

$$| \lambda_n | X_n = O(1) \tag{12}$$

and

$$\sum_{v=1}^{n} \frac{| t_v |^k}{v} = O(X_n) \quad as \quad n \to \infty, \tag{13}$$

where (t_n) is the n-th $(C, 1)$ mean of the sequence (na_n). Suppose further, the sequence (p_n) is such that

$$P_n = O(np_n), \tag{14}$$

$$P_n \Delta p_n = O(p_n p_{n+1}), \tag{15}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{np_n}$ is summable $| \bar{N}, p_n |_k$, $k \geq 1$.

Remark 1.2. It should be noted that, from the hypotheses of the Theorem 1.1, (λ_n) is bounded and $\Delta\lambda_n = O(1/n)$ (see [3]).

2. THE MAIN RESULT

The aim of this paper is to generalize Theorem 1.1 for absolute matrix summability.

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Before stating the main theorem we must first introduce some further notations. Given a normal matrix $A = (a_{nv})$, we associate two lover semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ... \tag{16}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ... \tag{17}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v \tag{18}$$

and

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{19}$$

Now, we shall prove the following theorem.

Theorem 2.1. Let $A = (a_{nv})$ be a positive normal matrix such that

$$\bar{a}_{no} = 1, \ n = 0, 1, ..., \tag{20}$$

$$a_{n-1,v} \geq a_{nv}, \ for \ n \geq v + 1, \tag{21}$$

$$a_{nn} = O(\frac{p_n}{P_n}), \tag{22}$$

$$\mid \widehat{a}_{n,v+1} \mid = O(v \mid \Delta_v(\widehat{a}_{nv}) \mid) \tag{23}$$

Let (X_n) be an almost increasing sequence and $(\frac{\varphi_n p_n}{P_n})$ be a non-increasing sequence. If conditions (9)-(15) of the Theorem 1.1 and

$$\sum_{n=1}^{m} \varphi_n^{k-1} (\frac{p_n}{P_n})^k |t_n|^k = O(X_m) \quad as \quad m \to \infty \tag{24}$$

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are satisfied, then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{n p_n}$ is summable $\varphi - |A, p_n|_k$, $k \geq 1$.

We need the following lemmas for the proof of our theorem.

Lemma 2.2. **([9])** If (X_n) an almost increasing sequence, then under the conditions (10)-(11) we have that

$$nX_n\beta_n = O(1), \tag{25}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{26}$$

Lemma 2.3. **([4])** If the conditions (14) and (15) are satisfied, then $\Delta(P_n/p_n n^2) = O(1/n^2)$.

3. PROOF OF THEOREM 2.1

Let (T_n) denotes A-transform of the series $\sum_{n=1}^{\infty} \frac{a_n P_n \lambda_n}{n p_n}$. Then we have by (18) and (19)

$$\bar{\Delta} T_n \;=\; \sum_{v=1}^{n} \hat{a}_{nv} \frac{a_v P_v \lambda_v}{v p_v}.$$

Applying Abel's transformation to this sum, we get that

$$\begin{aligned}
\bar{\Delta} T_n &= \sum_{v=1}^{n} \hat{a}_{nv} \frac{v a_v P_v \lambda_v}{v^2 p_v} \\
&= \sum_{v=1}^{n-1} \Delta_v \Big(\frac{\hat{a}_{nv} P_v \lambda_v}{v^2 p_v}\Big) \sum_{r=1}^{v} r a_r + \frac{\hat{a}_{nn} P_n \lambda_n}{n^2 p_n} \sum_{r=1}^{n} r a_r \\
&= \sum_{v=1}^{n-1} \Delta_v \Big(\frac{\hat{a}_{nv} P_v \lambda_v}{v^2 p_v}\Big)(v+1) t_v + \frac{a_{nn} P_n \lambda_n}{n^2 p_n}(n+1) t_n \\
&= \frac{a_{nn} P_n \lambda_n}{n^2 p_n}(n+1) t_n + \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{nv}) \frac{(v+1)}{v^2} \frac{P_v \lambda_v}{p_v} t_v \\
&\quad + \sum_{v=1}^{n-1} \frac{\hat{a}_{n,v+1} P_v}{p_v} \Delta\lambda_v t_v \frac{(v+1)}{v^2} + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_{v+1} \Delta\Big(\frac{P_v}{v^2 p_v}\Big) t_v (v+1) \\
&= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \quad say.
\end{aligned}$$

Since

$$|T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}|^k \leq 4^k (|T_{n,1}|^k + |T_{n,2}|^k + |T_{n,3}|^k + |T_{n,4}|^k)$$

to complete the proof of Theorem 2.1, it is sufficient to show that

$$\sum_{n=1}^{m} \varphi_n^{k-1} \mid T_{n,r} \mid^k < \infty, \quad for \quad r = 1, 2, 3, 4. \tag{27}$$

Firstly, by using Abel's transformation, we have that

$$\begin{aligned}
\sum_{n=1}^{m} \varphi_n^{k-1} \mid T_{n,1} \mid^k &= O(1) \sum_{n=1}^{m} \varphi_n^{k-1} a_{nn}^k (\frac{P_n}{p_n})^k |\lambda_n|^k \frac{|t_n|^k}{n^k} \\
&= O(1) \sum_{n=1}^{m} \varphi_n^{k-1} (\frac{p_n}{P_n})^k |\lambda_n|^{k-1} |\lambda_n| |t_n|^k \\
&= O(1) \sum_{n=1}^{m} \varphi_n^{k-1} (\frac{p_n}{P_n})^k |\lambda_n| |t_n|^k \\
&= O(1) \sum_{n=1}^{m-1} \Delta |\lambda_n| \sum_{v=1}^{n} \varphi_v^{k-1} (\frac{p_v}{P_v})^k |t_v|^k + O(1) |\lambda_m| \sum_{n=1}^{m} \varphi_n^{k-1} (\frac{p_n}{P_n})^k |t_n|^k \\
&= O(1) \sum_{n=1}^{m-1} |\Delta \lambda_n| X_n + O(1) |\lambda_m| X_m \\
&= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1) |\lambda_m| X_m \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 2.1 and Lemma 2.2.

Now, using the fact that $P_v = O(vp_v)$ by (14), we have that

$$\sum_{n=1}^{m} \varphi_n^{k-1} \mid T_{n,2} \mid^k \quad = \quad O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\sum_{v=1}^{n-1} |\Delta_v (\hat{a}_{nv})| \, |\lambda_v| \, |t_v| \right)^k$$

Now, applying Hölder's inequality with indices k and k', where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$, as in $T_{n,1}$, we have that

$$\begin{aligned}
\sum_{n=1}^{m} \varphi_n^{k-1} \mid T_{n,2} \mid^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\sum_{v=1}^{n-1} |\Delta_v (\hat{a}_{nv})| \, |\lambda_v|^k \, |t_v|^k \right) \\
&\quad \times (\sum_{v=1}^{n-1} |\Delta_v (\hat{a}_{nv})|)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} a_{nn}^{k-1} \left(\sum_{v=1}^{n-1} |\Delta_v (\hat{a}_{nv})| \, |\lambda_v|^k \, |t_v|^k \right) \\
&= O(1) \sum_{n=2}^{m+1} (\frac{\varphi_n p_n}{P_n})^{k-1} \left(\sum_{v=1}^{n-1} |\Delta_v (\hat{a}_{nv})| \, |\lambda_v|^k \, |t_v|^k \right)
\end{aligned}$$

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$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m}(\frac{\varphi_v p_v}{P_v})^{k-1}|\lambda_v|^k|t_v|^k\sum_{n=v+1}^{m+1}|\Delta_v(\hat{a}_{nv})| \\
&= O(1)\sum_{v=1}^{m}(\frac{\varphi_v p_v}{P_v})^{k-1}|\lambda_v|^{k-1}|\lambda_v||t_v|^k a_{vv} \\
&= O(1)\sum_{v=1}^{m}\varphi_v^{k-1}(\frac{p_v}{P_v})^k\,|\lambda_v|\,|t_v|^k \\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 2.1 and Lemma 2.2.

Now, using Hölder's inequality we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1}\varphi_n^{k-1}\mid T_{n,3}\mid^k &= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v||t_v|\right)^k \\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\beta_v|t_v|^k\right)\times\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\beta_v\right)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}a_{nn}^{k-1}(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\beta_v|t_v|^k) \\
&= O(1)\sum_{n=2}^{m+1}(\frac{\varphi_n p_n}{P_n})^{k-1}(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\beta_v|t_v|^k) \\
&= O(1)\sum_{v=1}^{m}\beta_v|t_v|^k\sum_{n=v+1}^{m+1}(\frac{\varphi_n p_n}{P_n})^{k-1}|\hat{a}_{n,v+1}| \\
&= O(1)\sum_{v=1}^{m}(\frac{\varphi_v p_v}{P_v})^{k-1}\beta_v|t_v|^k\sum_{n=v+1}^{m+1}|\hat{a}_{n,v+1}| \\
&= O(1)\sum_{v=1}^{m}\varphi_v^{k-1}(\frac{p_v}{P_v})^k v\beta_v|t_v|^k \\
&= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}\varphi_r^{k-1}(\frac{p_r}{P_r})^k|t_r|^k+O(1)m\beta_m\sum_{v=1}^{m}\varphi_v^{k-1}(\frac{p_v}{P_v})^k|t_v|^k \\
&= O(1)\sum_{v=1}^{m-1}|\Delta(v\beta_v)|X_v+O(1)m\beta_m X_m \\
&= O(1)\sum_{v=1}^{m-1}v|\Delta\beta_v|X_v+O(1)\sum_{v=1}^{m-1}\beta_{v+1}X_{v+1}+O(1)m\beta_m X_m \\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 2.1 and Lemma 2.2.

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Finally, since $\Delta(\frac{P_v}{v^2 p_v}) = O(\frac{1}{v^2})$, as in $T_{n,1}$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{k-1} |T_n(4)|^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}||\lambda_{v+1}| \frac{|t_v|}{v} \right)^k \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}||\lambda_{v+1}|^k \frac{|t_v|^k}{v} \right) \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| \frac{1}{v} \right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}||\lambda_{v+1}|^k \frac{|t_v|^k}{v} \right) \left(\sum_{v=1}^{n-1} |\Delta_v \hat{a}_{nv}| \right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} a_{nn}^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}||\lambda_{v+1}|^{k-1} |\lambda_{v+1}| \frac{|t_v|^k}{v} \right) \\
&= O(1) \sum_{n=2}^{m+1} (\frac{\varphi_n p_n}{P_n})^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}||\lambda_{v+1}| \frac{|t_v|^k}{v} \right) \\
&= O(1) \sum_{v=1}^{m} |\lambda_{v+1}| \frac{|t_v|^k}{v} \sum_{n=v+1}^{m+1} (\frac{\varphi_n p_n}{P_n})^{k-1} |\hat{a}_{n,v+1}| \\
&= O(1) \sum_{v=1}^{m} (\frac{\varphi_v p_v}{P_v})^{k-1} |\lambda_{v+1}| \frac{|t_v|^k}{v} \sum_{n=v+1}^{m+1} |\hat{a}_{n,v+1}| \\
&= O(1) \sum_{v=1}^{m} (\frac{\varphi_v p_v}{P_v})^{k-1} |\lambda_{v+1}| \frac{|t_v|^k}{v} \\
&= O(1) \sum_{v=1}^{m} \varphi_v^{k-1} (\frac{p_v}{P_v})^k |\lambda_{v+1}||t_v|^k \\
&= O(1) \quad as \quad m \to \infty.
\end{aligned}$$

by virtue of hypotheses of Theorem 2.1 and Lemma 2.3

Therefore we get

$$\sum_{n=1}^{m} \varphi_n^{k-1} \mid T_{n,r} \mid^k = O(1) \quad as \quad m \to \infty, \quad for \quad r = 1, 2, 3, 4.$$

This completes the proof of Theorem 2.1

Corollary 3.1. If we take $\varphi_n = \frac{P_n}{p_n}$, then we get a theorem dealing with $|A, p_n|_k$ summability.

Corollary 3.2. If we take $\varphi_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$, then we get Theorem 1.1.

Corollary 3.3. If we take $a_{nv} = \frac{p_v}{P_n}$, then we have another a result dealing with $\varphi - |\bar{N}, p_n|_k$ summability.

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Corollary 3.4. If we take $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then we get a result dealing with $\varphi - |C, 1|_k$ summability.

Corollary 3.5. If we take $\varphi_n = n$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then we get a result for $|C, 1|_k$ summability.

Corollary 3.6. If we take $k = 1$ and $a_{nv} = \frac{p_v}{P_n}$, then we get a result for $\left|\bar{N}, p_n\right|$ summability and in this case the condition "$\left(\frac{\varphi_n p_n}{P_n}\right)$ is a non-increasing sequence" is not needed.

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Certain Chebyshev type inequalities involving the generalized fractional integral operator

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Abstract: In this paper, we establish certain new Chebyshev type fractional integral inequalities involving the Gauss hypergeometric function. Several special cases as Chebyshev type fractional integral inequalities involving Saigo, Erdélyi-Kober, and Riemann-Liouville type fractional integral operators are presented. Furthermore, we also consider their relevance with other related known results. An example is also given to show the applications of obtained results.

Keywords: Chebyshev type inequalities; fractional integral inequalities; hypergeometric fractional integrals; synchronous (asynchronous) functions

2010 Mathematics Subject Classification: 26D10; 26A33; 33C05

1 Introduction and preliminaries

Due to the fact that the tools of fractional integral inequalities have many applications in establishing uniqueness of solutions in fractional boundary value problems and in fractional partial differential equations, fractional integral inequalities involving the fractional operators (like Saigo, Erdélyi-Kober, Riemann-Liouville type fractional integral operators, etc.) has gained considerable attention, attracting the interest of several researchers. For some recent developments on fractional integral inequalities, we refer the reader to [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] and the references cited therein. Belarbi and Dahmani [13] gave the following integral inequality, using the Riemann-Liouville fractional integrals: if f and g are two synchronous functions (see Definition 1.4) on $C[0,\infty)$, then

$$J^{\alpha}(fg)(t) \geq \frac{\Gamma(\alpha+1)}{t^{\alpha}} J^{\alpha} f(t) J^{\alpha} g(t), \tag{1.1}$$

and

$$\frac{t^{\alpha}}{\Gamma(\alpha+1)} J^{\beta}(fg)(t) + \frac{t^{\beta}}{\Gamma(\beta+1)} J^{\alpha}(fg)(t) \geq J^{\alpha} f(t) J^{\beta} g(t) + J^{\beta} f(t) J^{\alpha} g(t), \tag{1.2}$$

for all $t>0$, $\alpha>0$, and $\beta>0$. Öğünmez and Özkan [14], Chinchane and Pachpatte [15] and Purohit and Raina [16] obtained the Riemann-Liouville fractional q-integral inequalities, the Hadamard fractional integral inequalities and the Saigo fractional integral and q-integral inequalities similar to the inequalities (1.1) and (1.2), respectively.

Dahmani in [17] established the following fractional integral inequalities which are generalizations of the inequalities (1.1) and (1.2), by using the Riemann-Liouville fractional integrals. Let f and g be two synchronous functions on $[0,\infty)$ and let $u,v:[0,\infty)\to[0,\infty)$. Then

$$J^{\alpha}u(t)J^{\alpha}(vfg)(t) + J^{\alpha}v(t)J^{\alpha}(ufg)(t) \geq J^{\alpha}(uf)(t)J^{\alpha}(vg)(t) + J^{\alpha}(vf)(t)J^{\alpha}(ug)(t), \tag{1.3}$$

and

$$J^{\alpha}u(t)J^{\beta}(vfg)(t) + J^{\beta}v(t)J^{\alpha}(ufg)(t) \geq J^{\alpha}(uf)(t)J^{\beta}(vg)(t) + J^{\beta}(vf)(t)J^{\alpha}(ug)(t), \tag{1.4}$$

for all $t>0$, $\alpha>0$ and $\beta>0$. Yang [18], Brahim and Taf [19] and Chinchane and Pachpatte [20] and Agarwal *et al.* [21] gave the fractional q-integral inequalities, the fractional integral inequalities with two parameters of deformation q_1 and q_2, the Hadamard fractional integral inequalities and generalized Erdélyi-Kober fractional q-integral inequalities similar to inequalities (1.3) and (1.4), respectively.

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Let us consider the celebrated Chebyshev functional (see [22])

$$T(f,g)=\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{b-a}\int_a^b f(x)dx\cdot\frac{1}{b-a}\int_a^b g(x)dx$$

where f and g are two integrable functions on $[a,b]$. In [23], Grüss proved the well known inequality:

$$|T(f,g)|\leq\frac{1}{4}(\Phi-\phi)(\Psi-\psi), \tag{1.5}$$

where f and g are two integrable functions on $[a,b]$ satisfying the conditions

$$\phi\leq f(x)\leq\Phi,\quad \psi\leq g(x)\leq\Psi,\quad \phi,\Phi,\psi,\Psi\in\mathbb{R},\quad x\in[a,b]. \tag{1.6}$$

The inequality (1.5) is known as Grüss inequality. By using the Riemann-Liouville fractional integral and q-integral operators, Dahmani *et al.* [26] and Zhu *et al.* [27] gave the fractional integral and q-integral inequality similar to inequality (1.5) satisfying the conditions (1.6), respectively. Wang *et al.* [29] and Baleanu [30] *et al.* obtained some q-integral inequality of Grüss type for the Saigo fractional q-integral operator, respectively.

Throughout the present paper, we shall investigate a fractional integral over the space C_λ introduced in [31] and defined as follows.

Definition 1.1. For each real number λ, let C_λ define the space of all functions $f:\mathbb{R}^+\to\mathbb{R}$ that can be represented in the form $f(x)=x^p f_1(x)$ with $p>\lambda$ and $f_1\in C[0,\infty)$, where $C[0,\infty)$ denotes the set of all continuous real functions defined in $[0,\infty)$.

We give the generalized fractional integral operator $K_t^{\alpha,\beta,\eta,\mu}$ associated with the Gauss hypergeometric function as follows.

Definition 1.2. *[28]* Consider $\lambda\in\mathbb{R}$ and $f\in C_\lambda$. For $\alpha>\max\{0,-(\mu+\eta+1)\}$, $\beta<1$, $\mu>-1$ and $\beta-1<\eta<0$, we define the fractional integral

$$K_t^{\alpha,\beta,\eta,\mu}f(x)=\frac{\Gamma(1-\beta)\Gamma(\alpha+\mu+\eta+1)}{\Gamma(\eta-\beta+1)\Gamma(\mu+1)}x^{\beta+\mu}I_t^{\alpha,\beta,\eta,\mu}\{f(x)\}, \tag{1.7}$$

where $I_t^{\alpha,\beta,\eta,\mu}$ is the Gauss hypergeometric fractional integral of order α and is defined in the following.

Definition 1.3. Let $\alpha>0$, $\mu>-1$, $\beta,\eta\in\mathbb{R}$. Then the generalized fractional integral $I_t^{\alpha,\beta,\eta,\mu}$ (in terms of the Gauss hypergeometric function) of order α for real-valued continuous function $f(t)$ is defined by [31] (see also [32])

$$I_t^{\alpha,\beta,\eta,\mu}\{f(x)\}=\frac{x^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)}\int_0^x t^\mu(x-t)^{\alpha-1}{}_2F_1\left(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{t}{x}\right)f(t)dt, \tag{1.8}$$

where the function ${}_2F_1(\cdot)$ appearing as a kernel for the operator (1.7) is the Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;t)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{t^n}{n!},$$

and $(a)_n$ is the Pochhammer symbol defined by

$$(a)_0=1;\quad (a)_n=a(a+1)\cdots(a+n-1),\quad \text{for}\quad n\in\mathbb{N}.$$

Here $\mathbb{N}$ denotes the set of positive integers.

The above integral (1.8) has the following commutative property(see also [32, 33]):

$$I_t^{\alpha,\beta,\eta,\mu}I_t^{\gamma,\delta,\zeta,\nu}f(x)=I_t^{\gamma,\delta,\zeta,\nu}I_t^{\alpha,\beta,\eta,\mu}f(x).$$

Definition 1.4. Two functions f and g are said to be synchronous (asynchronous) functions on $[0,\infty)$ if

$$A(u,v)=(f(u)-f(v))(g(u)-g(v))\geq(\leq)0,\quad u,v\in[0,\infty).$$

In [31], Baleanu *et al.* obtained the following fractional integral inequalities involving the Gauss hypergeometric function: Let f and g be two synchronous functions on $[0,\infty)$. Then

$$I_t^{\alpha,\beta,\eta,\mu}\{f(t)g(t)\} \geq \frac{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)t^{\beta+\mu}}{\Gamma(1+\mu)\Gamma(1-\beta+\eta)} I_t^{\alpha,\beta,\eta,\mu}\{f(t)\} I_t^{\alpha,\beta,\eta,\mu}\{g(t)\},$$

for all $t>0$, where α,β,η,μ are real constants satisfying $\alpha > \max\{0,-\beta,-\mu\}$, $\beta<1$, $\mu>-1$ and $\beta-1<\eta<0$, and also

$$\frac{\Gamma(1+\mu)\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)t^{\beta+\mu}} I_t^{\gamma,\delta,\zeta,\nu}\{f(t)g(t)\} + \frac{\Gamma(1+\nu)\Gamma(1-\delta+\zeta)}{\Gamma(1-\delta)\Gamma(1+\nu+\gamma+\zeta)t^{\delta+\nu}} I_t^{\alpha,\beta,\eta,\mu}\{f(t)g(t)\}$$
$$\geq I_t^{\alpha,\beta,\eta,\mu}\{f(t)\} I_t^{\gamma,\delta,\zeta,\nu}\{g(t)\} + I_t^{\gamma,\delta,\zeta,\nu}\{f(t)\} I_t^{\alpha,\beta,\eta,\mu}\{g(t)\},$$

for all $t>0$, where α,β,η,μ satisfies the above inequalities and the constants γ,δ,ζ,ν satisfies $\gamma>\max\{0,-\delta,-\nu\}$, $\delta<1$, $\nu>-1$, $\delta-1<\zeta<0$.

In [28], Wang *et al.* gave the following integral inequalities by using the generalized fractional integral operator: Let f and g be two integrable functions with $f,g\in C_\lambda$ and satisfying the condition (1.6) on $[0,\infty)$. Thus we have

$$|K_t^{\alpha,\beta,\eta,\mu}(fg)(x) - K_t^{\alpha,\beta,\eta,\mu}f(x)K_t^{\alpha,\beta,\eta,\mu}g(x)| \leq \frac{1}{4}(\Phi-\phi)(\Psi-\psi),$$

for all $x\in[0,\infty)$, where α,β,η,μ are real constants with $\alpha>0$, $\mu>-1$, $\eta\leq 0$ and $\alpha+\beta+\mu\geq 0$. And Let f and g be two synchronous functions on $[0,\infty)$. Then the following inequality holds:

$$K_t^{\alpha,\beta,\eta,\mu}(fg)(x) \geq K_t^{\alpha,\beta,\eta,\mu}f(x)K_t^{\alpha,\beta,\eta,\mu}g(x),$$

for all $x\in[0,\infty)$, where α,β,η,μ are real constants such that $\alpha>0$, $\mu>-1$, $\eta\leq 0$ and $\alpha+\beta+\mu\geq 0$.

Motivated by the results mentioned above and using the generalized fractional integral operator, we establish certain new Chebyshev type fractional integral inequalities and some related inequalities. Furthermore, several special cases as Chebyshev type fractional integral inequalities involving Saigo, Erdélyi-Kober, and Riemann-Liouville type fractional integral operators are given. Then we present an example to show the applications of obtained results. At last, concluding remarks are also given.

2 Generalized fractional integral inequalities

In this section, we establish certain new Chebyshev type fractional integral inequalities and some related inequalities involving the generalized fractional integral operator.

For the sake of simplicity, we always assume that $K_t^{\alpha,\beta,\eta,\mu}u$ denotes $K_t^{\alpha,\beta,\eta,\mu}u(x)$ and all of the generalized fractional integral operator holds in this work.

Lemma 2.1. *Let f and g be two synchronous functions on $[0,\infty)$ and let u and v be two nonnegative functions on $[0,\infty)$. Then we have*

$$K_t^{\alpha,\beta,\eta,\mu}uK_t^{\alpha,\beta,\eta,\mu}(vfg) + K_t^{\alpha,\beta,\eta,\mu}vK_t^{\alpha,\beta,\eta,\mu}(ufg) \geq K_t^{\alpha,\beta,\eta,\mu}(vf)K_t^{\alpha,\beta,\eta,\mu}(ug) + K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\alpha,\beta,\eta,\mu}(vg), \tag{2.1}$$

for all $x\in[0,\infty)$, and real constants α,β,η,μ with $\alpha>0$, $\mu>-1$, $\eta\leq 0$ and $\alpha+\beta+\mu\geq 0$.

Proof. Since f and g are two synchronous functions on $[0,\infty)$, for all $\tau>0$ and $\rho>0$, then we have

$$(f(\tau)-f(\rho))(g(\tau)-g(\rho)) \geq 0. \tag{2.2}$$

Rewriting (2.2), we obtain

$$f(\tau)g(\tau) + f(\rho)g(\rho) \geq f(\tau)g(\rho) + f(\rho)g(\tau). \tag{2.3}$$

Multiplying both side of (2.3) by $v(\tau)\frac{\tau^\mu(x-\tau)^{\alpha-1}}{\Gamma(\alpha)}{}_2F_1(\alpha+\mu+\beta,-\eta;\alpha;1-\frac{\tau}{x})$, where $x>0$ and $\tau\in(0,x)$, when we integrate the inequality with respect to τ from 0 to x, we obtain by Definition 1.2 that

$$K_t^{\alpha,\beta,\eta,\mu}(vfg)(x) + f(\rho)g(\rho)K_t^{\alpha,\beta,\eta,\mu}v(x) \geq g(\rho)K_t^{\alpha,\beta,\eta,\mu}(vf)(x) + f(\rho)K_t^{\alpha,\beta,\eta,\mu}(vg)(x). \tag{2.4}$$

Again, by multiplying both side of (2.4) by $u(\rho)\frac{\rho^{\mu}(x-\rho)^{\alpha-1}}{\Gamma(\alpha)}{}_2F_1(\alpha+\mu+\beta,-\eta;\alpha;1-\frac{\rho}{x})$, where $x>0$ and $\rho\in(0,x)$, and integrating the resulting identity with respect to ρ from 0 to x, and then applying Definition 1.2, we conclude

$$\begin{aligned}K_t^{\alpha,\beta,\eta,\mu}u(x)K_t^{\alpha,\beta,\eta,\mu}(vfg)(x)+K_t^{\alpha,\beta,\eta,\mu}v(x)K_t^{\alpha,\beta,\eta,\mu}(ufg)(x)&\\ \geq K_t^{\alpha,\beta,\eta,\mu}(vf)(x)K_t^{\alpha,\beta,\eta,\mu}(ug)(x)+K_t^{\alpha,\beta,\eta,\mu}(uf)(x)K_t^{\alpha,\beta,\eta,\mu}(vg)(x),&\end{aligned}$$

which implies (2.1). □

Theorem 2.2. *Let f and g be two synchronous functions on $[0,\infty)$ and let p, q and r be three nonnegative functions on $[0,\infty)$. Then we have*

$$\begin{aligned}&2K_t^{\alpha,\beta,\eta,\mu}p\Big(K_t^{\alpha,\beta,\eta,\mu}qK_t^{\alpha,\beta,\eta,\mu}(rfg)+K_t^{\alpha,\beta,\eta,\mu}rK_t^{\alpha,\beta,\eta,\mu}(qfg)\Big)+2K_t^{\alpha,\beta,\eta,\mu}qK_t^{\alpha,\beta,\eta,\mu}rK_t^{\alpha,\beta,\eta,\mu}(pfg)\\ &\geq K_t^{\alpha,\beta,\eta,\mu}p\Big(K_t^{\alpha,\beta,\eta,\mu}(qf)K_t^{\alpha,\beta,\eta,\mu}(rg)+K_t^{\alpha,\beta,\eta,\mu}(rf)K_t^{\alpha,\beta,\eta,\mu}(qg)\Big)+K_t^{\alpha,\beta,\eta,\mu}q\Big(K_t^{\alpha,\beta,\eta,\mu}(pf)K_t^{\alpha,\beta,\eta,\mu}(rg)\\ &\quad+K_t^{\alpha,\beta,\eta,\mu}(rf)K_t^{\alpha,\beta,\eta,\mu}(pg)\Big)+K_t^{\alpha,\beta,\eta,\mu}r\Big(K_t^{\alpha,\beta,\eta,\mu}(pf)K_t^{\alpha,\beta,\eta,\mu}(qg)+K_t^{\alpha,\beta,\eta,\mu}(qf)K_t^{\alpha,\beta,\eta,\mu}(pg)\Big),\end{aligned}\tag{2.5}$$

for all $x\in[0,\infty)$, and real constants α,β,η,μ with $\alpha>0$, $\mu>-1$, $\eta\leq0$ and $\alpha+\beta+\mu\geq0$.

Proof. Putting $u=q$, $v=r$ and using Lemma 2.1, we can write

$$K_t^{\alpha,\beta,\eta,\mu}qK_t^{\alpha,\beta,\eta,\mu}(rfg)+K_t^{\alpha,\beta,\eta,\mu}rK_t^{\alpha,\beta,\eta,\mu}(qfg)\geq K_t^{\alpha,\beta,\eta,\mu}(rf)K_t^{\alpha,\beta,\eta,\mu}(qg)+K_t^{\alpha,\beta,\eta,\mu}(qf)K_t^{\alpha,\beta,\eta,\mu}(rg).\tag{2.6}$$

Multiplying both sides of (2.6) by $K_t^{\alpha,\beta,\eta,\mu}p$, we obtain

$$\begin{aligned}&K_t^{\alpha,\beta,\eta,\mu}p\Big(K_t^{\alpha,\beta,\eta,\mu}qK_t^{\alpha,\beta,\eta,\mu}(rfg)+K_t^{\alpha,\beta,\eta,\mu}rK_t^{\alpha,\beta,\eta,\mu}(qfg)\Big)\\ &\qquad\geq K_t^{\alpha,\beta,\eta,\mu}p\Big(K_t^{\alpha,\beta,\eta,\mu}(rf)(x)K_t^{\alpha,\beta,\eta,\mu}(qg)+K_t^{\alpha,\beta,\eta,\mu}(qf)K_t^{\alpha,\beta,\eta,\mu}(rg)\Big).\end{aligned}\tag{2.7}$$

Putting $u=p$, $v=r$ and using Lemma 2.1, we can state that

$$K_t^{\alpha,\beta,\eta,\mu}pK_t^{\alpha,\beta,\eta,\mu}(rfg)+K_t^{\alpha,\beta,\eta,\mu}rK_t^{\alpha,\beta,\eta,\mu}(pfg)\geq K_t^{\alpha,\beta,\eta,\mu}(rf)K_t^{\alpha,\beta,\eta,\mu}(pg)+K_t^{\alpha,\beta,\eta,\mu}(pf)K_t^{\alpha,\beta,\eta,\mu}(rg).\tag{2.8}$$

Multiplying both sides of (2.8) by $I_{0,t}^{\alpha,\beta,\eta}y(t)$, one verifies that

$$\begin{aligned}&K_t^{\alpha,\beta,\eta,\mu}q\Big(K_t^{\alpha,\beta,\eta,\mu}pK_t^{\alpha,\beta,\eta,\mu}(rfg)+K_t^{\alpha,\beta,\eta,\mu}r(x)K_t^{\alpha,\beta,\eta,\mu}(pfg)\Big)\\ &\qquad\geq K_t^{\alpha,\beta,\eta,\mu}q\Big(K_t^{\alpha,\beta,\eta,\mu}(rf)K_t^{\alpha,\beta,\eta,\mu}(pg)+K_t^{\alpha,\beta,\eta,\mu}(pf)K_t^{\alpha,\beta,\eta,\mu}(rg)\Big).\end{aligned}\tag{2.9}$$

With the same arguments as before, we can get

$$\begin{aligned}&K_t^{\alpha,\beta,\eta,\mu}r\Big(K_t^{\alpha,\beta,\eta,\mu}pK_t^{\alpha,\beta,\eta,\mu}(qfg)+K_t^{\alpha,\beta,\eta,\mu}q(x)K_t^{\alpha,\beta,\eta,\mu}(pfg)\Big)\\ &\qquad\geq K_t^{\alpha,\beta,\eta,\mu}r\Big(K_t^{\alpha,\beta,\eta,\mu}(qf)K_t^{\alpha,\beta,\eta,\mu}(pg)+K_t^{\alpha,\beta,\eta,\mu}(pf)K_t^{\alpha,\beta,\eta,\mu}(qg)\Big).\end{aligned}\tag{2.10}$$

The required inequality (2.5) follows on adding the inequalities (2.7), (2.9) and (2.10). □

Lemma 2.3. *Let f and g be two synchronous functions on $[0,\infty)$ and let u and v be two nonnegative functions on $[0,\infty)$. Then we have*

$$\begin{aligned}&K_t^{\alpha,\beta,\eta,\mu}u(x)K_t^{\gamma,\delta,\zeta,\nu}(vfg)(x)+K_t^{\gamma,\delta,\zeta,\nu}v(x)K_t^{\alpha,\beta,\eta,\mu}(ufg)(x)\\ &\qquad\geq K_t^{\alpha,\beta,\eta,\mu}(uf)(x)K_t^{\gamma,\delta,\zeta,\mu}(vg)(x)+K_t^{\gamma,\delta,\zeta,\mu}(vf)(x)K_t^{\alpha,\beta,\eta,\nu}(ug)(x),\end{aligned}\tag{2.11}$$

for all $x\in[0,\infty)$, and real constants $\alpha,\gamma,\beta,\delta,\eta,\zeta,\mu,\nu$ satisfying $\alpha,\gamma>0$, $\mu,\nu>-1$, $\eta,\zeta\leq0$ and $\alpha+\beta+\mu,\gamma+\delta+\zeta\geq0$.

Proof. Multiplying both sides of (2.3) by $v(\rho)\frac{\rho^{\nu}(x-\rho)^{\gamma-1}}{\Gamma(\gamma)}{}_2F_1(\gamma+\nu+\delta,-\zeta;\gamma;1-\frac{\rho}{x})$, where $x>0$ and $\rho\in(0,x)$, when we integrate the inequality with respect to ρ from 0 to x, we obtain by Definition 1.2 that

$$f(\tau)g(\tau)K_t^{\gamma,\delta,\zeta,\nu}v(x)+K_t^{\gamma,\delta,\zeta,\nu}(vfg)(x)\geq f(\tau)K_t^{\gamma,\delta,\zeta,\nu}(vg)(x)+g(\tau)K_t^{\gamma,\delta,\zeta,\nu}(vf)(x). \qquad (2.12)$$

Again, by multiplying both side of (2.12) by $u(\tau)\frac{\tau^{\mu}(x-\tau)^{\alpha-1}}{\Gamma(\alpha)}{}_2F_1(\alpha+\mu+\beta,-\eta;\alpha;1-\frac{\tau}{x})$, where $x>0$ and $\tau\in(0,x)$, and integrating the resulting identity with respect to τ from 0 to x, and then applying Definition 1.2, we obtain

$$\begin{aligned}&K_t^{\alpha,\beta,\eta,\mu}u(x)K_t^{\gamma,\delta,\zeta,\nu}(vfg)(x)+K_t^{\gamma,\delta,\zeta,\nu}v(x)K_t^{\alpha,\beta,\eta,\mu}(ufg)(x)\\&\qquad\geq K_t^{\alpha,\beta,\eta,\mu}(uf)(x)K_t^{\gamma,\delta,\zeta,\mu}(vg)(x)+K_t^{\gamma,\delta,\zeta,\mu}(vf)(x)K_t^{\alpha,\beta,\eta,\nu}(ug)(x),\end{aligned}$$

which implies (2.11). □

Theorem 2.4. *Let f and g be two synchronous functions on $[0,\infty)$ and let p, q and r be three nonnegative functions on $[0,\infty)$. Then we have*

$$\begin{aligned}&K_t^{\alpha,\beta,\eta,\mu}p\Big(K_t^{\alpha,\beta,\eta,\mu}rK_t^{\gamma,\delta,\zeta,\nu}(qfg)+2K_t^{\alpha,\beta,\eta,\mu}qK_t^{\gamma,\delta,\zeta,\nu}(rfg)+K_t^{\gamma,\delta,\zeta,\nu}rK_t^{\alpha,\beta,\eta,\mu}(qfg)\Big)\\&\qquad+\Big(K_t^{\alpha,\beta,\eta,\mu}qI_{0,t}^{\gamma,\delta,\zeta}r+K_t^{\gamma,\delta,\zeta,\nu}qK_t^{\alpha,\beta,\eta,\mu}r\Big)K_t^{\alpha,\beta,\eta,\mu}(pfg)\\&\geq K_t^{\alpha,\beta,\eta,\mu}p\Big(K_t^{\alpha,\beta,\eta,\mu}(qf)K_t^{\gamma,\delta,\zeta,\nu}(rg)+K_t^{\gamma,\delta,\zeta,\nu}(rf)K_t^{\alpha,\beta,\eta,\mu}(qg)\Big)+K_t^{\gamma,\delta,\zeta,\nu}q\Big(K_t^{\alpha,\beta,\eta,\mu}(pf)K_t^{\gamma,\delta,\zeta,\nu}(rg)\\&\quad+K_t^{\gamma,\delta,\zeta,\nu}(rf)(t)K_t^{\alpha,\beta,\eta,\mu}(pg)\Big)+K_t^{\gamma,\delta,\zeta,\nu}r\Big(K_t^{\alpha,\beta,\eta,\mu}(pf)K_t^{\gamma,\delta,\zeta,\nu}(qg)+K_t^{\gamma,\delta,\zeta,\nu}(qf)K_t^{\alpha,\beta,\eta,\mu}(pg)\Big),\end{aligned} \qquad (2.13)$$

for all $x\in[0,\infty)$, and real constants $\alpha,\gamma,\beta,\delta,\eta,\zeta,\mu,\nu$ satisfying $\alpha,\gamma>0$, $\mu,\nu>-1$, $\eta,\zeta\leq 0$ and $\alpha+\beta+\mu,\gamma+\delta+\zeta\geq 0$.

Proof. Putting $u=q$, $v=r$ and using Lemma 2.3, we can write

$$K_t^{\alpha,\beta,\eta,\mu}qK_t^{\gamma,\delta,\zeta,\nu}(rfg)+K_t^{\gamma,\delta,\zeta,\nu}rK_t^{\alpha,\beta,\eta,\mu}(qfg)\geq K_t^{\alpha,\beta,\eta,\mu}(qf)K_t^{\gamma,\delta,\zeta,\mu}(rg)+K_t^{\gamma,\delta,\zeta,\mu}(rf)K_t^{\alpha,\beta,\eta,\nu}(qg). \qquad (2.14)$$

Multiplying both sides of (2.14) by $K_t^{\alpha,\beta,\eta,\mu}p$, we obtain

$$\begin{aligned}&K_t^{\alpha,\beta,\eta,\mu}p\Big(K_t^{\alpha,\beta,\eta,\mu}qK_t^{\gamma,\delta,\zeta,\nu}(rfg)+K_t^{\gamma,\delta,\zeta,\nu}rK_t^{\alpha,\beta,\eta,\mu}(qfg)\Big)\\&\qquad\geq K_t^{\alpha,\beta,\eta,\mu}p\Big(K_t^{\alpha,\beta,\eta,\mu}(qf)K_t^{\gamma,\delta,\zeta,\mu}(rg)+K_t^{\gamma,\delta,\zeta,\mu}(rf)K_t^{\alpha,\beta,\eta,\nu}(qg)\Big).\end{aligned} \qquad (2.15)$$

Putting $u=p$, $v=r$ and using Lemma 2.3, we can state that

$$K_t^{\alpha,\beta,\eta,\mu}pK_t^{\gamma,\delta,\zeta,\nu}(rfg)+K_t^{\gamma,\delta,\zeta,\nu}rK_t^{\alpha,\beta,\eta,\mu}(pfg)\geq K_t^{\alpha,\beta,\eta,\mu}(pf)K_t^{\gamma,\delta,\zeta,\mu}(rg)+K_t^{\gamma,\delta,\zeta,\mu}(rf)K_t^{\alpha,\beta,\eta,\nu}(pg).$$

Multiplying both sides of (2.14) by $K_t^{\alpha,\beta,\eta,\mu}q$, one verifies that

$$\begin{aligned}&K_t^{\alpha,\beta,\eta,\mu}q\Big(K_t^{\alpha,\beta,\eta,\mu}pK_t^{\gamma,\delta,\zeta,\nu}(rfg)+K_t^{\gamma,\delta,\zeta,\nu}rK_t^{\alpha,\beta,\eta,\mu}(pfg)\Big)\\&\qquad\geq K_t^{\alpha,\beta,\eta,\mu}q\Big(K_t^{\alpha,\beta,\eta,\mu}(pf)K_t^{\gamma,\delta,\zeta,\mu}(rg)+K_t^{\gamma,\delta,\zeta,\mu}(rf)K_t^{\alpha,\beta,\eta,\nu}(pg)\Big).\end{aligned} \qquad (2.16)$$

With the same arguments as before, we can get

$$\begin{aligned}&K_t^{\alpha,\beta,\eta,\mu}r\Big(K_t^{\alpha,\beta,\eta,\mu}qK_t^{\gamma,\delta,\zeta,\nu}(pfg)+K_t^{\gamma,\delta,\zeta,\nu}pK_t^{\alpha,\beta,\eta,\mu}(qfg)\Big)\\&\qquad\geq K_t^{\alpha,\beta,\eta,\mu}r\Big(K_t^{\alpha,\beta,\eta,\mu}(qf)K_t^{\gamma,\delta,\zeta,\mu}(pg)+K_t^{\gamma,\delta,\zeta,\mu}(pf)K_t^{\alpha,\beta,\eta,\nu}(qg)\Big).\end{aligned} \qquad (2.17)$$

The required inequality (2.13) follows on adding the inequalities (2.15), (2.16) and (2.17). □

Remark 2.5. The inequalities (2.5) and (2.13) are reversed in the following cases: (a) The functions f and g are synchronous on $[0,\infty)$. (b) The functions p, q and r are negative on $[0,\infty)$. (c) Two of he functions p, q and r are positive and the third one is negative on $[0,\infty)$.

Theorem 2.6. *Let f,g and h be three synchronous functions on $[0,\infty)$ and let u be a nonnegative function on $[0,\infty)$. Then we have*

$$\begin{aligned}&K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(ufgh)+K_t^{\alpha,\beta,\eta,\mu}(uh)K_t^{\gamma,\delta,\zeta,\nu}(ufg)+K_t^{\alpha,\beta,\eta,\mu}(ufg)K_t^{\gamma,\delta,\zeta,\nu}(uh)\\&\quad+K_t^{\alpha,\beta,\eta,\mu}(ufgh)K_t^{\gamma,\delta,\zeta,\nu}u\geq K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\gamma,\delta,\zeta,\nu}(ugh)+K_t^{\alpha,\beta,\eta,\mu}(ug)K_t^{\gamma,\delta,\zeta,\nu}(ufh)\\&\quad+K_t^{\alpha,\beta,\eta,\mu}(ugh)K_t^{\gamma,\delta,\zeta,\nu}(uf)+K_t^{\alpha,\beta,\eta,\mu}(ufh)K_t^{\gamma,\delta,\zeta,\nu}(ug),\end{aligned}\tag{2.18}$$

for all $x\in[0,\infty)$, and real constants $\alpha,\gamma,\beta,\delta,\eta,\zeta,\mu,\nu$ satisfying $\alpha,\gamma>0$, $\mu,\nu>-1$, $\eta,\zeta\leq 0$ and $\alpha+\beta+\mu,\gamma+\delta+\zeta\geq 0$.

Proof. Let f,g and h be three synchronous functions on $[0,\infty)$, Then, for all $\tau,\rho\geq 0$, we have

$$(f(\tau)-f(\rho))(g(\tau)-g(\rho))(h(\tau)+h(\rho))\geq 0,$$

which implies that

$$\begin{aligned}&f(\tau)g(\tau)h(\tau)+f(\rho)g(\rho)h(\rho)+f(\tau)g(\tau)h(\rho)+f(\rho)g(\rho)h(\tau)\\&\quad\geq f(\tau)g(\rho)h(\tau)+f(\tau)g(\rho)h(\rho)+f(\rho)g(\tau)h(\tau)+f(\rho)g(\tau)h(\rho).\end{aligned}\tag{2.19}$$

Multiplying both side of (2.19) by $u(\tau)\frac{\tau^{\nu}(x-\tau)^{\gamma-1}}{\Gamma(\gamma)}{}_2F_1(\gamma+\nu+\delta,-\zeta;\gamma;1-\frac{\tau}{x})$, where $x>0$ and $\tau\in(0,x)$, and integrating the resulting identity with respect to τ from 0 to x, and then applying Definition 1.2, we obtain

$$\begin{aligned}&K_t^{\gamma,\delta,\zeta,\nu}(ufgh)+f(\rho)g(\rho)h(\rho)K_t^{\gamma,\delta,\zeta,\nu}u+h(\rho)K_t^{\gamma,\delta,\zeta,\nu}(ufg)+f(\rho)g(\rho)K_t^{\gamma,\delta,\zeta,\nu}(uh)\\&\quad\geq g(\rho)K_t^{\gamma,\delta,\zeta,\nu}(ufh)+g(\rho)h(\rho)K_t^{\gamma,\delta,\zeta,\nu}(uf)+f(\rho)K_t^{\gamma,\delta,\zeta,\nu}(ugh)+f(\rho)h(\rho)K_t^{\gamma,\delta,\zeta,\nu}(ug).\end{aligned}\tag{2.20}$$

Again, by multiplying both sides of (2.20) by $u(\rho)\frac{\rho^{\mu}(x-\rho)^{\alpha-1}}{\Gamma(\alpha)}{}_2F_1(\alpha+\mu+\beta,-\eta;\alpha;1-\frac{\rho}{x})$ where $x>0$ and $\rho\in(0,x)$, when we integrate the inequality with respect to ρ from 0 to x, we obtain by Definition 1.2 that

$$\begin{aligned}&K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(ufgh)+K_t^{\alpha,\beta,\eta,\mu}(uh)K_t^{\gamma,\delta,\zeta,\nu}(ufg)+K_t^{\alpha,\beta,\eta,\mu}(ufg)K_t^{\gamma,\delta,\zeta,\nu}(uh)\\&\quad+K_t^{\alpha,\beta,\eta,\mu}(ufgh)K_t^{\gamma,\delta,\zeta,\nu}u\geq K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\gamma,\delta,\zeta,\nu}(ugh)+K_t^{\alpha,\beta,\eta,\mu}(ug)K_t^{\gamma,\delta,\zeta,\nu}(ufh)\\&\quad+K_t^{\alpha,\beta,\eta,\mu}(ugh)K_t^{\gamma,\delta,\zeta,\nu}(uf)+K_t^{\alpha,\beta,\eta,\mu}(ufh)K_t^{\gamma,\delta,\zeta,\nu}(ug),\end{aligned}$$

which implies (2.18). □

Theorem 2.7. *Let f,g and h be three synchronous functions on $[0,\infty)$ and let u and v be two nonnegative functions on $[0,\infty)$. Then we have*

$$\begin{aligned}&K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(vfgh)+K_t^{\alpha,\beta,\eta,\mu}(uh)K_t^{\gamma,\delta,\zeta,\nu}(vfg)+K_t^{\alpha,\beta,\eta,\mu}(ufg)K_t^{\gamma,\delta,\zeta,\nu}(vh)\\&\quad+K_t^{\alpha,\beta,\eta,\mu}(ufgh)K_t^{\gamma,\delta,\zeta,\nu}v\geq K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\gamma,\delta,\zeta,\nu}(vgh)+K_t^{\alpha,\beta,\eta,\mu}(ug)K_t^{\gamma,\delta,\zeta,\nu}(vfh)\\&\quad+K_t^{\alpha,\beta,\eta,\mu}(ugh)K_t^{\gamma,\delta,\zeta,\nu}(vf)+K_t^{\alpha,\beta,\eta,\mu}(ufh)K_t^{\gamma,\delta,\zeta,\nu}(vg),\end{aligned}\tag{2.21}$$

for all $x\in[0,\infty)$, and real constants $\alpha,\gamma,\beta,\delta,\eta,\zeta,\mu,\nu$ satisfying $\alpha,\gamma>0$, $\mu,\nu>-1$, $\eta,\zeta\leq 0$ and $\alpha+\beta+\mu,\gamma+\delta+\zeta\geq 0$.

Proof. Multiplying both side of (2.19) by $v(\tau)\frac{\tau^{\nu}(x-\tau)^{\gamma-1}}{\Gamma(\gamma)}{}_2F_1(\gamma+\nu+\delta,-\zeta;\gamma;1-\frac{\tau}{x})$, where $x>0$ and $\tau\in(0,x)$, and integrating the resulting identity with respect to τ from 0 to x, and then applying Definition 1.2, we obtain

$$\begin{aligned}&K_t^{\gamma,\delta,\zeta,\nu}(vfgh)+f(\rho)g(\rho)h(\rho)K_t^{\gamma,\delta,\zeta,\nu}v+h(\rho)K_t^{\gamma,\delta,\zeta,\nu}(vfg)+f(\rho)g(\rho)K_t^{\gamma,\delta,\zeta,\nu}(vh)\\&\quad\geq g(\rho)K_t^{\gamma,\delta,\zeta,\nu}(vfh)+g(\rho)h(\rho)K_t^{\gamma,\delta,\zeta,\nu}(vf)+f(\rho)K_t^{\gamma,\delta,\zeta,\nu}(vgh)+f(\rho)h(\rho)K_t^{\gamma,\delta,\zeta,\nu}(vg).\end{aligned}\tag{2.22}$$

Again, by multiplying both sides of (2.22) by $u(\rho)\frac{\rho^{\mu}(x-\rho)^{\alpha-1}}{\Gamma(\alpha)}{}_2F_1(\alpha+\mu+\beta,-\eta;\alpha;1-\frac{\rho}{x})$ where $x>0$ and $\rho\in(0,x)$, when we integrate the inequality with respect to ρ from 0 to x, we obtain by Definition 1.2 that

$$\begin{aligned} &K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(vfgh)+K_t^{\alpha,\beta,\eta,\mu}(ufgh)K_t^{\gamma,\delta,\zeta,\nu}v+K_t^{\alpha,\beta,\eta,\mu}(uh)K_t^{\gamma,\delta,\zeta,\nu}(vfg)\\ &\quad+K_t^{\alpha,\beta,\eta,\mu}(ufg)K_t^{\gamma,\delta,\zeta,\nu}(vh)\geq K_t^{\alpha,\beta,\eta,\mu}(ug)K_t^{\gamma,\delta,\zeta,\nu}(vfh)+K_t^{\alpha,\beta,\eta,\mu}(ugh)K_t^{\gamma,\delta,\zeta,\nu}(vf)\\ &\qquad+K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\gamma,\delta,\zeta,\nu}(vgh)+K_t^{\alpha,\beta,\eta,\mu}(ufh)K_t^{\gamma,\delta,\zeta,\nu}(vg),\end{aligned}$$

which implies (2.21). □

Remark 2.8. It may be noted that the inequalities in (2.18) and (2.21) are reversed if functions f,g and h are asynchronous. It is also easily seen that the special case $u=v$ of (2.21) in Theorem 2.7 reduces to Theorem 2.6.

Lemma 2.9. *Let f and u be two functions defined on $[0,\infty)$ satisfying the condition (1.6). Then we have*

$$\begin{aligned} K_t^{\alpha,\beta,\eta,\mu}uK_t^{\alpha,\beta,\eta,\mu}(uf^2)-\left(K_t^{\alpha,\beta,\eta,\mu}(uf)\right)^2&=\left(\Phi K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\alpha,\beta,\eta,\mu}(uf)\right)\left(K_t^{\alpha,\beta,\eta,\mu}(xf)(t)-\phi K_t^{\alpha,\beta,\eta,\mu}u\right)\\ &\quad-K_t^{\alpha,\beta,\eta,\mu}uK_t^{\alpha,\beta,\eta,\mu}\left(u(x)(\Phi-f(x))(f(x)-\phi)\right), \end{aligned}\tag{2.23}$$

for all $x\in[0,\infty)$, and real constants α,β,η,μ with $\alpha>0$, $\mu>-1$, $\eta\leq 0$ and $\alpha+\beta+\mu\geq 0$.

Proof. Let f be a function defined on $[0,\infty)$ satisfying the condition (1.6) on $[0,\infty)$. For any $\rho,\tau\in[0,\infty)$, we have

$$\begin{aligned} &(\Phi-f(\rho))(f(\tau)-\phi)+(\Phi-f(\tau))(f(\rho)-\phi)-(\Phi-f(\tau))(f(\tau)-\phi)\\ &\qquad-(\Phi-f(\rho))(f(\rho)-\phi)=f^2(\tau)+f^2(\rho)-2f(\rho)f(\tau). \end{aligned}\tag{2.24}$$

Multiplying both sides of (2.24) by $u(\rho)\frac{\rho^{\mu}(x-\rho)^{\alpha-1}}{\Gamma(\alpha)}{}_2F_1(\alpha+\mu+\beta,-\eta;\alpha;1-\frac{\rho}{x})$ where $x>0$ and $\rho\in(0,x)$, when we integrate the inequality with respect to ρ from 0 to x, we obtain by Definition 1.2 that

$$\begin{aligned} &(f(\tau)-\phi)\left(\Phi K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\alpha,\beta,\eta,\mu}(uf)\right)+(\Phi-f(\tau))\left(K_t^{\alpha,\beta,\eta,\mu}(uf)-\phi K_t^{\alpha,\beta,\eta,\mu}u\right)\\ &\qquad-(\Phi-f(\tau))(f(\tau)-\phi)K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\alpha,\beta,\eta,\mu}\left(u(x)(\Phi-f(x))(f(x)-\phi)\right)\\ &\qquad=f^2(\tau)K_t^{\alpha,\beta,\eta,\mu}u+K_t^{\alpha,\beta,\eta,\mu}(uf^2)-2f(\tau)K_t^{\alpha,\beta,\eta,\mu}(uf). \end{aligned}\tag{2.25}$$

Again, by multiplying both sides of (2.25) by $u(\rho)\frac{\rho^{\mu}(x-\rho)^{\alpha-1}}{\Gamma(\alpha)}{}_2F_1(\alpha+\mu+\beta,-\eta;\alpha;1-\frac{\rho}{x})$ where $x>0$ and $\rho\in(0,x)$, when we integrate the inequality with respect to ρ from 0 to x, we obtain by Definition 1.2 that

$$\begin{aligned} &\left(K_t^{\alpha,\beta,\eta,\mu}(uf)-\phi K_t^{\alpha,\beta,\eta,\mu}u\right)\left(\Phi K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\alpha,\beta,\eta,\mu}(uf)\right)\\ &\qquad+\left(\Phi K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\alpha,\beta,\eta,\mu}(uf)\right)\left(K_t^{\alpha,\beta,\eta,\mu}(uf)-\phi K_t^{\alpha,\beta,\eta,\mu}u\right)\\ &\quad-K_t^{\alpha,\beta,\eta,\mu}\left(u(x)(\Phi-f(x))(f(x)-\phi)\right)K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\alpha,\beta,\eta,\mu}uK_t^{\alpha,\beta,\eta,\mu}\left(u(x)(\Phi-f(x))(f(x)-\phi)\right)\\ &\qquad=K_t^{\alpha,\beta,\eta,\mu}(uf^2)K_t^{\alpha,\beta,\eta,\mu}u+K_t^{\alpha,\beta,\eta,\mu}uK_t^{\alpha,\beta,\eta,\mu}(uf^2)-2K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\alpha,\beta,\eta,\mu}(uf), \end{aligned}$$

which gives (2.23) and proves the lemma. □

Theorem 2.10. *Let f and g be two functions defined satisfying the condition (1.6) on $[0,\infty)$ and let u be a nonnegative function on $[0,\infty)$. Then we have*

$$\left|K_t^{\alpha,\beta,\eta,\mu}uK_t^{\alpha,\beta,\eta,\mu}(ufg)-K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\alpha,\beta,\eta,\mu}(ug)\right|\leq\frac{1}{4}(\Phi-\phi)(\Psi-\psi)\left(K_t^{\alpha,\beta,\eta,\mu}u\right)^2,\tag{2.26}$$

for all $x\in[0,\infty)$, and real constants α,β,η,μ with $\alpha>0$, $\mu>-1$, $\eta\leq 0$ and $\alpha+\beta+\mu\geq 0$.

Proof. Let f and g be two functions satisfying the conditions of Theorem 2.10. Let $H(\tau,\rho)$ be defined by

$$H(\tau,\rho)=(f(\tau)-f(\rho))(g(\tau)-g(\rho)),\quad \tau,\rho\in(0,x),\quad x>0. \tag{2.27}$$

Multiplying both sides of (2.27) by $u(\tau)F(x,\tau)u(\rho)F(x,\rho)$, where

$$F(x,\tau)=\frac{\Gamma(1-\beta)\Gamma(\alpha+\mu+\eta+1)}{\Gamma(\eta-\beta+1)\Gamma(\mu+1)}x^{\alpha+\beta}\frac{x^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)}\tau^{\mu}(x-\tau)^{\alpha-1}{}_2F_1(\alpha+\mu+\beta,-\eta;\alpha;1-\frac{\tau}{x}), \tag{2.28}$$

where $x>0$ and $\tau\in(0,x)$, and integrating the resulting inequality obtained with respect to τ and ρ from 0 to x, we have

$$\int_0^x\int_0^x u(\tau)F(x,\tau)u(\rho)F(x,\rho)H(\tau,\rho)d\tau d\rho=2K_t^{\alpha,\beta,\eta,\mu}uK_t^{\alpha,\beta,\eta,\mu}(ufg)-2K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\alpha,\beta,\eta,\mu}(ug). \tag{2.29}$$

Thanks to the weighted Cauchy-Schwartz integral inequality for double integrals, we can write that

$$\begin{aligned}&\Big(\int_0^x\int_0^x u(\tau)F(x,\tau)u(\rho)F(x,\rho)H(\tau,\rho)d\tau d\rho\Big)^2\\ &\le\Big(\int_0^x\int_0^x u(\tau)F(x,\tau)u(\rho)F(x,\rho)(f(\tau)-f(\rho))d\tau d\rho\Big)\Big(\int_0^x\int_0^x u(\tau)F(x,\tau)u(\rho)F(x,\rho)(g(\tau)-g(\rho))d\tau d\rho\Big)\\ &=4\Big(K_t^{\alpha,\beta,\eta,\mu}uK_t^{\alpha,\beta,\eta,\mu}(uf^2)-\Big(K_t^{\alpha,\beta,\eta,\mu}(uf)\Big)^2\Big)\Big(K_t^{\alpha,\beta,\eta,\mu}uK_t^{\alpha,\beta,\eta,\mu}(ug^2)-\Big(K_t^{\alpha,\beta,\eta,\mu}(ug)\Big)^2\Big).\end{aligned} \tag{2.30}$$

Since $(\Phi-f(\tau))(f(\tau)-\phi)\ge0$ and $(\Psi-g(\tau))(g(\tau)-\psi)\ge0$, we have

$$K_t^{\alpha,\beta,\eta,\mu}uK_t^{\alpha,\beta,\eta,\mu}\Big(u(x)(\Phi-f(x))(f(x)-\phi)\Big)\ge0, \tag{2.31}$$

and

$$K_t^{\alpha,\beta,\eta,\mu}uK_t^{\alpha,\beta,\eta,\mu}\Big(u(x)(\Psi-g(x))(g(x)-\psi)\Big)\ge0. \tag{2.32}$$

Thus, from (2.31), (2.32) and Lemma 2.9,we get

$$K_t^{\alpha,\beta,\eta,\mu}uK_t^{\alpha,\beta,\eta,\mu}(uf^2)-\Big(K_t^{\alpha,\beta,\eta,\mu}(uf)\Big)^2\le\Big(\Phi K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\alpha,\beta,\eta,\mu}(uf)\Big)\Big(K_t^{\alpha,\beta,\eta,\mu}(uf)-\phi K_t^{\alpha,\beta,\eta,\mu}u\Big), \tag{2.33}$$

and

$$K_t^{\alpha,\beta,\eta,\mu}uK_t^{\alpha,\beta,\eta,\mu}(ug^2)-\Big(K_t^{\alpha,\beta,\eta,\mu}(ug)\Big)^2\le\Big(\Psi K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\alpha,\beta,\eta,\mu}(ug)\Big)\Big(K_t^{\alpha,\beta,\eta,\mu}(ug)-\phi K_t^{\alpha,\beta,\eta,\mu}u\Big). \tag{2.34}$$

Combining (2.29), (2.30), (2.33) and (2.34), we deduce that

$$\begin{aligned}\Big(K_t^{\alpha,\beta,\eta,\mu}uK_t^{\alpha,\beta,\eta,\mu}(ufg)-K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\alpha,\beta,\eta,\mu}(ug)\Big)^2&\le\Big(\Phi K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\alpha,\beta,\eta,\mu}(uf)\Big)\\ \times\Big(K_t^{\alpha,\beta,\eta,\mu}(uf)-\phi K_t^{\alpha,\beta,\eta,\mu}u\Big)&\Big(\Psi K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\alpha,\beta,\eta,\mu}(ug)\Big)\Big(K_t^{\alpha,\beta,\eta,\mu}(ug)-\phi K_t^{\alpha,\beta,\eta,\mu}u\Big).\end{aligned} \tag{2.35}$$

Now using the elementary inequality $4xy\le(x+y)^2$, $x,y\in\mathbb{R}$, we can state that

$$4\Big(\Phi K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\alpha,\beta,\eta,\mu}(uf)\Big)\Big(K_t^{\alpha,\beta,\eta,\mu}(uf)-\phi K_t^{\alpha,\beta,\eta,\mu}u\Big)\le\Big((\Phi-\phi)K_t^{\alpha,\beta,\eta,\mu}u\Big)^2, \tag{2.36}$$

and

$$4\Big(\Psi K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\alpha,\beta,\eta,\mu}(ug)\Big)\Big(K_t^{\alpha,\beta,\eta,\mu}(ug)-\phi K_t^{\alpha,\beta,\eta,\mu}u\Big)\le\Big((\Psi-\psi)K_t^{\alpha,\beta,\eta,\mu}u\Big)^2. \tag{2.37}$$

From (2.35)-(2.37), we abtain (2.26). This complete the proof of Theorem 2.10. □

Lemma 2.11. *Let f and g be two functions defined on $[0,\infty)$ and let u and v be two nonnegative functions on $[0,\infty)$. Then we have*

$$\begin{aligned}\Big(K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(vfg)+K_t^{\gamma,\delta,\zeta,\nu}vK_t^{\alpha,\beta,\eta,\mu}(ufg)-K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\gamma,\delta,\zeta,\nu}(vg)-K_t^{\gamma,\delta,\zeta,\nu}(vf)K_t^{\alpha,\beta,\eta,\mu}(ug)\Big)^2\\ \leq\Big(K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(vf^2)+K_t^{\gamma,\delta,\zeta,\nu}vK_t^{\alpha,\beta,\eta,\mu}(uf^2)-2K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\gamma,\delta,\zeta,\nu}(vf)\Big)\\ \times\Big(K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(vg^2)+K_t^{\gamma,\delta,\zeta,\nu}uK_t^{\alpha,\beta,\eta,\mu}(ug^2)-2K_t^{\alpha,\beta,\eta,\mu}(ug)K_t^{\gamma,\delta,\zeta,\nu}(vg)\Big),\end{aligned} \tag{2.38}$$

for all $x\in[0,\infty)$, and real constants $\alpha,\gamma,\beta,\delta,\eta,\zeta,\mu,\nu$ satisfying $\alpha,\gamma>0$, $\mu,\nu>-1$, $\eta,\zeta\leq 0$ and $\alpha+\beta+\mu,\gamma+\delta+\zeta\geq 0$.

Proof. Multiplying (2.27) by $u(\tau)F(t,\tau)v(\rho)G(t,\rho)$, where $F(t,\tau)$ is defined by (2.28), and

$$G(x,\rho)=\frac{\Gamma(1-\delta)\Gamma(\gamma+\nu+\zeta+1)}{\Gamma(\zeta-\delta+1)\Gamma(\nu+1)}x^{\gamma+\delta}\frac{x^{-\gamma-\delta-2\nu}}{\Gamma(\gamma)}\rho^{\nu}(x-\rho)^{\gamma-1}{}_2F_1(\gamma+\nu+\delta,-\zeta;\gamma;1-\frac{\rho}{x}), \tag{2.39}$$

where $x>0$ and $\rho\in(0,x)$, and integrating the resulting inequality obtained with respect to τ and ρ from 0 to x, we have

$$\begin{aligned}\int_0^x\int_0^x u(\tau)F(x,\tau)v(\rho)G(t,\rho)H(\tau,\rho)d\tau d\rho=K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(vfg)+K_t^{\gamma,\delta,\zeta,\nu}vK_t^{\alpha,\beta,\eta,\mu}(ufg)\\ -K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\gamma,\delta,\zeta,\nu}(vg)-K_t^{\gamma,\delta,\zeta,\nu}(vf)K_t^{\alpha,\beta,\eta,\mu}(ug).\end{aligned} \tag{2.40}$$

Then, thanks to the weighted Cauchy-Schwartz integral inequality for double integrals, we can obtain (2.38). □

Lemma 2.12. *Let f be a function defined on $[0,\infty)$ and let u and v be two nonnegative functions on $[0,\infty)$. Then we have*

$$\begin{aligned}K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(vf^2)+K_t^{\gamma,\delta,\zeta,\nu}vK_t^{\alpha,\beta,\eta,\mu}(uf^2)-2K_t^{\gamma,\delta,\zeta,\nu}(vf)K_t^{\alpha,\beta,\eta,\mu}(uf)=\Big(\Phi K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\alpha,\beta,\eta,\mu}(uf)\Big)\\ \times\Big(K_t^{\gamma,\delta,\zeta,\nu}(vf)-\phi K_t^{\gamma,\delta,\zeta,\nu}v\Big)+\Big(K_t^{\alpha,\beta,\eta,\mu}(uf)-\phi K_t^{\alpha,\beta,\eta,\mu}u\Big)\Big(\Phi K_t^{\gamma,\delta,\zeta,\nu}v-K_t^{\gamma,\delta,\zeta,\nu}(vf)\Big)\\ -K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}\Big(v(x)(\Phi-f(x))(f(x)-\phi)\Big)-K_t^{\gamma,\delta,\zeta,\nu}vK_t^{\alpha,\beta,\eta,\mu}\Big(u(x)(\Phi-f(x))(f(x)-\phi)\Big),\end{aligned} \tag{2.41}$$

for all $x\in[0,\infty)$, and real constants $\alpha,\gamma,\beta,\delta,\eta,\zeta,\mu,\nu$ satisfying $\alpha,\gamma>0$, $\mu,\nu>-1$, $\eta,\zeta\leq 0$ and $\alpha+\beta+\mu,\gamma+\delta+\zeta\geq 0$.

Proof. Multiplying both sides of (2.25) by $v(\tau)G(t,\tau)$ ($G(t,\tau)$ defined by (2.39)), and integrating the resulting inequality obtained with respect to τ from 0 to x, we have

$$\begin{aligned}\Big(K_t^{\gamma,\delta,\zeta,\nu}(vf)-\phi K_t^{\gamma,\delta,\zeta,\nu}v\Big)\Big(\Phi K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\alpha,\beta,\eta,\mu}(uf)\Big)\\ +\Big(\Phi K_t^{\gamma,\delta,\zeta,\nu}v-K_t^{\gamma,\delta,\zeta,\nu}(vf)\Big)\Big(K_t^{\alpha,\beta,\eta,\mu}(uf)-\phi K_t^{\alpha,\beta,\eta,\mu}u\Big)\\ -K_t^{\gamma,\delta,\zeta,\nu}\Big(v(x)(\Phi-f(x))(f(x)-\phi)\Big)K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\gamma,\delta,\zeta,\nu}vK_t^{\alpha,\beta,\eta,\mu}\Big(u(x)(\Phi-f(x))(f(x)-\phi)\Big)\\ =K_t^{\gamma,\delta,\zeta,\nu}(vf^2)K_t^{\alpha,\beta,\eta,\mu}u+K_t^{\gamma,\delta,\zeta,\nu}vK_t^{\alpha,\beta,\eta,\mu}(uf^2)-2K_t^{\gamma,\delta,\zeta,\nu}(vf)K_t^{\alpha,\beta,\eta,\mu}(uf),\end{aligned} \tag{2.42}$$

which gives (2.41) and proves the lemma. □

Theorem 2.13. *Let f and g be two functions satisfying the condition (1.6) on $[0,\infty)$ and let u and v be two nonnegative functions on $[0,\infty)$. Then we have*

$$\begin{aligned}&\left(K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(vfg)+K_t^{\gamma,\delta,\zeta,\nu}vK_t^{\alpha,\beta,\eta,\mu}(ufg)-K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\gamma,\delta,\zeta,\nu}(vg)-K_t^{\gamma,\delta,\zeta,\nu}(vf)K_t^{\alpha,\beta,\eta,\mu}(ug)\right)^2\\&\le\left[\left(\Phi K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\alpha,\beta,\eta,\mu}(uf)\right)\left(K_t^{\gamma,\delta,\zeta,\nu}(vf)-\phi K_t^{\gamma,\delta,\zeta,\nu}v\right)+\left(K_t^{\alpha,\beta,\eta,\mu}(uf)-\phi K_t^{\alpha,\beta,\eta,\mu}u\right)\right.\\&\quad\left.\times\left(\Phi K_t^{\gamma,\delta,\zeta,\nu}v-K_t^{\gamma,\delta,\zeta,\nu}(vf)\right)\right]\left[\left(\Psi K_t^{\alpha,\beta,\eta,\mu}u-K_t^{\alpha,\beta,\eta,\mu}(ug)\right)\left(K_t^{\gamma,\delta,\zeta,\nu}(vg)-\psi K_t^{\gamma,\delta,\zeta,\nu}v\right)\right.\\&\quad\left.+\left(K_t^{\alpha,\beta,\eta,\mu}(ug)-\psi K_t^{\alpha,\beta,\eta,\mu}u\right)\left(\Psi K_t^{\gamma,\delta,\zeta,\nu}v-K_t^{\gamma,\delta,\zeta,\nu}(vg)\right)\right],\end{aligned}\tag{2.43}$$

for all $x\in[0,\infty)$, and real constants $\alpha,\gamma,\beta,\delta,\eta,\zeta,\mu,\nu$ satisfying $\alpha,\gamma>0$, $\mu,\nu>-1$, $\eta,\zeta\le 0$ and $\alpha+\beta+\mu,\gamma+\delta+\zeta\ge 0$.

Proof. Since $(\Phi-f(\tau))(f(\tau)-\phi)\ge 0$ and $(\Psi-g(\tau))(g(\tau)-\psi)\ge 0$, we have

$$-K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}\Big(v(x)(\Phi-f(x))(f(x)-\phi)\Big)-K_t^{\gamma,\delta,\zeta,\nu}vK_t^{\alpha,\beta,\eta,\mu}\Big(u(x)(\Phi-f(x))(f(x)-\phi)\Big)\le 0,\tag{2.44}$$

and

$$-K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}\Big(v(x)(\Phi-g(x))(g(x)-\phi)\Big)-K_t^{\gamma,\delta,\zeta,\nu}vK_t^{\alpha,\beta,\eta,\mu}\Big(u(x)(\Phi-g(x))(g(x)-\phi)\Big)\le 0,\tag{2.45}$$

Applying Lemma 2.12 to f and g, and using Lemma 2.11 and the formulas (2.44), (2.45), we obtain (2.43). □

Theorem 2.14. *Let u be a nonnegative function on $[0,\infty)$ and let f,g and h be three functions defined on $[0,\infty)$, satisfying the following condition*

$$\phi\le f(x)\le\Phi,\quad \psi\le g(x)\le\Psi,\quad \omega\le h(x)\le\Omega,\quad \phi,\Phi,\psi,\Psi,\omega,\Omega\in\mathbb{R},\quad x\in[0,\infty).\tag{2.46}$$

Then we have

$$\begin{aligned}&\Big|K_t^{\alpha,\beta,\eta,\mu}(ufgh)K_t^{\gamma,\delta,\zeta,\nu}u+K_t^{\alpha,\beta,\eta,\mu}(uh)K_t^{\gamma,\delta,\zeta,\nu}(ufg)+K_t^{\alpha,\beta,\eta,\mu}(ug)K_t^{\gamma,\delta,\zeta,\nu}(ufh)\\&\quad+K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\gamma,\delta,\zeta,\nu}(ugh)-K_t^{\alpha,\beta,\eta,\mu}(ugh)K_t^{\gamma,\delta,\zeta,\nu}(uf)-K_t^{\alpha,\beta,\eta,\mu}(ufh)K_t^{\gamma,\delta,\zeta,\nu}(ug)\\&\quad-K_t^{\alpha,\beta,\eta,\mu}(ufg)K_t^{\gamma,\delta,\zeta,\nu}(uh)-K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(ufgh)\Big|\le K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}u(\Phi-\phi)(\Psi-\psi)(\Omega-\omega),\end{aligned}$$

for all $x\in[0,\infty)$, and real constants $\alpha,\gamma,\beta,\delta,\eta,\zeta,\mu,\nu$ satisfying $\alpha,\gamma>0$, $\mu,\nu>-1$, $\eta,\zeta\le 0$ and $\alpha+\beta+\mu,\gamma+\delta+\zeta\ge 0$.

Proof. From the condition (2.46), we have

$$|f(\tau)-f(\rho)|\le\Phi-\phi,\quad |g(\tau)-g(\rho)|\le\Psi-\psi,\quad |h(\tau)-h(\rho)|\le\Omega-\omega,\quad \tau,\rho\in[0,\infty),$$

which implies that

$$|(f(\tau)-f(\rho))(g(\tau)-g(\rho))(h(\tau)-h(\rho))|\le(\Phi-\phi)(\Psi-\psi)(\Omega-\omega).\tag{2.47}$$

Let us define a function

$$\begin{aligned}A(\tau,\rho)&=(f(\tau)-f(\rho))(g(\tau)-g(\rho))(h(\tau)-h(\rho))=f(\tau)g(\tau)h(\tau)+f(\rho)g(\rho)h(\tau)+f(\tau)g(\rho)h(\rho)\\&\quad+f(\rho)g(\tau)h(\rho)-f(\tau)g(\rho)h(\tau)-f(\rho)g(\rho)h(\rho)-f(\tau)g(\tau)h(\rho)-f(\rho)g(\tau)h(\tau).\end{aligned}\tag{2.48}$$

Multiplying (2.48) by $u(\tau)F(t,\tau)$, where $F(t,\tau)$ is defined by (2.28), and integrating the resulting inequality obtained with respect to τ from 0 to x, we have

$$\begin{aligned}\int_0^x u(\tau)F(x,\tau)A(\tau,\rho)d\tau = K_t^{\alpha,\beta,\eta,\mu}(ufgh) + f(\rho)g(\rho)K_t^{\alpha,\beta,\eta,\mu}(uh) + f(\rho)h(\rho)K_t^{\alpha,\beta,\eta,\mu}(ug)\\ + g(\rho)h(\rho)K_t^{\alpha,\beta,\eta,\mu}(uf) - h(\rho)K_t^{\alpha,\beta,\eta,\mu}(ufg) - g(\rho)K_t^{\alpha,\beta,\eta,\mu}(ufh)\\ - f(\rho)K_t^{\alpha,\beta,\eta,\mu}(ugh) - f(\rho)g(\rho)h(\rho)K_t^{\alpha,\beta,\eta,\mu}u. \quad (2.49)\end{aligned}$$

Again, by multiplying (2.49) by $u(\rho)G(t,\rho)$, where $G(t,\tau)$ is defined by (2.39), and integrating the resulting inequality obtained with respect to ρ from 0 to x, we have

$$\begin{aligned}\int_0^x\int_0^x u(\tau)F(x,\tau)u(\rho)G(t,\rho)A(\tau,\rho)d\tau d\rho = K_t^{\alpha,\beta,\eta,\mu}(ufgh)K_t^{\gamma,\delta,\zeta,\nu}u + K_t^{\alpha,\beta,\eta,\mu}(uh)K_t^{\gamma,\delta,\zeta,\nu}(ufg)\\ + K_t^{\alpha,\beta,\eta,\mu}(ug)K_t^{\gamma,\delta,\zeta,\nu}(ufh) + K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\gamma,\delta,\zeta,\nu}(ugh) - K_t^{\alpha,\beta,\eta,\mu}(ugh)K_t^{\gamma,\delta,\zeta,\nu}(uf)\\ - K_t^{\alpha,\beta,\eta,\mu}(ufh)K_t^{\gamma,\delta,\zeta,\nu}(ug) - K_t^{\alpha,\beta,\eta,\mu}(ufg)K_t^{\gamma,\delta,\zeta,\nu}(uh) - K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(ufgh). \quad (2.50)\end{aligned}$$

Finally, by using (2.47) on to (2.50), we arrive at the desired result (**??**), involved in Theorem 2.14, after a little simplification. This concludes the proof. ☐

Theorem 2.15. *Let u and v be two nonnegative functions on $[0,\infty)$ and let f, g and h be three functions defined on $[0,\infty)$, satisfying the condition* (2.46)*. Then we have*

$$\begin{aligned}\Big|K_t^{\alpha,\beta,\eta,\mu}(ufgh)K_t^{\gamma,\delta,\zeta,\nu}v + K_t^{\alpha,\beta,\eta,\mu}(uh)K_t^{\gamma,\delta,\zeta,\nu}(vfg) + K_t^{\alpha,\beta,\eta,\mu}(ug)K_t^{\gamma,\delta,\zeta,\nu}(vfh)\\ + K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\gamma,\delta,\zeta,\nu}(vgh) - K_t^{\alpha,\beta,\eta,\mu}(ugh)K_t^{\gamma,\delta,\zeta,\nu}(vf) - K_t^{\alpha,\beta,\eta,\mu}(ufh)K_t^{\gamma,\delta,\zeta,\nu}(vg)\\ - K_t^{\alpha,\beta,\eta,\mu}(ufg)K_t^{\gamma,\delta,\zeta,\nu}(vh) - K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(vfgh)\Big| \leq K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}v(\Phi-\phi)(\Psi-\psi)(\Omega-\omega), \quad (2.51)\end{aligned}$$

for all $x\in[0,\infty)$, and real constants $\alpha,\gamma,\beta,\delta,\eta,\zeta,\mu,\nu$ satisfying $\alpha,\gamma>0$, $\mu,\nu>-1$, $\eta,\zeta\leq 0$ and $\alpha+\beta+\mu,\gamma+\delta+\zeta\geq 0$.

Proof. Multiplying (2.49) by $v(\rho)G(t,\rho)$, where $G(t,\tau)$ is defined by (2.39), and integrating the resulting inequality obtained with respect to ρ from 0 to x, and then applying (2.47) on the resulting inequality, we get the desired result (2.51). This concludes the proof. ☐

Remark 2.16. It is not difficult to notice that the spacial case $u=v$ of (2.51) in Theorem 2.15 reduces to Theorem 2.14.

Theorem 2.17. *Let f and g be two integrable functions satisfying the condition M-g-Lipschitzian on $[0,\infty)$, i.e., $|f(x)-f(y)|\leq M|g(x)-g(y)|$, $M>0$, $x,y\in\mathbb{R}$, and let u and v be two nonnegative continuous functions on $[0,\infty)$. Then we have*

$$\begin{aligned}\left|K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(vfg) + K_t^{\gamma,\delta,\zeta,\nu}vK_t^{\alpha,\beta,\eta,\mu}(ufg) - K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\gamma,\delta,\zeta,\nu}(yg) - K_t^{\gamma,\delta,\zeta,\nu}(vf)K_t^{\alpha,\beta,\eta,\mu}(xg)\right|\\ \leq M\left(K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(vg^2) + K_t^{\gamma,\delta,\zeta,\nu}vK_t^{\alpha,\beta,\eta,\mu}(ug^2) - 2K_t^{\alpha,\beta,\eta,\mu}(ug)K_t^{\gamma,\delta,\zeta,\nu}(vg)\right), \quad (2.52)\end{aligned}$$

for all $x\in[0,\infty)$, and real constants $\alpha,\gamma,\beta,\delta,\eta,\zeta,\mu,\nu$ satisfying $\alpha,\gamma>0$, $\mu,\nu>-1$, $\eta,\zeta\leq 0$ and $\alpha+\beta+\mu,\gamma+\delta+\zeta\geq 0$.

Proof. Let us define the following relations

$$|f(\tau)-f(\rho)|\leq M|g(\tau)-g(\rho)| \quad \tau,\rho\in[0,\infty), \quad (2.53)$$

which implies that

$$|H(\tau,\rho)| = |f(\tau)-f(\rho)||g(\tau)-g(\rho)| \leq M(g(\tau)-g(\rho))^2. \quad (2.54)$$

Multiplying (2.27) by $u(\tau)F(t,\tau)u(\rho)G(t,\rho)$, where $F(t,\tau)$ and $G(t,\rho)$ are defined by (2.28) and (2.39), respectively, and integrating the resulting inequality obtained with respect to τ and ρ from 0 to x, then applying (2.40) and (2.54) on the resulting inequality, we get the desired result (2.52). This concludes the proof of the theorem. □

Theorem 2.18. *Let u and v be two nonnegative functions on $[0,\infty)$ and let f and g be two Lipschitzian functions defined on $[0,\infty)$ with the constants L_1 and L_2, respectively. Then we have*

$$\begin{aligned}&\left|K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(vfg)+K_t^{\gamma,\delta,\zeta,\nu}vK_t^{\alpha,\beta,\eta,\mu}(ufg)-K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\gamma,\delta,\zeta,\nu}(yg)-K_t^{\gamma,\delta,\zeta,\nu}(vf)K_t^{\alpha,\beta,\eta,\mu}(xg)\right|\\&\le L_1L_2\Big(K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(x^2v(x))+K_t^{\gamma,\delta,\zeta,\nu}vK_t^{\alpha,\beta,\eta,\mu}(x^2u(x))-2K_t^{\alpha,\beta,\eta,\mu}(xu(x))K_t^{\gamma,\delta,\zeta,\nu}(xv(x))\Big),\end{aligned}\tag{2.55}$$

for all $x\in[0,\infty)$, and real constants $\alpha,\gamma,\beta,\delta,\eta,\zeta,\mu,\nu$ satisfying $\alpha,\gamma>0$, $\mu,\nu>-1$, $\eta,\zeta\le 0$ and $\alpha+\beta+\mu,\gamma+\delta+\zeta\ge 0$.

Proof. From the conditions of Theorem 2.18, we have

$$|f(\tau)-f(\rho)|\le L_1|\tau-\rho|,\quad |g(\tau)-g(\rho)|\le L_2|\tau-\rho|,\quad \tau,\rho\in[0,\infty),$$

which implies that

$$|H(\tau,\rho)|=|f(\tau)-f(\rho)||g(\tau)-g(\rho)|\le L_1L_2(\tau-\rho)^2.\tag{2.56}$$

Multiplying (2.27) by $u(\tau)F(t,\tau)v(\rho)G(t,\rho)$, where $F(t,\tau)$ and $G(t,\rho)$ are defined by (2.28) and (2.39), respectively, and integrating the resulting inequality obtained with respect to τ and ρ from 0 to x, then applying (2.40) and (2.56), on the resulting inequality, we get the desired result (2.55). This completes the proof. □

Corollary 2.19. *Let u and v be two nonnegative functions on $[0,\infty)$ and let f and g be two differentiable functions on $[0,\infty)$ with $\sup_{t\ge0}|f'(t)|,\sup_{t\ge0}|g'(t)|<\infty$. Then we have*

$$\begin{aligned}&\left|K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(vfg)+K_t^{\gamma,\delta,\zeta,\nu}vK_t^{\alpha,\beta,\eta,\mu}(ufg)-K_t^{\alpha,\beta,\eta,\mu}(uf)K_t^{\gamma,\delta,\zeta,\nu}(yg)-K_t^{\gamma,\delta,\zeta,\nu}(vf)K_t^{\alpha,\beta,\eta,\mu}(xg)\right|\\&\le\|f'\|_\infty\|g'\|_\infty\Big(K_t^{\alpha,\beta,\eta,\mu}uK_t^{\gamma,\delta,\zeta,\nu}(x^2v(x))+K_t^{\gamma,\delta,\zeta,\nu}vK_t^{\alpha,\beta,\eta,\mu}(x^2u(x))-2K_t^{\alpha,\beta,\eta,\mu}(xu(x))K_t^{\gamma,\delta,\zeta,\nu}(xv(x))\Big),\end{aligned}$$

for all $x\in[0,\infty)$, and real constants $\alpha,\gamma,\beta,\delta,\eta,\zeta,\mu,\nu$ satisfying $\alpha,\gamma>0$, $\mu,\nu>-1$, $\eta,\zeta\le 0$ and $\alpha+\beta+\mu,\gamma+\delta+\zeta\ge 0$.

Proof. We have $f(\tau)-f(\rho)=\int_\rho^\tau f'(t)dt$ and $g(\tau)-g(\rho)=\int_\rho^\tau g'(t)dt$. That is, $|f(\tau)-f(\rho)|\le\|f'\|_\infty|\tau-\rho|$, $|g(\tau)-g(\rho)|\le\|g'\|_\infty|\tau-\rho|$, $\tau,\rho\in[0,\infty)$, and the result follows from Theorem 2.18. This ends the proof. □

3 An example

In this section we present a way for constructing the four bounding functions, and use them to give some estimates of Chebyshev type inequalities involving the generalized fractional integral operator of two unknown functions.

For $0=x_0<x_1<x_2<\cdots<x_n<x_{n+1}=T$, we define a notation of sub-integrals of generalized fractional integral $I_x^{\alpha,\beta,\eta,\mu}$ as

$$I_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}\{f(T)\}=\frac{x^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)}\int_{x_j}^{x_{j+1}}t^\mu(T-t)^{\alpha-1}{}_2F_1\left(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{t}{T}\right)f(t)dt,\quad j=0,1,\dots,n.\tag{3.1}$$

Note that

$$I_{0,T}^{\alpha,\beta,\eta,\mu}\{f(T)\} = \sum_{j=0}^{n} I_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}\{f(T)\} = \frac{x^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)}\int_0^{x_1} t^{\mu}(T-t)^{\alpha-1}{}_2F_1\left(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{t}{T}\right)f(t)dt$$
$$+\frac{x^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)}\int_{x_1}^{x_2} t^{\mu}(T-t)^{\alpha-1}{}_2F_1\left(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{t}{T}\right)f(t)dt+\cdots$$
$$+\frac{x^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)}\int_{x_n}^{T} t^{\mu}(T-t)^{\alpha-1}{}_2F_1\left(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{t}{T}\right)f(t)dt. \quad (3.2)$$

So, from (3.2), we can rewrite (1.7) as

$$K_{0,T}^{\alpha,\beta,\eta,\mu}f(T) = \frac{\Gamma(1-\beta)\Gamma(\alpha+\mu+\eta+1)}{\Gamma(\eta-\beta+1)\Gamma(\mu+1)}T^{\beta+\mu}\, I_{0,T}^{\alpha,\beta,\eta,\mu}\{f(T)\}$$
$$=\frac{\Gamma(1-\beta)\Gamma(\alpha+\mu+\eta+1)}{\Gamma(\eta-\beta+1)\Gamma(\mu+1)}T^{\beta+\mu}\sum_{j=0}^{n} I_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}\{f(T)\} = \frac{\Gamma(1-\beta)\Gamma(\alpha+\mu+\eta+1)}{\Gamma(\eta-\beta+1)\Gamma(\mu+1)}x^{\beta+\mu}$$
$$\times\left\{\frac{T^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)}\int_0^{x_1} t^{\mu}(T-t)^{\alpha-1}{}_2F_1\left(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{t}{T}\right)f(t)dt\right.$$
$$\frac{x^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)}\int_{x_1}^{x_2} t^{\mu}(T-t)^{\alpha-1}{}_2F_1\left(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{t}{T}\right)f(t)dt$$
$$\left.\cdots+\frac{x^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)}\int_{x_n}^{T} t^{\mu}(T-t)^{\alpha-1}{}_2F_1\left(\alpha+\beta+\mu,-\eta;\alpha;1-\frac{t}{T}\right)f(t)dt\right\}. \quad (3.3)$$

Let φ be a unit step function defined by

$$\varphi(x) = \begin{cases} 1, & x>0,\\ 0, & x\le 0,\end{cases}$$

and let $\varphi_a(x)$ the Heaviside unit step function defined by

$$\varphi_a(x) = \varphi(x-a) = \begin{cases} 1, & x>a,\\ 0, & x\le a.\end{cases}$$

Let u be a piecewise continuous function on $[0,T]$ defined by

$$u(x) = U_1(\varphi_0(x)-\varphi_{x_1}(x)) + U_2(\varphi_{x_1}(x)-\varphi_{x_2}(x)) + U_3(\varphi_{x_2}(x)-\varphi_{x_3}(x)) + \cdots + U_{m+1}\varphi_{x_m}(x) = U_1\varphi_0(x)$$
$$+(U_2-U_1)\varphi_{x_1}(x) + (U_3-U_2)\varphi_{x_2}(x) + \cdots + (U_{m+1}-U_m)\varphi_{x_m}(x) = \sum_{j=0}^{m}(U_{j+1}-U_j)\varphi_{x_j}(x), \quad (3.4)$$

where $U_0\equiv 0$ and $0 = x_0 < x_1 < x_2 < \cdots < x_m < x_{m+1} = T$. Similarly, we have

$$v(x) = \sum_{j=0}^{m}(V_{j+1}-V_j)\varphi_{x_j}(x). \quad (3.5)$$

where constants $U_0 = V_0 \equiv 0$.

Proposition 3.1. *Let f and g be two synchronous functions on $[0,T)$. Assume that let u and v defined by (3.4) and (3.5), respectively. Then for $\alpha>0, \mu>-1, \eta\le 0$ and $\alpha+\beta+\mu\ge 0$, the following inequality holds:*

$$\left(\sum_{j=0}^{m}U_{j+1}\right)\left(\sum_{j=0}^{m}V_{j+1}K_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}(fg)(T)\right) + \left(\sum_{j=0}^{m}V_{j+1}\right)\left(\sum_{j=0}^{m}U_{j+1}K_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}(fg)(T)\right)$$
$$\ge\left(\sum_{j=0}^{m}U_{j+1}K_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}g(T)\right)\left(\sum_{j=0}^{m}V_{j+1}K_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}f(T)\right)+\left(\sum_{j=0}^{m}V_{j+1}K_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}g(T)\right)\left(\sum_{j=0}^{m}U_{j+1}K_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}f(T)\right). \quad (3.6)$$

Proof. By using the definition (3.1) and (3.3), we have

$$K_{0,T}^{\alpha,\beta,\eta,\mu}u(T) = \sum_{j=0}^{m} U_{j+1}K_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}(1)(T) = \sum_{j=0}^{m} U_{j+1},$$

and

$$K_{0,T}^{\alpha,\beta,\eta,\mu}v(T) = \sum_{j=0}^{m} V_{j+1}K_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}(1)(T) = \sum_{j=0}^{m} V_{j+1},$$

where $K_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}(1)(T) = 1$. Similarly, we have

$$K_{0,T}^{\alpha,\beta,\eta,\mu}(ufg)(T) = \sum_{j=0}^{m} U_{j+1}K_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}(fg)(T), \quad K_{0,T}^{\alpha,\beta,\eta,\mu}(vfg)(T) = \sum_{j=0}^{m} V_{j+1}K_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}(fg)(T),$$

$$K_{0,T}^{\alpha,\beta,\eta,\mu}(uf)(T) = \sum_{j=0}^{m} U_{j+1}K_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}f(T), \quad K_{0,T}^{\alpha,\beta,\eta,\mu}(vf)(T) = \sum_{j=0}^{m} V_{j+1}K_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}f(T),$$

$$K_{0,T}^{\alpha,\beta,\eta,\mu}(ug)(T) = \sum_{j=0}^{m} U_{j+1}K_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}g(T), \quad K_{0,T}^{\alpha,\beta,\eta,\mu}(vg)(T) = \sum_{j=0}^{m} V_{j+1}K_{x_j,x_{j+1}}^{\alpha,\beta,\eta,\mu}g(T),$$

By applying Lemma 2.1, the desired inequality (3.6) is established. □

4 Concluding remarks

In this section, we consider some consequences of the main results derived in the previous section. Following Curiel and Galue [33], the operator would reduce immediately to the extensively investigated Saigo, Erdélyi-Kober, and Riemann-Liouville type fractional integral operators, respectively, given by the following relationships (see also [32, 34]):

$$I_{0,x}^{\alpha,\beta,\eta}\{f(x)\} = I_{x}^{\alpha,\beta,\eta,0}\{f(x)\} = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x}\right)f(\tau)dt, \quad (\alpha>0;\beta,\eta\in\mathbb{R}), \tag{4.1}$$

$$I^{\alpha,\eta}\{f(x)\} = I_{x}^{\alpha,0,\eta,0}\{f(x)\} = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}t^{\alpha-1}f(t)dt, \quad (\alpha>0;\eta\in\mathbb{R}), \tag{4.2}$$

and

$$J^{\alpha}\{f(x)\} = I_{x}^{\alpha,-\alpha,\eta,0}\{f(x)\} = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt, \quad (\alpha>0). \tag{4.3}$$

By setting $\mu=0$, $\mu=\beta=0$, and $\mu=0$ and $\beta=-\alpha$ in (1.7), Definition 1.2 would immediately reduce to the Saigo, Erdélyi-Kober, and Riemann-Liouville type fractional integral operators, respectively, given as follows:

$$K_x^{\alpha,\beta,\eta}f(x) = \frac{\Gamma(1-\beta)\Gamma(\alpha+\eta+1)}{\Gamma(\eta-\beta+1)}x^{\beta}I_{0,x}^{\alpha,\beta,\eta}\{f(x)\}, \tag{4.4}$$

$$K_x^{\alpha,\eta}f(x) = \frac{\Gamma(\eta+\alpha+1)}{\Gamma(1+\eta)}I^{\alpha,\eta}\{f(x)\}, \tag{4.5}$$

and

$$K_x^{\alpha}f(x) = \frac{\Gamma(\alpha+1)}{x^{\alpha}}J^{\alpha}\{f(x)\}, \tag{4.6}$$

where $I_{0,x}^{\alpha,\beta,\eta}\{f(x)\}$, $I^{\alpha,\eta}\{f(x)\}$ and $J^{\alpha}\{f(x)\}$ are given by (4.1), (4.2), and (4.3), respectively.

Similar to main results in the preceding section, by using the fractional integral operators (4.1)-(4.6), we obtain various fractional integral inequalities involving such relatively more familiar fractional integral operators (4.1)-(4.6). Therefore, we omit the further details. For example, by (4.1), Theorem 2.2 and 2.4 yield the known

results in [24, 25]. If we consider $u = v = 1$ and make use of fractional integral operator $I_x^{\alpha,\beta,\eta,\mu}\{f(x)\}$, Lemma 2.1 and 2.3 provides respectively, the known fractional integral inequalities due to Baleanu *et al.* [31].

Let $u = 1$, Theorem 2.10 corresponds to the known results due to Wang *et al.* [28]. Taking $u = 1$, $\mu = 0$ and $\beta = -\alpha$ in Theorem 2.10 yields the known result due to Dahmani *et al.* [26]. Make use of fractional integral operator (4.3), Lemma 2.1 and 2.3 provides respectively, the known fractional integral inequalities due to Dahmani [17]. At the end of this paper, generalized fractional integral inequalities obtained in the previous section are expected to find more applications, for example, applications for establishing the solutions in fractional differential equations and fractional integral equations boundary value problems.

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Estimates for the Green's Function of 3D Elliptic Equations

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This article will first introduce the definition of the Green's function of 3D elliptic equations, which plays important roles in local superconvergence estimates for the finite element approximation. Then, using the weighted-norm methods, we derive some estimates for the 3D Green's function.

1 Introduction

It is well known that estimates for the Green's function play very important roles in the study of the superconvergence (especially, pointwise superconvergence) of the finite element method (see [1–9]). For dimensions three and up, we have obtained the estimates for discrete Green's functions and discrete derivative Green's functions, which were used to the global superconvergence estimates of the finite element approximation. However, the fact is that the high generalization conditions to the true solution is difficult to satisfy for the global superconvergence estimates. Thus the global superconvergence results is only theoretical. In order to study local superconvergence properties of the finite element approximation, we need to introduce a Green's function, which will play important roles in the study of local superconvergence properties.

we shall use the symbol C to denote a generic constant, which is independent from the discretization parameter h and which may not be the same in each occurrence and also use the standard notations for the Sobolev spaces and their norms.

In this article, we consider the following elliptic equation:

$$\mathcal{L}u \equiv -\sum_{i,j=1}^{3} \partial_j(a_{ij}\partial_i u) + a_0 u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where $\Omega \subset \mathcal{R}^3$ is a bounded polytopic domain. The weak formulation of (1.1) reads,

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ satisfying} \\ a(u\,,\,v) = (f\,,\,v) \text{ for all } v \in H_0^1(\Omega), \end{cases}$$

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where

$$a(u\,,\,v)\equiv\int_\Omega(\sum_{i,j=1}^{3}a_{ij}\partial_iu\partial_jv+a_0uv)\,dxdydz,\ (f\,,\,v)\equiv\int_\Omega fv\,dxdydz.$$

We assume that the given functions $a_{ij}\in W^{1,\infty}(\Omega)$, $a_{ij}=a_{ji}$, $a_0\in L^\infty(\Omega)$, and $f\in L^2(\Omega)$. In addition, we write $\partial_1u=\frac{\partial u}{\partial x}$, $\partial_2u=\frac{\partial u}{\partial y}$, and $\partial_3u=\frac{\partial u}{\partial z}$, which are usual partial derivatives. Let $\{\mathcal{T}^h\}$ be a regular family of partitions of $\bar\Omega$. Denote by $S^h(\Omega)$ a continuous finite elements space of degree $m(m\ge1)$ regarding this kind of partitions and let $S_0^h(\Omega)=S^h(\Omega)\cap H_0^1(\Omega)$. Discretizing the above weak formulation using $S_0^h(\Omega)$ as approximating space means,

$$\begin{cases}\text{Find } u_h\in S_0^h(\Omega)\text{ satisfying}\\ a(u_h\,,\,v)=(f\,,\,v)\text{ for all } v\in S_0^h(\Omega).\end{cases}$$

For every $Z\in\Omega$, we define the discrete δ function $\delta_Z^h\in S_0^h(\Omega)$, the discrete derivative δ function $\partial_{Z,\ell}\delta_Z^h\in S_0^h(\Omega)$, the regularized Green's function $G_Z^*\in H^2(\Omega)\cap H_0^1(\Omega)$, the regularized derivative Green's function $\partial_{Z,\ell}G_Z^*\in H^2(\Omega)\cap H_0^1(\Omega)$, the discrete Green's function $G_Z^h\in S_0^h(\Omega)$, the discrete derivative Green's function $\partial_{Z,\ell}G_Z^h\in S_0^h(\Omega)$, and the L^2-projection $P_hu\in S_0^h(\Omega)$ such that (see [9])

$$(v,\,\delta_Z^h)=v(Z)\quad\forall\,v\in S_0^h(\Omega),\tag{1.2}$$

$$(v,\,\partial_{Z,\ell}\delta_Z^h)=\partial_\ell v(Z)\quad\forall\,v\in S_0^h(\Omega),\tag{1.3}$$

$$a(G_Z^*,\,v)=(\delta_Z^h,\,v)\quad\forall\,v\in H_0^1(\Omega),\tag{1.4}$$

$$a(\partial_{Z,\ell}G_Z^*,\,v)=(\partial_{Z,\ell}\delta_Z^h,\,v)\quad\forall\,v\in H_0^1(\Omega),\tag{1.5}$$

$$a(G_Z^h,\,v)=v(Z)\quad\forall\,v\in S_0^h(\Omega),\tag{1.6}$$

$$a(\partial_{Z,\ell}G_Z^h,\,v)=\partial_\ell v(Z)\quad\forall\,v\in S_0^h(\Omega),\tag{1.7}$$

$$(u-P_hu,\,v)=0\quad\forall\,v\in S_0^h(\Omega).\tag{1.8}$$

Here, for any direction $\ell\in R^3$, $|\ell|=1$, $\partial_{Z,\ell}\delta_Z^h$, $\partial_{Z,\ell}G_Z^h$, and $\partial_\ell v(Z)$ stand for the following onesided directional derivatives, respectively.

$$\partial_{Z,\ell}\delta_Z^h=\lim_{|\Delta Z|\to0}\frac{\delta_{Z+\Delta Z}^h-\delta_Z^h}{|\Delta Z|},\ \partial_{Z,\ell}G_Z^h=\lim_{|\Delta Z|\to0}\frac{G_{Z+\Delta Z}^h-G_Z^h}{|\Delta Z|},$$

$$\partial_\ell v(Z)=\lim_{|\Delta Z|\to0}\frac{v(Z+\Delta Z)-v(Z)}{|\Delta Z|},\ \Delta Z=|\Delta Z|\ell.$$

As for G_Z^*, $\partial_{Z,\ell}G_Z^*$, G_Z^h, and $\partial_{Z,\ell}G_Z^h$, we have obtained some optimal estimates (see [4–6]), which will be used in next section. From (1.4)–(1.7), we easily find G_Z^h and $\partial_{Z,\ell}G_Z^h$ are the finite element approximations to G_Z^* and $\partial_{Z,\ell}G_Z^*$, respectively.

For the L^2-projection operator P_h, we have (see [4])

Lemma 1.1. *For $P_h w$ the L^2-projection of $w \in L^p(\Omega)$, we have the following stability estimate:*

$$\|P_h w\|_{0,\,p,\,\Omega} \leq C^t \|w\|_{0,\,p,\,\Omega}, \tag{1.9}$$

where $t = \left|1 - \frac{2}{p}\right|$, and $1 \leq p \leq \infty$.

Further, by Lemma 1.1, we easily obtain the following result:

$$\begin{aligned} \|w - P_h w\|_{0,\,p,\,\Omega} &\leq (1 + C^t) \inf_{v \in S_0^h \Omega} \|w - v\|_{0,\,p,\,\Omega} \\ &\leq C\|w - \Pi w\|_{0,\,p,\,\Omega} \leq C h^{m+1} \|w\|_{m+1,\,p,\,\Omega}, \end{aligned} \tag{1.10}$$

where $1 \leq p \leq \infty$.

In addition, we also assume the following a priori estimate holds.

Lemma 1.2. *For the true solution u of (1.1), there exists a $q_0(1 < q_0 \leq \infty)$ such that for every $1 < q < q_0$,*

$$\|u\|_{2,\,q,\,\Omega} \leq C(q)\|\mathcal{L}u\|_{0,\,q,\,\Omega}. \tag{1.11}$$

2 Definition of the 3D Green's Function

For $Z \in \Omega$, we introduce the definition of the 3D Green's function G_Z as follows

$$a(G_Z, v) = v(Z) \ \ \forall\, v \in C_0^\infty(\Omega).$$

In the following, we will prove the existence and uniqueness of the Green's function.

Lemma 2.1. *For G_Z^* and G_Z^h defined by (1.4) and (1.6), respectively, we have*

$$\left\|G_Z^* - G_Z^h\right\|_{1,1} \leq Ch\,|\ln h|^{\frac{2}{3}}\,. \tag{2.1}$$

This result can be seen in [4].

Theorem 2.1. *There exists a unique $G_Z \in W_0^{1,1}(\Omega)$ such that*

$$a(G_Z, v) = v(Z) \ \ \forall\, v \in W_0^{1,\infty}(\Omega). \tag{2.2}$$

Proof. We first prove the uniqueness of G_Z. Suppose there exists another Green's function $H_Z \in W_0^{1,1}(\Omega)$ satisfying (2.2). Set $E_Z = G_Z - H_Z$, thus

$$a(E_Z, v) = 0 \ \ \forall\, v \in W_0^{1,\infty}(\Omega). \tag{2.3}$$

Let $w \in W^{2,4}(\Omega) \cap W_0^{1,4}(\Omega)$ and $\mathcal{L}w = \mathrm{sgn} E_Z |E_Z|^{\frac{1}{4}}$. We have

$$\|E_Z\|_{0,\frac{5}{4}}^{\frac{5}{4}} = (E_Z, \mathrm{sgn} E_Z |E_Z|^{\frac{1}{4}}) = a(E_Z, w), \tag{2.4}$$

By the Sobolev Embedding Theorem [10], $W^{2,4}(\Omega) \hookrightarrow W^{1,\infty}$. Thus $w \in W_0^{1,\infty}(\Omega)$. From (2.3) and (2.4), $E_Z = 0$, i.e., $G_Z = H_Z$. The proof of the uniqueness is completed.

Next, we prove the existence of G_Z. We give a series of finite element spaces $S_0^{h_i}(\Omega)$, $i=0,1,2,\cdots$ satisfying $S_0^{h_i}(\Omega)\subset S_0^{h_j}(\Omega)$ when $i<j$, where $h_0\equiv h$ and $\frac{1}{4}h_{i-1}\leq h_i\leq\frac{1}{2}h_{i-1}$. Let $G^*_{Z,i}$ be the regularized Green's function for the finite element space $S_0^{h_i}(\Omega)$, and $G_Z^{h_i}$ the discrete Green's function. Their definitions can be seen in Section 1. Obviously, we have

$$a(G_Z^{h_i},v)=v(Z),\ a(G^*_{Z,i+1},v)=v(Z),\ \forall\, v\in S_0^{h_i}(\Omega).$$

Thus,

$$a(G^*_{Z,i+1}-G_Z^{h_i},v)=0\ \ \forall\, v\in S_0^{h_i}(\Omega). \tag{2.5}$$

Similar to the proof of Lemma 2.1, we have

$$\left\|G^*_{Z,i+1}-G_Z^{h_i}\right\|_{1,1}\leq Ch_i\left|\ln h_i\right|^{\frac{2}{3}}. \tag{2.6}$$

In addition, from (2.1),

$$\left\|G^*_{Z,i}-G_Z^{h_i}\right\|_{1,1}\leq Ch_i\left|\ln h_i\right|^{\frac{2}{3}}. \tag{2.7}$$

By (2.6), (2.7), and the triangular inequality, we immediately obtain

$$\left\|G^*_{Z,i+1}-G^*_{Z,i}\right\|_{1,1}\leq Ch_i\left|\ln h_i\right|^{\frac{2}{3}}.$$

Thus,

$$\sum_{i=0}^{\infty}\left\|G^*_{Z,i+1}-G^*_{Z,i}\right\|_{1,1}\leq C\sum_{i=0}^{\infty}\frac{h}{2^i}\left|\ln\frac{h}{2^i}\right|^{\frac{2}{3}}\leq Ch\left|\ln h\right|^{\frac{2}{3}}. \tag{2.8}$$

Set

$$G_Z\equiv G^*_Z+\sum_{i=0}^{\infty}(G^*_{Z,i+1}-G^*_{Z,i}).$$

Thus we have $G_Z\in W_0^{1,1}(\Omega)$. From (2.8),

$$\left\|G_Z-G^*_Z\right\|_{1,1}\leq Ch\left|\ln h\right|^{\frac{2}{3}}. \tag{2.9}$$

Thus, we have

$$G^*_{Z,i}\longrightarrow G_Z\ \text{in}\, W^{1,1}(\Omega)\ \text{when}\, i\to\infty.$$

Hence, for $v\in W_0^{1,\infty}(\Omega)$, we have

$$a(G_Z,v)=\lim_{i\to\infty}a(G^*_{Z,i},v)=\lim_{i\to\infty}P_{h_i}v(Z). \tag{2.10}$$

From (1.10),

$$\lim_{i\to\infty}P_{h_i}v(Z)=v(Z). \tag{2.11}$$

Combining (2.10) and (2.11) yields the result (2.2).

Finally, we show G_Z is independent of h. Suppose there exists a Green's function $\tilde{G}_Z$ for the mesh-size $\tilde{h}$. In addition, $\frac{1}{4}\tilde{h}_{i-1}\leq\tilde{h}_i\leq\frac{1}{2}\tilde{h}_{i-1}$ and $\tilde{h}_0=\tilde{h}$. Thus, for every $f\in L^\infty(\Omega)$, we choose $v\in W^{2,\infty}(\Omega)\cap W_0^{1,\infty}(\Omega)$ such that $\mathcal{L}v=f$. Then we get $(G_Z,f)=a(G_Z,v)=v(Z)$ and $(\tilde{G}_Z,f)=a(\tilde{G}_Z,v)=v(Z)$. Thus, $(G_Z,f)=(\tilde{G}_Z,f)$, i.e., $(G_Z-\tilde{G}_Z,f)=0$. So we get $G_Z=\tilde{G}_Z$. The proof of Theorem 2.1 is completed.

3 Estimates for the 3D Green's Function

Lemma 3.1. *Suppose* $1 < p < \min\{2, q_0\}$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *For* G_Z^*, $\partial_{Z,\ell} G_Z^*$, G_Z^h, *and* $\partial_{Z,\ell} G_Z^h$ *defined by (1.4)–(1.7), we have*

$$\left\|G_Z^* - G_Z^h\right\|_{0,q} + h\left\|\partial_{Z,\ell} G_Z^* - \partial_{Z,\ell} G_Z^h\right\|_{0,q} \le Ch^{2-\frac{3}{p}}. \tag{3.1}$$

Proof. Obviously, by the interpolation error estimate and the a priori estimate (1.11), we have

$$\begin{array}{rcl} \left\|G_Z^* - G_Z^h\right\|_1 & \le & C\inf_{v\in S_0^h(\Omega)} \|G_Z^* - v\|_1 \le \|G_Z^* - \Pi G_Z^*\|_1 \\ & \le & Ch^{2.5-\frac{3}{p}} \|G_Z^*\|_{2,p} \le Ch^{2.5-\frac{3}{p}} \left\|\delta_Z^h\right\|_{0,p}. \end{array} \tag{3.2}$$

For $\varphi \in L^p(\Omega)$, we choose $\Phi \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that $\mathcal{L}\Phi = \varphi$. Then we have

$$\begin{array}{rcl} \left|(G_Z^* - G_Z^h, \varphi)\right| & = & \left|a(G_Z^* - G_Z^h, \Phi)\right| = \left|a(G_Z^* - G_Z^h, \Phi - \Pi\Phi)\right| \\ & \le & C\left\|G_Z^* - G_Z^h\right\|_1 \|\Phi - \Pi\Phi\|_1 . \end{array} \tag{3.3}$$

From (3.2), (3.3), and the interpolation error estimate, we get

$$\left|(G_Z^* - G_Z^h, \varphi)\right| \le Ch^{5-\frac{6}{p}} \left\|\delta_Z^h\right\|_{0,p} \|\varphi\|_{0,p}. \tag{3.4}$$

Thus

$$\left\|G_Z^* - G_Z^h\right\|_{0,q} \le Ch^{5-\frac{6}{p}} \left\|\delta_Z^h\right\|_{0,p}. \tag{3.5}$$

In addition, for $1 \le p \le \infty$, we easily prove

$$\left\|\delta_Z^h\right\|_{0,p} + h\left\|\partial_{Z,\ell}\delta_Z^h\right\|_{0,p} \le Ch^{-3+\frac{3}{p}}. \tag{3.6}$$

From (3.5) and (3.6),

$$\left\|G_Z^* - G_Z^h\right\|_{0,q} \le Ch^{2-\frac{3}{p}}.$$

Similarly, we have

$$\left\|\partial_{Z,\ell} G_Z^* - \partial_{Z,\ell} G_Z^h\right\|_{0,q} \le Ch^{1-\frac{3}{p}}.$$

The result (3.1) is proved. We now introduce a weight function defined by

$$\phi \equiv \phi(X) = \left(|X - \bar{X}|^2 + \theta^2\right)^{-\frac{3}{2}} \ \ \forall\, X \in \bar{\Omega},$$

where $\bar{X} \in \bar{\Omega}$ is a fixed point, $\theta = \gamma h$, and $\gamma \in [3, +\infty)$ is a suitable real number. As for the function ϕ, it is easy to prove the following properties hold.

$$\int_\Omega \phi^k(X)dX \le C(k-1)^{-1}\theta^{-3(k-1)} \ \ \forall\, k > 1, \tag{3.7}$$

$$\int_\Omega \phi^k(X)dX \le \frac{C}{1-k} \ \ \forall\, 0 < k < 1, \tag{3.8}$$

$$\int_\Omega \phi(X)dX \le C(\beta)|\ln\theta|, \ \ \theta \le \beta < 1. \tag{3.9}$$

Similar to the arguments of Lemma 2.4 in [4], we can get the following Lemma 3.2.

Lemma 3.2. *For δ_Z^h and $\partial_{Z,\ell}\delta_Z^h$, the discrete δ function and the discrete derivative δ function defined by (1.2) and (1.3), respectively, we have the following weighted-norm estimate:*

$$\left\|\delta_Z^h\right\|_{\phi^{-\alpha}} + h\left\|\nabla\delta_Z^h\right\|_{\phi^{-\alpha}} + h\left\|\partial_{Z,\ell}\delta_Z^h\right\|_{\phi^{-\alpha}} \le Ch^{\frac{3(\alpha-1)}{2}} \quad \forall\alpha > 0. \tag{3.10}$$

Lemma 3.3. *For δ_Z^h and G_Z^*, the discrete δ function and the regularized Green's function defined by (1.2) and (1.4), respectively, we have the following weighted-norm estimate:*

$$\|\nabla G_Z^*\|_{\phi^{-\alpha}} \le C\left\|\delta_Z^h\right\|_{\phi^{-\alpha-\frac{2}{3}}} + C\left\|G_Z^*\right\|_{\phi^{-\alpha+\frac{2}{3}}} \quad \forall\alpha\in R. \tag{3.11}$$

Proof. First, we find

$$\|\nabla G_Z^*\|^2_{\phi^{-\alpha}} \le a(G_Z^*\,,\,\phi^{-\alpha}G_Z^*) + C\left\|G_Z^*\right\|^2_{\phi^{-\alpha+\frac{2}{3}}}. \tag{3.12}$$

Moreover,

$$\begin{array}{rcl} a(G_Z^*\,,\,\phi^{-\alpha}G_Z^*) &=& (\delta_Z^h\,,\,\phi^{-\alpha}G_Z^*) \\ &\le& \|\delta_Z^h\|_{\phi^{-\alpha-\frac{2}{3}}}\|G_Z^*\|_{\phi^{-\alpha+\frac{2}{3}}} \\ &\le& \frac{1}{2}(\|\delta_Z^h\|^2_{\phi^{-\alpha-\frac{2}{3}}} + \|G_Z^*\|^2_{\phi^{-\alpha+\frac{2}{3}}}). \end{array} \tag{3.13}$$

Combining (3.12) and (3.13) immediately yields the result (3.11).

Theorem 3.1. *Suppose $q_0 > \frac{3}{2}$, $\frac{3}{2} < p < \min\{2, q_0\}$, and $\frac{1}{p} + \frac{1}{q} = 1$, then we have*

$$\|G_Z - G_Z^*\|_{0,q} \le Ch^{2-\frac{3}{p}} = Ch^{\frac{3-q}{q}}. \tag{3.14}$$

Remark 1. Similar to the arguments of (2.9) and with the result (3.1), we easily obtain the result (3.14). Obviously, we have $\max\{2, q_0'\} < q < 3$ and $\frac{1}{q_0} + \frac{1}{q_0'} = 1$.

Theorem 3.2. *Suppose $q_0 > \frac{3}{2}$. For G_Z, the Green's function defined by (2.2), and the weight function $\tau = |X - Z|^{-3}$, we have*

$$\|G_Z\|_{0,q} \le C(q), \ \ 1 \le q \le 3. \tag{3.15}$$

$$\|G_Z\|_{1,\tau^{-\epsilon}} \le C(\epsilon), \ \ \frac{1}{3} < \epsilon < \infty. \tag{3.16}$$

$$\|G_Z\|_{1,q} \le C(q), \ \ 1 \le q < \frac{3}{2}. \tag{3.17}$$

Proof. Obviously, from (3.14), $G_Z \in L^q(\Omega)$ and $1 \le q < 3$. In addition, we have proved $\|G_Z^*\|_{0,3} \le C$ in [4]. Moreover, $L^3(\Omega)$ is a reflexive space. Thus, $\{G_{Z,i}^*\}$

is weakly convergent to $Q_Z \in L^3(\Omega) \subset L^q(\Omega)$, where $\max\{2, q_0'\} < q < 3$. From (3.14),

$$G^*_{Z,i} \longrightarrow G_Z \text{ in } L^q(\Omega) \text{ when } i \to \infty.$$

Thus $G_Z = Q_Z \in L^3(\Omega)$. So we have $G_Z \in L^q(\Omega)(1 \le q \le 3)$.

When $\max\{2, q_0'\} < q < 3$, we have $\frac{3}{2} < p < \min\{2, q_0\}$, where $\frac{1}{p} + \frac{1}{q} = 1$. For every $\varphi \in C_0^\infty(\Omega)$, we can find a function $\tilde{\varphi} \in C_0^\infty(\Omega)$ such that $\mathcal{L}\tilde{\varphi} = \varphi$. Moreover, by the Sobolev Embedding Theorem [10] and the a priori estimate (1.11), we get

$$(G_Z, \varphi) = a(G_Z, \tilde{\varphi}) = \tilde{\varphi}(Z) \le \|\tilde{\varphi}\|_{0,\infty} \le C(q)\, \|\tilde{\varphi}\|_{2,p} \le C(q)\, \|\varphi\|_{0,p}\,.$$

Thus,

$$\|G_Z\|_{0,q} \le C(q).$$

Since $\left\|G^*_{Z,i}\right\|_{0,3} \le C$, and $\{G^*_{Z,i}\}$ is weakly convergent to $G_Z \in L^3(\Omega)$, thus, $\|G_Z\|_{0,3} \le C$. In addition, when $1 \le q \le \max\{2, q_0'\}$, we have $\|G_Z\|_{0,q} \le C(q)\,\|G_Z\|_{0,3} \le C(q)$. Thus we have finished the proof of the result (3.15).

Now we prove the result (3.16). We have obtained the result $\|G^*_Z\|_{\phi^{\frac{1}{3}}} \le C\,|\ln h|^{\frac{1}{6}}$ in [4]. When $0 < r < \frac{1}{3}$, we have by (3.8) and $\|G^*_Z\|_{0,3} \le C$,

$$\|G^*_Z\|^2_{\phi^r} = \int_\Omega \phi^r\, |G^*_Z|^2\, dX \le \left(\int_\Omega \phi^{3r}\, dX\right)^{\frac{1}{3}} \|G^*_Z\|^2_{0,\,3} \le C(r)\, \|G^*_Z\|^2_{0,\,3} \le C(r).$$

Namely, $\|G^*_Z\|_{\phi^r} \le C(r)\ \forall\, 0 < r < \frac{1}{3}$. Obviously, when $s < t$, we have $\phi^s \le C\phi^t$. Thus, $\|G^*_Z\|_{\phi^r} \le C(r)\ \forall\, r \le 0$. So we have

$$\|G^*_Z\|_{\phi^r} \le C(r)\ \ \forall\, r < \frac{1}{3}. \tag{3.18}$$

From (3.10) and (3.11),

$$\begin{aligned} \|\nabla G^*_Z\|_{\phi^{-\epsilon}} &\le C\left\|\delta^h_Z\right\|_{\phi^{-\epsilon-\frac{2}{3}}} + C\,\|G^*_Z\|_{\phi^{-\epsilon+\frac{2}{3}}} \\ &\le Ch^{\frac{3\epsilon-1}{2}} + C\,\|G^*_Z\|_{\phi^{-\epsilon+\frac{2}{3}}}\,. \end{aligned} \tag{3.19}$$

Combining (3.18) and (3.19) yields

$$\|G^*_Z\|_{1,\phi^{-\epsilon}} \le C(\epsilon)\ \ \forall\, \epsilon > \frac{1}{3}. \tag{3.20}$$

By the Hölder inequality, we have for $1 \le q < \frac{3}{2}$

$$\|\nabla G^*_Z\|^q_{0,q} = \int_\Omega \phi^{\frac{q\epsilon}{2}}\phi^{-\frac{q\epsilon}{2}}\,|\nabla G^*_Z|^q\, dX \le \left(\int_\Omega \phi^{\frac{q\epsilon}{2-q}}\, dX\right)^{\frac{2-q}{2}} \|\nabla G^*_Z\|^q_{\phi^{-\epsilon}}\,.$$

Choosing a suitable ϵ such that $\frac{q\epsilon}{2-q} < 1$, we have by (3.8) and (3.20),

$$\|\nabla G^*_Z\|_{0,q} \le C(q). \tag{3.21}$$

Obviously,

$$\|G_Z^*\|_{1,\tau^{-\epsilon}} \leq \|G_Z^*\|_{1,\phi^{-\epsilon}} \leq C(\epsilon) \ \forall \epsilon > \frac{1}{3}. \tag{3.22}$$

Since G_Z^* is bounded according to the weighted-norm $\|\cdot\|_{1,\tau^{-\epsilon}}$, thus, $\{G_{Z,i}^*\}$ is weakly convergent to a function F_Z with $\|F_Z\|_{1,\tau^{-\epsilon}} < \infty$. Further, we have $\|F_Z\|_{1,1,\tau^{-\epsilon}} < \infty$. From (2.9),

$$\|G_Z - G_Z^*\|_{1,1,\tau^{-\epsilon}} \leq C(\epsilon) \|G_Z - G_Z^*\|_{1,1} \leq C(\epsilon) h \left|\ln h\right|^{\frac{2}{3}},$$

which shows $\{G_{Z,i}^*\}$ is convergent to the function G_Z with $\|G_Z\|_{1,1,\tau^{-\epsilon}} < \infty$. Thus, $F_Z = G_Z$. Namely,

$$\|G_Z\|_{1,\tau^{-\epsilon}} \leq C(\epsilon) \ \forall \epsilon > \frac{1}{3}.$$

Up to now, the result (3.16) is thoroughly proved. Similar to the arguments of (3.16), from (3.21), we can obtain the result (3.17).

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The structure of the zeros and fixed point for Genocchi polynomials

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Abstract We find the behavior of complex roots and fixed point for Genocchi polynomials by using numerical investigation. By means of numerical experiments, we display a remarkably regular structure of the complex roots and fixed point for the Genocchi polynomials.

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Key words- Genocchi polynomials, Newton method, complex roots, fixed point

1. Introduction

Mathematicians have studied various kinds of the Euler, Bernoulli, Tangent, and Genocchi polynomials. Recently, many authors have studied the relations between these polynomials and Stirling numbers of the second kind(see [1-24]). Numerical experiments of Bernoulli, Euler, and Genocchi polynomials also have been made the subject of extensive research.

The computing environment will be making more and more rapid advance and this environment has been increasing the interest in solving mathematical problems with the aid of computers. The zeros of Genocchi polynomials $G_n(x)$ is very interesting a realistic study by using computer(see [2,16-20,23]).

The Genocchi numbers G_n and Genocchi polynomials $G_n(x)$ are usually defined by the following generating functions.

Definition 1.1.[5,14,17] Let $n \in \mathbb{N}_0$. Then we define

$$\sum_{n=0}^{\infty} G_n \frac{t^n}{n!} = \frac{2t}{e^t+1}, \qquad |t| < \pi,$$

$$\sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} = \left(\frac{2t}{e^t+1}\right) e^{tx},$$

1

where we use the notation by replacing $G(x)^n$ by $G_n(x)$ symbolically. Clearly, $G_n = G_n(0)$. In general, it satisfies $G_3 = G_5 = G_7 = G_9 = \cdots = 0$, and even coefficients are given $G_n = 2nE_{2n-1} = 2(1-2^{2n})B_{2n}$, where E_n are the Euler numbers and B_n are the Bernoulli numbers(see [4-5, 6, 8, 12, 15]).

These polynomials and numbers play important roles in many different areas of mathematics such as combinatorics, number theory, special function and analysis, and numerous interesting results for them have been explored. The following elementary properties of Genocchi polynomials $G_n(x)$ are readily derived from the Definition 1.1. Therefore we choose to omit the details involved. More studies and results in this subject we may see references(see [5-6,14-20]).

Throughout this paper, we always make use of the following notations: $\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the set of natural numbers, $\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$ denotes the set of nonnegative integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers, and $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

Theorem 1.2.[5,6,17,19] For $n \in \mathbb{N}_0$, we know

$$G_n(x) = \sum_{k=0}^{n} \binom{n}{k} G_k x^{n-k}.$$

Theorem 1.3.[5,6,15] Let $x \in \mathbb{N}_0$. Then we have

$$(G+1)^n + G_n = \begin{cases} 2 & \text{if } n = 1 \\ 0 & \text{if } n \neq 1 \end{cases}.$$

From the Theorem 1.2 and Theorem 1.3, it is easy to deduce that $G_n(x)$ are polynomials of degree n. The Genocchi polynomials are as follows.

$$\begin{aligned}
&G_1(x) = 1,\\
&G_2(x) = 2x - 1,\\
&G_3(x) = 3x^2 - 3x,\\
&G_4(x) = 4x^3 - 6x^2 + 1,\\
&G_5(x) = 5x^4 - 10x^3 + 5x,\\
&G_6(x) = 6x^5 - 15x^4 + 15x^2 - 3,\\
&G_7(x) = 7x^6 - 21x^5 + 35x^3 - 21x,\\
&G_8(x) = 8x^7 - 28x^6 + 70x^4 - 84x^2 + 17,\\
&\cdots.
\end{aligned}$$

Definition 1.4. Let $f : D \to D$ be a complex function, with D a subset of $\mathbb{C}$. We define the iterated maps of the complex function as the following:

$$f_n : z_0 \mapsto \underbrace{f(f(\cdots(f}_{n-\text{times}}(z_0)\cdots))$$

The iterates of f are the functions $f, f \circ f, f \circ f \circ f, ...$, which are denoted $f^1, f^2, f^3, ...$ If $z \in \mathbb{C}$, then the orbit of z_0 under f is the sequence $(z_0, f(z_0), f(f(z_0)), \cdots)$.

We consider the Newton's dynamical system as the follows:

$$\left\{\mathbb{C}_\infty : R(x) = x - \frac{S(x)}{S'(x)}\right\}.$$

R is called the Newton iteration function of S. It can be shown that the fixed points of R are zeros of S and all fixed points of R are attracting. R may also have one or more attracting cycles(see [2, 23-24]).

This paper is organized as follows. In Section 2, we study some properties of zeros for Genocchi polynomials from Newtons'method. In section 3, we find some distributions and properties of fixed point for Genocchi polynomials by using iterating map.

2. The observation for scattering of zeros of the Genocchi polynomials

In this section, we can see the several conjecture from the Tables. we also find the approximate zeros of the Genocchi polynomials. Using the Mathematica software, we can see the structure of the zeros of the Genocchi polynomials in various viewpoints.

From the Definition of Genocchi polynomials, we get

$$\sum_{n=0}^{\infty} G_n(1-x)\frac{(-t)^n}{n!} = \frac{-2t}{e^{-t}+1}e^{-t(1-x)} = -\frac{2t}{e^t-1}e^{tx} = -\sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}.$$

From the above equation, we find the following theorem.

Theorem 2.1.[14,-15,17,19-20]. For $n \in \mathbb{N}_0$, we have

$$G_n(x) = (-1)^{n+1} G_n(1-x).$$

Conjecture 2.2. $G_n(x) = 0$ has n distinct solutions.

We find a counterexample of the conjecture 2.2. When $n = 6$, there exist five numbers, $x_i (i = 1, 2, 3, 4, 5)$ such that $G_6(x_i) = 0$. That is, we can find $x_1 = \frac{1}{2}, x_2 = \frac{1}{2}\left(1-\sqrt{5}\right), x_3 = \frac{1}{2}\left(1-\sqrt{5}\right), x_4 = \frac{1}{2}\left(1+\sqrt{5}\right), x_5 = \frac{1}{2}\left(1+\sqrt{5}\right)$. Therefore, the conjecture 2.3 is not true for all n. Using computers, many more values of n have been checked. It still remains unknown if the conjecture fails or holds for any value $n \neq 6$.

See Table 1 for tabulated values of $R_{G_n(x)}$ and $C_{G_n(x)}$, where $R_{G_n(x)}$ denote the numbers of real zeros and $C_{G_n(x)}$ denotes the numbers of complex zeros. Our numerical results, that is the numbers of real and complex zeros of $G_n(x)$ for $29 \leq n \leq 60$ are displayed in the Table 1.

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Table 1. Numbers of real and complex zeros of $G_n(x)$

degree n	$R_{G_n(x)}$	$C_{G_n(x)}$	degree n	$R_{G_n(x)}$	$C_{G_n(x)}$
29	8	20	45	12	32
30	9	20	46	13	32
31	10	20	47	14	32
32	11	20	48	15	32
33	8	24	49	12	36
34	9	24	50	13	36
35	10	24	51	14	36
36	11	24	52	15	36
37	12	24	53	16	36
38	9	28	54	13	40
39	10	28	55	14	40
40	11	28	56	15	40
41	12	28	57	16	40
42	13	28	58	17	40
43	11	32	59	14	44
44	11	32	60	15	44

If we consider $G_n(x)$ for $2 \leq n \leq 100$, we then find the Figure 1. The x-axis means the numbers of real zeros and the y-axis means the numbers of complex zeros in the Genocchi polynomials in Figure 1. From Table 1 and Figure 1, we can suggest a below conjecture.

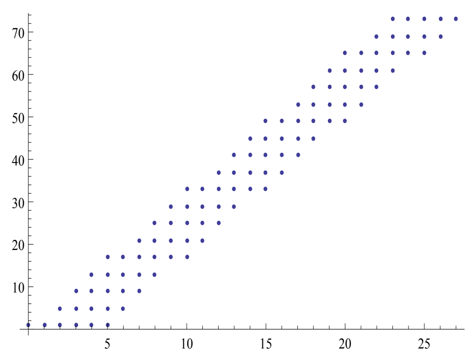

Figure 1: Numbers of real and complex zeros of $G_n(x)$ for $2 \leq n \leq 100$

Conjecture 2.3. When $Im(x) \neq 0$, we find that
(1) the numbers of $R_{G_n(x)}$ of $G_n(x)$:

$$R_{G_n(x)} = n - 1 - C_{G_n(x)}.$$

(2) the numbers of $C_{G_n(x)}$ of $G_n(x)$:

$$C_{G_n(x)} = 4\left[\frac{n-1-\alpha}{5}\right], \quad \alpha = \left[\frac{n+19}{21}\right],$$

4

where $[x]$ is the greatest integer not exceeding x.

By using the Theorem 2.1, we also have the following theorem.

Theorem 2.4. For $n \in \mathbb{N}_0$, if $n \equiv 0 \pmod 2$, then $G_n\left(\frac{1}{2}\right) = 0$.

By Theorem 2.4, we can know the center of the structure of zeros in Genocchi polynomials is $\frac{1}{2}$(see the Figure2). The forms of 3D structure which is stacks of zeros of $G_n(x)$ for $2 \le n \le 60$ are presented in the top-left of Figure 2. We can draw the top-right figure and bottom-left figure when we look at the top-left Figure 2 in the above position and left orthographic viewpoint, respectively.

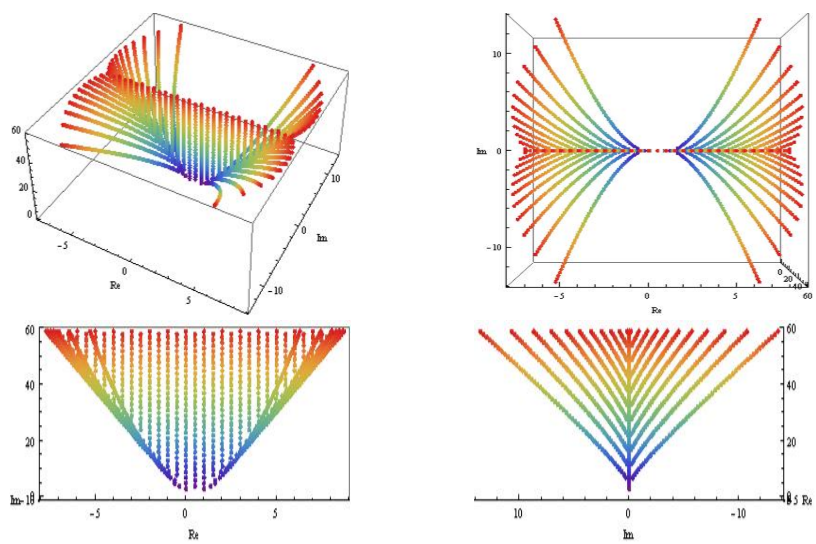

Figure 2: Stacks of zeros of $G_n(x)$ for $2 \le n \le 60$

From Definition of Genocchi polynomials, we get

$$\begin{aligned}\sum_{n=0}^{\infty}\left(G_n(x+1)+G_n(x)\right)\frac{t^n}{n!} &= \frac{2t}{e^t+1}e^{t(x+1)} + \frac{2t}{e^t+1}e^{tx}\\ &= 2te^{tx} = 2\sum_{n=0}^{\infty}(n+1)x^n\frac{t^n}{n!}.\end{aligned}$$

By comparing the coefficients of $\dfrac{t^n}{n!}$ on both sides of the above equation, we find the theorem 2.5.

Theorem 2.5. For $n \in \mathbb{N}_0$ we find

$$G_n(x+1)+G_n(x) = 2nx^{n-1}.$$

5

Substituting $x = 0$ in the Theorem 2.5, we find the following corollary 2.6.

Corollary 2.6. For $n \in \mathbb{N}$, one has

$$G_n = -G_n(1).$$

We consider the Newton's dynamical system at numbers of roots in $G_{10}(x)$. We can obtain roots in the $G_{10}(x)$, that is,

$$\begin{aligned}
&x_1 = -1.31362 - 0.876373i, && x_2 = -1.31362 + 0.876373i,\\
&x_3 = -1.21973, && x_4 = -0.50008,\\
&x_5 = 0.5, && x_6 = 1.50008,\\
&x_7 = 2.21973, && x_8 = 2.31362 - 0.876373i,\\
&x_9 = 2.31362 + 0.876373i.
\end{aligned}$$

The orbit of x_0 from the Newton method appears by calculating until 30 iterations or the absolute difference value of the last two iterations is within 10^{-6}. We hope to determine whether the orbit of x_0 under the action of Newton's dynamical system converges to one of roots when it is given a point x_0 in the complex plane.

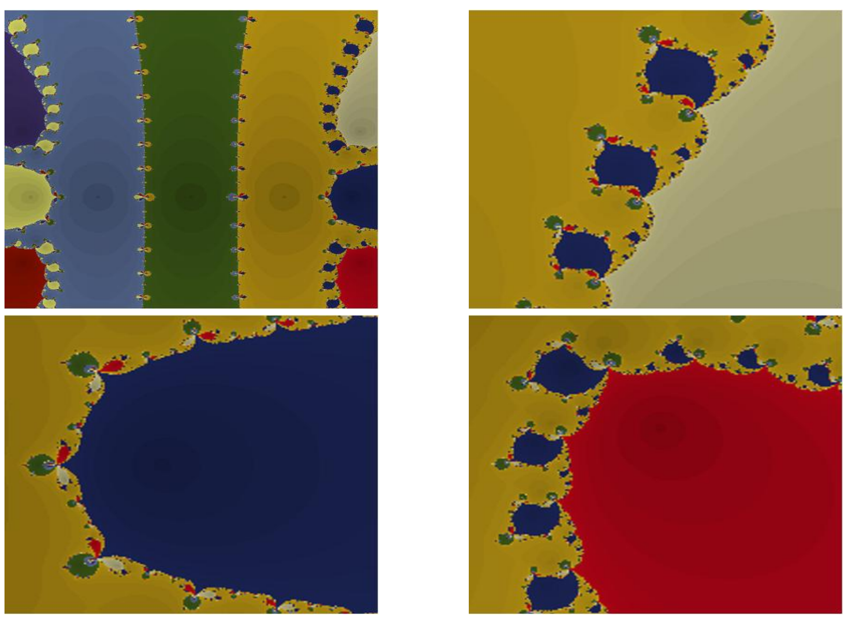

Figure 3: General structure of orbits for $\{-1.5 \leq x \leq 2.5\}, \{-1.5 \leq y \leq 2.5\}$

The output of Figure 3 is the orbit values by using the above method. We plot the blue, brown, yellow, skyblue, green, ocher, navy blue, red, or gray to x_0 in the Figure 3, when an

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orbit of x_0 converge to $-1.31362-0.876373i, -1.31362+0.876373i, -1.21973, -0.50008, 0.5, 1.50008, 2.21973, 2.31362-0.876373i, 2.31362+0.876373i$, respectively. From the top-left figure, we can observe general structure for $\{-1.5 \le x \le 2.5\}, \{-1.5 \le y \le 2.5\}$. Moreover, we can observe property of complex conjugate from the top-right figure and bottom-figures in the right part of general structure by narrowing range. The interesting result is the fact that each boundaries of range parts have every colors and self-similarity.

3. The fixed points of Genocchi polynomials

In this section, we present distributions of fixed points and period points from iterating map. From definition and property of fixed point, we find it and construct structure of this points in the complex plane. By expanding method of previous section we can discuss the fixed points and period points of the Genocchi polynomials.

Definition 3.1. The orbit of the point $z_0 \in \mathbb{C}$ under the action of the function f is said to be bounded if there exists $M \in \mathbb{R}$ such that $|f^n(z_0)| < M$ for all $n \in \mathbb{N}$. If the orbit is not bounded, it is said to be unbounded.

Definition 3.2. Let $f : D \to D$ be a transformation on a metric space. A point $z_0 \in D$ such that $f(z_0) = z_0$ is called a fixed point of the transformation.

Suppose that the complex function f is analytic in a region D of $\mathbb{C}$, and f has a fixed point at $z_0 \in D$. Then z_0 is said to be:
an attracting fixed point if $|f'(z_0)| < 1$;
a repelling fixed point if $|f'(z_0)| > 1$;
a neutral fixed point if $|f'(z_0)| = 1$.

For example, $G_4(x) - 1.01 - 0.1i$ have three points satisfying $G_4(x) - 1.01 - 0.1i = x$. That is, $x_0 = -0.174314+0.0695883i, 0.0220059-0.0779681i, 1.65231+0.00837978i$. Since

$$\left|\frac{d}{dz}G_4(0.0220059 - 0.0779681i) - 1.01 - 0.1i\right| = 0.953792 < 1,$$

we obtain the following theorem.

Theorem 3.3. The Genocchi polynomials $G_4(x) - 1.01 - 0.1i$ has the only one attracting fixed point at

$$\alpha = 0.0220059 - 0.0779681i.$$

We can separate the numerical results for fixed point of $G_n(x)$ by using Mathematica software. In the Table 2, we can look for numbers of fixed points of $G_n(x)$ for $3 \le n \le 10$ and find property of their points.

Table 2. Numbers of attracting, repelling, and neutral fixed points of $G_n(x)$

degree n	attractor	repellor	neutral
3	0	2	0
4	0	3	0
5	0	4	0
6	0	5	0
7	0	6	0
8	0	7	0
9	0	8	0
10	0	9	0

Conjecture 3.4. The Genocchi polynomials $G_n(x)$ has no attracting and neutral fixed point except for infinity.

In the Table 3, we consider $G_4^r(x)$ by using iterating map. We can know the numbers of real roots of $G_4^r(x)$ using iterated function are less than 3^r. In addition, we observe the numbers of real roots will be $2^{r+1}-1$ for $r \geq 1$ and find there is no the real number which is related to fixed point.

Table 3. Numbers of roots and fixed points of $G_4^r(x)$ for $1 \leq r \leq 9$

r	numbers of real roots	numbers of real numbers in fixed points
$G_4^1(x)$	3	3
$G_4^2(x)$	7	5
$G_4^3(x)$	15	15
$G_4^4(x)$	31	51
$G_4^5(x)$	63	0
$G_4^6(x)$	127	0
$G_4^7(x)$	255	0
$G_4^8(x)$	511	0
$G_4^9(x)$	1023	0
$\cdots$	$\cdots$	$\cdots$

In the top-left Figure 4, we can see the forms of 3D structure which is related to stacks of fixed points of iterated $G_4^r(x)$ for $1 \leq r \leq 6$. We can draw the top-right figure when we look at the top-left Figure 4 in the below position. The bottom-left of Figure 4 represent that image and n axes are exist but there is no real axis. The bottom-right of Figure 4 is the right orthographic viewpoint for the top-left figure, that is, there exist real and n axes but don't exist image axis.

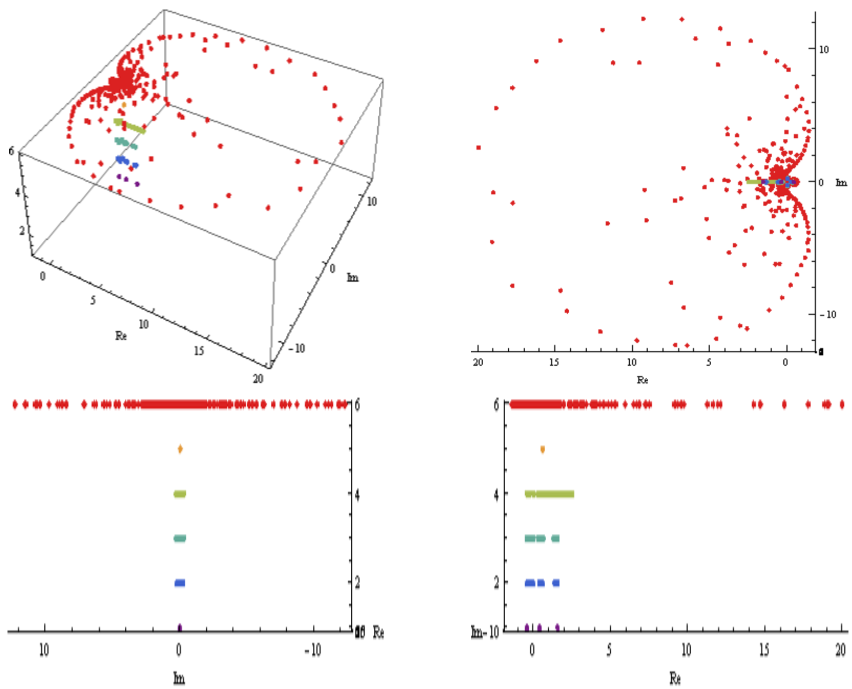

Figure 4: Stacks of fixed points of $G_4^r(x)$ for $1 \leq n \leq 6$

We consider $G_4^2(x)$ for $x \in \mathbb{C}$. This polynomial has nine distinct complex numbers, $a_i(i = 1, 2, 3, 4, 5, 6, 7, 8, 9)$ such that $G_4^2(a_i) = a_i$. We obtain $a_1 = -0.430403, a_2 = -0.244653, a_3 = -0.0322871-0.240632i, a_4 = -0.0322871+0.240632i, a_5 = 0.372949, a_6 = 0.582294, a_7 = 1.36347 - 0.0405081i, a_8 = 1.36347 + 0.0405081i, a_9 = 1.55745$. By combining Newton's method in the $G_4^2(x)$, we have

$$\left\{ \mathbb{C}_\infty : \widetilde{R}(x) = x - \frac{G_4^2(x)}{(G_4^2(x))'} \right\}.$$

The general expectation is a typical orbit $\{\widetilde{R}(x)\}$ will converge to one of the fixed points of $G_4^2(x)$ for $x_0 \in \mathbb{C}$. If we choose x_0 close enough to a_i then it is readily proved that

$$\lim_{n\to\infty} \widetilde{R}(x_0) = a_i, \text{ for } i = 1, 2, 3, 4, 5, 6, 7, 8, 9.$$

Given a point x_0 in the complex plane, we want to find out if the orbit of x_0 under the action of $\widetilde{R}(x)$ does or does not converge to one of the fixed points, and if so, which one. When $\widetilde{R}(x)$ is applied to x_0, the orbit of x_0 under the action of $\widetilde{R}(x)$ is calculated until the absolute value of the last 2 iterations differs by an amount less than 10^{-6}, or until 30 iteration have been carried out.

The Figure 5 is the last orbit value calculated. We construct a function which assigns one of nine colors to each point in the plane, according to the outcome of $\widetilde{R}$. We allocate the red, violet, yellow, skyblue, green, ocher, blue, navy blue, or gray to x_0 if its orbit converges to $-0.430403, -0.244653, -0.0322871 - 0.240632i, -0.0322871 + 0.240632i, 0.372949, 0.582294, 1.36347 - 0.0405081i, 1.36347 + 0.0405081i, 1.55745$, respectively. We make the range which is $\{(x, y) : -4 \leq x \leq 4, -4 \leq y \leq 4\}$. For example, the red region represents part of the attracting basin of $a_1 = -0.430403$

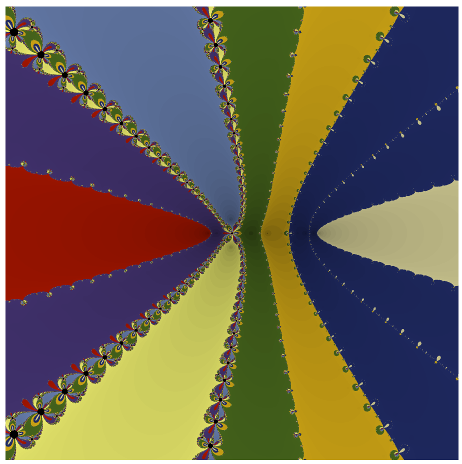

Figure 5: Orbit of x_0 under the action of $\widetilde{R}$ for $G_4^2(x)$

The Figure 6 express the coloring of the next Figure 7. Points which escape after 1 to 30 iterations are colored red to green.

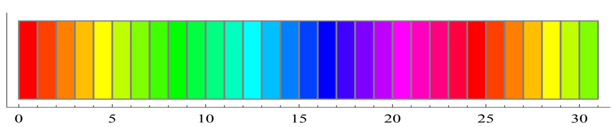

Figure 6: Palette for escaping points

In the Figure 7, the above-mentioned rapid change can be illustrated by applying the three-dimensional structure to the escape-time function. We consturct the range of left figure which is $\{(x, y) : -3 \leq x \leq 3, -3 \leq y \leq 3\}$ and the range of right figure which is $\{(x, y) : -4 \leq x \leq 4, -4 \leq y \leq 4\}$. From this figure, we can see the same color regions which are the orbit of point, z_0, approached an one of fixed points at the equivalent iterated step.

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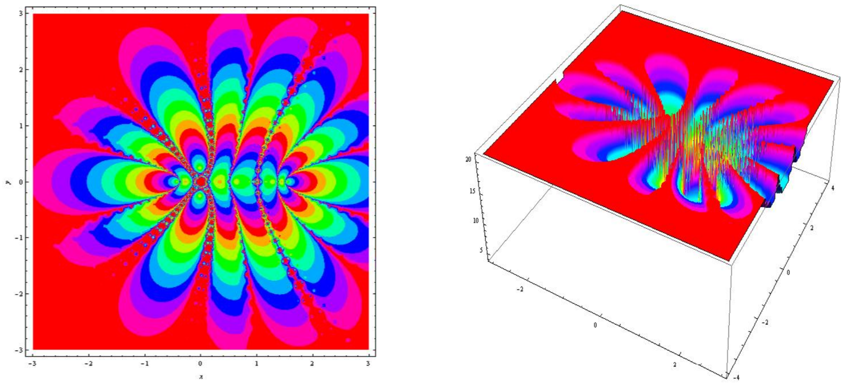

Figure 7: Escape-time map of $\widetilde{R}(x)$ for $G_4^2(x)$

Acknowledgements

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ADDITIVE ρ-FUNCTIONAL EQUATIONS

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Abstract. In this paper, we solve the additive ρ-functional equations

$$f(x+y)-f(x)-f(y) = \rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right), \tag{0.1}$$

$$2f\left(\frac{x+y}{2}\right)-f(x)-f(y) = \rho\left(f(x+y)-f(x)-f(y)\right), \tag{0.2}$$

where ρ is a fixed non-Archimedean number or a fixed real or complex number with $\rho \neq 1$.

Using the direct method, we prove the Hyers-Ulam stability of the additive ρ-functional equations (0.1) and (0.2) in non-Archimedean Banach spaces and in Banach spaces.

1. Introduction and preliminaries

A *valuation* is a function $|\cdot|$ from a field K into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|rs| = |r|\cdot|s|$ and the triangle inequality holds, i.e.,

$$|r+s| \le |r| + |s|, \qquad \forall r, s \in K.$$

A field K is called a *valued field* if K carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s| \le \max\{|r|, |s|\}, \qquad \forall r, s \in K,$$

then the function $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly $|1| = |-1| = 1$ and $|n| \le 1$ for all $n \in \mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$.

Throughout this paper, we assume that the base field is a non-Archimedean field, hence call it simply a field.

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1

Definition 1.1. ([12]) Let X be a vector space over a field K with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \to [0, \infty)$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|rx\| = |r|\|x\| \qquad (r \in K, x \in X)$;

(iii) the strong triangle inequality

$$\|x + y\| \le \max\{\|x\|, \|y\|\}, \qquad \forall x, y \in X$$

holds. Then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space.*

Definition 1.2. (i) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *Cauchy* if for a given $\varepsilon > 0$ there is a positive integer N such that

$$\|x_n - x_m\| \le \varepsilon$$

for all $n, m \ge N$.

(ii) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *convergent* if for a given $\varepsilon > 0$ there are a positive integer N and an $x \in X$ such that

$$\|x_n - x\| \le \varepsilon$$

for all $n \ge N$. Then we call $x \in X$ a limit of the sequence $\{x_n\}$, and denote by $\lim_{n\to\infty} x_n = x$.

(iii) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a *non-Archimedean Banach space.*

The stability problem of functional equations originated from a question of Ulam [17] concerning the stability of group homomorphisms.

The functional equation $f(x + y) = f(x) + f(y)$ is called the *Cauchy equation.* In particular, every solution of the Cauchy equation is said to be an *additive mapping.* Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [15] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [8] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The functional equation $f\left(\frac{x+y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$ is called the *Jensen equation.* See [2, 3, 4, 5, 6, 7, 9, 10, 13, 14, 16, 18] for more information on functional equations.

In Section 2, we solve the additive ρ-functional equation (0.1) in vector spaces and prove the Hyers-Ulam stability of the additive ρ-functional equation (0.1) in non-Archimedean Banach spaces.

In Section 3, we solve the additive ρ-functional equation (0.2) in vector spaces and prove the Hyers-Ulam stability of the additive ρ-functional equation (0.2) in non-Archimedean Banach spaces.

In Section 4, we prove the Hyers-Ulam stability of the additive ρ-functional equation (0.1) in Banach spaces.

In Section 5, we prove the Hyers-Ulam stability of the additive ρ-functional equation (0.2) in Banach spaces.

2. Additive ρ-functional equation (0.1) in non-Archimedean Banach spaces

Throughout Sections 2 and 3, assume that X is a non-Archimedean normed space and that Y is a non-Archimedean Banach space. Let $|2| \neq 1$ and let ρ be a fixed non-Archimedean number with $\rho \neq 1$.

We solve the additive ρ-functional equation (0.1) in vector spaces.

Lemma 2.1. *Let X and Y be vector spaces. If a mapping $f : X \to Y$ satisfies*

$$f(x+y) - f(x) - f(y) = \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right) \tag{2.1}$$

for all $x, y \in X$, then $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $-f(0) = 0$. So $f(0) = 0$.

Letting $y = x$ in (2.1), we get $f(2x) - 2f(x) = 0$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2}f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} f(x+y) - f(x) - f(y) &= \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right) \\ &= \rho(f(x+y) - f(x) - f(y)) \end{aligned}$$

and so

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$. □

We prove the Hyers-Ulam stability of the additive ρ-functional equation (2.1) in non-Archimedean Banach spaces.

Theorem 2.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be a mapping such that*

$$\Psi(x,y) := \sum_{j=1}^{\infty} |2|^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) \quad < \quad \infty, \tag{2.3}$$

$$\left\| f(x+y) - f(x) - f(y) - \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)\right\| \quad \leq \quad \varphi(x,y) \tag{2.4}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{|2|}\Psi(x,x) \tag{2.5}$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.4), we get

$$\|f(2x) - 2f(x)\| \leq \varphi(x,x) \tag{2.6}$$

for all $x \in X$. So

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \leq \varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x \in X$. Hence

$$\begin{aligned}
&\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \qquad (2.7)\\
&\leq \max\left\{\left\|2^l f\left(\frac{x}{2^l}\right) - 2^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, \left\|2^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|\right\}\\
&\leq \max\left\{|2|^l \left\|f\left(\frac{x}{2^l}\right) - 2f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, |2|^{m-1}\left\|f\left(\frac{x}{2^{m-1}}\right) - 2f\left(\frac{x}{2^m}\right)\right\|\right\}\\
&\leq \sum_{j=l}^{\infty} |2|^j \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right)
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.7) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a non-Archimedean Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.7), we get (2.5).

Now, let $T : X \to Y$ be another additive mapping satisfying (2.5). Then we have

$$\begin{aligned}
\|A(x) - T(x)\| &= \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right)\right\|\\
&\leq \max\left\{\left\|2^q A\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\|, \left\|2^q T\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\|\right\}\\
&\leq |2|^{q-1}\Psi\left(\frac{x}{2^q}, \frac{x}{2^q}\right),
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A.

It follows from (2.3) and (2.4) that

$$\begin{aligned}
&\left\| A(x+y) - A(x) - A(y) - \rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)\right\| \\
&= \lim_{n\to\infty} \left\| 2^n \left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - \rho\left(2f\left(\frac{x+y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right)\right\| \\
&\le \lim_{n\to\infty} |2|^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0
\end{aligned}$$

for all $x, y \in X$. So

$$A(x+y) - A(x) - A(y) = \rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)$$

for all $x, y \in X$. By Lemma 2.1, the mapping $A : X \to Y$ is additive. □

Corollary 2.3. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\left\| f(x+y) - f(x) - f(y) - \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)\right\| \le \theta(\|x\|^r + \|y\|^r) \quad (2.8)$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2\theta}{|2|^r - |2|}\|x\|^r$$

for all $x \in X$.

Theorem 2.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying (2.4) and*

$$\Psi(x, y) := \sum_{j=0}^{\infty} \frac{1}{|2|^j}\varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{|2|}\Psi(x, x) \quad (2.9)$$

for all $x \in X$.

Proof. It follows from (2.6) that

$$\left\| f(x) - \frac{1}{2}f(2x)\right\| \le \frac{1}{|2|}\varphi(x, x)$$

for all $x \in X$. Hence

$$\begin{aligned}
&\left\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\right\| \qquad (2.10)\\
&\le \max\left\{\left\|\frac{1}{2^l}f\left(2^l x\right) - \frac{1}{2^{l+1}}f\left(2^{l+1}x\right)\right\|, \cdots, \left\|\frac{1}{2^{m-1}}f\left(2^{m-1}x\right) - \frac{1}{2^m}f\left(2^m x\right)\right\|\right\}\\
&\le \max\left\{\frac{1}{|2|^l}\left\|f\left(2^l x\right) - \frac{1}{2}f\left(2^{l+1}x\right)\right\|, \cdots, \frac{1}{|2|^{m-1}}\left\|f\left(2^{m-1}x\right) - \frac{1}{2}f\left(2^m x\right)\right\|\right\}\\
&\le \sum_{j=l}^{\infty}\frac{1}{|2|^{j+1}}\varphi(2^j x, 2^j x)
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.10) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.10), we get (2.9).

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.8). *Then there exists a unique additive mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \le \frac{2\theta}{|2| - |2|^r}\|x\|^r$$

for all $x \in X$.

3. Additive ρ-functional equation (0.2) in non-Archimedean Banach spaces

We solve the additive ρ-functional equation (0.2) in vector spaces.

Lemma 3.1. *Let X and Y be vector spaces. If a mapping $f : X \to Y$ satisfis $f(0) = 0$ and*

$$2f\left(\frac{x+y}{2}\right) - f(x) - f(y) = \rho\left(f(x+y) - f(x) - f(y)\right) \qquad (3.1)$$

for all $x, y \in X$, then $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $y = 0$ in (3.1), we get

$$2f\left(\frac{x}{2}\right) - f(x) = 0 \qquad (3.2)$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{2}f(x)$ for all $x \in X$.

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It follows from (3.1) and (3.2) that

$$\begin{aligned} f(x+y)-f(x)-f(y) &= 2f\left(\frac{x+y}{2}\right)-f(x)-f(y) \\ &= \rho(f(x+y)-f(x)-f(y)) \end{aligned}$$

and so

$$f(x+y)=f(x)+f(y)$$

for all $x, y \in X$. $\square$

Now, we prove the Hyers-Ulam stability of the additive ρ-functional equation (3.1) in non-Archimedean Banach spaces.

Theorem 3.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\Psi(x,y) := \sum_{j=0}^{\infty} |2|^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) \quad < \quad \infty,$$

$$\left\|2f\left(\frac{x+y}{2}\right)-f(x)-f(y)-\rho\left(f(x+y)-f(x)-f(y)\right)\right\| \quad \leq \quad \varphi(x,y) \tag{3.3}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x)-A(x)\| \leq \Psi(x,0) \tag{3.4}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (3.3), we get

$$\left\|f(x)-2f\left(\frac{x}{2}\right)\right\| = \left\|2f\left(\frac{x}{2}\right)-f(x)\right\| \leq \varphi(x,0) \tag{3.5}$$

for all $x \in X$. So

$$\begin{aligned} &\left\|2^l f\left(\frac{x}{2^l}\right)-2^m f\left(\frac{x}{2^m}\right)\right\| \qquad (3.6) \\ &\leq \max\left\{\left\|2^l f\left(\frac{x}{2^l}\right)-2^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, \left\|2^{m-1} f\left(\frac{x}{2^{m-1}}\right)-2^m f\left(\frac{x}{2^m}\right)\right\|\right\} \\ &\leq \max\left\{|2|^l \left\|f\left(\frac{x}{2^l}\right)-2f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, |2|^{m-1}\left\|f\left(\frac{x}{2^{m-1}}\right)-2f\left(\frac{x}{2^m}\right)\right\|\right\} \\ &\leq \sum_{j=l}^{\infty} |2|^j \varphi\left(\frac{x}{2^j}, 0\right) \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.6) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a non-Archimedean Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.6), we get (3.4).

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 3.3. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) - \rho(f(x+y) - f(x) - f(y)) \right\| \le \theta(\|x\|^r + \|y\|^r) \quad (3.7)$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{|2|^r \theta}{|2|^r - |2|} \|x\|^r$$

for all $x \in X$.

Theorem 3.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (3.3) and*

$$\Psi(x, y) := \sum_{j=1}^{\infty} \frac{1}{|2|^j} \varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \Psi(x, 0) \quad (3.8)$$

for all $x \in X$.

Proof. It follows from (3.5) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\| \le \frac{1}{|2|} \varphi(2x, 0)$$

for all $x \in X$. Hence

$$\begin{aligned} &\left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \right\| \quad (3.9) \\ &\le \max\left\{ \left\| \frac{1}{2^l} f\left(2^l x\right) - \frac{1}{2^{l+1}} f\left(2^{l+1} x\right) \right\|, \cdots, \left\| \frac{1}{2^{m-1}} f\left(2^{m-1} x\right) - \frac{1}{2^m} f\left(2^m x\right) \right\| \right\} \\ &\le \max\left\{ \frac{1}{|2|^l} \left\| f\left(2^l x\right) - \frac{1}{2} f\left(2^{l+1} x\right) \right\|, \cdots, \frac{1}{|2|^{m-1}} \left\| f\left(2^{m-1} x\right) - \frac{1}{2} f\left(2^m x\right) \right\| \right\} \\ &\le \sum_{j=l+1}^{\infty} \frac{1}{|2|^j} \varphi(2^j x, 0) \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.10) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n x)$$

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for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.10), we get (3.9).

The rest of the proof is similar to the proof of Theorem 2.2. $\square$

Corollary 3.5. *Let $r > 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.7). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{|2|^r\theta}{|2| - |2|^r}\|x\|^r$$

for all $x \in X$.

4. Additive ρ-functional equation (0.1) in Banach spaces

Throughout Sections 4 and 5, assume that X is a normed space and that Y is a Banach space. Let ρ be a fixed real or complex number with $\rho \neq 1$.

We prove the Hyers-Ulam stability of the additive ρ-functional equation (2.1) in Banach spaces.

Theorem 4.1. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping such that*

$$\Psi(x,y) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) \ < \ \infty, \tag{4.1}$$

$$\left\|f(x+y) - f(x) - f(y) - \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)\right\| \ \leq \ \varphi(x,y) \tag{4.2}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{2}\Psi(x,x) \tag{4.3}$$

for all $x \in X$.

Proof. Letting $y = x$ in (4.2), we get

$$\|f(2x) - 2f(x)\| \leq \varphi(x,x) \tag{4.4}$$

for all $x \in X$. So

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \leq \varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x \in X$. Hence

$$\begin{aligned} \left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \ &\leq \ \sum_{j=l}^{m-1} \left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\leq \ \sum_{j=l}^{m-1} 2^j \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) \end{aligned} \tag{4.5}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (4.5) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.5), we get (4.3).

Now, let $T : X \to Y$ be another additive mapping satisfying (4.3). Then we have

$$\begin{aligned}
\|A(x) - T(x)\| &= \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right)\right\| \\
&\le \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| + \left\|2^q T\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| \\
&\le 2^q \Psi\left(\frac{x}{2^q}, \frac{x}{2^q}\right),
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A.

It follows from (4.1) and (4.2) that

$$\begin{aligned}
&\left\|A(x+y) - A(x) - A(y) - \rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)\right\| \\
&= \lim_{n\to\infty} \left\|2^n \left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - \rho\left(2f\left(\frac{x+y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right)\right\| \\
&\le \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0
\end{aligned}$$

for all $x, y \in X$. So

$$A(x+y) - A(x) - A(y) = \rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)$$

for all $x, y \in X$. By Lemma 2.1, the mapping $A : X \to Y$ is additive. □

Corollary 4.2. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\left\|f(x+y) - f(x) - f(y) - \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)\right\| \le \theta(\|x\|^r + \|y\|^r) \quad (4.6)$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2\theta}{2^r - 2}\|x\|^r$$

for all $x \in X$.

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Theorem 4.3. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be a mapping satisfying* (4.2) *and*

$$\Psi(x,y) := \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2}\Psi(x,x) \tag{4.7}$$

for all $x \in X$.

Proof. It follows from (4.4) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \le \frac{1}{2}\varphi(x,x)$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f\left(2^j x\right) - \frac{1}{2^{j+1}}f\left(2^{j+1}x\right)\right\| \\ &\le \sum_{j=l}^{m-1}\frac{1}{2^{j+1}}\varphi(2^j x, 2^j x)\end{aligned} \tag{4.8}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (4.8) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.8), we get (4.7).

The rest of the proof is similar to the proof of Theorem 4.1. □

Corollary 4.4. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying* (4.6)*. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{2\theta}{2 - 2^r}\|x\|^r$$

for all $x \in X$.

5. Additive ρ-functional equation (0.2) in Banach spaces

In this section, we prove the Hyers-Ulam stability of the additive ρ-functional equation (3.1) in Banach spaces.

Theorem 5.1. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\Psi(x,y) := \sum_{j=0}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) \quad < \quad \infty,$$

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y) - \rho\left(f(x+y) - f(x) - f(y)\right)\right\| \quad \leq \quad \varphi(x,y) \qquad (5.1)$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \Psi(x,0) \qquad (5.2)$$

for all $x \in X$.

Proof. Letting $y = 0$ in (5.1), we get

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| = \left\|2f\left(\frac{x}{2}\right) - f(x)\right\| \leq \varphi(x,0) \qquad (5.3)$$

for all $x \in X$. So

$$\begin{aligned} \left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \quad &\leq \quad \sum_{j=l}^{m-1} \left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\leq \quad \sum_{j=l}^{m-1} 2^j \varphi\left(\frac{x}{2^j}, 0\right) \end{aligned} \qquad (5.4)$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (5.4) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (5.4), we get (5.2).

The rest of the proof is similar to the proof of Theorem 4.1. □

Corollary 5.2. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y) - \rho(f(x+y) - f(x) - f(y))\right\| \leq \theta(\|x\|^r + \|y\|^r) \qquad (5.5)$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2^r\theta}{2^r - 2}\|x\|^r$$

for all $x \in X$.

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Theorem 5.3. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (5.1) and*

$$\Psi(x,y) := \sum_{j=1}^{\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \Psi(x,0) \tag{5.6}$$

for all $x \in X$.

Proof. It follows from (5.3) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \le \frac{1}{2}\varphi(2x,0)$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\right\| &\le \sum_{j=l+1}^{m}\left\|\frac{1}{2^j}f\left(2^j x\right) - \frac{1}{2^{j+1}}f\left(2^{j+1}x\right)\right\| \\ &\le \sum_{j=l+1}^{m}\frac{1}{2^j}\varphi(2^j x, 0) \end{aligned} \tag{5.7}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (5.7) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (5.7), we get (5.6).

The rest of the proof is similar to the proof of Theorem 4.1. □

Corollary 5.4. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (5.5). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{2^r\theta}{2 - 2^r}\|x\|^r$$

for all $x \in X$.

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HYPERSTABILITY OF A GENERALIZED CAUCHY FUNCTIONAL EQUATION

ABBAS NAJATI, DARYOUSH MOLAEE, AND CHOONKIL PARK

ABSTRACT. The aim of this paper is to present some results concerning the hyperstability of the generalized Cauchy functional equation

$$f(ax+by) = Af(x) + Bf(y) + C$$

Namely, we show, under some assumptions, that a function satisfying the equation approximately must be actually a solution to it.

1. Introduction and preliminaries

Throughout the paper $\mathbb{F}$ and $\mathbb{K}$ denote the fields of real or complex numbers. Let X and Y be linear spaces over $\mathbb{F}$ and $\mathbb{K}$, respectively. In this paper we give some hyperstability results for the generalized Cauchy functional equation

$$f(ax+by) = Af(x) + Bf(y) + C \tag{1.1}$$

where $f : X \to Y$ and $a, b \in \mathbb{F}\backslash\{0\}, A, B \in \mathbb{K}, C \in Y$. In [10], Piszczek proved the hyperstability of the generalized Cauchy functional equation (1.1).

Theorem 1.1. [10] *Let X be a normed space over a field $\mathbb{F}$, Y be a Banach space over $\mathbb{K}$, $a, b \in \mathbb{F} \setminus \{0\}, A, B \in \mathbb{K}, p < 0$ and $g : X \to Y$ satisfy*

$$\|g(ax+by) - Ag(x) - Bg(y)\| \leqslant \varepsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in X \setminus \{0\}$. Then g satisfies

$$g(ax+by) = Ag(x) + Bg(y)$$

for all $x, y \in X \setminus \{0\}$.

The method of the proof used in Theorem 1.1 is based on a fixed point theorem in [3]. Let us recall that the study of stability problems of functional equations was motivated by a question of Ulam [15] asked in 1940. The first result of stability proved by Hyers [6] in 1941. For more details about various results concerning such problems the reader is referred to [4, 5, 8, 9, 11, 12, 13, 14].

It seems the first hyperstability result was published in [1] and concerned ring homomorphisms. However the term hyperstability was used for the first time in [7].

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2. Hyperstability results

In this part, we will prove a general version of Theorem 1.1. Let us start with a result. A version of the next result was proved in [2]. But we give another simple proof.

Proposition 2.1. *Assume that $\mathcal{X}$ and $\mathcal{Y}$ are linear spaces over $\mathbb{F}$ and $\mathbb{K}$, respectively. Let $a, b \in \mathbb{F} \setminus \{0\}$, $A, B \in \mathbb{K}$, $C \in Y$ and $f : \mathcal{X} \to \mathcal{Y}$ satisfy*

$$f(ax + by) = Af(x) + Bf(y) + C \tag{2.1}$$

for all $x, y \in \mathcal{X} \setminus \{0\}$. Then f satisfies $f(ax + by) = Af(x) + Bf(y) + C$ for all $x, y \in \mathcal{X}$.

Proof. Let $x \in \mathcal{X} \setminus \{0\}$. Then in view of (2.1), we get

$$\begin{aligned}
f(0) &= Af(bx) + Bf(-ax) + C \\
&= A\Big[Af(2a^{-1}bx) + Bf(-x) + C\Big] + B\Big[Af(-2x) + Bf(ab^{-1}x) + C\Big] + C \\
&= A\Big[Af(2a^{-1}bx) + Bf(-2x) + C\Big] + B\Big[Af(-x) + Bf(ab^{-1}x) + C\Big] + C \\
&= Af(0) + Bf(0) + C.
\end{aligned}$$

Therefore we have

$$f(0) = Af(bx) + Bf(-ax) + C \tag{2.2}$$

for all $x \in \mathcal{X}$. Consequently, by (2.1) and (2.2), we get

$$\begin{aligned}
f(2a^2bx) &= Af(abx + b^2y) + Bf(a^2x - aby) + C \\
&= A\Big[Af(bx) + Bf(by) + C\Big] + B\Big[Af(ax) + Bf(-ay) + C\Big] + C \\
&= A\Big[Af(bx) + Bf(by) + C\Big] + B\Big[Af(ax) + f(0) - Af(by)\Big] + C \\
&= A\Big[Af(bx) + Bf(ax) + C\Big] + Bf(0) + C \\
&= Af(2abx) + Bf(0) + C
\end{aligned}$$

Hence $f(2a^2bx) = Af(2abx) + Bf(0) + C$ for all $x \in \mathcal{X} \setminus \{0\}$. Replacing x by $(2ab)^{-1}x$, we infer that $f(ax) = Af(x) + Bf(0) + C$ holds for $x \in \mathcal{X}$ by (2.2). Similarly, one can prove that $f(by) = Af(0) + Bf(y) + C$ holds for $y \in \mathcal{X}$. Thus we have proved that f satisfies $f(ax + by) = Af(x) + Bf(y) + C$ for all $x, y \in \mathcal{X}$. $\square$

In the following results we assume that X is a vector space over $\mathbb{F}$ and Y is a normed space over $\mathbb{K}$.

Theorem 2.2. *Let $a, b \in \mathbb{F} \setminus \{0\}$ and $\varphi : X \times X \to [0, +\infty)$ be a function such that*

$$\lim_{m\to\infty} \varphi(a^{-1}(m+1)x, -b^{-1}mx) = 0, \quad \lim_{m\to\infty} \varphi(mx, my) = 0 \tag{2.3}$$

for all $x, y \in X \setminus \{0\}$. Let $A, B \in \mathbb{K}, C \in Y$ and $f : X \to Y$ satisfy

$$\|f(ax + by) - Af(x) - Bf(y) - C\| \leqslant \varphi(x, y) \tag{2.4}$$

for all $x, y \in X \setminus \{0\}$. Then f satisfies

$$f(ax + by) = Af(x) + Bf(y) + C, \tag{2.5}$$

for all $x, y \in X$. Moreover,

$$(A + B)f(0) = Af(x) + Bf(-ab^{-1}x) \tag{2.6}$$

for all $x \in X$.

Proof. Replacing x by $a^{-1}(m+1)x$ and y by $-b^{-1}mx$ in (2.4), we get

$$\Big\|f(x) - Af(a^{-1}(m+1)x) - Bf(-b^{-1}mx) - C\Big\| \leqslant \varphi(a^{-1}(m+1)x, -b^{-1}mx), \tag{2.7}$$

for all $x \in X \setminus \{0\}$ and positive integers m. Letting $m \to \infty$ in (2.7) and using (2.3), we obtain

$$f(x) = \lim_{m\to\infty} \Big[Af(a^{-1}(m+1)x) + Bf(-b^{-1}mx) + C\Big] \tag{2.8}$$

for all $x \in X \setminus \{0\}$. If $x \in X \setminus \{0\}$, then we get from (2.3) and (2.8)

$$\begin{aligned}
&\Big\|(A+B)f(0) - Af(x) - Bf(-ab^{-1}x)\Big\| \\
&= \lim_{m\to\infty} \Big\|(A+B)f(0) - A^2 f(a^{-1}(m+1)x) - ABf(-b^{-1}mx) - AC \\
&\qquad - ABf(-b^{-1}(m+1)x) - B^2 f(ab^{-2}mx) - BC\Big\| \\
&\leqslant |A| \lim_{m\to\infty} \Big\|f(0) - Af(a^{-1}(m+1)x) - Bf(-b^{-1}(m+1)x) - C\Big\| \\
&\quad + |B| \lim_{m\to\infty} \Big\|f(0) - Af(-b^{-1}mx) - Bf(ab^{-2}mx) - C\Big\| \\
&\leqslant |A| \lim_{m\to\infty} \varphi(a^{-1}(m+1)x, -b^{-1}(m+1)x) + |B| \lim_{m\to\infty} \varphi(-b^{-1}mx, ab^{-2}mx) = 0.
\end{aligned}$$

Hence we get

$$(A+B)f(0) = Af(x) + Bf(-ab^{-1}x)$$

for all $x \in X$. If we replace x by bmx and y by $-amx$ in (2.4), we get

$$\Big\|f(0) - Af(bmx) - Bf(-amx) - C\| \leqslant \varphi(bmx, -amx), \tag{2.9}$$

for all $x \in X \setminus \{0\}$ and positive integers m. Thus

$$f(0) = \lim_{m\to\infty} \Big[Af(bmx) + Bf(-amx) + C\Big] \tag{2.10}$$

for all $x \in X \setminus \{0\}$. Replacing x by bmx in (2.9) and letting $m \to \infty$, we get from (2.10)

$$(1 - A - B)f(0) = C.$$

Therefore (2.8) holds for all $x \in X$.

To prove (2.5), let $x, y \in X \setminus \{0\}$. Then

$$\begin{aligned}
&\|f(ax+by) - Af(x) - Bf(y) - C\| \\
&= \lim_{m\to\infty} \Big\|Af(a^{-1}(m+1)(ax+by)) + Bf(-b^{-1}m(ax+by)) \\
&\qquad - A^2 f(a^{-1}(m+1)x) - ABf(-b^{-1}mx) - AC \\
&\qquad - ABf(a^{-1}(m+1)y) - B^2 f(-b^{-1}my) - BC\Big\| \\
&\leqslant |A| \lim_{m\to\infty} \Big\|f(a^{-1}(m+1)(ax+by)) - Af(a^{-1}(m+1)x) - Bf(a^{-1}(m+1)y) - C\Big\| \\
&\quad + |B| \lim_{m\to\infty} \Big\|f(-b^{-1}m(ax+by)) - Af(-b^{-1}mx) - Bf(-b^{-1}my) - C\Big\| \\
&\leqslant |A| \lim_{m\to\infty} \varphi(a^{-1}(m+1)x, -a^{-1}(m+1)y) + |B| \lim_{m\to\infty} \varphi(-b^{-1}mx, -b^{-1}my) = 0.
\end{aligned}$$

Therefore f satisfies (2.5) for all $x, y \in X \setminus \{0\}$. Hence f satisfies (2.5) for all $x, y \in X$ by Proposition 2.1. $\square$

Remark 2.3. If f satisfies (2.4) with $A + B = 1$, then $C = 0$ and f satisfies $f(ax + by) = Af(x) + Bf(y)$ for all $x, y \in X$.

When X is a normed linear space, Theorem 1.1 is a corollary of Theorem 2.2. In the following results, we assume that X and Y are normed linear spaces.

Corollary 2.4. *Let $\varepsilon > 0$ and $p, q < 0$. If $a, b \in \mathbb{F} \setminus \{0\}, A, B \in \mathbb{K}, C \in Y$ and $f : X \to Y$ satisfies*

$$\|f(ax + by) - Af(x) - Bf(y) - C\| \leqslant \varepsilon(\|x\|^p + \|y\|^q)$$

for all $x, y \in X \setminus \{0\}$. Then f satisfies (2.5) and (2.6) for all $x, y \in X$.

Corollary 2.5. *Let $\varepsilon > 0$ and p, q be real numbers such that $p + q < 0$. If $a, b \in \mathbb{F} \setminus \{0\}, A, B \in \mathbb{K}, C \in Y$ and $f : X \to Y$ satisfies*

$$\|f(ax + by) - Af(x) - Bf(y) - C\| \leqslant \varepsilon\|x\|^p\|y\|^q$$

for all $x, y \in X \setminus \{0\}$. Then f satisfies (2.5) and (2.6) for all $x, y \in X$.

Corollary 2.6. *Let $\delta, \varepsilon \geqslant 0$, $p, q < 0$ and l, r, s be real numbers such that $l > 0$ and $r + s < 0$. If $a, b \in \mathbb{F} \setminus \{0\}, A, B \in \mathbb{K}, C \in Y$ and $f : X \to Y$ satisfies*

$$\|f(ax + by) - Af(x) - Bf(y) - C\| \leqslant \varepsilon(\|x\|^p + \|y\|^q)^l + \delta\|x\|^r\|y\|^s$$

for all $x, y \in X \setminus \{0\}$. Then f satisfies (2.5) and (2.6) for all $x, y \in X$.

Corollary 2.7. *Let $\theta, \delta, \varepsilon \geqslant 0$, $p, q < 0$ and r, s be real numbers such that $r + s < 0$. If $a, b \in \mathbb{F} \setminus \{0\}, A, B \in \mathbb{K}, C \in Y$ and $f : X \to Y$ satisfies*

$$\|f(ax + by) - Af(x) - Bf(y) - C\| \leqslant \varepsilon\|x + y\|^p + \delta\|x - y\|^q + \theta\|x\|^r\|y\|^s \tag{2.11}$$

for all $x, y \in X \setminus \{0\}$ with $x \pm y \neq 0$. Then we have

(*i*) *if $a \neq \pm b$, then f satisfies (2.5) and (2.6) for all $x, y \in X$;*
(*ii*) *if $a = \pm b$ and $A, B \in \mathbb{K} \setminus \{0\}$, then f satisfies (2.5) for all $x, y \in X \setminus \{0\}$ with $x \pm y \neq 0$.*

Proof. Let $\varphi(x, y) = \|x + y\|^p + \delta\|x - y\|^q + \theta\|x\|^r\|y\|^s$. If $a \neq \pm b$, then φ satisfies (2.3). Therefore the result follows from Theorem 2.2. If $a = \pm b$, then (2.11) implies that

$$Af(x) = \lim_{m\to\infty} \Big[f((a + bm)x) - Bf(mx) - C\Big]$$

for all $x \in X \setminus \{0\}$. Therefore

$$\begin{aligned}
&\Big\|f(ax + by) - Af(x) - Bf(y) - C\Big\| \\
&= |A|^{-1} \lim_{m\to\infty} \Big\|f((a + bm)(ax + by)) - Bf(m(ax + by)) - C \\
&\qquad\qquad - Af((a + bm)x) + ABf(mx) - Bf((a + bm)y) + B^2f(my) + BC\Big\| \\
&\leqslant |A|^{-1} \lim_{m\to\infty} \Big\|f((a + bm)(ax + by)) - Af((a + bm)x) - Bf((a + bm)y) - C\Big\| \\
&\quad + |B||A|^{-1} \lim_{m\to\infty} \Big\|f(m(ax + by)) - Af(mx) - Bf(my) - C\Big\| \\
&\leqslant |A|^{-1} \lim_{m\to\infty} \varphi((a + bm)x, (a + bm)y) + |B||A|^{-1} \lim_{m\to\infty} \varphi(mx, my) = 0.
\end{aligned}$$

Hence $f(ax + by) = Af(x) + Bf(y) + C$ for all $x, y \in X \setminus \{0\}$ with $x \pm y \neq 0$. $\square$

In the next result we will derive from Theorem 2.2 a hyperstability result for the inhomogeneous version of the generalized Cauchy functional equation.

Theorem 2.8. *Let $a, b \in \mathbb{F} \setminus \{0\}$, $A, B \in \mathbb{K}$ and $\varphi : X \times X \to [0, +\infty)$ be a function satisfy (2.3) for all $x, y \in X \setminus \{0\}$. Assume that $d : X \times X \to Y$ and $f : X \to Y$ satisfy the inequality*

$$\|f(ax + by) - Af(x) - Bf(y) - d(x, y)\| \leqslant \varphi(x, y) \tag{2.12}$$

for all $x, y \in X \setminus \{0\}$. If the functional equation

$$g(ax + by) = Ag(x) + Bg(y) + d(x, y), \quad x, y \in X \tag{2.13}$$

has a solution $f_0 : X \to Y$, then f is a solution to (2.13).

Proof. It follows from (2.12) that $h := f - f_0$ satisfies (2.4) with $C = 0$. Consequently, Theorem 2.2 implies that h is a solution to (2.5) with $C = 0$, which means that f is a solution to (2.13). □

In the following results, we assume that $a, b \in \mathbb{F} \setminus \{0\}$, $A, B \in \mathbb{K}$, X and Y are normed linear spaces.

Corollary 2.9. *Let $\varepsilon > 0$ and $p, q < 0$. Assume that $d : X \times X \to Y$ and $f : X \to Y$ satisfy*

$$\|f(ax + by) - Af(x) - Bf(y) - d(x, y)\| \leqslant \varepsilon(\|x\|^p + \|y\|^q)$$

for all $x, y \in X \setminus \{0\}$. If the functional equation (2.13) has a solution $f_0 : X \to Y$, then f is a solution to (2.13).

Corollary 2.10. *Let $\varepsilon > 0$ and p, q be real numbers such that $p + q < 0$. Assume that $d : X \times X \to Y$ and $f : X \to Y$ satisfy*

$$\|f(ax + by) - Af(x) - Bf(y) - d(x, y)\| \leqslant \varepsilon\|x\|^p\|y\|^q$$

for all $x, y \in X \setminus \{0\}$. If the functional equation (2.13) has a solution $f_0 : X \to Y$, then f is a solution to (2.13).

Corollary 2.11. *Let $\delta, \varepsilon \geqslant 0$, $p, q < 0$ and l, r, s be real numbers such that $l > 0$ and $r + s < 0$. Assume that $d : X \times X \to Y$ and $f : X \to Y$ satisfy*

$$\|f(ax + by) - Af(x) - Bf(y) - d(x, y)\| \leqslant \varepsilon(\|x\|^p + \|y\|^q)^l + \delta\|x\|^r\|y\|^s$$

for all $x, y \in X \setminus \{0\}$. If the functional equation (2.13) has a solution $f_0 : X \to Y$, then f is a solution to (2.13).

Corollary 2.12. *Let $\theta, \delta, \varepsilon \geqslant 0$, $p, q < 0$ and r, s be real numbers such that $r + s < 0$. Assume that the functional equation (2.13) has a solution $f_0 : X \to Y$. Let $d : X \times X \to Y$ and $f : X \to Y$ satisfy*

$$\|f(ax + by) - Af(x) - Bf(y) - d(x, y)\| \leqslant \varepsilon\|x + y\|^p + \delta\|x - y\|^q + \theta\|x\|^r\|y\|^s$$

for all $x, y \in X \setminus \{0\}$ with $x \pm y \neq 0$. Then we have

(i) *if $a \neq \pm b$, then f satisfies (2.13) for all $x, y \in X$;*
(ii) *if $a = \pm b$ and $A, B \in \mathbb{K} \setminus \{0\}$, then f satisfies (2.13) for all $x, y \in X \setminus \{0\}$ with $x \pm y \neq 0$.*

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Stability analysis and optimal control of a cholera model with time delay

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Abstract

An optimal control method for cholera epidemic with time delay is developed in this paper. We first explore the local stability of both the disease-free and endemic equilibria of ODE model by analyzing the corresponding characteristic equations, whose global stability is established by constructing two suitable Lyapunov functionals. Furthermore, in order to, we use optimal control theory via the Pontryagin's Maximum Principle and genetic algorithm based on the forward and backward difference approximation to minimize the infected populations and the costs. Numerical simulations demonstrate that the time delay and multiple optimal controls can bring different effects on the dynamics behaviors of the proposed cholera model.

Cholera; optimal control; time delay; global asymptotical stability; Pontryagin's Maximum Principle.

1 Introduction

Cholera, a waterborne gastroenteric infection, caused by a number of types of Vibrio cholerae, remains a significant threat to public health for most of the developing countries in the past few years. Since 1961, cholera has become an acute disease throughout the world, according to the World Health Organization (WHO) report (2010), with an estimated 3-5 million cases worldwide and causes 58,000-130,000 deaths a year, children and the senior are being most affected. It was found in Congo (2008), in Iraq (2008), in Zimbabwe (2008-2009), in Vietnam (2009), in Kenya (2010), in Nigeria (2010), in Haiti (2010), in Mexico (2013), and most recently in South Sudan (2014). In the last few decades, enormous attention is being paid to the cholera disease and a number of mathematical models have been contributed to a better understanding of the transmission of cholera. In 2001, Codeço [1] put an emphasis on the decisive importance of the environmental component and proposed a $SIRB$ epidemic model in which B represents the V. cholerae concentration in water. Meanwhile, according to the laboratory results, Hartley Morris and Smith [2] in 2006 discovered a representitive hyperinfectious state of the pathogen-the explosive infectivity of freshly shed V. cholerae. Tien and Earn later [3] proposed a water-borne disease model with multiple transmission pathways, accounting both direct human-to-human and indirect water-to-human transmissions, they identified how these transmission routes influence disease dynamics. Mukandavire *et al.* [4] in 2011 simplified Hartley's model to understand transmission dynamics of cholera outbreak in Zimbabwe. Liao and Wang [5] conducted a dynamical analysis of the Hartley's model to study the stability of both the disease-free and endemic equilibria so as to explore the complex epidemic and endemic dynamics of the disease.

These epidemiological models above often take the form of a system of ordinary differential equations and ignore the time delay by assuming that the infectious process is instantaneous. However, it may make these models more biologically reasonable and mathematically challenging to consider incorporating suitable delay terms. Time delay plays an important role to reflect the real dynamical behaviors of models, many researchers have proposed and analyzed more realistic models including delays to model different mechanisms in the dynamics of epidemics. Wei *et al.* [6] considered a differential delay model of a vector-borne disease which has direct mode of transmission in addition to the vector-mediated transmission. The delay in their model accounts for the incubation time the vectors need to become infectious. They studied the effect of that delay on the stability of the equilibria and investigated that the introduction of a time delay in the host-to-vector transmission term can destabilize the system. McCluskey [7] in 2010 studied two $SIRS$ models with distributed delay and with discrete delay, respectively. They solved the global stability of the endemic equilibrium for larger delay when $R_0 > 1$. Misra *et al.* [8] in 2012 proposed a delay model to explore the dynamics of water borne diseases like cholera by using disinfectants to control the disease. Their analysis showed that under certain conditions, the cholera disease can be controlled by using disinfectants but a longer delay in their use may destabilize the system. Misra *et al.* [9] in 2013 analyzed a nonlinear delay mathematical model for the control of carrier-dependent infectious diseases, they suggested that as delay in using insecticides exceeds some critical value, the system loses its stability and Hopf-bifurcation occurs. Wang and Wei [10] investigated the global dynamics of a cholera model with delay to demonstrate the impact of the time lag.

Optimal control method [11] as a powerful tool has been applied to control various kinds of diseases [12–16]. Sunmi *et al.* [17] in 2010 studied a model for the transmission dynamics of influenza to evaluate the impact of isolation and/or antiviral drug delivery measures. They compared five control strategies to show the optimal control strategy involving antiviral treatment and/or isolation measures can reduce significantly the number of clinical cases of influenza. Ding *et al.* [18] studied the control problem of maximizing the total payoff in the conservation of a single species with a fixed amount of resource. The existence of an optimal control was established while its uniqueness and characterization was investigated as well. Okosun *et al.* [19] in 2011 derived and analyzed a deterministic model for the transmission of malaria disease with mass action form of infection. They obtained the conditions under which it is optimal to eradicate the disease and examined the impact of a possible combination of vaccination and treatment strategy on the disease transmission by using optimal control theory and the Pontryagin's Maximum Principle. Kar and Jana [20] in 2013 proposed an epidemic model and used the optimal control strategy to minimize both the infected populations and the associated costs. They compared the numerical results with no controls, with only vaccination control, with only treatment control and with both vaccination as well as treatment controls. It is observed that the best result comes out from the application of both vaccination and treatment controls in this case that the number of infected individuals would be the least in number. Wang and Modnak [21] presented a cholera epidemiological model with three control measures. Equilibrium analysis was conducted in the cases with constant controls and with optimal controls, respectively.

According to the above collection of works, an optimal control model including time delay in the context has been not completely understood yet. There are only few papers that tackle

this problem. In recent years, Laarabi *et al.* [22] studied an epidemic model with optimal control strategies and time delay, the optimality system was numerically solved by using an algorithm based on the forward and backward difference approximation in their work. Mohamed *et al.* [23] investigated an optimally controlled SIR epidemic model with time delay in state and control variables, they used optimal control approach via Pontryagin's Maximum Principle to minimize the number of susceptible and infected individuals and to maximize the number of recovered individuals during the course of an epidemic.

In this paper, we will consider an optimally controlled cholera model with time delay based on the model originally suggested by Wang and Modnak [21], which involves both the environment-to-human and human-to-human transmission modes. Our main aim is to explore the role of time delay and optimal control on the spread of cholera in the model. Note most of the delay epidemic models mentioned above are only concerned with local stability of equilibria, we will pay attention to global stability of our model in this paper. The rest of the paper is organized as follows. In the next section, we formulate the mathematical model and determine the basic reproductive number R_0. Section 3 is devoted to the local and global stability analysis of both the disease-free and endemic equilibria of our model. The analysis of optimization problem is presented in Section 4. In Section 5 we present genetic algorithm based on the forward and backward difference approximation and carry out the numerical study of the model, which confirms our theoretical results. Finally, the conclusions are summarized in Section 6.

2 The model formulation

Cholera has been found in multiple transmission pathways including both direct human-to-human and indirect environment-to-human transmissions pathways, which distinct cholera from many other infectious diseases. It is important to notice that, it takes a period for the infected individual to affect the bacterial concentration of cholera, and its size may be very influential in controlling the outbreak of cholera. Thus the delay τ is used to describe the period during the person being infected to his pathogenic bacteria of V. cholera being given off to the aquatic environment. Motivated by the works of Wang and Modnak [21], the deterministic model is given by the following system of ODE:

$$\frac{dS}{dt} = \mu N - \beta_W \frac{SW}{\kappa + W} - \beta_I SI - \mu S - u_1 S, \tag{1}$$

$$\frac{dI}{dt} = \beta_W \frac{SW}{\kappa + W} - \beta_I SI - (\gamma + \mu) I - u_2 I, \tag{2}$$

$$\frac{dW}{dt} = \xi I(t-\tau) - \delta W - u_3 W, \tag{3}$$

$$\frac{dR}{dt} = \gamma I - \mu R + u_2 I + u_1 S. \tag{4}$$

In the equations above, let N be the total population which is divided into three epidemiological compartments, susceptible compartment S, infectious compartment I, recovered compartment R. Let W be the density of V. cholerae in the aquatic environment. The parameter κ is the concentration of vibrios in contaminated water in the environment,

β_W and β_I are rates of ingesting vibrios from the contaminated environment and through human-to-human interaction, respectively. μ represents the natural human birth/death rate, ξ the shedding rate, γ the recovery rate, δ the bacterial death rate. All the parameters are strictly positive constants. Intervention strategies are modeled by the control variables $u_i(t)$ $(i = 1, 2, 3)$, which are bounded, Lebesgue integrable functions. The control $u_1(t)$ represents the rate of vaccination, $u_2(t)$ represents the rate of therapeutic treatment, water sanitation leads to the death of vibrios at a rate $u_3(t)$. Based on biological assumption, we assume that for $\theta \in [-\tau, 0]$, $S(\theta)$, $I(\theta)$ and $R(\theta)$ are non negative real valued functions. Let $C = C([-\tau, 0], R^3)$ be the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into R^3 with the topology of uniform convergence. For ecological reasons, we assume that the initial conditions for system (1-4) satisfies:

$$S_0(\theta) \geq 0\,, I_0(\theta) \geq 0\,, R_0(\theta) \geq 0\,, \theta \in [-\tau, 0]. \tag{5}$$

In order to determine the dynamics of each class, we only need to study the first three equations in model (1-4), thereby reducing the order of the system through eliminating R to obtain the following system:

$$\begin{aligned} \frac{dS}{dt} &= \mu N - \beta_W \frac{SW}{\kappa + W} - \beta_I SI - \mu S - u_1 S, & (6) \\ \frac{dI}{dt} &= \beta_W \frac{SW}{\kappa + W} - \beta_I SI - (\gamma + \mu) I - u_2 I, & (7) \\ \frac{dW}{dt} &= \xi I(t - \tau) - \delta W - u_3 W. & (8) \end{aligned}$$

As the study of model system (1-4) is equivalent to study model system (6-8), so we study model system (6-8).

Based on the next-generation matrix approach [25], we define the basic reproduction number R_0, representing the average number of secondary infections that occurs when one infective is introduced into a completely susceptible host population, as:

$$R_0 = \frac{\mu N[\xi \beta_W + (\delta + u_3)\kappa \beta_I]}{\kappa(\mu + u_1)(\delta + u_3)(\gamma + \mu + u_2)}. \tag{9}$$

3 Mathematical analysis of the epidemic model

In particular, when the time delay is set to zero, i.e. $\tau = 0$, the above system (6-8) is reduced to the original model developed in Wang and Modnak [21]. Based on their work, the results below directly follows:

Theorem 1 *The disease-free equilibrium (DFE) of the model (6-8)* $E_0 = \left(\frac{\mu N}{\mu + u_1}\,,\ 0\,,\ 0\,,\ 0\right)^T$, *is both locally and globally asymptotically stable if* $R_0 < 1$ *with* $\tau = 0$.

Theorem 2 *The endemic equilibrium of the model (6-8)* $E^* = (S^*, I^*, W^*)$ *is locally asymptotically stable and globally asymptotically stable if* $R_0 > 1$ *with* $\tau = 0$.

3.1 The stability of the disease-free equilibrium

Our primary focus is on the stability analysis of the model when $\tau \neq 0$ in this section. First, we prove the local and global stability of the disease-free equilibrium E_0 with $\tau > 0$.

Theorem 3 *The disease-free equilibrium (DFE) of the model (6-8) is locally asymptotically stable if $R_0 < 1$ with $\tau > 0$.*

Proof After linearizing the ODE system (6-8) around the disease-free equilibrium E_0, we obtain one negative characteristic solution $\lambda = -\mu - u_1$ and the following transcendental characteristic equation is:

$$\lambda^2 + a_1\lambda + a_2 + b_1 e^{-\lambda\tau} = 0, \tag{10}$$

where

$$\begin{aligned} a_1 &= \delta + \gamma + \mu + u_2 + u_3 - \beta_I \frac{\mu N}{\mu + u_1}, \\ a_2 &= (\delta + u_3)(\gamma + \mu + u_2 - \beta_I \frac{\mu N}{\mu + u_1}), \\ b_1 &= -\frac{\xi\beta_W}{\kappa}\frac{\mu N}{\mu + u_1}. \end{aligned}$$

We can rearrange equation (10) in the form:

$$\begin{aligned} \lambda^2 + a_1\lambda &= (\delta + u_3)(\gamma + \mu + u_2)[(\frac{\mu N \kappa \beta_I}{\kappa(\mu + u_1)(\gamma + \mu + u_2)} - 1) \\ &\quad + \frac{\mu N \xi \beta_W}{\kappa(\mu + u_1)(\delta + u_3)(\gamma + \mu + u_2)} e^{-\lambda\tau}]. \end{aligned} \tag{11}$$

Let the left-hand side and right-hand side of equation (11) be $F(\lambda)$ and $H(\lambda)$, respectively. It is easy to see that $F(0) = 0$ and $lim_{\lambda\to\infty} F(\lambda) = \infty$, therefore, $F(\lambda)$ is an increasing function of λ. On the other hand, $H(\lambda)$ is a decreasing function of λ and $H(0) = (\delta + u_3)(\gamma + \mu + u_2)(R_0 - 1)$ is less than zero when $R_0 < 1$. Thus, equation (11) has no non-negative real roots. If equation (10) has roots with non-negative real parts, they must be complex and obtained from a pair of complex conjugate roots which cross the imaginary axis. As a result, a pair of purely imaginary solution may come out from the equation (10) for $\tau > 0$. Assume that $i\omega$ ($\omega > 0$) is the root of equation (10) and ω satisfies the following equation:

$$-\omega^2 + a_1 i\omega + a_2 + b_1(cos(\omega\tau) - isin(\omega\tau)) = 0. \tag{12}$$

Separating the real and imaginary parts of equation (12) gives

$$-\omega^2 + a_2 = -b_1 cos(\omega\tau), \qquad -a_1\omega = -b_1 sin(\omega\tau). \tag{13}$$

To eliminate the trigonometric functions, we add up the squares of equation (13) above, and obtain the following forth order equation in ω:

$$\omega^4 + (a_1^2 - 2a_2)\omega^2 + a_2^2 - b_1^2 = 0. \tag{14}$$

We can solve that

$$\omega^2 \;=\; \frac{1}{2}[-(a_1^2-2a_2)\pm\sqrt{(a_1^2-2a_2)^2-4(a_2^2-b_2^2)}]. \tag{15}$$

This implies equation (14) has no positive roots, which leads to the conclusion that there is no ω such that $i\omega$ is a solution of equation (10) for time delay $\tau > 0$. Based on Rouche's theorem [26], E_0 is locally asymptotically stable if $R_0 < 1$. ■ Next, we will analyze the global stability of the disease-free equilibrium of the model system (6-8) for time delay $\tau > 0$.

Theorem 4 *The disease-free equilibrium (DFE) of the model (6-8) is globally asymptotically stable with time delay $\tau > 0$ if $R_0 < 1$.*

Proof

Adding equations (1) and (2), we obtain

$$S^{'}+I^{'}=\mu N-(\mu+u_1)S-(\gamma+\mu+u_2)I\leq \mu N-\eta(S+I), \tag{16}$$

and equation (3) yields

$$W^{'}=\xi I(t-\tau)-(\delta+u_3)W\leq \xi\frac{\mu N}{\eta}-(\delta+u_3)W, \tag{17}$$

where $\eta=\min\{(\mu+u_1),(\gamma+\mu+u_2)\}$. These imply

$$\limsup_{t\to\infty} I(t) \;\leq\; \frac{\mu N}{\eta}. \tag{18}$$

and

$$\limsup_{t\to\infty} W(t) \;\leq\; \frac{\xi\mu N}{\eta(\delta+u_3)}. \tag{19}$$

We consider the following Lyapunov function:

$$V_1(t) \;=\; \xi[S(t)-\frac{\mu N}{\mu+u_1}ln\frac{S(t)}{\frac{\mu N}{\mu+u_1}}]+\xi I_t(0)+(\gamma+\mu+u_2)W(t)+\xi(\gamma+\mu+u_2)\int_{-\tau}^{0} I_t(\theta)d\theta. \tag{20}$$

Here, $I_t(\theta) = I(t+\theta)$ for $\theta \in [-\tau, 0]$, therefore, $I_t(0) = I(t)$ in this equation (20). Calculating the time derivative of $V_1(t)$ along solutions of system (6-8),

$$\begin{aligned}
\frac{dV_1(t)}{dt} &= \xi(S'(t) - \frac{\mu N}{\mu+u_1}\frac{S'(t)}{S(t)}) + \xi I'(t) + (\gamma+\mu+u_2)W'(t) + \xi(\gamma+\mu+u_2)[\int_{t-\tau}^{t} I(t)dS]' \\
&= \xi[\mu N - \beta_W\frac{S(t)W(t)}{\kappa+W(t)} - \beta_I S(t)I(t) - (\mu+u_1)S(t) \\
&\quad + \frac{\mu N}{\mu+u_1}(\frac{\beta_W W(t)}{\kappa+W(t)} + \beta_I I(t) + \mu + u_1 - \frac{\mu N}{S(t)})] + \xi\beta_W\frac{S(t)W(t)}{\kappa+W(t)} \\
&\quad + \xi\beta_I S(t)I(t) - \xi(\gamma+\mu+u_2)I(t) + (\gamma+\mu+u_2)\xi I(t-\tau) \\
&\quad - (\gamma+\mu+u_2)(\delta+u_3)W(t) + \xi(\gamma+\mu+u_2)I(t) - (\gamma+\mu+u_2)\xi I(t-\tau) \\
&= 2\xi\mu N - \xi(\mu+u_1)S(t) + \frac{\xi\mu N}{\mu+u_1}(\frac{\beta_W W(t)}{\kappa+W(t)} + \beta_I I(t) - \frac{\mu N}{S(t)}) \\
&\quad - (\gamma+\mu+u_2)(\delta+u_3)W(t) \\
&= \xi\mu N(2 - \frac{\mu N}{\mu+u_1}\frac{1}{S(t)} - \frac{\mu+u_1}{\mu N}S(t)) + [\frac{\xi\mu N}{\mu+u_1}(\frac{\beta_W W(t)}{\kappa+W(t)} + \beta_I I(t)) \\
&\quad - (\gamma+\mu+u_2)(\delta+u_3)W(t)].
\end{aligned} \tag{21}$$

Obviously, $2 - \frac{\mu N}{\mu+u_1}\frac{1}{S(t)} - \frac{\mu+u_1}{\mu N}S(t) \le 0$, thus, $\frac{dV_1(t)}{dt} = 0$ if and only if $S = \frac{\mu N}{\mu+u_1}$. In addition, if $R_0 < 1$, it is sufficient to verify that the second term of equation (21) is less than 0 by combining equations (18) and (19). Therefore, $\frac{dV_1(t)}{dt} \le 0$. This completes the proof. ∎

3.2 The stability of the endemic equilibrium

To study the stability of the endemic equilibrium $E^*(S^*, I^*, W^*)$, we linearize the system (6-8) at the point E^* by Letting $S = S^* + s$, $I = I^* + i$, $W = W^* + w$, here s, i and w are small perturbations around the equilibrium E^*. To make the algebraic manipulation simpler, we set $P^* = \frac{\beta_W W^*}{\kappa+W^*} + \beta_I I^*$. When $\tau > 0$, the characteristic polynomial for linearized equation is obtained as:

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 + (b_1\lambda + b_2)e^{-\lambda\tau} = 0, \tag{22}$$

where

$$\begin{aligned}
&a_1 = -\beta_I S^* + P^* + \gamma + 2\mu + \delta + u_1 + u_2 + u_3, \\
&a_2 = (P^* + \mu + u_1)(-\beta_I S^* + \gamma + \mu + u_2) + P^* S^* \beta_I + (\delta + u_3)\times \\
&(-\beta_I S^* + P^* + \gamma + 2\mu + u_1 + u_2), \\
&a_3 = (\delta + u_3)(P^* + \mu + u_1)(-\beta_I S^* + \gamma + \mu + u_2) + \beta_I(\delta + u_3)P^* S^*, \\
&b_1 = -\xi\beta_W S^*\frac{\kappa}{(\kappa+W^*)^2}, \\
&b_2 = -\xi(\mu+u_1)\beta_W S^*\frac{\kappa}{(\kappa+W^*)^2}.
\end{aligned}$$

Now we suppose λ is a root of equation (22), and substitute $\lambda = i\omega$ ($\omega > 0$) into equation (22), after separating real and imaginary parts, we finally obtain the following two transcendental equations:

$$-a_1\omega^2 + a_3 = -b_2cos(\omega\tau) - b_1\omega sin(\omega\tau), \tag{23}$$
$$-\omega^3 + a_2\omega = -b_1\omega cos(\omega\tau) + b_2sin(\omega\tau). \tag{24}$$

By adding up the squares of both the equations (23) and (24), the following sixth degree equation for ω is obtained:

$$\omega^6 + \omega^4(a_1^2 - 2a_2) + \omega^2(a_2^2 - 2a_1a_3 - b_1^2) + a_3^2 - b_2^2 = 0. \tag{25}$$

Letting $\omega^2 = x$ gives:

$$F(x) = x^3 + B_1x^2 + B_2x + B_3 = 0, \tag{26}$$

where

$$B_1 = a_1^2 - 2a_2\,, B_2 = a_2^2 - 2a_1a_3 - b_1^2\,, B_3 = a_3^2 - b_2^2.$$

Here, we establish the following theorem.

Theorem 5 *When $R_0 > 1$, the endemic equilibrium E^* of ODE system (6-8) is locally asymptotically stable for the delay $\tau > 0$ if $B_1 \geq 0$, $B_3 \geq 0$ and $B_2 > 0$.*

Proof In order to show that the endemic equilibrium E^* is locally stable, we have to show that equation (26) does not have a positive real root. In fact, if we take the derivative of $F(x)$ with respect to x, $F^{'}(x) = 3x^2 + 2B_1x + B_2$. The roots of equation $F^{'}(x) = 0$ can be solved as $x_{1,2} = \frac{-B_1 \pm \sqrt{B_1^2 - 3B_2}}{3}$. If $B_2 > 0$, then $\sqrt{B_1^2 - 3B_2} < B_1$. Hence, neither x_1 nor x_2 is positive, it follows that equation $F^{'}(x) = 0$ has no positive roots. Also, a simple assumption that $F(0) = B_3 \geq 0$, implies that equation (26) will have no positive real roots. Therefore, there is no ω such that $i\omega$ is an eigenvalue of the characteristic equation (22). By Rouch's theorem [26], the real parts of all the eigenvalues of (22) are negative for time delay $\tau \geq 0$. This completes the proof. ■

Next, we turn our attention to the global stability of the ODE system (6-8) if $R_0 > 1$ for all values of the delay $\tau > 0$.

Theorem 6 *When $R_0 > 1$, the positive endemic equilibrium E^* of ODE system (6-8) is globally asymptotically stable for all delay $\tau > 0$.*

Proof We consider the following Lyapunov function:

$$\begin{aligned} V_2(t) = {} & S^*(\frac{S(t)}{S^*} - 1 - ln\frac{S(t)}{S^*}) + I^*(\frac{I_t(0)}{I^*} - 1 - ln\frac{I_t(0)}{I^*}) + \frac{\gamma + \mu + u_2}{\xi}W^* \times \\ & (\frac{W(t)}{W^*} - 1 - ln\frac{W(t)}{W^*}) + (\gamma + \mu + u_2)I^*\int_{-\tau}^{0}(\frac{I_t(s)}{I^*} - 1 - ln\frac{I_t(s)}{I^*})ds. \end{aligned} \tag{27}$$

Differentiating $V_2(t)$ along solutions of (6-8), we can obtain:

$$\begin{aligned}
\frac{dV_2(t)}{dt} &= \mu N - \mu S(t) - u_1 S(t) - S^* \frac{\mu N}{S(t)} + S^* P + 2\mu S^* + 2u_1 S^* - \frac{\beta_W S^* S(t) W(t)}{\kappa + W(t)} \\
&\quad -\beta_I S(t) I^* + 2(\gamma+\mu+u_2) I^* - \frac{(\gamma+\mu+u_2)(\delta+u_3)W(t)}{\xi} \\
&\quad -\frac{(\gamma+\mu+u_2)W^* I(t-\tau)}{W(t)} + \frac{(\gamma+\mu+u_2)(\delta+u_3)W^*}{\xi} \\
&\quad +(\gamma+\mu+u_2)I^*(ln\frac{I(t-\tau)}{I^*} - ln\frac{I(t)}{I^*}) \\
&= \mu S^*(2 - \frac{S(t)}{S^*} - \frac{S^*}{S(t)}) + u_1 S^*(2 - \frac{S^*}{S(t)} - \frac{S(t)}{S^*}) + (\gamma+\mu+u_2)I^* \times \\
&\quad [(\frac{P(t)}{P^*} - 1)(1 - \frac{P^*}{P(t)}\frac{W(t)}{W^*})] - (\gamma+\mu+u_2)I^*(\frac{S^*}{S(t)} - 1 - ln\frac{S^*}{S(t)}) \\
&\quad -(\gamma+\mu+u_2)I^*[\frac{P(t)}{P^*}\frac{I^*}{S^*}\frac{S(t)}{I(t)} - 1 - ln(\frac{P(t)}{P^*}\frac{I^*}{S^*}\frac{S(t)}{I(t)})] \\
&\quad -(\gamma+\mu+u_2)I^*[\frac{W^*}{W(t)}\frac{I(t-\tau)}{I^*} - 1 - ln(\frac{W^*}{W(t)}\frac{I(t-\tau)}{I^*})]. \qquad (28)
\end{aligned}$$

Clearly, $2-\frac{S(t)}{S^*}-\frac{S^*}{S(t)} \le 0$ for $S(t) > 0$. Furthermore, note that at the endemic equilibrium E^*, the right-hand side of equation (8) becomes 0, which yields $\xi I^* = (\delta + u_3)W^*$, and combine the facts (18) and (19), we can get $(\frac{P(t)}{P^*} - 1)(1 - \frac{P^*}{P(t)}\frac{W(t)}{W^*}) < 0$ if $R_0 > 1$. Also, for all $t \ge 0$, the function $g(t) = t - 1 - lnt$ is always non-negative, and $g(t) = 0$ if and only if $t = 1$, then the fourth term, the fifth term and the last term in (28) are non-negative. Therefore, we can finally show $\frac{dV_2(t)}{dt} \le 0$. This completes the proof. ■

4 Optimal control analysis

In this section, we seek to minimize the objective functional defined by decreasing the number of infected and the costs of time-related controls,the method is described in [28]. We choose a linear function for the cost on infection I, and quadratic forms for the cost on the controls u_1, u_2 and u_3. The objective function subject to the differential equations (1-4) is constructed as follows:

$$J = \int_0^{t_f} (A_0 I + A_1 u_1^2 + A_2 u_2^2 + A_3 u_3^2) dt.$$

We assume t_f is the fixed final time, the parameters A_0, A_1, A_2 and A_3 are weight parameters describing the comparative importance of the all terms on control cost. The optimal control problem is that of finding optimal functions u_1^*, u_2^* and u_3^* such that

$$J(u_1^*, u_2^*, u_3^*) = \min_{u_1,u_2,u_3 \in \Theta} J(u_1, u_2, u_3), \qquad (29)$$

where Θ is measurable on $[0, 1]$ and $\Theta = \{u_i | 0 \le u_i \le 1\}$ for the controls.

The Lagrangian of this object is given by

$$L(I, u_1, u_2, u_3) \;=\; A_0 I + A_1 u_1^2 + A_2 u_2^2 + A_3 u_3^2, \tag{30}$$

and the Hamiltonian H for the control problem is:

$$H(S, I, W, R, u_1, u_2, u_3, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = L + \lambda_1(t)\frac{dS}{dt} + \lambda_2(t)\frac{dI}{dt} + \lambda_3(t)\frac{dW}{dt} + \lambda_4(t)\frac{dR}{dt}, \tag{31}$$

where $\lambda_i(t)$ for $i = 1, 2, 3, 4$ are the adjoint variables, which determine the adjoint system, and can be solved by the following system:

$$\begin{aligned}
\dot{\lambda_1}(t) &= -\frac{\partial H}{\partial S} - \chi_{[0,t_f-\tau]}\frac{\partial H}{\partial S_\tau}(t+\tau) \\
&= \lambda_1(\frac{\beta_W W}{\kappa + W} + \beta_I + \mu + u_1) - \lambda_2(\frac{\beta_W W}{\kappa + W} + \beta_I) - \lambda_4\mu, && (32) \\
\dot{\lambda_2}(t) &= -\frac{\partial H}{\partial I} - \chi_{[0,t_f-\tau]}\frac{\partial H}{\partial I_\tau}(t+\tau) \\
&= -A_0 + \lambda_1\beta_I S - \lambda_2[\beta_I S - (\gamma + \mu + u_2)] - \lambda_4(\gamma + u_2) - \lambda_2(t+h)\xi, && (33) \\
\dot{\lambda_3}(t) &= -\frac{\partial H}{\partial W} - \chi_{[0,t_f-\tau]}\frac{\partial H}{\partial W_\tau}(t+\tau) \\
&= \lambda_1\frac{\beta_W S\kappa}{(\kappa + W)^2} - \lambda_2\frac{\beta_W S\kappa}{(K + W)^2} + \lambda_3(\delta + u_3), && (34) \\
\dot{\lambda_4}(t) &= -\frac{\partial H}{\partial R} - \chi_{[0,t_f-\tau]}\frac{\partial H}{\partial R_\tau}(t+\tau) \\
&= \lambda_4\mu. && (35)
\end{aligned}$$

Satisfying the transversality conditions:

$$\lambda_i(t_f) = 0, \qquad\qquad i = 1, 2, 3, 4. \tag{36}$$

The combination of the ODE system (1-4) and the state system (32-35) is the optimality system, which describes how the system behaves minimize J under the control applications. By applying Pontryagin's Maximum theory and the existence result for the optimal control [27], we thus establish the following theorem:

Theorem 7 *There is a triplet of optimal control* (u_1^*, u_2^*, u_3^*) *such that* $J(u_1^*, u_2^*, u_3^*) = \min_{u_1,u_2,u_3\in\Theta} J(u_1, u_2, u_3)$ *subject to the optimality control system.*

Theorem 8 *There is a triplet of optimal control* (u_1^*, u_2^*, u_3^*) *which minimizes* J *over the region* Θ *given by*

$$u_1^* = \min\{\max\{0, u_1\}, 1\}\,,\; u_2^* = \min\{\max\{0, u_2\}, 1\}\,,\; u_3^* = \min\{\max\{0, u_3\}, 1\}, \tag{37}$$

where

$$u_1 = \frac{(\lambda_1(t) - \lambda_4(t))S^*}{2A_1}\,,\; u_2 = \frac{(\lambda_2(t) - \lambda_4(t))I^*}{2A_2}\,,\; u_3 = \frac{\lambda_3(t)W^*}{2A_3}. \tag{38}$$

Proof The optimal controls u_1^*, u_2^* and u_3^* can be solved by setting the partial derivatives of H equal to zero,

$$\frac{\partial H}{\partial u_1} = 2A_1u_1 - \lambda_1(t)S^* + \lambda_4(t)S^* = 0, \tag{39}$$

$$\frac{\partial H}{\partial u_2} = 2A_2u_2 - \lambda_2(t)I^* + \lambda_4(t)I^* = 0, \tag{40}$$

$$\frac{\partial H}{\partial u_3} = 2A_3u_3 - \lambda_3(t)W^* = 0. \tag{41}$$

After a simple manipulation, the optimal control pair $(u_1^*, u_2^*,\ u_3^*)$ is characterized as (37) and (38). ■

By standard control arguments involving the bounds on the controls, we conclude

$$u_1^* = \begin{cases} \frac{(\lambda_1(t)-\lambda_4(t))S^*}{2A_1} & \text{if } 0 < \frac{(\lambda_1(t)-\lambda_4(t))S^*}{2A_1} < 1, \\ 0 & \text{if } \frac{(\lambda_1(t)-\lambda_4(t))S^*}{2A_1} \leq 0, \\ 1 & \text{if } \frac{(\lambda_1(t)-\lambda_4(t))S^*}{2A_1} \geq 1. \end{cases}$$

$$u_2^* = \begin{cases} \frac{(\lambda_2(t)-\lambda_4(t))I^*}{2A_2} & \text{if } 0 < \frac{(\lambda_2(t)-\lambda_4(t))I^*}{2A_2} < 1, \\ 0 & \text{if } \frac{(\lambda_2(t)-\lambda_4(t))I^*}{2A_2} \leq 0, \\ 1 & \text{if } \frac{(\lambda_2(t)-\lambda_4(t))I^*}{2A_2} \geq 1 \end{cases}$$

$$u_3^* = \begin{cases} \frac{\lambda_3(t)W^*}{2A_3} & \text{if } 0 < \frac{\lambda_3(t)W^*}{2A_3} < 1, \\ 0 & \text{if } \frac{\lambda_3(t)W^*}{2A_3} \leq 0, \\ 1 & \text{if } \frac{\lambda_3(t)W^*}{2A_3} \geq 1. \end{cases}$$

5 Numerical results

In this section, we work out the optimality system which is combined by the ODE system (1-4) and the adjoint system (32-35) by using the data regarding the course of the cholera in Zimbabwe (2008-2009). It began in August 2008, not only swept to all of Zimbabwe's ten provinces but also spread to Botswana, Mozambique, South Africa and Zambia quickly. The principal cause of the outbreak was the collapse of Zimbabwe's public health system. By the end of November 2008, three of Zimbabwe's four major hospitals had shut down, and many places had no basic drugs, medicines and water supply for such a long enough period during the outbreak period. On 4 December 2008, the Zimbabwe government declared the outbreak to be a national emergency. By March 2009, the World Health Organization (WHO) estimated that 4,011 people had succumbed to this waterborne disease and 91,164 cases were infected. The total population in Zimbabwe is 12,347,240, in order to make the calculation simpler, we scale down all data numbers by a factor of 1,200. All epidemiological parameter values for cholera in literature are given as $N = 10000$, $\mu = 0.000442$, $\gamma = 1.4$, $\xi = 70$, $\delta = 0.023$, $\beta_W = 0.12$, $\beta_I = 0.00075$. We use the initial values as $S_0 = 9999$, $I_0 = 1$, $W_0 = 0$, $R_0 = 0$. The weight constants are set as $A_0 = A_1 = A_2 = A_3 = 10$.

We note that the optimality system is a two-point boundary value problem, with separated boundary conditions at initial time $t = 0$ and final time $t = t_f$. Solving this optimality

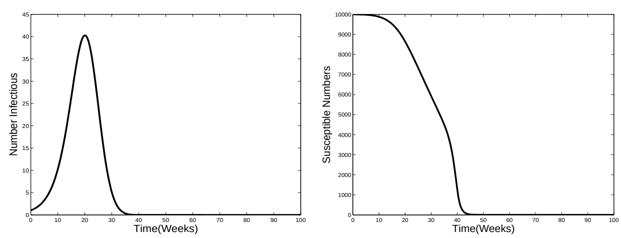
system requires an iterative scheme which is combination of forward and backward difference approximation developed by [22, 24], we show this procedure in the following algorithm. In the programming, let there exist a uniform step size $h > 0$ and $(n, m) \in N^2$, $\tau = mh$ and $t_f = nh$. We can obtain the following partition by setting m knots to left of 0 and right of t_f.

$$\Delta = (t_{-m} = -\tau < \cdots < t_{-1} < 0 < t_1 < \cdots < t_n = t_f < \cdots < t_{n+m}).$$

Therefore, $t_i = ih(-m \leq i \leq n+m)$. The state and adjoint variables and control variables, such as $S(t)$, $I(t)$, $W(t)$, $R(t)$, λ_i and u_i in terms of nodal points S_i, I_i, W_i, R_i, λ_i^i and u_i.

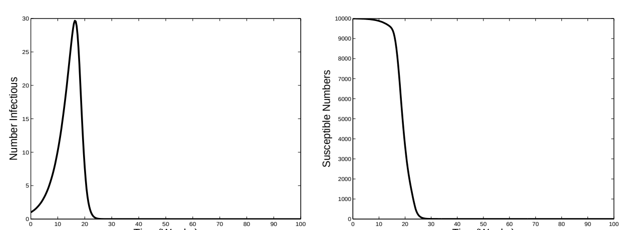
Fig.1 (a) represents the number of infected individuals as a function of time when $\tau = 5$, epidemic outbreak increases rapidly and reaches the peak at $t = 22$ weeks with value 40, the controls take some time to react with the infected individuals, it then starts to gradually drop to almost zero, meaning the disease is gradually eradicated from the population. Fig.1 (b) shows the susceptible population S vs. time (weeks), we observe that there is a significant decrease in the number of susceptible after around 40 weeks.

In order to clearly see the effect of the time lag on the dynamical behavior of the system, we take a smaller time delay as $\tau = 1$ in Fig.2. By comparison with Fig.1, we can observe the smaller the time delay, the shorter it takes the equilibrium points to settle to their state value, which implies that the disease will be more serious if the delay lag is shorter.



(a) (b)

Figure 1: (a)The plot shows the infected population I vs. time (weeks) for time delay $\tau = 5$. (b)The plot shows the susceptible population S vs. time (weeks) for time delay $\tau = 5$.



(a) (b)

Figure 2: (a)The plot shows the infected population I vs. time (weeks) for time delay $\tau = 1$. (b)The plot shows the susceptible population S vs. time (weeks) for time delay $\tau = 1$.

We have plotted the controls $u_i(t)$ $(i = 1, 2, 3)$ as a function of time in Fig.3, representing the optimal controls in blocking new infection and inhibiting viral production under two

Algorithm

Step1

for $i=-m,...,0,$ **do**

$S_i=S(0),\ I_i=I(0),\ W_i=W(0),\ R_i=R(0),\ u_1^i=0,\ u_2^i=0,\ u_3^i=0,$

end for

for $i=n,...,n+m,$ **do** $\lambda_1^i=0,\ \lambda_2^i=0,\ \lambda_3^i=0,$

end for

Step2

for $i=0,...,n-1,$ **do**

$S_{i+1}=\frac{S_i+h\mu N}{1+h(\frac{\beta_W W_{i+1}}{\kappa+W_{i+1}}+\beta_I I_{i+1}+\mu+u_1)},$

$I_{i+1}=\frac{I_i+h\beta_W\frac{S_{i+1}W_{i+1}}{\kappa+W_{i+1}}}{1+h(\gamma+\mu+u_2-\beta_I S_{i+1})},$

$W_{i+1}=\frac{W_i+h\xi I_{i-m}}{1+h(\delta+u_3)},$

$R_{i+1}=\frac{R_i+h(\gamma I_{i+1}+u_2 I_{i+1}+u_1 S_{i+1})}{1+h\mu},$

$\lambda_1^{n-i-1}=\frac{\lambda_1^{n-i}+h(\frac{\beta_W W_{i+1}}{\kappa+W_{i+1}}+\beta_I I_{i+1})\lambda_2^{n-i}+h\mu\lambda_4^{n-i}}{1+h(\frac{\beta_W W_{i+1}}{\kappa+W_{i+1}}+\beta_I I_{i+1}+\mu+u_1)},$

$\lambda_2^{n-i-1}=\frac{\lambda_2^{n-i}+h-h\lambda_1^{n-i-1}\beta_I S_{i+1}+h\lambda_4^{n-i}(\gamma+u_2)+h\lambda_2^{n-i+m}\chi_{[0,t_f-\tau]}(t_{n-i})\xi}{1+h[\beta_I S_{i+1}-(\gamma+\mu+u_2)]},$

$\lambda_3^{n-i-1}=\frac{\lambda_3^{n-i}-h\lambda_1^{n-i-1}\frac{\beta_W\kappa S_{i+1}}{(\kappa+W_{i+1})^2}+h\lambda_2^{n-i-1}\frac{\beta_W\kappa S_{i+1}}{(\kappa+W_{i+1})^2}}{1+h(\delta+u_3)},$

$\lambda_4^{n-i-1}=\frac{\lambda_4^{n-i}}{1+h\mu},$

$T_1^{i+1}=\frac{(\lambda_1^{n-i}-\lambda_4^{n-i})S_{i+1}}{2A_1},$

$T_2^{i+1}=\frac{(\lambda_2^{n-i}-\lambda_4^{n-i})I_{i+1}}{2A_2},$

$T_3^{i+1}=\frac{\lambda_3^{n-i}W_{i+1}}{2A_1},$

$u_1^{i+1}=min(max(0,T_1^{i+1}),1),$

$u_2^{i+1}=min(max(0,T_2^{i+1}),1),$

$u_2^{i+1}=min(max(0,T_3^{i+1}),1),$

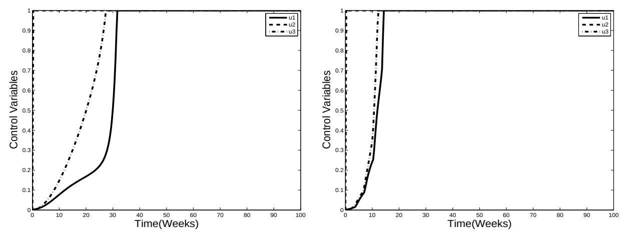
Step3

for $i=0,...,n,$ **write**

$S^*(t_i)=S_i, I^*(t_i)=I_i, W^*(t_i)=W_i, R^*(t_i)=R_i, u_1^*(t_i)=u_1^i, u_2^*(t_i)=u_2^i, u_3^*(t_i)=u_3^i,$

end for

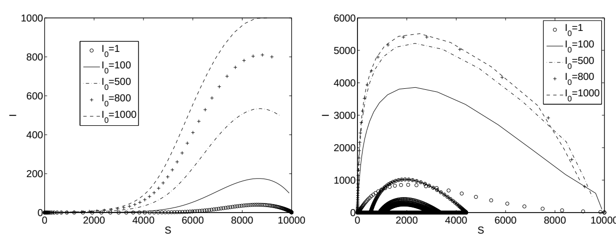
different cases: $\tau = 6$ and $\tau = 3$, respectively. From Fig.3, it is apparent that a larger value of optimal control variables is necessary in case of smaller time delay. It is also clear to see that the control u_2 in both cases always needs to be the maximal while the other two controls u_1 and u_3, which need not to be the maximal at very first, increase gradually and reach the maximal until certain weeks. Hence, we can firstly apply more of the therapeutic treatment in order to effectively reduce the number of infectious individuals.



(a) (b)

Figure 3: (a)The plot represents the controls u_1, u_2 and u_3 vs. time (weeks) for time delay $\tau = 6$. (b)The plot represents the controls u_1, u_2 and u_3 vs. time (weeks) for time delay $\tau = 3$.

To verify the global asymptotic stability of the ODE system analyzed in Sections 3, we pick five different initial conditions with $I(0) = 1, 100, 500, 800, 1000$, respectively, and plot these five solution curves by the phase plane portrait of I vs. S in Fig. 4. We clearly see that all these five orbits converge to the disease-free equilibrium E_0 when $R_0 < 1$ in Fig. 4(a) and converge to endemic equilibrium E^* when $R_0 > 1$ in Fig. 4(b), respectively.

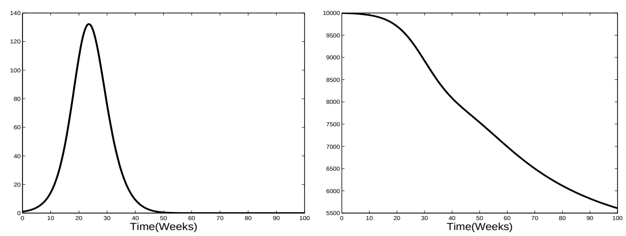


(a) (b)

Figure 4: (a)The phase plane portrait of I vs. S for $R_0 < 1$, all these orbits converge to the disease-free equilibrium E_0. (b)The phase plane portrait of I vs. S for $R_0 > 1$, all these orbits converge to the endemic equilibrium E^*.

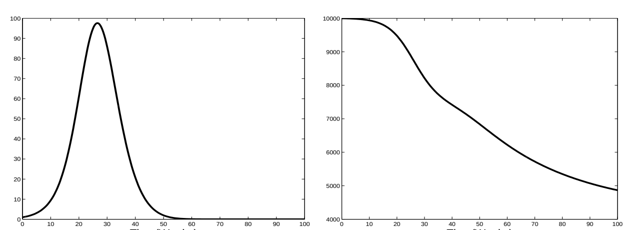
In order to illustrate the impacts of the different optimal control strategies, we investigate and compare numerical results in the following four strategies for the control of the disease: (1)when the objective function J is optimized through the control u_1, while u_2 and u_3 are set to be zero; (2)when the objective function J is optimized through the control u_2, while u_1 and u_3 are set to be zero; (3)when the objective function J is optimized through the control u_3, while u_1 and u_2 are set to be zero; (4)without any controls, while u_1, u_2 and u_3 are all set to be zero. We observe from Fig.5, as can be expected, there is a significant increase

in the number of infected individuals and susceptible individuals controlled compared with optimal controlled, so that the infected population is affected very much due to the lack of all the three controls. Compared with Fig.6, Fig.7 and Fig.8, the number of infectious does not differ significantly by applying either the strategies with control u_1 only or with control u_3 only, but does make greater significance when only treatment control u_2 is employed, thus the application of therapeutic treatment control gives better result than the application of u_1 or u_3 only. This simulation indicates that therapeutic treatment is more effective in reducing the infection level, which highlights the effectiveness of treatment measure in controlling the diseases. In a word, the use of a single optimal control method does not make a significant impact, while the use of multi-strategies is more efficient. However, if the budget is limited, it is much better to apply the treatment well before the occurrence of the outbreak.



(a) (b)

Figure 5: (a)The plot shows the infected population I vs. time (weeks) for time delay $\tau = 5$ if there are no controls. (b)The plot shows the susceptible population S vs. time (weeks) for time delay $\tau = 5$ if there are no controls.

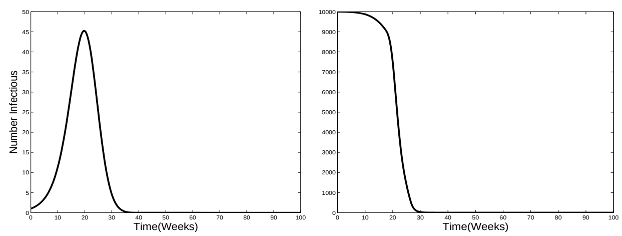


(a) (b)

Figure 6: (a)The plot shows the infected population I vs. time (weeks) for time delay $\tau = 5$ if there is only control u_1. (b)The plot shows the susceptible population S vs. time (weeks) for time delay $\tau = 5$ if there is only control u_1.

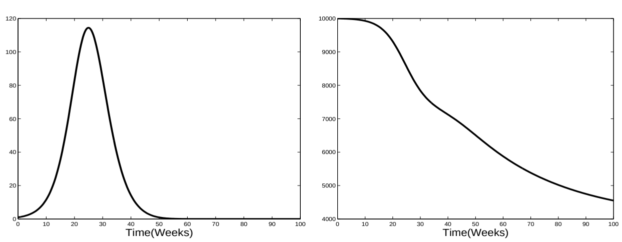
6 Conclusions and discussions

In this paper, we have presented a cholera epidemiological model by incorporating three types of intervention strategies and time delay inspired by the work in Wang and Modnak [21]. We have mainly investigated that by applying both an optimal control and a time delay to a



(a) (b)

Figure 7: (a)The plot shows the infected population I vs. time (weeks) for time delay $\tau = 5$ if there is only control u_2. (b)The plot shows the susceptible population S vs. time (weeks) for time delay $\tau = 5$ if there is only control u_2.



(a) (b)

Figure 8: (a)The plot shows the infected population I vs. time (weeks) for time delay $\tau = 5$ if there is only control u_3. (b)The plot shows the susceptible population S vs. time (weeks) for time delay $\tau = 5$ if there is only control u_3.

cholera model in order to eliminate the infectious disease. First of all, both the disease-free equilibrium E_0 and endemic equilibrium E^* of the model were obtained. By analyzing the corresponding characteristic equations, the local stability of E_0 and E^* was investigated. In particular, we have established the global stability analysis of the disease-free and endemic equilibria of ODE system by constructing two suitable Lyapunov functionals. Moreover, we used the Pontryagins Maximum Principle with delay to characterize optimal controls and derived the optimality system at the same time. Finally, we presented an efficient numerical simulation based on a specific algorithm to show that the optimal control strategy is much more effective for reducing the number of infected individuals than using of any single control, which highlights the effectiveness of treatment measure in controlling the diseases. However, if the budget is limited, it is much better to apply the therapeutic treatment well before the occurrence of the outbreak.

Since the choice of the weights A_i reflects the different scales of the costs for different controls, it is important to notice that the ideal weights are very difficult to obtain in the real world. We only use theoretical weights to propose the simulations in this paper, thus the appropriate data is a difficult problem and it still remains for our further work. We also need to pay attention to that different choices of final time t_f lead to different results, because there is an opposite time orientations for the optimality system when we carry out the simulations. Mathematically speaking, the control is very sensitive to the final time. In the work of [19] in 2011, it was mentioned that the shorter the period of control programme is, the smaller the marginal cost of control will be.

7 Acknowledgments

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Effect of antibodies and latently infected cells on HIV dynamics with differential drug efficacy in cocirculating target cells

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Abstract

In this paper, we investigate the qualitative behaviors of three viral infection models with two types of cocirculating target cells. The models take into account both antibodies and latently infected cells. The incidence rate is represented by bilinear, saturation and general function. For the first two models, we have derived two threshold parameters, R_0 and R_1 which completely determined the global properties of the models. Lyapunov functions are constructed and LaSalle's invariance principle is applied to prove the global asymptotic stability of all equilibria of the models. For the third model, we have established a set of conditions on the general incidence rate function which are sufficient for the global stability of the equilibria of the model. Theoretical results have been checked by numerical simulations.

Keywords: Virus infection; Global stability; Latently infected cells; cocirculating target cells; Lyapunov function.

1 Introduction

Mathematical modeling and model analysis of virus infection in vivo have attracted the interests of mathematicians during the recent years. Such virus infection models can be very useful in the control of epidemic diseases and provide insights into the dynamics of viral load in vivo. Therefore, mathematical analysis of the virus infection models can play a significant role in the development of a better understanding of diseases and various drug therapy strategies.Many authors have formulated mathematical models to describe the population dynamics of several viruses such as, human immunodeficiency virus (HIV) (see e.g. [1]-[10]), hepatitis B virus (HBV) [11]-[13], hepatitis C virus (HCV) [14]-[15], human T cell leukemia HTLV [16] and dengue virus [17], etc. During viral infections, the host immune system reacts with antigen-specific immune response. The immune system has two main responses to viral infections. The first is based on the Cytotoxic T Lymphocyte (CTL) cells which are responsible to attack and kill the infected cells. The second immune response is based on the antibodies that are produced by the B cells. The function of the antibodies is to attack the viruses [1]. In some infections such as in malaria, the CTL immune response is less effective than the antibody immune response [18]. Several mathematical models have been proposed to consider the antibody immune response into the viral infection models ([19]-[24]). The basic model of viral infection with antibody immune response has been

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introduced by Murase et. al. [19] and Wang and Zou [21] as:

$$\dot{x} = \lambda - dx - \bar{\beta}xv, \tag{1}$$
$$\dot{y} = \bar{\beta}xv - ay, \tag{2}$$
$$\dot{v} = ky - cv - rzv, \tag{3}$$
$$\dot{z} = gzv - \mu z, \tag{4}$$

where x, y, v and z represent, respectively, the concentrations of uninfected cells, infected cells, free viruses and the antibody immune cells. Parameters λ, k and g represent respectively, the rate of new uninfected cells that are generated from sources within the body, the rate of free virus production and the proliferation rate constant of the antibody immune cells. Parameters d, a, c and μ are the natural death rate constant of uninfected cells, infected cells, free virus particles and the antibody immune cells respectively. Parameter $\bar{\beta}$ is the infection rate constant at which a target cell becomes infected via contacting with virus and r is the removal rate constant of the virus due to the antibodies. Model (1)-(4) is based on the assumption that the infection could occur and that the viruses are produced from infected cells instantaneously, once the uninfected cells are contacted by the virus particles. Other accurate models incorporate the latently infected cells which are due to the delay between the time of infection and the time when the infected cell becomes active to produce infectious viruses. In [26], model (1)-(4) was extended to take into consideration both latently and actively infected cells as:

$$\dot{x} = \lambda - dx - \bar{\beta}xv, \tag{5}$$
$$\dot{w} = (1-\alpha)\bar{\beta}xv - (e+b)w, \tag{6}$$
$$\dot{y} = \alpha\bar{\beta}xv + bw - ay, \tag{7}$$
$$\dot{v} = ky - cv - rvz, \tag{8}$$
$$\dot{z} = gvz - \mu z, \tag{9}$$

where w and y are the concentrations of latently infected and actively infected cells, respectively. Eq. (6) describes the population dynamics of the latently infected cells and show that they are converted to actively infected cells with rate constant b. The parameters e and a are the death rate constants of the latently and actively infected cells, respectively. The fractions $(1-\alpha)$ where, $0 < \alpha < 1$ are the probabilities that upon infection, an uninfected cell will become either latently infected or actively infected. Model (5)-(9) it have been assumed that, the HIV has one class of target cells, CD4$^+$T cells. However, Perelson et al. in [25] have shown that, HIV infects the macrophages in addition to the CD4$^+$T cells. Recently, many efforts have been devoted to study various mathematical models of HIV dynamics with two classes of target cells (see e.g. [3]).

Our primary goal of the present paper is to propose the global stability analysis of three viral infection models with two types of target cells, CD4$^+$T cells and macrophages taking into consideration the latently, actively infected cells and antibody immune response. The infection rate is represented by bilinear incidence and saturated incidence in the first and the second models, respectively, while it is given by a general function in the third one. The global stability of the three models is established using Lyapunov functionals.

2 HIV model with bilinear incidence rate

In this section, we introduce an HIV dynamics model which describes two cocirculation populations of target cells, CD4$^+$ T cells and macrophages and takes into account the antibody immune response. We consider two types of infected cells, the latently infected and actively infected cells.

$$\dot{x}_i = \lambda_i - d_i x_i - \beta_i x_i v, \qquad i = 1, 2, \tag{10}$$
$$\dot{w}_i = (1-\alpha_i)\beta_i x_i v - (e_i + b_i)w_i, \qquad i = 1, 2, \tag{11}$$
$$\dot{y}_i = \alpha_i \beta_i x_i v + b_i w_i - a_i y_i, \qquad i = 1, 2, \tag{12}$$
$$\dot{v} = \sum_{i=1}^{2} k_i y_i - cv - rvz, \tag{13}$$
$$\dot{z} = gvz - \mu z. \tag{14}$$

Here $i=1,2$ correspond to the CD4$^+$ T cells and macrophages and $\beta_1=(1-\varepsilon)\bar{\beta}_1,\ \beta_2=(1-\varepsilon f)\bar{\beta}_2$. The model incorporates RTI drug therapy where in the CD4$^+$T cells, the drug efficacy is ε and $0\leqslant\varepsilon<1$, while in the macrophages the drug efficacy εf is reduced by a factor f and $0<f<1$. All the parameters and variables of the model have the same meanings as given in (5)-(9).

2.1 Properties of solutions

One can easily show that the non-negative orthant $\mathbb{R}^8\geq 0$ by model (10)-(14).

Proposition 1. There exist positive numbers $L_j,\ j=1,2,3,4$ such that the compact set $\Omega=\left\{(x_i,w_i,y_i,v,z)\in\mathbb{R}^8\geq 0: 0\leq x_i,w_i,y_i\leq L_i,\ 0\leq v\leq L_3,\ 0\leq z\leq L_4,\ i=1,2\right\}$ is positively invariant.

Proof. To show the boundedness of the solutions of system (10)-(14) we let $T_i(t)=x_i(t)+w_i(t)+y_i(t)$, then

$$\dot{T}_i(t)=\lambda_i-d_ix_i(t)-e_iw_i(t)-a_iy_i(t)\leq\lambda_i-\rho_iT_i(t),$$

where $\rho_i=\min\{d_i,a_i,e_i\},i=1,2$. Hence $T_i(t)\leq L_i$, if $T_i(0)\leq L_i$, where $L_i=\dfrac{\lambda_i}{\rho_i}$. Since $x_i(t)$, $w_i(t)$ and $y(t)$ are all non-negative, then $0\leq x_i(t),\ w_i(t),\ y_i(t)\leq L_i$, for all $t\geq 0$, if $0\leq x_i(0)+w_i(0)+y_i(0)\leq L_i$, $i=1,2$. On the other hand, let $G(t)=v(t)+\dfrac{r}{g}z(t)$, then

$$\dot{G}(t)=\sum_{i=1}^{2}k_iy_i-cv-\frac{r\mu}{g}z\leq\sum_{i=1}^{2}k_iL_i-\delta\left(v+\frac{r}{g}z\right)=\sum_{i=1}^{2}k_iL_i-\delta G(t),$$

where $\delta=\min\{c,\mu\}$. Hence $G(t)\leq L_3$, if $G(0)\leq L_3$, where $L_3=\dfrac{1}{\delta}\sum_{i=1}^{2}k_iL_i$. Since $v(t)\geq 0$ and $z(t)\geq 0$, then $0\leq v(t)\leq L_3$ and $0\leq z(t)\leq L_4$ if $0\leq v(0)+\dfrac{r}{g}z(0)\leq L_3$, where $L_4=\dfrac{gL_3}{r}$.

2.2 Equilibria and biological thresholds

Let $\overset{\circ}{\Omega}$ be the interior of Ω.

Lemma 1. For system (10)-(14) we have (i) There exist only one uninfected equilibrium $E_0=(x_1^0,x_2^0,0,0,0,0,0,0)\in\Omega$, when $R_0\leq 1$.

(ii) There exist E_0 and a chronic-infection equilibrium without antibody immune response $E_1=(\tilde{x}_1,\tilde{x}_2,\tilde{w}_1,\tilde{w}_2,\tilde{y}_1,\tilde{y}_2,\tilde{v},0,)\in\Omega$, when $R_1\leq 1<R_0$.

(iii) There exist E_0, E_1 and a chronic-infection equilibrium with antibody immune response $E_2=(\bar{x}_1,\bar{x}_2,\bar{w}_1,\bar{w}_2,\bar{y}_1,\bar{y}_2,\bar{v},\bar{z})\in\overset{\circ}{\Omega}$, when $R_1>1$.

Proof. The equilibria of (10)-(14) satisfy the following equations:

$$\lambda_i-d_ix_i-\beta_ix_iv=0, \tag{15}$$

$$(1-\alpha_i)\beta_ix_iv-(e_i+b_i)w_i=0, \tag{16}$$

$$\alpha_i\beta_ix_iv+b_iw_i-a_iy_i=0, \tag{17}$$

$$\sum_{i=1}^{2}k_iy_i-cv-rvz=0, \tag{18}$$

$$gvz-\mu z=0. \tag{19}$$

Eq. (19) has two possible solutions $z=0$ or $v=\dfrac{\mu}{g}$. If $z=0$, then from Eqs.(15)-(17) we get

$$x_i=\frac{x_i^0}{(1+\eta_iv)},\qquad w_i=\frac{(1-\alpha_i)\beta_ix_i^0}{(e_i+b_i)(1+\eta_iv)}v,\qquad y_i=\frac{(e_i\alpha_i+b_i)\beta_ix_i^0}{a_i(e_i+b_i)(1+\eta_iv)}v, \tag{20}$$

where $x_i^0=\dfrac{\lambda_i}{d_i},\ \eta_i=\dfrac{\beta_i}{d_i},\ i=1,2$. From Eq. (18) we obtain

$$\left(\sum_{i=1}^{2}\frac{(e_i\alpha_i+b_i)k_i\beta_ix_i^0}{a_ic(e_i+b_i)(1+\eta_iv)}-1\right)cv=0. \tag{21}$$

We note that $v = 0$ is a solution for Eq. (21) which leads to the disease-free equilibrium $E_0 = (x_1^0, x_2^0, 0, 0, 0, 0, 0, 0)$. If $v \neq 0$, we have

$$\sum_{i=1}^{2} \frac{\Phi_i}{1+\eta_i v} = 1. \tag{22}$$

where $\Phi_i = \dfrac{(e_i\alpha_i + b_i)k_i\beta_i x_i^0}{a_i c(e_i + b_i)}$. Equation (22) can be written as:

$$Av^2 + Bv - C = 0, \tag{23}$$

where

$$A = \eta_1\eta_2, \quad B = \eta_1\Phi_1 + \eta_2\Phi_2 + (1 - \Phi_1 - \Phi_2)(\eta_1 + \eta_2), \quad C = \Phi_1 + \Phi_2 - 1$$

The solutions of Eq. (23) is given by

$$v^{\pm} = \frac{-B \pm \sqrt{B^2 + 4AC}}{2A}.$$

We have $A > 0$, therefore if $C > 0$, then $v^+ > 0$ and $v^- < 0$. Let $\tilde{v} = v^+$, then from Eq. (20) we get

$$\tilde{x}_i = \frac{x_i^0}{1+\eta_i\tilde{v}}, \quad \tilde{w}_i = \frac{(1-\alpha_i)\beta_i x_i^0}{(e_i + b_i)(1+\eta_i\tilde{v})}\tilde{v}, \quad \tilde{y}_i = \frac{(e_i\alpha_i + b_i)\beta_i x_i^0}{a_i(e_i + b_i)(1+\eta_i\tilde{v})}\tilde{v}, \quad i = 1, 2. \tag{24}$$

Therefore, a chronic-infection equilibrium without antibody immune response $E_1 = (\tilde{x}_1, \tilde{x}_2, \tilde{w}_1, \tilde{w}_2, \tilde{y}_1, \tilde{y}_2, \tilde{v}, 0)$ exists when $C > 0$ or $(\Phi_1 + \Phi_2 > 1)$. Now we are ready to define the basic infection reproduction number R_0 as

$$R_0 = \Phi_1 + \Phi_2 = \sum_{i=1}^{2} R_{0i} = \sum_{i=1}^{2} \frac{k_i\beta_i x_i^0(e_i\alpha_i + b_i)}{a_i c(e_i + b_i)}.$$

If $v = \dfrac{\mu}{g}$, then we obtain the chronic-infection equilibrium with antibody immune response $E_2 = (\bar{x}_1, \bar{x}_2, \bar{w}_1, \bar{w}_2, \bar{y}_1, \bar{y}_2, \bar{v}, \bar{z})$, where

$$\bar{x}_i = \frac{g\lambda_i}{gd_i + \mu\beta_i}, \quad \bar{w}_i = \frac{(1-\alpha_i)\lambda_i\beta_i\mu}{(e_i + b_i)(gd_i + \mu\beta_i)}, \quad \bar{y}_i = \frac{(e_i\alpha_i + b_i)\lambda_i\beta_i\mu}{a_i(e_i + b_i)(gd_i + \mu\beta_i)}, \quad i = 1, 2,$$
$$\bar{v} = \frac{\mu}{g}, \qquad \bar{z} = \frac{c}{r}\left(\sum_{i=1}^{2} \frac{gk_i\beta_i\lambda_i(e_i\alpha_i + b_i)}{a_i c(e_i + b_i)(gd_i + \mu\beta_i)} - 1\right).$$

We note that E_2 exists when $\sum_{i=1}^{2} \dfrac{gk_i\beta_i\lambda_i(e_i\alpha_i + b_i)}{a_i c(e_i + b_i)(gd_i + \mu\beta_i)} > 1$. Let us define the antibody immune response activation number as

$$R_1 = \sum_{i=1}^{2} \frac{gk_i\beta_i\lambda_i(e_i\alpha_i + b_i)}{a_i c(e_i + b_i)(gd_i + \mu\beta_i)} = \sum_{i=1}^{2} \frac{R_{0i}}{1 + \frac{\mu\beta_i}{gd_i}},$$

which determines whether or not a persistent antibody immune response can be established. Then we can write $\bar{z} = \dfrac{c}{r}(R_1 - 1)$. Clearly $R_1 < R_0$.

Now, we show that E_0, $E_1 \in \Omega$ and $E_2 \in \mathring{\Omega}$. Clearly, $E_0 \in \Omega$. Let $R_0 > 1$, then from Eq. (20) we have $\tilde{x}_i < x_i^0$, then

$$0 < \tilde{x}_i < \frac{\lambda_i}{d_i} \leq \frac{\lambda_i}{\rho_i} = L_i.$$

From Eqs. (10)-(12), we get

$$\lambda_i = d_i\tilde{x}_i + e_i\tilde{w}_i + a_i\tilde{y}_i.$$

Thus,

$$0 < \tilde{w}_i < \frac{\lambda_i}{e_i} \leq \frac{\lambda_i}{\rho_i} = L_i, \qquad 0 < \tilde{y}_i < \frac{\lambda_i}{a_i} \leq \frac{\lambda_i}{\rho_i} = L_i.$$

Also, $\tilde{v}=\frac{1}{c}\sum_{i=1}^{2}k_i\tilde{y}_i<\frac{1}{c}\sum_{i=1}^{2}k_iL_i\leq\frac{1}{\delta}\sum_{i=1}^{2}k_iL_i=L_3$. Moreover, $\tilde{z}=0$, and then, $E_1\in\Omega$. Let $R_1>1$, then one can show that $0<\bar{x}_i<L_i$, $0<\bar{w}_i<L_i$ and $0<\bar{y}_i<L_i$. Now we show that $0<\bar{v}<L_3$ and $0<\bar{z}<L_4$. From Eq. (13), we have $c\bar{v}+r\bar{v}\bar{z}=\sum_{i=1}^{2}k_i\bar{y}_i$. Then

$$c\bar{v}<\sum_{i=1}^{2}k_i\bar{y}_i\Rightarrow 0<\bar{v}<\frac{1}{c}\sum_{i=1}^{2}k_iL_i\leqslant\frac{1}{\delta}\sum_{i=1}^{2}k_iL_i=L_3,$$
$$r\bar{v}\bar{z}<\sum_{i=1}^{2}k_i\bar{y}_i\Rightarrow 0<\bar{z}<\frac{g}{r\mu}\sum_{i=1}^{2}k_i\bar{y}_i<\frac{g}{r\delta}\sum_{i=1}^{2}k_iL_i=\frac{gL_3}{r}=L_4.$$

It follows that, $E_2\in\mathring{\Omega}$.

2.3 Global stability

Let us define the function $F(s)=s-1-\ln s$.

Theorem 1. The infection-free equilibrium E_0 of system (10)-(14) is GAS when $R_0\leq 1$.

Proof. Define a Lyapunov function W_0 as follows:

$$W_0=\sum_{i=1}^{2}\gamma_i\left[x_i^0F\left(\frac{x_i}{x_i^0}\right)+\frac{b_i}{e_i\alpha_i+b_i}w_i+\frac{e_i+b_i}{e_i\alpha_i+b_i}y_i\right]+v+\frac{r}{g}z, \tag{25}$$

where $\gamma_i=\dfrac{k_i(e_i\alpha_i+b_i)}{a_i(e_i+b_i)}$, $\quad i=1,2$. The time derivative of W_0 along the trajectories of (10)-(14) satisfies

$$\begin{aligned}\frac{dW_0}{dt}&=\sum_{i=1}^{2}\gamma_i\left[\left(1-\frac{x_i^0}{x_i}\right)(\lambda_i-d_ix_i-\beta_ix_iv)+\frac{b_i}{e_i\alpha_i+b_i}\left((1-\alpha_i)\beta_ix_iv-(e_i+b_i)w_i\right)\right.\\&\left.+\frac{e_i+b_i}{e_i\alpha_i+b_i}(\alpha_i\beta_ix_iv+b_iw_i-a_iy_i)\right]+\sum_{i=1}^{2}k_iy_i-cv-rvz+\frac{r}{g}(gvz-\mu z).\end{aligned}\tag{26}$$

Collecting terms of Eq. (26) we get

$$\begin{aligned}\frac{dW_0}{dt}&=\sum_{i=1}^{2}\gamma_i\left[d_i\left(1-\frac{x_i^0}{x_i}\right)(x_i^0-x_i)+\beta_ix_i^0v\right]-cv-\frac{r\mu}{g}z\\&=-\sum_{i=1}^{2}\gamma_id_i\frac{(x_i-x_i^0)^2}{x_i}+\sum_{i=1}^{2}\frac{k_i(e_i\alpha_i+b_i)}{a_i(e_i+b_i)}\beta_ix_i^0v-cv-\frac{r\mu}{g}z\\&=-\sum_{i=1}^{2}\gamma_id_i\frac{(x_i-x_i^0)^2}{x_i}+\left(\sum_{i=1}^{2}\frac{k_i\beta_ix_i^0(e_i\alpha_i+b_i)}{a_ic(e_i+b_i)}-1\right)cv-\frac{r\mu}{g}z\\&=-\sum_{i=1}^{2}\gamma_id_i\frac{(x_i-x_i^0)^2}{x_i}+(R_0-1)cv-\frac{r\mu}{g}z.\end{aligned}\tag{27}$$

If $R_0\leq 1$ then $\frac{dW_0}{dt}\leq 0$ for all $x_i,v,z>0$. Thus, the solutions of system (10)-(14) converge to Ω, the largest invariant subset of $\left\{\frac{dW_0}{dt}=0\right\}$ [27]. Clearly, it follows from Eq. (26) that $\frac{dW_0}{dt}=0$ if and only if $x_i=x_i^0$, $v=0$ and $z=0$. The set Ω is invariant and for any element belongs to Ω satisfies $v=0$ and $z=0$, then $\dot{v}=0$. We can see from Eq. (13) that $0=\dot{v}=\sum_{i=1}^{2}k_iy_i$, and thus $y_i=0$. Moreover, from Eq. (12) we get $w_i=0$. Hence $\frac{dW_0}{dt}=0$ occurs at E_0. From LaSalle's invariance principle, E_0 is GAS.

Theorem 2. The chronic-infection equilibrium without antibody immune response E_1 of system (10)-(14) is GAS when $R_1\leq 1<R_0$.

Proof. We construct the following Lyapunov function

$$W_1 = \sum_{i=1}^{2} \gamma_i \left[\tilde{x}_i F(\frac{x_i}{\tilde{x}_i}) + \frac{b_i}{e_i\alpha_i + b_i} \tilde{w}_i F\left(\frac{w_i}{\tilde{w}_i}\right) + \frac{e_i + b_i}{e_i\alpha_i + b_i} \tilde{y}_i F\left(\frac{y_i}{\tilde{y}_i}\right) \right] + \tilde{v} F\left(\frac{v}{\tilde{v}}\right) + \frac{r}{g} z.$$

Calculating $\frac{dW_1}{dt}$ along the trajectories of (10)-(14) we get

$$\begin{aligned}\frac{dW_1}{dt} &= \sum_{i=1}^{2} \gamma_i \left[\left(1 - \frac{\tilde{x}_i}{x_i}\right)(\lambda_i - d_i x_i - \beta_i x_i v) + \frac{b_i}{e_i\alpha_i + b_i}\left(1 - \frac{\tilde{w}_i}{w_i}\right)((1-\alpha_i)\beta_i x_i v - (e_i + b_i) w_i) \right.\\ &\left. + \frac{e_i + b_i}{e_i\alpha_i + b_i}\left(1 - \frac{\tilde{y}_i}{y_i}\right)(\alpha_i\beta_i x_i v + b_i w_i - a_i y_i) \right] + \left(1 - \frac{\tilde{v}}{v}\right)\left(\sum_{i=1}^{2} k_i y_i - cv - rvz\right) + \frac{r}{g}(gvz - \mu z). \end{aligned} \quad (28)$$

Collecting terms of Eq. (28) we get

$$\begin{aligned}\frac{dW_1}{dt} &= \sum_{i=1}^{2} \gamma_i \left[\left(1 - \frac{\tilde{x}_i}{x_i}\right)(\lambda_i - d_i x_i) + \beta_i \tilde{x}_i v - \frac{b_i(1-\alpha_i)}{e_i\alpha_i + b_i}\frac{\beta_i x_i v \tilde{w}_i}{w_i} + \frac{e_i + b_i}{e_i\alpha_i + b_i} b_i \tilde{w}_i - \frac{\alpha_i(e_i + b_i)}{e_i\alpha_i + b_i}\frac{\beta_i x_i v \tilde{y}_i}{y_i} \right.\\ &\left. - \frac{b_i(e_i + b_i)}{e_i\alpha_i + b_i}\frac{w_i \tilde{y}_i}{y_i} + \frac{e_i + b_i}{e_i\alpha_i + b_i} a_i \tilde{y}_i \right] - cv - \frac{\tilde{v}}{v}\sum_{i=1}^{2} k_i y_i + c\tilde{v} + r\tilde{v}z - \frac{r\mu}{g} z. \end{aligned} \quad (29)$$

Using the value of $\tilde{x}_i$ given in Eq. (24) we get $\left(\sum_{i=1}^{2} \gamma_i \beta_i \tilde{x}_i - c\right) v = 0$. Applying $\lambda_i = d_i \tilde{x}_i + \beta_i \tilde{x}_i \tilde{v}$, we obtain

$$\begin{aligned}\frac{dW_1}{dt} &= \sum_{i=1}^{2} \gamma_i \left[\left(1 - \frac{\tilde{x}_i}{x_i}\right)(d_i \tilde{x}_i - d_i x_i) + \beta_i \tilde{x}_i \tilde{v}\left(1 - \frac{\tilde{x}_i}{x_i}\right) - \frac{b_i(1-\alpha_i)}{e_i\alpha_i + b_i}\frac{\beta_i x_i v \tilde{w}_i}{w_i} + \frac{e_i + b_i}{e_i\alpha_i + b_i} b_i \tilde{w}_i \right.\\ &\left. - \frac{\alpha_i(e_i + b_i)}{e_i\alpha_i + b_i}\frac{\beta_i x_i v \tilde{y}_i}{y_i} - \frac{b_i(e_i + b_i)}{e_i\alpha_i + b_i}\frac{w_i \tilde{y}_i}{y_i} + \frac{e_i + b_i}{e_i\alpha_i + b_i} a_i \tilde{y}_i \right] - \frac{\tilde{v}}{v}\sum_{i=1}^{2} k_i y_i + c\tilde{v} + r\tilde{v}z - \frac{r\mu}{g} z. \end{aligned} \quad (30)$$

Using the equilibrium condition for E_1

$$(1-\alpha_i)\beta_i \tilde{x}_i \tilde{v} = (e_i + b_i)\tilde{w}_i, \quad \alpha_i \beta_i \tilde{x}_i \tilde{v} + b_i \tilde{w}_i = a_i \tilde{y}_i, \quad c\tilde{v} = \sum_{i=1}^{2} k_i \tilde{y}_i = \sum_{i=1}^{2} \gamma_i \beta_i \tilde{x}_i \tilde{v},$$

$$\frac{e_i + b_i}{e_i\alpha_i + b_i} a_i \tilde{y}_i = \beta_i \tilde{x}_i \tilde{v} = \frac{b_i(1-\alpha_i)}{e_i\alpha_i + b_i}\beta_i \tilde{x}_i \tilde{v} + \frac{(e_i + b_i)\alpha_i}{e_i\alpha_i + b_i}\beta_i \tilde{x}_i \tilde{v}.$$

we have

$$\begin{aligned}\frac{dW_1}{dt} &= \sum_{i=1}^{2} \gamma_i \left[-d_i \frac{(x_i - \tilde{x}_i)^2}{x_i} + \beta_i \tilde{x}_i \tilde{v}\left(1 - \frac{\tilde{x}_i}{x_i}\right)\left(\frac{b_i(1-\alpha_i)}{e_i\alpha_i + b_i} + \frac{(e_i + b_i)\alpha_i}{e_i\alpha_i + b_i}\right) - \frac{b_i(1-\alpha_i)}{e_i\alpha_i + b_i}\beta_i \tilde{x}_i \tilde{v}\frac{x_i \tilde{w}_i v}{\tilde{x}_i w_i \tilde{v}} \right.\\ &+ \frac{b_i(1-\alpha_i)}{e_i\alpha_i + b_i}\beta_i \tilde{x}_i \tilde{v} - \frac{(e_i + b_i)\alpha_i}{e_i\alpha_i + b_i}\beta_i \tilde{x}_i \tilde{v}\frac{x_i \tilde{y}_i v}{\tilde{x}_i y_i \tilde{v}} - \frac{b_i(1-\alpha_i)}{e_i\alpha_i + b_i}\beta_i \tilde{x}_i \tilde{v}\frac{w_i \tilde{y}_i}{\tilde{w}_i y_i} + \frac{b_i(1-\alpha_i)}{e_i\alpha_i + b_i}\beta_i \tilde{x}_i \tilde{v} + \frac{(e_i + b_i)\alpha_i}{e_i\alpha_i + b_i}\beta_i \tilde{x}_i \tilde{v}\\ &\left. - \left(\frac{b_i(1-\alpha_i)}{e_i\alpha_i + b_i} + \frac{(e_i + b_i)\alpha_i}{e_i\alpha_i + b_i}\right)\beta_i \tilde{x}_i \tilde{v}\frac{y_i \tilde{v}}{\tilde{y}_i v} + \left(\frac{b_i(1-\alpha_i)}{e_i\alpha_i + b_i} + \frac{(e_i + b_i)\alpha_i}{e_i\alpha_i + b_i}\right)\beta_i \tilde{x}_i \tilde{v} \right] + (\tilde{v} - \bar{v}) rz.\\ &= \sum_{i=1}^{2} \gamma_i \left[-d_i \frac{(x_i - \tilde{x}_i)^2}{x_i} + \frac{b_i(1-\alpha_i)}{e_i\alpha_i + b_i}\beta_i \tilde{x}_i \tilde{v}\left(4 - \frac{\tilde{x}_i}{x_i} - \frac{x_i \tilde{w}_i v}{\tilde{x}_i w_i \tilde{v}} - \frac{y_i \tilde{v}}{\tilde{y}_i v} - \frac{w_i \tilde{y}_i}{\tilde{w}_i y_i}\right) \right.\\ &\left. + \frac{(e_i + b_i)\alpha_i}{e_i\alpha_i + b_i}\beta_i \tilde{x}_i \tilde{v}\left(3 - \frac{\tilde{x}_i}{x_i} - \frac{y_i \tilde{v}}{\tilde{y}_i v} - \frac{x_i \tilde{y}_i v}{\tilde{x}_i y_i \tilde{v}}\right) \right] + (\tilde{v} - \bar{v}) rz. \end{aligned} \quad (31)$$

We have $x_i, w_i, y_i, v > 0$ when $R_0 > 1$. Since the geometrical mean is less than or equal to the arithmetical mean, the second and the third terms are less than or equal to zero. Now we show that if $R_1 \leq 1$ then $\tilde{v} \leq \frac{\mu}{g} = \bar{v}$.

Using the steady state conditions for E_1 we have $\sum_{i=1}^{2} \frac{k_i\beta_i\lambda_i(e_i\alpha_i+b_i)}{a_icd_i(e_i+b_i)(1+\eta_i\tilde{v})}=1$, then

$$R_1-1=\sum_{i=1}^{2}\frac{gk_i\beta_i\lambda_i(e_i\alpha_i+b_i)}{a_ic(e_i+b_i)(gd_i+\mu\beta_i)}-\sum_{i=1}^{2}\frac{k_i\beta_i\lambda_i(e_i\alpha_i+b_i)}{a_id_ic(e_i+b_i)(1+\eta_i\tilde{v})}$$
$$=\sum_{i=1}^{2}\frac{k_i\beta_i\lambda_i(e_i\alpha_i+b_i)}{a_id_ic(e_i+b_i)(1+\eta_i\bar{v})}-\sum_{i=1}^{2}\frac{k_i\beta_i\lambda_i(e_i\alpha_i+b_i)}{a_id_ic(e_i+b_i)(1+\eta_i\tilde{v})}=(\tilde{v}-\bar{v})\chi, \tag{32}$$

where $\chi=\sum_{i=1}^{2}\frac{k_i\beta_i\lambda_i\eta_i(e_i\alpha_i+b_i)}{a_id_ic(e_i+b_i)(1+\eta_i\bar{v})(1+\eta_i\tilde{v})}$. It follows that, if $R_1\le 1$ then $\frac{dW_1}{dt}\le 0$ for all $x_i,w_i,y_i,v,z>0$. Thus, the solutions of system (10)-(14) limit to Ω, the largest invariant subset of $\left\{\frac{dW_1}{dt}=0\right\}$ [27]. It can be seen that, $\frac{dW_1}{dt}=0$ occurs at E_1. Applying LaSalle's invariance principle we obtain that E_1 is GAS.

Theorem 3. The chronic-infection equilibrium with antibody immune response E_2 of system (10)-(14) is GAS when $R_1>1$.

Proof. Consider the following Lyapunov function

$$W_2=\sum_{i=1}^{2}\gamma_i\left[\bar{x}_iF(\frac{x_i}{\bar{x}_i})+\frac{b_i}{e_i\alpha_i+b_i}\bar{w}_iF\left(\frac{w_i}{\bar{w}_i}\right)+\frac{e_i+b_i}{e_i\alpha_i+b_i}\bar{y}_iF\left(\frac{y_i}{\bar{y}_i}\right)\right]+\bar{v}F\left(\frac{v}{\bar{v}}\right)+\frac{r}{g}\bar{z}F\left(\frac{z}{\bar{z}}\right).$$

Calculating the derivative of W_2 along the trajectories of (10)-(14) we get

$$\frac{dW_2}{dt}=\sum_{i=1}^{2}\gamma_i\left[\left(1-\frac{\bar{x}_i}{x_i}\right)(\lambda_i-d_ix_i-\beta_ix_iv)+\frac{b_i}{e_i\alpha_i+b_i}\left(1-\frac{\bar{w}_{\bar{\imath}}}{w_i}\right)((1-\alpha_i)\beta_ix_iv-(e_i+b_i)w_i)\right.$$
$$\left.+\frac{e_i+b_i}{e_i\alpha_i+b_i}\left(1-\frac{\bar{y}_i}{y_i}\right)(\alpha_i\beta_ix_iv+bw_i-a_iy_i)\right]+\left(1-\frac{\bar{v}}{v}\right)(\sum_{i=1}^{2}k_iy_i-cv-rvz)+\frac{r}{g}\left(1-\frac{\bar{z}}{z}\right)(gvz-\mu z). \tag{33}$$

Collecting terms of Eq. (33) we get

$$\frac{dW_2}{dt}=\sum_{i=1}^{2}\gamma_i\left[\left(1-\frac{\bar{x}_i}{x_i}\right)(\lambda_i-d_ix_i)+\beta_i\bar{x}_iv-\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}\frac{\beta_ix_iv\bar{w}_{\bar{\imath}}}{w_i}+\frac{e_i+b_i}{e_i\alpha_i+b_i}b_i\bar{w}_{\bar{\imath}}\right.$$
$$\left.-\frac{\alpha_i\left(e_i+b_i\right)}{e_i\alpha_i+b_i}\frac{\beta_ix_iv\bar{y}_i}{y_i}-\frac{b_i\left(e_i+b_i\right)}{e_i\alpha_i+b_i}\frac{w_i\bar{y}_i}{y_i}+\frac{e_i+b_i}{e_i\alpha_i+b_i}a_i\bar{y}_i\right]-cv-\frac{\bar{v}}{v}\sum_{i=1}^{2}k_iy_i+c\bar{v}-rv\bar{z}+\frac{r\mu}{g}\bar{z}. \tag{34}$$

Applying $\lambda_i=\ d_i\bar{x}_i+\beta\bar{x}_i\bar{v}$, we get

$$\frac{dW_2}{dt}=\sum_{i=1}^{2}\gamma_i\left[\left(1-\frac{\bar{x}_i}{x_i}\right)(d_i\bar{x}_i-d_ix_i)+\beta_i\bar{x}_i\bar{v}\left(1-\frac{\bar{x}_i}{x_i}\right)+\beta_i\bar{x}_iv-\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}\frac{\beta_ix_iv\bar{w}_{\bar{\imath}}}{w_i}+\frac{e_i+b_i}{e_i\alpha_i+b_i}b_i\bar{w}_{\bar{\imath}}\right.$$
$$\left.-\frac{\alpha_i\left(e_i+b_i\right)}{e_i\alpha_i+b_i}\frac{\beta_ix_iv\bar{y}_i}{y_i}-\frac{b_i\left(e_i+b_i\right)}{e_i\alpha_i+b_i}\frac{w_i\bar{y}_i}{y_i}+\frac{e_i+b_i}{e_i\alpha_i+b_i}a_i\bar{y}_i\right]-cv-\frac{\bar{v}}{v}\sum_{i=1}^{2}k_iy_i+c\bar{v}-rv\bar{z}+\frac{r\mu}{g}\bar{z}. \tag{35}$$

Using the equilibrium conditions for E_2

$$(1-\alpha_i)\beta_i\bar{x}_i\bar{v}=(e_i+b_i)\bar{w}_i,\quad \alpha_i\beta_i\bar{x}_i\bar{v}+b_i\bar{w}_i=a_i\bar{y}_i,\quad c\bar{v}+r\bar{v}\bar{z}=\sum_{i=1}^{2}k_i\bar{y}_i=\sum_{i=1}^{2}\gamma_i\beta_i\bar{x}_i\bar{v},$$
$$\frac{e_i+b_i}{e_i\alpha_i+b_i}a_i\bar{y}_i=\beta_i\bar{x}_i\bar{v}=\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}\beta_i\bar{x}_i\bar{v}+\frac{(e_i+b_i)\alpha_i}{(e_i\alpha_i+b_i)}\beta_i\bar{x}_i\bar{v},\quad \sum_{i=1}^{2}\gamma_i\beta_i\bar{x}_i\bar{v}-c\bar{v}-r\bar{v}\bar{z}=0,$$

we have

$$\begin{aligned}\frac{dW_2}{dt} &= \sum_{i=1}^{2}\gamma_i\Bigg[-d_i\frac{(x_i-\bar{x}_i)^2}{x_i}+\beta_i\bar{x}_i\bar{v}\left(1-\frac{\bar{x}_i}{x_i}\right)\left(\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}+\frac{(e_i+b_i)\alpha_i}{e_i\alpha_i+b_i}\right)-\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}\beta_i\bar{x}_i\bar{v}\frac{x_i\bar{w}_iv}{\bar{x}_iw_i\bar{v}}\\
&+\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}\beta_i\bar{x}_i\bar{v}-\frac{(e_i+b_i)\alpha_i}{e_i\alpha_i+b_i}\beta_i\bar{x}_i\bar{v}\frac{x_i\bar{y}_iv}{\bar{x}_iy_i\bar{v}}-\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}\beta_i\bar{x}_i\bar{v}\frac{w_i\bar{y}_i}{\bar{w}_iy_i}+\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}\beta_i\bar{x}_i\bar{v}+\frac{(e_i+b_i)\alpha_i}{e_i\alpha_i+b_i}\beta_i\bar{x}_i\bar{v}\\
&-\left(\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}+\frac{(e_i+b_i)\alpha_i}{e_i\alpha_i+b_i}\right)\beta_i\bar{x}_i\bar{v}\frac{y_i\bar{v}}{\bar{y}_iv}+\left(\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}+\frac{(e_i+b_i)\alpha_i}{e_i\alpha_i+b_i}\right)\beta_i\bar{x}_i\bar{v}\Bigg]\\
&= \sum_{i=1}^{2}\gamma_i\Bigg[-d_i\frac{(x_i-\bar{x}_i)^2}{x_i}+\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}\beta_i\bar{x}_i\bar{v}\left[4-\frac{\bar{x}_i}{x_i}-\frac{x_i\bar{w}_iv}{\bar{x}_iw_i\bar{v}}-\frac{y_i\bar{v}}{\bar{y}_iv}-\frac{w_i\bar{y}_i}{\bar{w}_iy_i}\right]\\
&+\frac{(e_i+b_i)\alpha_i}{(e_i\alpha_i+b_i)}\beta_i\bar{x}_i\bar{v}\left[3-\frac{\bar{x}_i}{x_i}-\frac{y_i\bar{v}}{\bar{y}_iv}-\frac{x_i\bar{y}_iv}{\bar{x}_iy_i\bar{v}}\right]\Bigg].\end{aligned}$$

Thus, if $R_1>1$, then $\bar{x}_i,\bar{w}_i,\bar{y}_i,\bar{v},\bar{z}>0$. Using the relation between arithmetical and geometrical means, we get $\frac{dW_2}{dt}\le 0$. Clearly, $\frac{dW_2}{dt}=0$ if and only if $x_i=\bar{x}_i$, $w_i=\bar{w}_i$, $y_i=\bar{y}_i$ and $v=\bar{v}$. If $v=\bar{v}$, then $\dot{v}=0$ and from Eq. (13) we have $0=\sum_{i=1}^{2}k_i\bar{y}_i-c\bar{v}-r\bar{v}\bar{z}$, which give $z=\bar{z}$. Therefore, $\frac{dW_2}{dt}$ equal to zero at E_2. The global stability of E_2 follows from LaSalle's invariance principle.

3 Model with saturation functional response

In this section, we modify model (10)-(14) by taking into account the saturation functional response as:

$$\dot{x}_i=\lambda_i-d_ix_i-\frac{\beta_ix_iv}{1+\sigma_iv},\qquad i=1,2,\tag{36}$$

$$\dot{w}_i=\frac{(1-\alpha_i)\beta_ix_iv}{1+\sigma_iv}-(e_i+b_i)w_i,\qquad i=1,2,\tag{37}$$

$$\dot{y}_i=\frac{\alpha_i\beta_ix_iv}{1+\sigma_iv}+b_iw_i-a_iy_i,\qquad i=1,2,\tag{38}$$

$$\dot{v}=\sum_{i=1}^{2}k_iy_i-cv-rvz,\tag{39}$$

$$\dot{z}=gvz-\mu z,\tag{40}$$

where $\sigma_i>0, i=1,2$, is the saturation constant, and all the variables and parameters of the model have the same definition as given in (10)-(14). We mention that the compact set Ω given in Section 2 is also positively invariant with respect to system (36)-(40).

3.1 Equilibria

Lemma 2. For system (36)-(40) we have (i) There exist only one uninfected equilibrium $E_0=(x_1^0,x_2^0,0,0,0,0,0,0)\in\Omega$, when $R_0\le 1$.

(ii) There exist E_0 and a chronic-infection equilibrium without antibody immune response $E_1=(\tilde{x}_1,\tilde{x}_2,\tilde{w}_1,\tilde{w}_2,\tilde{y}_1,\tilde{y}_2,\tilde{v},0,)\in\Omega$, when $R_1\le 1<R_0$.

(iii) There exist E_0, E_1 and a chronic-infection equilibrium with antibody immune response $E_2=(\bar{x}_1,\bar{x}_2,\bar{w}_1,\bar{w}_2,\bar{y}_1,\bar{y}_2,\bar{v},\bar{z})\in\mathring{\Omega}$, when $R_1>1$.

Proof. We let the right-hand side of Eqs.(36)-(40) equal zero, then we obtain the following: Eq. (40) has two possible solutions $z=0$ or $v=\frac{\mu}{g}$.

If $z=0$, then from Eqs.(36)-(38) we have

$$x_i=\frac{x_i^0(1+\sigma_iv)}{(1+\xi_iv)},\qquad w_i=\frac{(1-\alpha_i)\beta_ix_i^0}{(e_i+b_i)(1+\xi_iv)}v,\qquad y_i=\frac{(e_i\alpha_i+b_i)\beta_ix_i^0}{a_i(e_i+b_i)(1+\xi_iv)}v,\tag{41}$$

where $x_i^0 = \dfrac{\lambda_i}{d_i}$, $\xi_i = \sigma_i + \dfrac{\beta_i}{d_i}$, $i = 1, 2$. From Eq. (39) we find

$$\left(\sum_{i=1}^{2} \frac{(e_i\alpha_i + b_i)k_i\beta_i x_i^0}{a_i c(e_i + b_i)(1 + \xi_i v)} - 1 \right) cv = 0. \tag{42}$$

Eq. (42) has also two possible solutions $v = 0$ or $\sum_{i=1}^{2} \dfrac{(e_i\alpha_i + b_i)k_i\beta_i x_i^0}{a_i c(e_i + b_i)(1 + \xi_i v)} - 1 = 0$.

If $v = 0$, then substituting it in Eq. (41) we get the disease-free equilibrium $E_0 = (x_1^0, x_2^0, 0, 0, 0, 0, 0, 0)$.

If $v \neq 0$, we have

$$\sum_{i=1}^{2} \frac{\Psi_i}{(1 + \xi_i v)} = 1. \tag{43}$$

where $\Psi_i = \dfrac{(e_i\alpha_i + b_i)k_i\beta_i x_i^0}{a_i c(e_i + b_i)}$. Eq. (43) can be written as:

$$A_1 v^2 + B_1 v - C_1 = 0 \tag{44}$$

where

$$A_1 = \xi_1\xi_2, \quad B_1 = \xi_1\Psi_1 + \xi_2\Psi_2 + (1 - \Psi_1 - \Psi_2)(\xi_1 + \xi_2), \quad C_1 = \Psi_1 + \Psi_2 - 1$$

The solutions of Eq. (23) is given by:

$$v^{\pm} = \frac{-B_1 \pm \sqrt{B_1^2 + 4A_1C_1}}{2A}.$$

We have $A_1 > 0$, therefore $v^+ > 0$ and $v^- < 0$ when $C_1 > 0$. Let $\tilde{v} = v^+$, then from Eq. (41) we get

$$\tilde{x}_i = \frac{x_i^0(1 + \sigma_i\tilde{v})}{(1 + \xi_i\tilde{v})} > 0, \qquad \tilde{w}_i = \frac{(1 - \alpha_i)\beta_i x_i^0}{(e_i + b_i)(1 + \xi_i\tilde{v})}\tilde{v} > 0, \qquad \tilde{y}_i = \frac{(e_i\alpha_i + b_i)\beta_i x_i^0}{a_i(e_i + b_i)(1 + \xi_i\tilde{v})}\tilde{v} > 0, \quad i = 1, 2.$$

Therefore, an endemic equilibrium $E_1 = (\tilde{x}_1, \tilde{x}_2, \tilde{w}_1, \tilde{w}_2, \tilde{y}_1, \tilde{y}_2, \tilde{v}, 0,)$ exists when $C_1 > 0$ or $(\Psi_1 + \Psi_2 > 1)$.

Now we are ready to define the basic reproduction number R_0 as

$$R_0 = \sum_{i=1}^{2} R_{0i} = \sum_{i=1}^{2} \Psi_i = \sum_{i=1}^{2} \frac{(e_i\alpha_i + b_i)k_i\beta_i x_i^0}{a_i c(e_i + b_i)}.$$

If $v = \dfrac{\mu}{g}$, then we obtain the chronic-infection equilibrium with antibody immune response $E_2 = (\bar{x}_1, \bar{x}_2, \bar{w}_1, \bar{w}_2, \bar{y}_1, \bar{y}_2, \bar{v}, \bar{z})$, where

$$\bar{x}_i = \frac{(g + \mu\sigma_i)x_i^0}{g + \mu\xi_i}, \quad \bar{w}_i = \frac{(1 - \alpha_i)\beta_i\mu x_i^0}{(e_i + b_i)(g + \mu\xi_i)}, \quad \bar{y}_i = \frac{(e_i\alpha_i + b_i)\beta_i\mu x_i^0}{a_i(e_i + b_i)(g + \mu\xi_i)}, \quad i = 1, 2,$$
$$\bar{v} = \frac{\mu}{g}, \qquad \bar{z} = \frac{c}{r}\left(\sum_{i=1}^{2} \frac{(e_i\alpha_i + b_i)k_i\beta_i g x_i^0}{a_i c(e_i + b_i)(g + \mu\xi_i)} - 1 \right).$$

We note that E_2 exists when $\sum_{i=1}^{2} \dfrac{(e_i\alpha_i + b_i)k_i\beta_i g x_i^0}{a_i c(e_i + b_i)(g + \mu\xi_i)} > 1$. This equilibrium represents the state that both the viruses and antibodies are present. Let us define the antibody immune response activation number as

$$R_1 = \sum_{i=1}^{2} \frac{(e_i\alpha_i + b_i)k_i\beta_i g x_i^0}{a_i c(e_i + b_i)(g + \mu\xi_i)} = \sum_{i=1}^{2} \frac{R_{0i}}{\left(1 + \dfrac{\mu}{g}\xi_i\right)},$$

which determines whether a persistent antibody immune response can be established. Then we can write $\bar{z} = \dfrac{c}{r}(R_1 - 1)$. Clearly $R_1 < R_0$. Similar to Section 2.2, one can show that, E_0, $E_1 \in \Omega$ and $E_2 \in \mathring{\Omega}$

3.2 Global stability

Theorem 4. The disease-free equilibrium E_0 of system (36)-(40) is GAS when $R_0 \leq 1$.

Proof. We define a Lyapunov function W_0 as:

$$W_0 = \sum_{i=1}^{2} \gamma_i \left[x_i^0 F\left(\frac{x_i}{x_i^0}\right) + \frac{b_i}{e_i\alpha_i + b_i} w_i + \frac{e_i + b_i}{e_i\alpha_i + b_i} y_i \right] + v + \frac{r}{g} z. \tag{45}$$

We calculate $\frac{dW_0}{dt}$ along the trajectories of (36)-(40)

$$\frac{dW_0}{dt} = \sum_{i=1}^{2} \gamma_i \Bigg[\left(1 - \frac{x_i^0}{x_i}\right)\left(\lambda_i - d_i x_i - \frac{\beta_i x_i v}{1+\sigma_i v}\right) + \frac{b_i}{e_i\alpha_i + b_i}\left(\frac{(1-\alpha_i)\beta_i x_i v}{1+\sigma_i v} - (e_i + b_i) w_i\right)$$
$$+ \frac{e_i + b_i}{e_i\alpha_i + b_i}\left(\frac{\alpha_i\beta_i x_i v}{1+\sigma_i v} + b_i w_i - a_i y_i\right)\Bigg] + \sum_{i=1}^{2} k_i y_i - cv - rvz + \frac{r}{g}(gvz - \mu z). \tag{46}$$

Collecting terms of Eq. (46) we get

$$\begin{aligned}
\frac{dW_0}{dt} &= \sum_{i=1}^{2} \gamma_i \left[d_i \left(1 - \frac{x_i^0}{x_i}\right)(x_i^0 - x_i) + \frac{\beta_i x_i^0 v}{1+\sigma_i v}\right] - cv - \frac{r\mu}{g} z \\
&= -\sum_{i=1}^{2} \gamma_i d_i \frac{(x_i - x_i^0)^2}{x_i} + \sum_{i=1}^{2} \frac{(e_i\alpha_i + b_i) k_i \beta_i x_i^0}{a_i(e_i + b_i)(1+\sigma_i v)} v - cv - \frac{r\mu}{g} z \\
&= -\sum_{i=1}^{2} \gamma_i d_i \frac{(x_i - x_i^0)^2}{x_i} + \left(\sum_{i=1}^{2} \frac{R_{0i}}{(1+\sigma_i v)} - 1\right) cv - \frac{r\mu}{g} z \\
&= -\sum_{i=1}^{2} \gamma_i d_i \frac{(x_i - x_i^0)^2}{x_i} + (R_0 - 1) cv - \sum_{i=1}^{2} \frac{c\sigma_i R_{0i} v^2}{(1+\sigma_i v)} - \frac{r\mu}{g} z.
\end{aligned} \tag{47}$$

If $R_0 \leq 1$ then $\frac{dW_0}{dt} \leq 0$ for all $x_i, v, z > 0$. Similar to the proof of Theorem 1, one can easily show that $\frac{dW_0}{dt} = 0$ at E_0. Then using LaSalle's invariance principle, we can show the global stability of E_0.

Next, we show that the endemic equilibrium E_1 is GAS.

Theorem 5. The chronic-infection equilibrium without antibody immune response E_1 of system (36)-(40) is GAS when $R_1 \leq 1 < R_0$.

Proof. We consider the following Lyapunov function

$$W_1 = \sum_{i=1}^{2} \gamma_i \left[\tilde{x}_i F\left(\frac{x_i}{\tilde{x}_i}\right) + \frac{b_i}{e_i\alpha_i + b_i} \tilde{w}_i F\left(\frac{w_i}{\tilde{w}_i}\right) + \frac{e_i + b_i}{e_i\alpha_i + b_i} \tilde{y}_i F\left(\frac{y_i}{\tilde{y}_i}\right)\right] + \tilde{v} F\left(\frac{v}{\tilde{v}}\right) + \frac{r}{g} z.$$

Calculating $\frac{dW_1}{dt}$ along the solutions of (36)-(40) we get

$$\frac{dW_1}{dt} = \sum_{i=1}^{2} \gamma_i \Bigg[\left(1 - \frac{\tilde{x}_i}{x_i}\right)\left(\lambda_i - d_i x_i - \frac{\beta_i x_i v}{1+\sigma_i v}\right) + \frac{b_i}{e_i\alpha_i + b_i}\left(1 - \frac{\tilde{w}_i}{w_i}\right)\left(\frac{(1-\alpha_i)\beta_i x_i v}{1+\sigma_i v} - (e_i + b_i) w_i\right)$$
$$+ \frac{e_i + b_i}{e_i\alpha_i + b_i}\left(1 - \frac{\tilde{y}_i}{y_i}\right)\left(\frac{\alpha_i\beta_i x_i v}{1+\sigma_i v} + b_i w_i - a_i y_i\right)\Bigg] + \left(1 - \frac{\tilde{v}}{v}\right)\left(\sum_{i=1}^{2} k_i y_i - cv - rvz\right) + \frac{r}{g}(gvz - \mu z). \tag{48}$$

Collecting terms of Eq. (48) we have:

$$\frac{dW_1}{dt} = \sum_{i=1}^{2} \gamma_i \Bigg[\left(1 - \frac{\tilde{x}_i}{x_i}\right)(\lambda_i - d_i x_i) + \frac{\beta_i \tilde{x}_i v}{1+\sigma_i v} + \frac{b_i}{e_i\alpha_i + b_i}\left(-\frac{(1-\alpha_i)\beta_i x_i v \tilde{w}_i}{(1+\sigma_i v) w_i} + (e_i + b_i)\tilde{w}_i\right)$$
$$+ \frac{e_i + b_i}{e_i\alpha_i + b_i}\left(-\frac{\alpha_i\beta_i x_i v \tilde{y}_i}{(1+\sigma_i v) y_i} + \frac{b_i w_i \tilde{y}_i}{y_i} + a_i \tilde{y}_i\right)\Bigg] - cv - \frac{\tilde{v}}{v}\sum_{i=1}^{2} k_i y_i + c\tilde{v} + r\tilde{v}z - \frac{\mu r}{g} z.$$

Using the equilibrium condition for E_1:

$$\lambda_i = d_i\tilde{x}_i + \frac{\beta\tilde{x}_i\tilde{v}}{1+\sigma_i\tilde{v}}, \quad \frac{(1-\alpha_i)\beta_i\tilde{x}_i\tilde{v}}{1+\sigma_i\tilde{v}} = (e_i+b_i)\tilde{w}_i, \quad a_i\tilde{y}_i = \frac{\alpha_i\beta_i\tilde{x}_i\tilde{v}}{1+\sigma_i\tilde{v}} + b_i\tilde{w}_i = \frac{e_i\alpha_i+b_i}{e_i+b_i}\frac{\beta_i\tilde{x}_i\tilde{v}}{1+\sigma_i\tilde{v}},$$

$$c\tilde{v} = \sum_{i=1}^{2}k_i\tilde{y}_i = \sum_{i=1}^{2}\gamma_i\frac{\beta_i\tilde{x}_i\tilde{v}}{1+\sigma_i\tilde{v}}, \quad \frac{\tilde{v}}{v}\sum_{i=1}^{2}k_iy_i = \sum_{i=1}^{2}\gamma_i\frac{\beta_i\tilde{x}_i\tilde{v}}{1+\sigma_i\tilde{v}}\frac{y_i\tilde{v}}{\tilde{y}_iv}, \quad cv = \frac{v}{\tilde{v}}\sum_{i=1}^{2}\gamma_i\frac{\beta_i\tilde{x}_i\tilde{v}}{1+\sigma_i\tilde{v}},$$

$$\frac{\beta_i\tilde{x}_i\tilde{v}}{1+\sigma_i\tilde{v}} = \frac{b_i(1-\alpha_i)}{(e_i\alpha_i+b_i)}\frac{\beta_i\tilde{x}_i\tilde{v}}{(1+\sigma_i\tilde{v})} + \frac{(e_i+b_i)\alpha_i}{(e_i\alpha_i+b_i)}\frac{\beta_i\tilde{x}_i\tilde{v}}{(1+\sigma_i\tilde{v})},$$

we obtain

$$\begin{aligned}
\frac{dW_1}{dt} &= \sum_{i=1}^{2}\gamma_i\Bigg[\left(1-\frac{\tilde{x}_i}{x_i}\right)\left(d_i\tilde{x}_i + \frac{\beta_i\tilde{x}_i\tilde{v}}{1+\sigma_i\tilde{v}} - d_ix_i\right) + \frac{\beta_i\tilde{x}_iv}{1+\sigma_iv} + \frac{b_i}{e_i\alpha_i+b_i}\left(-\frac{(1-\alpha_i)\beta_ix_iv\tilde{w}_i}{(1+\sigma_iv)w_i} + \frac{(1-\alpha_i)\beta_i\tilde{x}_i\tilde{v}}{1+\sigma_i\tilde{v}}\right) \\
&\quad + \frac{e_i+b_i}{e_i\alpha_i+b_i}\left(-\frac{\alpha_i\beta_ix_iv\tilde{y}_i}{(1+\sigma_iv)y_i} + \frac{b_iw_i\tilde{y}_i\tilde{w}_i}{y_i\tilde{w}_i} + \frac{e_i\alpha_i+b_i}{e_i+b_i}\frac{\beta_i\tilde{x}_i\tilde{v}}{1+\sigma_i\tilde{v}}\right) - \frac{y_iv}{\tilde{y}_i\tilde{v}}\frac{\beta_i\tilde{x}_i\tilde{v}}{1+\sigma_i\tilde{v}} - \frac{v}{\tilde{v}}\frac{\beta_i\tilde{x}_i\tilde{v}}{1+\sigma_i\tilde{v}} + \frac{\beta_i\tilde{x}_i\tilde{v}}{1+\sigma_i\tilde{v}}\Bigg] + r\tilde{v}z - \frac{\mu r}{g}z. \\
&= \sum_{i=1}^{2}\gamma_i\Bigg[-d_i\frac{(x_i-\tilde{x}_i)^2}{x_i} + \frac{\beta_i\tilde{x}_i\tilde{v}}{1+\sigma_i\tilde{v}}\left(-1+\frac{v(1+\sigma_i\tilde{v})}{\tilde{v}(1+\sigma_iv)} - \frac{v}{\tilde{v}} + \frac{1+\sigma_iv}{1+\sigma_i\tilde{v}}\right) \\
&\quad + \frac{b_i(1-\alpha_i)}{(e_i\alpha_i+b_i)}\frac{\beta_i\tilde{x}_i\tilde{v}}{(1+\sigma_i\tilde{v})}\left(5-\frac{\tilde{x}_i}{x_i} - \frac{x_i\tilde{w}_iv(1+\sigma_i\tilde{v})}{\tilde{x}_iw_i\tilde{v}(1+\sigma_iv)} - \frac{y_i\tilde{v}}{\tilde{y}_iv} - \frac{w_i\tilde{y}_i}{\tilde{w}_iy_i} - \frac{1+\sigma_iv}{1+\sigma_i\tilde{v}}\right) \\
&\quad + \frac{(e_i+b_i)\alpha_i}{(e_i\alpha_i+b_i)}\frac{\beta_i\tilde{x}_i\tilde{v}}{(1+\sigma_i\tilde{v})}\left(4-\frac{\tilde{x}_i}{x_i} - \frac{x_i\tilde{y}_iv(1+\sigma_i\tilde{v})}{\tilde{x}_iy_i\tilde{v}(1+\sigma_iv)} - \frac{y_i\tilde{v}}{\tilde{y}_iv} - \frac{1+\sigma_iv}{1+\sigma_i\tilde{v}}\right)\Bigg] + \left(\tilde{v}-\frac{\mu}{g}\right)rz \\
&= \sum_{i=1}^{2}\gamma_i\Bigg[-d_i\frac{(x_i-\tilde{x}_i)^2}{x_i} - \frac{\beta_i\tilde{x}_i\tilde{v}}{(1+\sigma_i\tilde{v})}\frac{\sigma_i(v-\tilde{v})^2}{(1+\sigma_iv)(1+\sigma_i\tilde{v})\tilde{v}} \\
&\quad + \frac{b_i(1-\alpha_i)}{(e_i\alpha_i+b_i)}\frac{\beta_i\tilde{x}_i\tilde{v}}{(1+\sigma_i\tilde{v})}\left(5-\frac{\tilde{x}_i}{x_i} - \frac{x_i\tilde{w}_iv(1+\sigma_i\tilde{v})}{\tilde{x}_iw_i\tilde{v}(1+\sigma_iv)} - \frac{y_i\tilde{v}}{\tilde{y}_iv} - \frac{w_i\tilde{y}_i}{\tilde{w}_iy_i} - \frac{1+\sigma_iv}{1+\sigma_i\tilde{v}}\right) \\
&\quad + \frac{(e_i+b_i)\alpha_i}{(e_i\alpha_i+b_i)}\frac{\beta_i\tilde{x}_i\tilde{v}}{(1+\sigma_i\tilde{v})}\left(4-\frac{\tilde{x}_i}{x_i} - \frac{x_i\tilde{y}_iv(1+\sigma_i\tilde{v})}{\tilde{x}_iy_i\tilde{v}(1+\sigma_iv)} - \frac{y_i\tilde{v}}{\tilde{y}_iv} - \frac{1+\sigma_iv}{1+\sigma_i\tilde{v}}\right)\Bigg] + \left(\tilde{v}-\frac{\mu}{g}\right)rz. \qquad (49)
\end{aligned}$$

As the same proof of Eq. (32) we can show that $(\tilde{v}-\bar{v}) = \frac{1}{\omega}(R_1-1)$, where $\omega = \sum_{i=1}^{2}\frac{k_i\beta_i\lambda_i\xi_i(e_i\alpha_i+b_i)}{a_id_ic(e_i+b_i)(1+\xi_i\bar{v})(1+\xi_i\tilde{v})}$. So, if $R_1 \le 1$ then $\tilde{v} \le \frac{\mu}{g} = \bar{v}$. We have $x_i, w_i, y_i, v > 0$ when $R_0 > 1$. Since the geometrical mean is less than or equal to the arithmetical mean, then the third and fourth terms of Eq. (49) are less than or equal zero, then if $R_1 \le 1$ then $\frac{dW_1}{dt} \le 0$ for all $x_i, w_i, y_i, v, z > 0$. Clearly, $\frac{dW_1}{dt} = 0$ occurs at E_1. LaSalle's invariance principle implies global stability of E_1.

Theorem 6. The chronic-infection equilibrium with antibody immune response E_2 of system (36)-(40) is GAS when $R_1 > 1$.

Proof. Define Lyapunov function W_2 as:

$$W_2 = \sum_{i=1}^{2}\gamma_i\left[\bar{x}_iF\left(\frac{x_i}{\bar{x}_i}\right) + \frac{b_i}{e_i\alpha_i+b_i}\bar{w}_iF\left(\frac{w_i}{\bar{w}_i}\right) + \frac{e_i+b_i}{e_i\alpha_i+b_i}\bar{y}_iF\left(\frac{y_i}{\bar{y}_i}\right)\right] + \bar{v}F\left(\frac{v}{\bar{v}}\right) + \frac{r}{g}\bar{z}F\left(\frac{z}{\bar{z}}\right).$$

The time derivative of W_2 along the trajectories of (36)-(40) is given by

$$\begin{aligned}
\frac{dW_2}{dt} &= \sum_{i=1}^{2}\gamma_i\Bigg[\left(1-\frac{\bar{x}_i}{x_i}\right)\left(\lambda_i - d_ix_i - \frac{\beta_ix_iv}{1+\sigma_iv}\right) + \frac{b_i}{e_i\alpha_i+b_i}\left(1-\frac{\bar{w}_{\bar{i}}}{w_i}\right)\left(\frac{(1-\alpha_i)\beta_ix_iv}{1+\sigma_iv} - (e_i+b_i)w_i\right) \\
&\quad + \frac{e_i+b_i}{e_i\alpha_i+b_i}\left(1-\frac{\bar{y}_i}{y_i}\right)\left(\frac{\alpha_i\beta_ix_iv}{1+\sigma_iv} + bw_i - a_iy_i\right)\Bigg] + \left(1-\frac{\bar{v}}{v}\right)\left(\sum_{i=1}^{2}k_iy_i - cv - rvz\right) + \frac{r}{g}\left(1-\frac{\bar{z}}{z}\right)(gvz-\mu z).
\end{aligned} \qquad (50)$$

Collecting terms of Eq. (50) and using the equilibrium condition for E_2

$$\lambda_i = d_i\bar{x}_i + \frac{\beta\bar{x}_i\bar{v}}{1+\sigma_i\bar{v}}, \; \frac{(1-\alpha_i)\beta_i\bar{x}_i\bar{v}}{1+\sigma_i\bar{v}} = (e_i+b_i)\bar{w}_i, \; \frac{\alpha_i\beta_i\bar{x}_i\bar{v}}{1+\sigma_i\bar{v}} + b_i\bar{w}_i = a_i\bar{y}_i, \; c\bar{v} + r\bar{v}\bar{z} = \sum_{i=1}^{2}k_i\bar{y}_i,$$

$$\frac{e_i+b_i}{e_i\alpha_i+b_i}a_i\bar{y}_i=\frac{\beta_i\bar{x}_i\bar{v}}{1+\sigma_i\bar{v}}=\frac{b_i(1-\alpha_i)}{(e_i\alpha_i+b_i)}\frac{\beta_i\bar{x}_i\bar{v}}{(1+\sigma_i\bar{v})}+\frac{(e_i+b_i)\alpha_i}{(e_i\alpha_i+b_i)}\frac{\beta_i\bar{x}_i\bar{v}}{(1+\sigma_i\bar{v})}$$

Eq. (50) becomes

$$\begin{aligned}\frac{dW_2}{dt}&=\sum_{i=1}^{2}\gamma_i\Bigg[-d_i\frac{(x_i-\bar{x}_i)^2}{x_i}-\frac{\beta_i\bar{x}_i\bar{v}}{(1+\sigma_i\bar{v})}\frac{\sigma_i(v-\bar{v})^2}{\bar{v}(1+\sigma_iv)(1+\sigma_i\bar{v})}\\&+\frac{b_i(1-\alpha_i)}{(e_i\alpha_i+b_i)}\frac{\beta_i\bar{x}_i\bar{v}}{(1+\sigma_i\bar{v})}\left(5-\frac{\bar{x}_i}{x_i}-\frac{x_i\bar{w}_iv(1+\sigma_i\bar{v})}{\bar{x}_iw_i\bar{v}(1+\sigma_iv)}-\frac{y_i\bar{v}}{\bar{y}_iv}-\frac{w_i\bar{y}_i}{\bar{w}_iy_i}-\frac{1+\sigma_iv}{1+\sigma_i\bar{v}}\right)\\&+\frac{(e_i+b_i)\alpha_i}{(e_i\alpha_i+b_i)}\frac{\beta_i\bar{x}_i\bar{v}}{(1+\sigma_i\bar{v})}\left(4-\frac{\bar{x}_i}{x_i}-\frac{x_i\bar{y}_iv(1+\sigma_i\bar{v})}{\bar{x}_iy_i\bar{v}(1+\sigma_iv)}-\frac{y_i\bar{v}}{\bar{y}_iv}-\frac{1+\sigma_iv}{1+\sigma_i\bar{v}}\right)\Bigg]\end{aligned}$$

Thus, if $R_1>1$ then x_i, w_i, y_i, v and $z>0$. Similar to the proof of Theorem 3, one can show that E_2 is GAS.

4 Model with general incidence rate

In this section, we propose a viral infection model with latently infected cells and antibody immune response. The incidence rate of infection is represented by a general function of the populations of the uninfected target cells and free viruses.

$$\dot{x}_i=\lambda_i-d_ix_i-f_i(x_i,v),\qquad i=1,2,\tag{51}$$

$$\dot{w}_i=(1-\alpha_i)f_i(x_i,v)-(e_i+b_i)w_i,\qquad i=1,2,\tag{52}$$

$$\dot{y}_i=\alpha_if_i(x_i,v)+b_iw_i-a_iy_i,\qquad i=1,2,\tag{53}$$

$$\dot{v}=\sum_{i=1}^{2}k_iy_i-cv-rvz,\tag{54}$$

$$\dot{z}=gvz-\mu z,\tag{55}$$

where the function $f_i(x_i,v)$ represents the rate of the uninfected target cells to be infected by the viruses.

Assumption A1 For $i=1,2$, function f_i satisfies:

(i) $f_i(x_i,v)$ is positive, continuous, and differentiable,

(ii) $\dfrac{\partial f_i(x_i,v)}{\partial v}>0$ and $\dfrac{\partial f_i(x_i,v)}{\partial x_i}>0$ for any x_i,$v>0$. Furthermore, $\dfrac{\partial f_i(x_i,0)}{\partial v}>0$ for any $x_i>0$,

(iii) $f_i(x_i,0)=f_i(0,v)=0$, for all $x_i>0$ and $v>0$.

Assumption A2 For $i=1,2$, function f_i satisfies:

(i) $f_i(x_i,v)\le v\dfrac{\partial f_i(x_i,0)}{\partial v}$, for all $v>0$.

(ii) $\frac{d}{dx_i}\left(\dfrac{\partial f_i(x_i,0)}{\partial v}\right)>0$

4.1 Equilibria and biological thresholds

We define the basic infection reproduction number of system (51)-(55) as:

$$R_0=\sum_{i=1}^{2}\frac{k_i(e_i\alpha_i+b_i)}{a_ic(e_i+b_i)}\frac{\partial f_i(x_i^0,0)}{\partial v}.$$

The equilibria of (51)-(55) satisfy the following equations:

$$\lambda_i-d_ix_i-f_i(x_i,v)=0,\tag{56}$$

$$(1-\alpha_i)f_i(x_i,v)-(e_i+b_i)w_i=0,\tag{57}$$

$$\alpha_if_i(x_i,v)+b_iw_i-a_iy_i=0,\tag{58}$$

$$\sum_{i=1}^{2}k_iy_i-cv-rvz=0,\tag{59}$$

$$(gv-\mu)z=0.\tag{60}$$

Equation (60) has two possible solutions, $z = 0$ or $v = \mu/g$. When $z = 0$, we obtain two equilibria, the infection-free equilibrium $E_0 = (x_1^0, x_2^0, 0, 0, 0, 0, 0, 0)$, where $x_i^0 = \dfrac{\lambda_i}{d_i}$, $i = 1, 2$ and the infected steady state without antibody immune response $E_1 = (\tilde{x}_1, \tilde{x}_2, \tilde{w}_1, \tilde{w}_2, \tilde{y}_1, \tilde{y}_2, \tilde{v}, 0,)$, where the coordinates satisfy the equalities:

$$\lambda_i = d_i\tilde{x}_i + f_i(\tilde{x}_i, \tilde{v}), \quad (1-\alpha_i) f_i(\tilde{x}_i, \tilde{v}) = (e_i + b_i)\tilde{w}_i, \; \alpha_i f_i(\tilde{x}_i, \tilde{v}) + b_i\tilde{w}_i = a_i\tilde{y}_i, \; \sum_{i=1}^{2} k_i\tilde{y}_i = c\tilde{v}. \tag{61}$$

The other possibility of Eq. (60) $z \neq 0$ leads to $\bar{v} = \dfrac{\mu}{g}$. Substitute the value of $\bar{v}$ in Eq. (56) and let

$$\Pi(x_i) = \lambda_i - d_i x_i - f_i(x_i, \bar{v}) = 0.$$

According to Assumptions A1, Π is a strictly decreasing function of x_i. Besides, $\Pi(0) = \lambda_i > 0$ and $\Pi(x_i^0) = -f_i(x_i^0, \bar{v}) < 0$. Thus, there exists a unique $\bar{x}_i \in (0, x_i^0)$ such that $\Pi(\bar{x}_i) = 0$. From Eqs. (57)-(59) we have

$$\bar{w}_i = \frac{(1-\alpha_i) f_i(\bar{x}_i, \bar{v})}{(e_i + b_i)}, \quad \bar{y}_i = \frac{(e_i\alpha_i + b_i) f_i(\bar{x}_i, \bar{v})}{a_i(e_i + b_i)}, \quad \bar{z} = \frac{c}{r}\left[\sum_{i=1}^{2} \frac{k_i(e_i\alpha_i + b_i) f_i(\bar{x}_i, \bar{v})}{a_i c(e_i + b_i)\bar{v}} - 1\right].$$

Thus $\bar{w}_i > 0$ and $\bar{y}_i > 0$, moreover, $\bar{z} > 0$ when $\sum_{i=1}^{2} \dfrac{k_i(e_i\alpha_i + b_i) f_i(\bar{x}_i, \bar{v})}{a_i c(e_i + b_i)\bar{v}} > 1$. Now we define the antibody immune response activation number as:

$$R_1 = \sum_{i=1}^{2} \frac{k_i(e_i\alpha_i + b_i) f_i(\bar{x}_i, \bar{v})}{a_i c(e_i + b_i)\bar{v}}.$$

Hence, $\bar{z}$ can be rewritten as $\bar{z} = \frac{c}{r}(R_1 - 1)$. It follows that, there exists a chronic-infection equilibrium with antibody immune response $E_2 = (\bar{x}_1, \bar{w}_1, \bar{y}_1, \bar{x}_2, \bar{w}_2, \bar{y}_2, \bar{v}, \bar{z})$ when $R_1 > 1$. Clearly from **Assumptions A1** and **A2**, we have

$$R_1 = \sum_{i=1}^{2} \frac{k_i(e_i\alpha_i + b_i) f_i(\bar{x}_i, \bar{v})}{a_i c(e_i + b_i)\bar{v}} < \sum_{i=1}^{2} \frac{k_i(e_i\alpha_i + b_i)}{a_i c(e_i + b_i)\bar{v}} \frac{\partial f_i(\bar{x}_i, 0)}{\partial \bar{v}}\bar{v} < \sum_{i=1}^{2} \frac{k_i(e_i\alpha_i + b_i)}{a_i c(e_i + b_i)} \frac{\partial f_i(x_i^0, 0)}{\partial v} = R_0.$$

5 Global stability analysis

Theorem 7. Let Assumptions A1-A2 be hold true and $R_0 \leq 1$, then the infection-free equilibrium E_0 for system (51)-(55) is GAS.

Proof. Define a Lyapunov functional W_0 as follows:

$$W_0 = \sum_{i=1}^{2} \gamma_i \left[x_i - x_i^0 - \int_{x_i^0}^{x_i} \lim_{v \to 0^+} \frac{f_i(x_i^0, v)}{f_i(s_i, v)} ds_i + \frac{b_i}{e_i\alpha_i + b_i} w_i + \frac{e_i + b_i}{e_i\alpha_i + b_i} y_i \right] + v + \frac{r}{g} z.$$

Calculating $\dfrac{dW_0}{dt}$ along the trajectories of (51)-(55) as:

$$\begin{aligned}
\frac{dW_0}{dt} &= \sum_{i=1}^{2} \gamma_i \Bigg[\left(1 - \lim_{v \to 0^+} \frac{f_i(x_i^0, v)}{f_i(x_i, v)}\right) (\lambda_i - d_i x_i - f_i(x_i, v)) + \frac{b_i}{e_i\alpha_i + b_i} \left((1-\alpha_i) f_i(x_i, v) - (e_i + b_i) w_i\right) \\
&\quad + \frac{e_i + b_i}{e_i\alpha_i + b_i} \left(\alpha_i f_i(x_i, v) + b_i w_i - a_i y_i\right) \Bigg] + \sum_{i=1}^{2} k_i y_i - cv - rvz + \frac{r}{g}(gvz - \mu z) \\
&= \sum_{i=1}^{2} \gamma_i \lambda_i \left(1 - \frac{\partial f_i(x_i^0, 0)/\partial v}{\partial f_i(x_i, 0)/\partial v}\right) \left(1 - \frac{x_i}{x_i^0}\right) + (R_0 - 1)\, cv - \frac{r\mu}{g} z.
\end{aligned} \tag{62}$$

Based on Assumption A2, the first term of Eq. (62) is less than or equal zero. Therefore if $R_0 \leq 1$, then $\frac{dW_0}{dt} \leq 0$ for all $x_i, v, z > 0$. Similar to the previous sections, one can show that E_0 is GAS.

Now we need to the following Assumption to proof that, E_1 and E_2 for the system (51)-(55) are GAS.

Assumption A3 Function $f_i(x_i, v)$ satisfies the following:

$$\left(\frac{f_i(x_i,v)}{f_i(x_i,\tilde{v})}-\frac{v}{\tilde{v}}\right)\left(1-\frac{f_i(x_i,\tilde{v})}{f_i(x_i,v)}\right)\le 0,\quad \left(\frac{f_i(x_i,v)}{f_i(x_i,\bar{v})}-\frac{v}{\bar{v}}\right)\left(1-\frac{f_i(x_i,\bar{v})}{f_i(x_i,v)}\right)\le 0,\qquad x_i,v>0,$$

Theorem 8. Suppose that Assumptions A1-A3 are satisfied, E_1 exists and $R_1 \le 1$, then E_1 for system (51)-(55) is GAS.

Proof. We construct the following Lyapunov functional

$$W_1=\sum_{i=1}^{2}\gamma_i\left[x_i-\tilde{x}_i-\int_{\tilde{x}_i}^{x_i}\frac{f_i(\tilde{x}_i,\tilde{v})}{f_i(s_i,\tilde{v})}ds_i+\frac{b_i}{e_i\alpha_i+b_i}\tilde{w}_iF\left(\frac{w_i}{\tilde{w}_i}\right)+\frac{e_i+b_i}{e_i\alpha_i+b_i}\tilde{y}_iF\left(\frac{y_i}{\tilde{y}_i}\right)\right]+\tilde{v}F\left(\frac{v}{\tilde{v}}\right)+\frac{r}{g}z.$$

The time derivative of W_1 along the trajectories of (51)-(55) is given by

$$\begin{aligned}\frac{dW_1}{dt}=&\sum_{i=1}^{2}\gamma_i\Bigg[\left(1-\frac{f_i(\tilde{x}_i,\tilde{v})}{f_i(x_i,\tilde{v})}\right)(\lambda_i-d_ix_i-f_i(x_i,v))+\frac{b_i}{e_i\alpha_i+b_i}\left(1-\frac{\tilde{w}_i}{w_i}\right)((1-\alpha_i)f_i(x_i,v)-(e_i+b_i)w_i)\\&+\frac{e_i+b_i}{e_i\alpha_i+b_i}\left(1-\frac{\tilde{y}_i}{y_i}\right)(\alpha_if_i(x_i,v)+b_iw_i-a_iy_i)\Bigg]+\left(1-\frac{\tilde{v}}{v}\right)(\left(\sum_{i=1}^{2}k_iy_i-cv-rvz\right)+\frac{r}{g}(gvz-\mu z).\end{aligned}\tag{63}$$

Collecting terms of Eq. (63) we get

$$\begin{aligned}\frac{dW_1}{dt}=&\sum_{i=1}^{2}\gamma_i\Bigg[\left(1-\frac{f_i(\tilde{x}_i,\tilde{v})}{f_i(x_i,\tilde{v})}\right)(\lambda_i-d_ix_i)+f_i(x_i,v)\frac{f_i(\tilde{x}_i,\tilde{v})}{f_i(x_i,\tilde{v})}-\frac{b_i(1-\alpha_i)}{e_i\alpha_i+b_i}f_i(x_i,v)\frac{\tilde{w}_i}{w_i}\\&+\frac{(e_i+b_i)}{e_i\alpha_i+b_i}b_i\tilde{w}_i-\frac{(e_i+b_i)\alpha_i}{e_i\alpha_i+b_i}f_i(x_i,v)\frac{\tilde{y}_i}{y_i}-\frac{(e_i+b_i)b_iw_i}{e_i\alpha_i+b_i}\frac{\tilde{y}_i}{y_i}-\frac{e_i+b_i}{e_i\alpha_i+b_i}a_i\tilde{y}_i\Bigg]\\&-cv-\sum_{i=1}^{2}k_iy_i\frac{\tilde{v}}{v}+c\tilde{v}+r\tilde{v}z-\frac{r\mu}{g}z.\end{aligned}$$

Using the equilibrium condition for E_1:

$$\lambda_i=d_i\tilde{x}_i+f_i(\tilde{x}_i,\tilde{v}),\quad (1-\alpha_i)f_i(\tilde{x}_i,\tilde{v})=(e_i+b_i)\tilde{w}_i,\quad a_i\tilde{y}_i=\alpha_if_i(\tilde{x}_i,\tilde{v})+b_i\tilde{w}_i,\quad cv=\frac{v}{\tilde{v}}\sum_{i=1}^{2}\gamma_if_i(\tilde{x}_i,\tilde{v}),$$

$$c\tilde{v}=\sum_{i=1}^{2}k_i\tilde{y}_i=\sum_{i=1}^{2}\gamma_if_i(\tilde{x}_i,\tilde{v}),\quad \frac{e_i+b_i}{e_i\alpha_i+b_i}a_i\tilde{y}_i=f_i(\tilde{x}_i,\tilde{v})=\frac{b_i(1-\alpha_i)}{(e_i\alpha_i+b_i)}f_i(\tilde{x}_i,\tilde{v})+\frac{(e_i+b_i)\alpha_i}{(e_i\alpha_i+b_i)}f_i(\tilde{x}_i,\tilde{v}),$$

we obtain

$$\begin{aligned}\frac{dW_1}{dt}=&\sum_{i=1}^{2}\gamma_i\Bigg[d_i\tilde{x}_i\left(1-\frac{f_i(\tilde{x}_i,\tilde{v})}{f_i(x_i,\tilde{v})}\right)\left(1-\frac{x_i}{\tilde{x}_i}\right)+\left(1-\frac{f_i(x_i,\tilde{v})}{f_i(x_i,v)}\right)\left(\frac{f_i(x_i,v)}{f_i(x_i,\tilde{v})}-\frac{v}{\tilde{v}}\right)\\&+\frac{b_i(1-\alpha_i)}{(e_i\alpha_i+b_i)}f_i(\tilde{x}_i,\tilde{v})\left(5-\frac{f_i(\tilde{x}_i,\tilde{v})}{f_i(x_i,\tilde{v})}-\frac{\tilde{w}_if_i(x_i,v)}{w_if_i(\tilde{x}_i,\tilde{v})}-\frac{w_i\tilde{y}_i}{\tilde{w}_iy_i}-\frac{y_i\tilde{v}}{\tilde{y}_iv}-\frac{vf_i(x_i,\tilde{v})}{\tilde{v}f_i(x_i,v)}\right)\\&+\frac{(e_i+b_i)\alpha_i}{(e_i\alpha_i+b_i)}f_i(\tilde{x}_i,\tilde{v})\left(4-\frac{f_i(\tilde{x}_i,\tilde{v})}{f_i(x_i,\tilde{v})}-\frac{\tilde{y}_if_i(x_i,v)}{y_if_i(\tilde{x}_i,\tilde{v})}-\frac{y_i\tilde{v}}{\tilde{y}_iv}-\frac{vf_i(x_i,\tilde{v})}{\tilde{v}f_i(x_i,v)}\right)\Bigg]+r\left(\tilde{v}-\frac{\mu}{g}\right)z.\end{aligned}\tag{64}$$

From **Assumptions A1 and A3**, we get that the first and second terms of Eq. (64) are less than or equal zero. Because the geometrical mean is less than or equal to the arithmetical mean, then the third and fourth terms of Eq. (64) are less than or equal zero. Now we show that if $R_1 \le 1$ then $\tilde{v} \le \frac{\mu}{r} = \bar{v}$. This can be achieved if we show that

$$sgn\left(\bar{x}_i-\tilde{x}_i\right)=sgn\left(\tilde{v}-\bar{v}\right)=sgn\left(R_1-1\right).$$

Applying **Assumptions A1-A2**, we have

$$(f_i(\bar{x}_i,\tilde{v})-f_i(\tilde{x}_i,\tilde{v}))(\bar{x}_i-\tilde{x}_i)>0, \tag{65}$$

$$(f_i(\tilde{x}_i,\bar{v})-f_i(\tilde{x}_i,\tilde{v}))(\bar{v}-\tilde{v})>0, \quad (f_i(\bar{x}_i,\bar{v})-f_i(\bar{x}_i,\tilde{v}))(\bar{v}-\tilde{v})>0. \tag{66}$$

Using **Assumption A3** with $x_i=\tilde{x}_i$ and $v=\bar{v}$, we get

$$(f_i(\tilde{x}_i,\bar{v})\tilde{v}-f_i(\tilde{x}_i,\tilde{v})\bar{v})(f_i(\tilde{x}_i,\bar{v})-f_i(\tilde{x}_i,\tilde{v}))\le 0$$

It follows from inequality (66) that

$$((f_i(\tilde{x}_i,\bar{v})\tilde{v}-f_i(\tilde{x}_i,\tilde{v})\bar{v}))(\tilde{v}-\bar{v})>0. \tag{67}$$

Suppose that, $sgn\,(\bar{x}_i-\tilde{x}_i)=sgn\,(\bar{v}-\tilde{v})$. Using the conditions of the equilibria E_1 and E_2 we have

$$(\lambda_i-d_i\bar{x}_i)-(\lambda_i-d_i\tilde{x}_i)=f_i(\bar{x}_i,\bar{v})-f_i(\tilde{x}_i,\tilde{v})=f_i(\bar{x}_i,\bar{v})-f_i(\bar{x}_i,\tilde{v})+f_i(\bar{x}_i,\tilde{v})-f_i(\tilde{x}_i,\tilde{v}),$$

and from inequalities (65) and (66) we get $sgn\,(\tilde{x}_i-\bar{x}_i)=sgn\,(\bar{x}_i-\tilde{x}_i)$, which leads to contradiction. Thus, $sgn\,(\bar{x}_i-\tilde{x}_i)=sgn\,(\tilde{v}-\bar{v})$. Using the equilibrium conditions for E_1 we have $\sum\limits_{i=1}^{2}\frac{k_i(e_i\alpha_i+b_i)f_i(\tilde{x}_i,\tilde{v})}{a_ic(e_i+b_i)\tilde{v}}=1$, then

$$\begin{aligned}R_1-1&=\sum_{i=1}^{2}\frac{k_i(e_i\alpha_i+b_i)}{a_ic(e_i+b_i)}\left(\frac{f_i(\bar{x}_i,\bar{v})}{\bar{v}}-\frac{f_i(\tilde{x}_i,\tilde{v})}{\tilde{v}}\right)\\&=\sum_{i=1}^{2}\frac{k_i(e_i\alpha_i+b_i)}{a_ic(e_i+b_i)}\left(\frac{1}{\bar{v}}(f_i(\bar{x}_i,\bar{v})-f_i(\tilde{x}_i,\bar{v}))+\frac{1}{\tilde{v}\bar{v}}(f_i(\tilde{x}_i,\bar{v})\tilde{v}-f_i(\tilde{x}_i,\tilde{v})\bar{v})\right).\end{aligned}$$

From inequalities (65) and (67) we get $sgn\,(R_1-1)=sgn\,(\tilde{v}-\bar{v})$. It follows that, if $R_1\le 1$ then $\tilde{v}\le\frac{\mu}{r}=\bar{v}$. Therefore, if $R_1\le 1$ then $\frac{dW_1}{dt}\le 0$ for all $x_i,w_i,y_i,v,z>0$, where the equality occurs at the equilibrium E_1. LaSalle's invariance principle implies the global stability of E_1.

Theorem 9. Let **Assumptions A1-A3** be hold true and $R_1>1$, then chronic-infection equilibrium with antibody immune response E_2 for system (51)-(55) is GAS.

Proof. We construct the following Lyapunov functional

$$W_2=\sum_{i=1}^{2}\gamma_i\left[x_i-\bar{x}_i-\int_{\bar{x}_i}^{x_i}\frac{f_i(\bar{x}_i,\bar{v})}{f_i(s_i,\bar{v})}ds_i+\frac{b_i}{e_i\alpha_i+b_i}\bar{w}_iF\left(\frac{w_i}{\bar{w}_i}\right)+\frac{e_i+b_i}{e_i\alpha_i+b_i}\bar{y}_iF\left(\frac{y_i}{\bar{y}_i}\right)\right]+\bar{v}F\left(\frac{v}{\bar{v}}\right)+\frac{r}{g}\bar{z}F\left(\frac{z}{\bar{z}}\right).$$

We calculate the time derivative of W_2 along the trajectories of (51)-(55) as:

$$\begin{aligned}\frac{dW_2}{dt}=&\sum_{i=1}^{2}\gamma_i\Bigg[\left(1-\frac{f_i(\bar{x}_i,\bar{v})}{f_i(x_i,\bar{v})}\right)(\lambda_i-d_ix_i-f_i(x_i,v))+\frac{b_i}{e_i\alpha_i+b_i}\left(1-\frac{\bar{w}_i}{w}\right)((1-\alpha_i)f_i(x_i,v)-(e_i+b_i)w_i)\\&+\frac{e_i+b_i}{e_i\alpha_i+b_i}\left(1-\frac{\bar{y}_i}{y}\right)(\alpha_if_i(x_i,v)+b_iw_i-a_iy_i)\Bigg]+\left(1-\frac{\bar{v}}{v}\right)(\sum_{i=1}^{2}k_iy_i-cv-rvz)+\frac{r}{g}\left(1-\frac{\bar{z}}{z}\right)(gvz-\mu z).\end{aligned} \tag{68}$$

Collecting terms of Eq. (68) and using the equilibrium conditions for E_2

$$\lambda_i=d_i\bar{x}_i+f_i(\bar{x}_i,\bar{v}),\quad (1-\alpha_i)f_i(\bar{x}_i,\bar{v})=(e_i+b_i)\bar{w}_i,\quad a_i\bar{y}_i=\alpha_if_i(\bar{x}_i,\bar{v})+b_i\bar{w}_i,\quad c\bar{v}=\sum_{i=1}^{2}\gamma_if_i(\bar{x}_i,\bar{v})-r\bar{v}\bar{z},$$

$$cv=\frac{v}{\bar{v}}\sum_{i=1}^{2}\gamma_if_i(\bar{x}_i,\bar{v})-rv\bar{z},\quad \frac{e_i+b_i}{e_i\alpha_i+b_i}a_i\bar{y}_i=f_i(\bar{x}_i,\bar{v})=\frac{b_i(1-\alpha_i)}{(e_i\alpha_i+b_i)}f_i(\bar{x}_i,\bar{v})+\frac{(e_i+b_i)\alpha_i}{(e_i\alpha_i+b_i)}f_i(\bar{x}_i,\bar{v}),$$

we get

$$\begin{aligned}\frac{dW_2}{dt} &= \sum_{i=1}^{2}\gamma_i\Bigg[d_i\bar{x}_i\left(1-\frac{f_i(\bar{x}_i,\bar{v})}{f_i(x_i,\bar{v})}\right)(1-\frac{x_i}{\bar{x}_i})+f_i(\bar{x}_i,\bar{v})\left(1-\frac{f_i(x_i,\bar{v})}{f_i(x_i,v)}\right)\left(\frac{f_i(x_i,v)}{f_i(x_i,\bar{v})}-\frac{v}{\bar{v}}\right)\\ &+\frac{b_i(1-\alpha_i)}{(e_i\alpha_i+b_i)}f_i(\bar{x}_i,\bar{v})\left(5-\frac{f_i(\bar{x}_i,\bar{v})}{f_i(x_i,\bar{v})}-\frac{\bar{w}_i f_i(x_i,v)}{w_i f_i(\bar{x}_i,\bar{v})}-\frac{\bar{y}_i w_i}{y_i\bar{w}_i}-\frac{y_i\bar{v}}{\bar{y}_i v}-\frac{v f_i(x_i,\bar{v})}{\bar{v} f_i(x_i,v)}\right)\\ &+\frac{(e_i+b_i)\alpha_i}{(e_i\alpha_i+b_i)}f_i(\bar{x}_i,\bar{v})\left(4-\frac{f_i(\bar{x}_i,\bar{v})}{f_i(x_i,\bar{v})}-\frac{\bar{y}_i f_i(x_i,v)}{y_i f_i(\bar{x}_i,\bar{v})}-\frac{y_i\bar{v}}{\bar{y}_i v}-\frac{v f_i(x_i,\bar{v})}{\bar{v} f_i(x_i,v)}\right)\Bigg] \qquad (69)\end{aligned}$$

Thus, if $R_1 > 1$ then $\bar{x}_i, \bar{w}_i, \bar{y}_i, \bar{v}$ and $\bar{z} > 0$. From Assumptions A1 and A3, we get that the first and second terms of Eq. (69) are less than or equal zero. Since the arithmetical mean is greater than or equal to the geometrical mean, then $\frac{dW_2}{dt} \le 0$. It can be seen that, $\frac{dW_2}{dt} = 0$ if and only if $x_i = \bar{x}_i$, $w_i = \bar{w}_i$ and $v = \bar{v}$. From Eq. (54), if $v = \bar{v}$ and $y_i = \bar{y}_i$ then $\dot{v} = 0$ and $0 = \sum_{i=1}^{2} k\bar{y}_i - c\bar{v} - r\bar{v}\bar{z} = 0$, which yields $z = \bar{z}$ and hence $\frac{dW_2}{dt}$ equal to zero at E_2. LaSalle's invariance principle implies global stability of E_2.

5.1 Special forms of the incidence rate

By using the Lyapunov direct method, we have established a set of conditions on $f_i(x_i, v),\ i = 1, 2$ ensuring the global asymptotic stability of the equilibria of model (51)-(55). Now we introduce some forms of the incidence rate and verify A1-A3.

(1) Bilinear incidence rate: $f_i(x_i, v) = \beta_i x_i v$,
(2) Saturation functional response: $f_i(x_i, v) = \frac{\beta_i x_i v}{1+\eta_i v}$,
(3) Beddington-DeAngelis functional response: $f_i(x_i, v) = \frac{\beta_i x_i v}{1+\gamma_i x_i+\eta_i v}$,
(4) Crowley-Martin functional response: $f_i(x_i, v) = \frac{\beta_i x_i v}{(1+\gamma_i x_i)(1+\eta_i v)}$,
(5) Hill type incidence rate: $f_i(x_i, v) = \frac{\beta_i x_i^n v}{\gamma_i^n + x_i^n}$, where $\beta_i, \gamma_i, n > 0$.

One can easily show that A1-A3 for the functions f_i, $i = 1, 2$ given above.

Now we verify Assumptions A1-A3 for the function $f_i(x_i, v) = \frac{\beta_i x_i^n v}{\gamma_i^n + x_i^n}$, $i = 1, 2$. We have $f_i(x_i, v) > 0$ for all $x_i > 0$, $v > 0$, $f_i(0, v) = f_i(x_i, 0) = 0$ and

$$\frac{\partial f_i(x_i,v)}{\partial x_i} = \frac{n\beta_i\gamma_i^n x_i^{n-1} v}{(\gamma_i^n + x_i^n)^2}, \qquad \frac{\partial f_i(x_i,v)}{\partial v} = \frac{\beta_i x_i^n}{\gamma_i^n + x_i^n} = \frac{\partial f_i(x_i,0)}{\partial v}.$$

Then, for all $x_i > 0, v > 0$, we have $\frac{\partial f_i(x_i,v)}{\partial x_i} > 0$, $\frac{\partial f_i(x_i,v)}{\partial v} > 0$ and $\frac{\partial f_i(x_i,0)}{\partial v} > 0$ if $n > 0$. Therefore Assumptions A1 is satisfied. We have also

$$f_i(x_i,v) = \frac{\beta_i x_i^n v}{\gamma_i^n + x_i^n} = v\frac{\beta_i x_i^n}{\gamma_i^n + x_i^n} = v\frac{\partial f_i(x_i,0)}{\partial v},$$

$$\frac{d}{dx_i}\left(\frac{\partial f_i(x_i^0,0)/\partial v}{\partial f_i(x_i,0)/\partial v}\right) = -\frac{n\gamma_i^n (x_i^0)^n}{(\gamma_i^n + (x_i^0)^n)x_i^{n+1}} < 0,$$

then, Assumptions A2 is satisfied. Moreover,

$$\left(\frac{f_i(x_i,v)}{f_i(x_i,\tilde{v})} - \frac{v}{\tilde{v}}\right)\left(1 - \frac{f_i(x_i,\tilde{v})}{f_i(x_i,v)}\right) = \left(\frac{v}{\tilde{v}} - \frac{v}{\tilde{v}}\right)\left(1 - \frac{\tilde{v}}{v}\right) = 0.$$

Thus, Assumptions A3 is satisfied. In this case, R_0 and R_1 are given by

$$R_0 = \sum_{i=1}^{2}\frac{k_i(e_i\alpha_i+b_i)}{a_i c(e_i+b_i)}\frac{\partial f_i(x_i^0,0)}{\partial v} = \sum_{i=1}^{2}\frac{k_i(e_i\alpha_i+b_i)}{a_i c(e_i+b_i)}\frac{\beta_i (x_i^0)^n}{\gamma_i^n + (x_i^0)^n},$$

$$R_1 = \sum_{i=1}^{2}\frac{k_i(e_i\alpha_i+b_i)f_i(\bar{x}_i,\bar{v})}{a_i c(e_i+b_i)\bar{v}} = \sum_{i=1}^{2}\frac{k_i(e_i\alpha_i+b_i)}{a_i c(e_i+b_i)}\frac{\beta_i \bar{x}_i^n}{\gamma_i^n + \bar{x}_i^n}.$$

6 Numerical simulations

In this section, we will perform some numerical simulations to confirm our theoretical results. Let us consider model (51)-(55) with the incidence rate $f_i(x_i, v) = \frac{\beta_i x_i^n v}{\gamma_i^n + x_i^n}$, $i = 1, 2$. In Table 1, we provide the values of some parameters of model (51)-(55) with the incidence rate given by the function f_i. The effect of the parameter ε on the dynamical behavior of the system will be discussed below in details.In order to investigate the theoretical

Table 1: The values of the parameters of model (51)-(55).

Parameter	λ_1	λ_2	$\bar{\beta}_1$	$\bar{\beta}_2$	d_1	d_2	α_1	α_2	e_1	e_2	b_1	b_2	γ_1
Value	6.03198	0.03198	0.05	0.08	0.01	0.01	0.5	0.5	0.02	0.02	0.2	0.2	0.1
Parameter	γ_2	k_1	k_2	a_1	a_2	f	r	c	μ	g	n	ε	
Value	0.5	10	5	0.3	0.1	0.3	0.5	3	0.07	0.1	1	Varied	

results involved in Theorems 7-9, we shall study the following cases:

Case (I): $R_0 \leq 1$. Choosing $\varepsilon = 0.85$ and using the data in Table 1, we have $R_0 = 0.899$ and $R_1 = 0.641$. Since $R_0 < 1$, then according to Theorem 7, the infection-free equilibrium E_0 is GAS. Evidently, Figures 1-8 show that, the numerical results are consistent with the theoretical results of Theorem 7. We can see that, the concentration of uninfected target cells tends to its normal value $\frac{\lambda_1}{d_1} = 603.198$, $\frac{\lambda_2}{d_2} = 3.198$, respectively, while the concentrations of latently infected cells, actively infected cells, free virus particles and antibody immune cells are decreasing and tend to zero. In this case, the treatment succeeded to eliminate the HIV viruses from the blood.

Case (II): $R_1 \leq 1$. By taking $\varepsilon = 0.40$, we have $R_1 = 0.915 < 1$ and E_1 exists where $E_1 = (601.504, 0.780, 0.038, 0.055, 0.054, 0.231, 0.565, 0.000)$. Based on Theorem 8, E_1 is GAS. Figures 1-8 show that the numerical simulations confirm our theoretical result presented in Theorem 8. We observe that, the trajectory of the system will converge to the chronic-infection equilibrium without antibody immune response E_1. In such situation, the infection becomes chronic but without antibody immune response.

Case (III): $R_1 > 1$.We choose, $\varepsilon = 0.0$. Then, we calculate $R_0 = 1.631$ and $R_1 = 1.149 > 1$, this means that, E_2 exists and it is GAS. From Figures 1-8, we can see that, our simulation results are consistent with the theoretical results of Theorem 9. We observe that, the trajectory of the system tend to the chronic-infection equilibrium with antibody immune response $E_2 = (599.699, 0.474, 0.079, 0.062, 0.111, 0.260, 0.700, 0.896)$. In this case, the infection becomes chronic but with persistent antibody immune response. Figures 1 and 7 demonstrate that, when $R_1 > 1$, the antibody immune response is activated and it reduces the concentration of free virus particles and increases the concentration of uninfected cells. In case (i) we calculate the critical drug efficacy (i.e, the efficacy needed in order stabilize the system around the disease-free equilibrium). For system (51)-(55), E_0 is GAS when $R_0 \leq 1$ i.e.

$$\varepsilon_1^{crit} \leq \varepsilon < 1, \qquad \varepsilon_1^{crit} = \max\left\{0, \frac{\bar{R}_0 - 1}{\bar{R}_{01} + f\bar{R}_{02}}\right\},$$

where, $\bar{R}_0 = R_0\,|_{\varepsilon=0}$ and $\bar{R}_{0i} = R_{0i}\,|_{\varepsilon=0}$, $i = 1, 2$. Using the data in Table 1, we have $\varepsilon_1^{crit} = 0.7332$. Also, in case (ii) we can calculate the critical drug efficacy $\varepsilon_2^{crit} = 0.2566$.

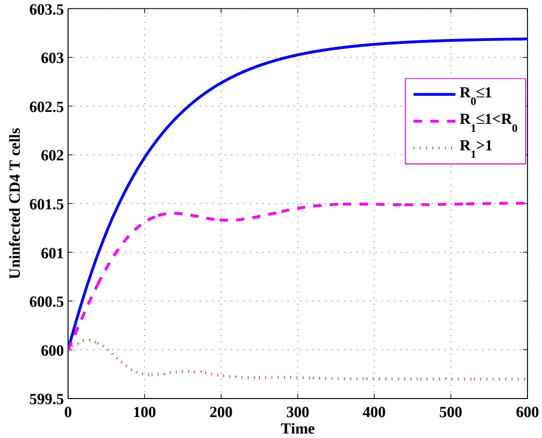

Figure 1: The evolution of uninfected CD4+T cells for model (51)-(55).

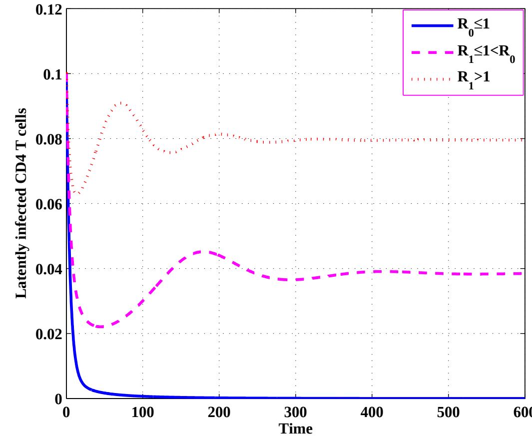

Figure 2: The evolution of uninfected macrophage cells for model (51)-(55).

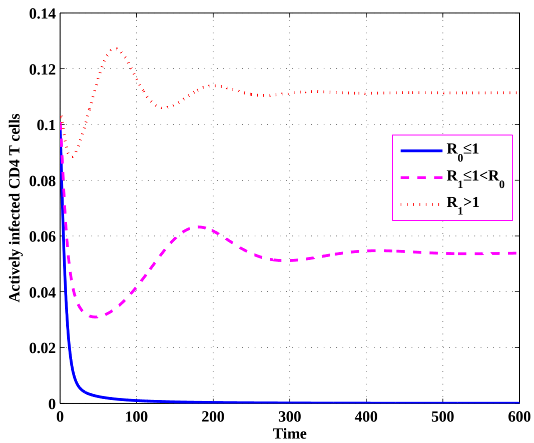

Figure 3: The evolution of actively infected CD4+T cells for model (51)-(55).

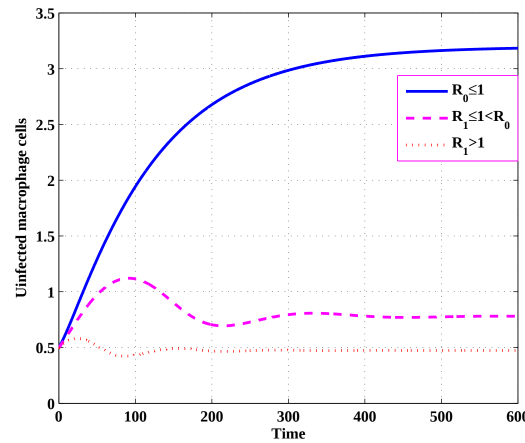

Figure 4: The evolution of uninfected macrophage cells for model (51)-(55).

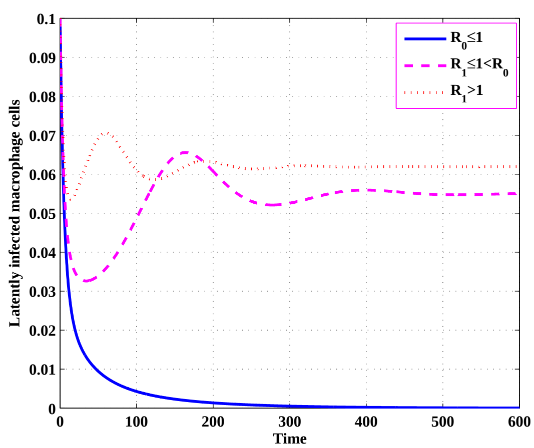

Figure 5: The evolution of latently infected macrophage cells for model (51)-(55).

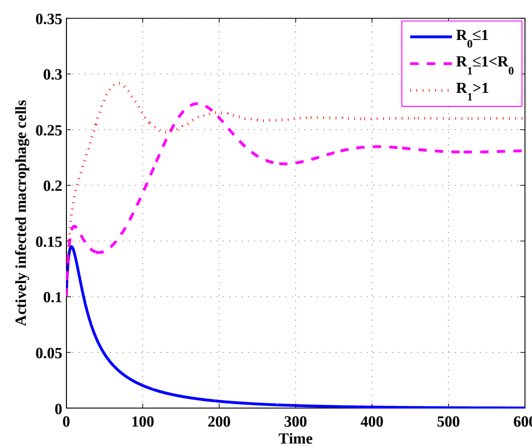

Figure 6: The evolution of actively infected macrophage cells for model (51)-(55).

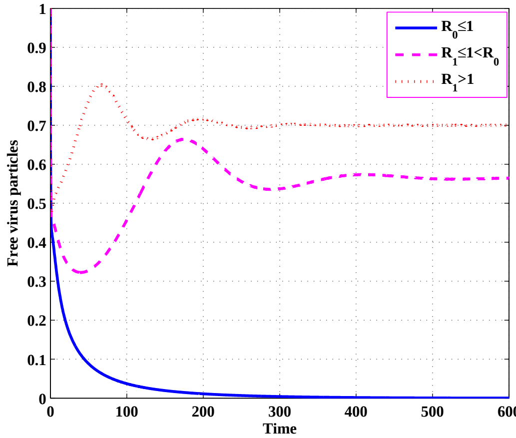

Figure 7: The evolution of free virus particles for model (51)-(55).

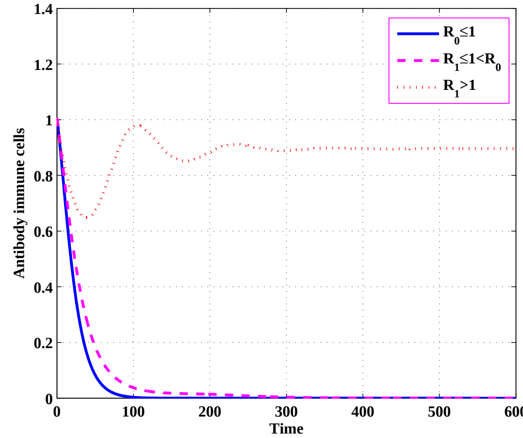

Figure 8: The evolution of antibody immune cells for model (51)-(55).

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A New Implicit Midpoint Iterative Scheme Involving Asymptotically Nonexpansive Mappings in Abstract Spaces

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Abstract

We establish the convergence properties of the implicit midpoint iterative scheme for solving the nonlinear equation $T\varrho = \varrho$ for asymptotically nonexpansive mappings in Hilbert and more general uniformly convex Banach spaces.

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Key words and phrases: asymptotically nonexpansive mappings, iterative scheme, Hilbert spaces, Banach spaces.

1 Introduction

In 2001, Xu and Ori [7] introduced the following implicit iteration process for a finite family of nonexpansive mappings $\{T_i : i \in I\}$ (here $I = \{1, 2, \ldots, N\}$), with $\{t_n\}$ a real

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sequence in $(0, 1)$, and an initial point $\varrho_0 \in K \subset X$, where X is an arbitrary Banach space:

$$\begin{aligned}
\varrho_1 &= (1-t_1)\varrho_0 + t_1 T_1 \varrho_1,\\
\varrho_2 &= (1-t_2)\varrho_1 + t_2 T_2 \varrho_2,\\
&\vdots\\
\varrho_N &= (1-t_N)\varrho_{N-1} + t_N T_N \varrho_N,\\
\varrho_{N+1} &= (1-t_{N+1})\varrho_N + t_{N+1} T_{N+1} \varrho_{N+1},\\
&\vdots,
\end{aligned}$$

which can be written in the following compact form:

$$\varrho_n = (1-t_n)\varrho_{n-1} + t_n T_n \varrho_n, \quad n \geq 1,$$

where $T_n = T_{n\,(mod\,N)}$ (here the $mod\,N$ function takes values in I). Xu and Ori [7] proved the weak convergence of this process to a common fixed point of the finite family defined in a Hilbert space.

Let H be the Hilbert space and T is, in general, a nonlinear operator. Recently Alghamdi et al. [1] defined the following algorithm:

Algorithm 1.1. *Initialize $\varrho_0 \in H$ arbitrarily and define*

$$\varrho_{n+1} = (1-t_n)\varrho_n + t_n T\left(\frac{\varrho_n + \varrho_{n+1}}{2}\right), \quad n \geq 0,$$

where $t_n \in (0, 1)$ for all n.

For the approximation of fixed points of nonexpansive mappings under the setting of Hilbert spaces. They proved the following results:

Lemma 1.2. ([1]) *Let $\{\varrho_n\}$ be the sequence generated by Algorithm 1.1. Then*

(i) $\|\varrho_{n+1} - p\| \leq \|\varrho_n - p\|$ *for all $n \geq 0$ and $p \in Fi\varrho(T)$,*

(ii) $\sum_{n=1}^{\infty} t_n \|\varrho_n - \varrho_{n+1}\|^2 < \infty$,

(iii) $\sum_{n=1}^{\infty} t_n(1-t_n)\|\varrho_n - T(\frac{\varrho_n+\varrho_{n+1}}{2})\|^2 < \infty$.

Lemma 1.3. ([1]) *Let $\{\varrho_n\}$ be the sequence generated by Algorithm I. Suppose that $t_{n+1}^2 \leq at_n$ for all $n \geq 0$ and $a > 0$. Then*

$$\lim_{n\to\infty} \|\varrho_{n+1} - \varrho_n\| = 0.$$

Lemma 1.4. ([1]) *Assume that,*

(i) $t_{n+1}^2 \leq at_n$ *for all $n \geq 0$ and $a > 0$,*

(ii) $\liminf_{n\to\infty} t_n > 0$.

Then the sequence $\{\varrho_n\}$ generated by Algorithm 1.1 satisfies the property

$$\lim_{n\to\infty} \|\varrho_n - T\varrho_n\| = 0.$$

Theorem 1.5. ([1]) *Let H be a Hilbert space and $T : H \to H$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Assume that $\{\varrho_n\}$ is generated by Algorithm 1.1, where the sequence $\{t_n\}$ of parameters satisfies the conditions:*

(i) $t_{n+1}^2 \leq a t_n$ *for all* $n \geq 0$ *and* $a > 0$,

(ii) $\limsup_{n\to\infty} t_n > 0$.

Then $\{\varrho_n\}$ converges weakly to a fixed point of T.

We establish the convergence properties of the implicit midpoint iterative scheme for solving the nonlinear equation $T\varrho = \varrho$ for asymptotically nonexpansive mappings in Hilbert and more general uniformly convex Banach, spaces.

2 Preliminaries

Throughout this section we always assume that H is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$ and that $T : H \to H$ is a nonexpansive mapping with a fixed point. We use $Fix(T)$ to denote the set of fixed points of T.

We establish the strong convergence of a new implicit midpoint iterative scheme for nonexpansive mappings under the setting of Hilbert and more general uniformly convex Banach spaces.

We need the following well known results:

Lemma 2.1. ([5]) *Let $\{\sigma_n\}$ and $\{\beta_n\}$ be sequences of nonnegative real numbers satisfying the following inequality*

$$\beta_{n+1} \leq (1 + \sigma_n)\,\beta_n, \quad n \geq 0.$$

If $\sum_{n=1}^{\infty} \sigma_n < \infty$, then $\lim_{n\to\infty} \beta_n$ exists.

Lemma 2.2. ([3]) *For all $\varrho, \varsigma \in H$ and $\lambda \in [0, 1]$, the following well-known identity holds:*

$$\|(1-\lambda)\varrho + \lambda\varsigma\|^2 = (1-\lambda)\|\varrho\|^2 + \lambda\|\varsigma\|^2 - \lambda(1-\lambda)\|\varrho - \varsigma\|^2.$$

For every ε with $0 \leq \varepsilon \leq 2$, we define the modulus $\delta(\varepsilon)$ of convexity of E by

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\|\varrho + \varsigma\|}{2} : \|\varrho\| \leq 1, \|\varsigma\| \leq 1, \|\varrho - \varsigma\| \geq \varepsilon,\ \varrho, \varsigma \in E\right\}.$$

The space E is said to be *uniformly convex* if

$$\delta(\varepsilon) > 0$$

for every $\varepsilon > 0$.

If E is uniformly convex, then for each r, ε with $r \geq \varepsilon > 0$, we have $\delta(\frac{\varepsilon}{r}) > 0$ and

$$\left\|\frac{\varrho + \varsigma}{2}\right\| \leq r\left(1 - \delta\left(\frac{\varepsilon}{r}\right)\right)$$

for every $\varrho, \varsigma \in E$ with $\|\varrho\| \leq r$, $\|\varsigma\| \leq r$ and $\|\varrho - \varsigma\| \geq \varepsilon$.

The space E is said to be *strictly convex* if

$$\left\|\frac{\varrho + \varsigma}{2}\right\| < 1$$

for every $\varrho, \varsigma \in E$ with $\|\varrho\| = \|\varsigma\| = 1$ and $\varrho \neq \varsigma$.

Lemma 2.3. ([6]) *Let X be the arbitrary Banach space and $p>1$, $r>0$ be two fixed numbers. Then X is uniformly convex if and only if there exists a continuous, strictly increasing and convex function $g:[0,\infty)\to[0,\infty)$, $g(0)=0$, such that*

$$\|\lambda\varrho+(1-\lambda)\varsigma\|^p \leq \lambda\|\varrho\|^p+(1-\lambda)\|\varsigma\|^p-w_p(\lambda)g(\|\varrho-\varsigma\|)$$

for all ϱ,ς in $B_r=\{\varrho\in X:\|\varrho\|\leq r\}$, $\lambda\in[0,1]$, where $w_p(\lambda)=\lambda(1-\lambda)^p+\lambda^p(1-\lambda)$.

3 Main results

Algorithm 3.1. *Initialize $\varrho_0\in H$ arbitrarily and define*

$$\varrho_n=(1-t_n)\frac{\varrho_{n-1}+\varrho_n}{2}+t_nT^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right),\quad n\geq 0,$$

where $t_n\in(0,1)$ for all n,

and T is asymptotically nonexpansive, that is,

$$\|T^n\varrho-T^n\varsigma\|\leq k_n\|\varrho-\varsigma\|,\quad \varrho,\varsigma\in H;$$

$\{k_n\}\in[0,\infty)$ satisfying $\sum_{n=1}^{\infty}(k_n-1)<\infty$.

Remark 3.2. The Algorithm 3.1 can be rewritten as

$$\varrho_n=e_n\varrho_{n-1}+(1-e_n)T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right),\quad n\geq 0,$$

where $e_n=\frac{1-t_n}{1+t_n}$.

Remark 3.3. The Algorithm 3.1 is well defined.

Indeed, for each fixed $u\in H$ and $t\in(0,1)$, the mapping

$$\varrho\mapsto T_u\varrho=tu+(1-t)T^n\left(\frac{u+\varrho}{2}\right),\quad n\geq 0,$$

is asymptotically nonexpansive with coefficient $\frac{1-t}{2}k_n\in[0,\infty)$. That is,

$$\begin{aligned}\|T_u\varrho-T_u\varsigma\|&=(1-t)\left\|T^n\left(\frac{u+\varrho}{2}\right)-T^n\left(\frac{u+\varsigma}{2}\right)\right\|\\&\leq\frac{1-t}{2}k_n\|\varrho-\varsigma\|,\quad \varrho,\varsigma\in H.\end{aligned}$$

Remark 3.4. Since $k_n\geq 1$, it is obvious that for any $q>0$, $\sum_{n=1}^{\infty}(k_n^q-1)<\infty$ implies $\sum_{n=1}^{\infty}(k_n-1)<\infty$.

Now we prove our main results.

Lemma 3.5. *The sequence $\{\varrho_n\}$ defined by the Algorithm 3.1, where $\{t_n\}\in(0,1)$ satisfying $\{t_n\}\in[\delta,1-\delta]$, is bounded.*

Proof. For $\varrho^* \in Fix(T)$, consider

$$\begin{aligned}
\|\varrho_n - \varrho^*\| &= \left\|(1-t_n)\frac{\varrho_{n-1}+\varrho_n}{2} + t_n T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right) - \varrho^*\right\| \\
&= \left\|(1-t_n)\left(\frac{\varrho_{n-1}+\varrho_n}{2} - \varrho^*\right) + t_n\left(T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right) - \varrho^*\right)\right\| \\
&\le (1-t_n)\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - \varrho^*\right\| + t_n\left\|T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right) - \varrho^*\right\| \\
&\le (1-t_n)\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - \varrho^*\right\| + t_n k_n\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - \varrho^*\right\| \\
&= (1-t_n+t_nk_n)\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - \varrho^*\right\| \\
&= (1-t_n+t_nk_n)\left\|\frac{1}{2}(\varrho_{n-1}-\varrho^*) + \frac{1}{2}(\varrho_n-\varrho^*)\right\| \\
&\le (1-t_n+t_nk_n)\left(\frac{1}{2}\|\varrho_{n-1}-\varrho^*\| + \frac{1}{2}\|\varrho_n-\varrho^*\|\right),
\end{aligned}$$

which implies that

$$\|\varrho_n - \varrho^*\| \le \frac{\frac{1}{2}(1-t_n+t_nk_n)}{1-\frac{1}{2}(1-t_n+t_nk_n)}\|\varrho_{n-1}-\varrho^*\|.$$

Let

$$\begin{aligned}
\frac{\frac{1}{2}(1-t_n+t_nk_n)}{1-\frac{1}{2}(1-t_n+t_nk_n)} &= 1 + \frac{t_n(k_n-1)}{1-\frac{1}{2}(1-t_n+t_nk_n)} \\
&= 1 + \frac{2t_n(k_n-1)}{1-t_n(k_n-1)}.
\end{aligned}$$

By $\sum_{n=1}^{\infty}(k_n-1) < \infty$, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, $k_n - 1 \le 1$ and

$$1 - t_n(k_n-1) \ge \delta,$$

which implies that

$$\frac{1}{1-t_n(k_n-1)} \le \frac{1}{\delta}.$$

Thus

$$\|\varrho_n - \varrho^*\| \le \left(1 + 2\frac{\delta}{1-\delta}(k_n-1)\right)\|\varrho_{n-1}-\varrho^*\|.$$

Hence according to Lemma 2.1, the sequence $\{\varrho_n\}$ is bounded. This completes the proof. □

Lemma 3.6. *Let $\{\varrho_n\}$ be the sequence generated by Algorithm 3.1 where $\{t_n\} \in (0,1)$ satisfying $\{t_n\} \in [\delta, 1-\delta]$. Then*

(i) $\lim_{n\to\infty}\|\varrho_{n-1}-\varrho_n\| = 0$,

(ii) $\lim_{n\to\infty}\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - T^n(\frac{\varrho_{n-1}+\varrho_n}{2})\right\| = 0$.

Proof. According to Lemma 2.2,

$$
\begin{aligned}
\|\varrho_n - \varrho^*\|^2 &= \left\|(1-t_n)\frac{\varrho_{n-1}+\varrho_n}{2} + t_n T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right) - \varrho^*\right\|^2 \\
&= \left\|(1-t_n)\left(\frac{\varrho_{n-1}+\varrho_n}{2} - \varrho^*\right) + t_n\left(T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right) - \varrho^*\right)\right\|^2 \\
&= (1-t_n)\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - \varrho^*\right\|^2 + t_n\left\|T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right) - \varrho^*\right\|^2 \\
&\quad - t_n(1-t_n)\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|^2 \\
&\le (1-t_n)\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - \varrho^*\right\|^2 + t_n k_n^2\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - \varrho^*\right\|^2 \\
&\quad - t_n(1-t_n)\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|^2 \\
&= (1-t_n+t_n k_n^2)\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - \varrho^*\right\|^2 \\
&\quad - t_n(1-t_n)\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|^2 \\
&\le (1-t_n+t_n k_n^2)\left(\frac{1}{2}\|\varrho_{n-1}-\varrho^*\|^2 + \frac{1}{2}\|\varrho_n-\varrho^*\|^2 - \frac{1}{4}\|\varrho_{n-1}-\varrho_n\|^2\right) \\
&\quad - t_n(1-t_n)\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|^2,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|\varrho_n - \varrho^*\|^2 \le& \frac{\frac{1}{2}(1-t_n+t_n k_n^2)}{1-\frac{1}{2}(1-t_n+t_n k_n^2)}\|\varrho_{n-1}-\varrho^*\|^2 \\
&- \frac{1}{4}\frac{(1-t_n+t_n k_n^2)}{1-\frac{1}{2}(1-t_n+t_n k_n^2)}\|\varrho_{n-1}-\varrho_n\|^2 \\
&- \frac{t_n(1-t_n)}{1-\frac{1}{2}(1-t_n+t_n k_n^2)}\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|^2.
\end{aligned}
$$

Let us assume that

$$
\begin{aligned}
\frac{\frac{1}{2}(1-t_n+t_n k_n^2)}{1-\frac{1}{2}(1-t_n+t_n k_n^2)} &= 1 + \frac{t_n(k_n^2-1)}{1-\frac{1}{2}(1-t_n+t_n k_n^2)} \\
&= 1 + \frac{2t_n(k_n^2-1)}{1-t_n(k_n^2-1)}.
\end{aligned}
$$

By $\sum_{n=1}^{\infty}(k_n^2-1) < \infty$, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, $k_n^2 - 1 \le 1$ and

$$1 - t_n(k_n^2-1) \ge \delta,$$

which implies that

$$\frac{1}{1-t_n(k_n^2-1)} \le \frac{1}{\delta}.$$

Also
$$1-t_n+t_nk_n^2=1+t_n(k_n^2-1)\geq 1$$
and
$$\begin{aligned}1-\frac{1}{2}(1-t_n+t_nk_n^2)&=1-\frac{1}{2}\left(1+t_n(k_n^2-1)\right)\\&=\frac{1}{2}\left(1-t_n(k_n^2-1)\right)\\&\leq\frac{1}{2},\end{aligned}$$
which yields that
$$\frac{1}{1-\frac{1}{2}(1-t_n+t_nk_n^2)}\geq 2.$$
Thus for $M>0$,
$$\begin{aligned}\|\varrho_n-\varrho^*\|^2&\leq\left(1+2\frac{\delta}{1-\delta}(k_n^2-1)\right)\|\varrho_{n-1}-\varrho^*\|^2-\frac{1}{2}\|\varrho_{n-1}-\varrho_n\|^2\\&\quad-2\delta^2\left\|\frac{\varrho_{n-1}+\varrho_n}{2}-T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|^2\\&\leq\|\varrho_{n-1}-\varrho^*\|^2+2M^2\frac{\delta}{1-\delta}(k_n^2-1)-\frac{1}{2}\|\varrho_{n-1}-\varrho_n\|^2\\&\quad-2\delta^2\left\|\frac{\varrho_{n-1}+\varrho_n}{2}-T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|^2,\end{aligned}$$
which implies that
$$\begin{aligned}&\frac{1}{2}\|\varrho_{n-1}-\varrho_n\|^2+2\delta^2\left\|\frac{\varrho_{n-1}+\varrho_n}{2}-T(\frac{\varrho_{n-1}+\varrho_n}{2})\right\|^2\\&\leq\|\varrho_{n-1}-\varrho^*\|^2-\|\varrho_n-\varrho^*\|^2+2M^2\frac{\delta}{1-\delta}(k_n^2-1).\end{aligned}$$
Thus
$$\begin{aligned}&\frac{1}{2}\sum_{j=1}^{m}\|\varrho_{j-1}-\varrho_j\|^2+2\delta^2\sum_{j=1}^{m}\left\|\frac{\varrho_{j-1}+\varrho_j}{2}-T^n\left(\frac{\varrho_{j-1}+\varrho_j}{2}\right)\right\|^2\\&\leq\sum_{j=1}^{m}\left(\|\varrho_{j-1}-\varrho^*\|^2-\|\varrho_j-\varrho^*\|^2+2M^2\frac{\delta}{1-\delta}(k_j^2-1)\right).\end{aligned}$$
Hence
$$\sum_{j=1}^{\infty}\|\varrho_{n-1}-\varrho_n\|^2<+\infty$$
and
$$\sum_{j=1}^{\infty}\left\|\frac{\varrho_{n-1}+\varrho_n}{2}-T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|^2<+\infty.$$
It implies that
$$\lim_{n\to\infty}\|\varrho_{n-1}-\varrho_n\|=0$$

and

$$\lim_{n\to\infty}\left\|\frac{\varrho_{n-1}+\varrho_n}{2}-T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|=0.$$

This completes the proof. □

Lemma 3.7. *Let $\{\varrho_n\}$ be the sequence generated by Algorithm 3.1, where $\{t_n\}\in(0,1)$ satisfying $\{t_n\}\in[\delta,1-\delta]$. Then $\lim_{n\to\infty}\|\varrho_n-T\varrho_n\|=0$.*

Proof. Consider

$$\begin{aligned}
\|\varrho_n-T^n\varrho_n\| &\le \left\|\varrho_n-\frac{\varrho_{n-1}+\varrho_n}{2}\right\|+\left\|\frac{\varrho_{n-1}+\varrho_n}{2}-T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\| \\
&\quad+\left\|T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)-T^n\varrho_n\right\| \\
&\le \left\|\varrho_n-\frac{\varrho_{n-1}+\varrho_n}{2}\right\|+\left\|\frac{\varrho_{n-1}+\varrho_n}{2}-T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\| \\
&\quad+k_n\left\|\varrho_n-\frac{\varrho_{n-1}+\varrho_n}{2}\right\| \\
&=(1+k_n)\left\|\varrho_n-\frac{\varrho_{n-1}+\varrho_n}{2}\right\|+\left\|\frac{\varrho_{n-1}+\varrho_n}{2}-T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\| \\
&=\frac{1+k_n}{2}\|\varrho_{n-1}-\varrho_n\|+\left\|\frac{\varrho_{n-1}+\varrho_n}{2}-T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\| \\
&\to 0 \quad \text{as } n\to\infty
\end{aligned}$$

and

$$\begin{aligned}
\|\varrho_n-T\varrho_n\| &\le \|\varrho_n-T^n\varrho_n\|+\|T^n\varrho_n-T^n\varrho_{n-1}\|+\|T^n\varrho_{n-1}-T\varrho_n\| \\
&\le \|\varrho_n-T^n\varrho_n\|+k_n\|\varrho_n-\varrho_{n-1}\|+k_1\|T^{n-1}\varrho_{n-1}-\varrho_n\| \\
&\le \|\varrho_n-T^n\varrho_n\|+k_n\|\varrho_n-\varrho_{n-1}\| \\
&\quad+k_1\left(\|T^{n-1}\varrho_{n-1}-\varrho_{n-1}\|+\|\varrho_{n-1}-\varrho_n\|\right) \\
&\to 0 \quad \text{as } n\to\infty.
\end{aligned}$$

This completes the proof. □

Theorem 3.8. *Let $T:H\to H$ be asymptotically nonexpansive. For arbitrary $\varrho_0\in K$, generate the sequence $\{\varrho_n\}$ by the Algorithm 3.1. If T is completely continuos, then $\{\varrho_n\}$ converges strongly to some fixed point of T in H.*

Proof. From Lemma 3.7, $\lim\limits_{n\to\infty}\|\varrho_n-T\varrho_n\|=0$. Therefore, there exists a subsequence $\{\varrho_{n_j}\}$ of $\{\varrho_n\}$ such that $\lim_{j\to\infty}\|\varrho_{n_j}-T\varrho_{n_j}\|=0$. Since $\{\varrho_{n_j}\}$ is bounded and T is completely continuous, then $\{T\varrho_{n_j}\}$ has a subsequence $\{T\varrho_{n_{j_k}}\}$ which converges strongly. Hence $\{\varrho_{n_{j_k}}\}$ converges strongly. Let $\lim_{j\to\infty}\varrho_{n_{j_k}}=p$. Then $\lim_{j\to\infty}T\varrho_{n_{j_k}}=Tp$. Thus we have $\lim_{j\to\infty}\|\varrho_{n_{j_k}}-T\varrho_{n_{j_k}}\|=\|p-Tp\|=0$. Hence $p\in F(T)$. From Lemma 2.1 and Lemma 3.7 it follows that $\lim_{n\to\infty}\|\varrho_n-p\|=0$. This completes the proof. □

Lemma 3.9. *Let E be the uniformly convex Banach space and $T : E \to E$ be asymptotically nonexpansive mapping. Let $\{\varrho_n\} \in E$ be the sequence generated by Algorithm 3.1 and $\{t_n\} \in (0,1)$ satisfying $\{t_n\} \in [\delta, 1-\delta]$. Then*

(i) $\lim_{n\to\infty} \|\varrho_{n-1} - \varrho_n\| = 0$,

(ii) $\lim_{n\to\infty} \left\|\frac{\varrho_{n-1}+\varrho_n}{2} - T^n(\frac{\varrho_{n-1}+\varrho_n}{2})\right\| = 0$.

Proof. According to Lemma 2.3,

$$\begin{aligned}
\|\varrho_n - \varrho^*\|^p &= \left\|(1-t_n)\frac{\varrho_{n-1}+\varrho_n}{2} + t_n T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right) - \varrho^*\right\|^p \\
&= \left\|(1-t_n)\left(\frac{\varrho_{n-1}+\varrho_n}{2} - \varrho^*\right) + t_n\left(T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right) - \varrho^*\right)\right\|^p \\
&\le (1-t_n)\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - \varrho^*\right\|^p + t_n\left\|T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right) - \varrho^*\right\|^p \\
&\quad - w_p(t_n) g\left(\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|\right) \\
&\le (1-t_n)\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - \varrho^*\right\|^p + t_n k_n^p\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - \varrho^*\right\|^p \\
&\quad - w_p(t_n) g\left(\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|\right) \\
&= (1-t_n+t_n k_n^p)\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - \varrho^*\right\|^p \\
&\quad - w_p(t_n) g\left(\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|\right),
\end{aligned}$$

where

$$\begin{aligned}
\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - \varrho^*\right\|^p &= \left\|\frac{1}{2}(\varrho_{n-1} - \varrho^*) + \frac{1}{2}(\varrho_n - \varrho^*)\right\|^p \\
&\le \left[\frac{1}{2}\|\varrho_{n-1} - \varrho^*\| + \frac{1}{2}\|\varrho_n - \varrho^*\|\right]^p \\
&\le \frac{1}{2}\|\varrho_{n-1} - \varrho^*\|^p + \frac{1}{2}\|\varrho_n - \varrho^*\|^p.
\end{aligned}$$

Thus

$$\begin{aligned}
\|\varrho_n - \varrho^*\|^p \le{}& (1-t_n+t_n k_n^p)\left(\frac{1}{2}\|\varrho_{n-1} - \varrho^*\|^p + \frac{1}{2}\|\varrho_n - \varrho^*\|^p\right) \\
&- w_p(t_n) g\left(\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|\right),
\end{aligned}$$

which implies that

$$\begin{aligned}
\|\varrho_n - \varrho^*\|^p \le{}& \frac{\frac{1}{2}(1-t_n+t_n k_n^p)}{1-\frac{1}{2}(1-t_n+t_n k_n^p)}\|\varrho_{n-1} - \varrho^*\|^p \\
&- \frac{w_p(t_n)}{1-\frac{1}{2}(1-t_n+t_n k_n^p)} g\left(\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|\right).
\end{aligned}$$

Let us assume that

$$\begin{aligned}\frac{\frac{1}{2}(1-t_n+t_nk_n^p)}{1-\frac{1}{2}(1-t_n+t_nk_n^p)} &= 1+\frac{t_n(k_n^p-1)}{1-\frac{1}{2}(1-t_n+t_nk_n^p)}\\ &= 1+\frac{2t_n(k_n^p-1)}{1-t_n(k_n^p-1)}.\end{aligned}$$

By $\sum_{n=1}^{\infty}(k_n^p-1)<\infty$, there exists $n_0\in\mathbb{N}$ such that for all $n\geq n_0$, $k_n^p-1\leq 1$, and

$$1-t_n(k_n^p-1)\geq\delta,$$

which implies that

$$\frac{1}{1-t_n(k_n^p-1)}\leq\frac{1}{\delta}.$$

Also

$$\begin{aligned}1-\frac{1}{2}(1-t_n+t_nk_n^p) &= 1-\frac{1}{2}\left(1+t_n(k_n^p-1)\right)\\ &= \frac{1}{2}\left(1-t_n(k_n^p-1)\right)\\ &\leq\frac{1}{2},\end{aligned}$$

which yields that

$$\frac{1}{1-\frac{1}{2}(1-t_n+t_nk_n^p)}\geq 2.$$

Hence

$$\begin{aligned}\|\varrho_n-\varrho^*\|^p \leq{}& \left(1+2\frac{\delta}{1-\delta}(k_n^p-1)\right)\|\varrho_{n-1}-\varrho^*\|^p\\ &-4\delta^{p+1}g\left(\left\|\frac{\varrho_{n-1}+\varrho_n}{2}-T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|\right).\end{aligned}$$

For $M>0$,

$$\begin{aligned}\|\varrho_n-\varrho^*\|^p \leq{}& \|\varrho_{n-1}-\varrho^*\|^p+2M^p\frac{\delta}{1-\delta}(k_n^p-1)\\ &-4\delta^{p+1}g\left(\left\|\frac{\varrho_{n-1}+\varrho_n}{2}-T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\|\right),\end{aligned}$$

which implies that

$$\begin{aligned}&4\delta^{p+1}g\left(\left\|\frac{\varrho_{n-1}+\varrho_n}{2}-T^n(\frac{\varrho_{n-1}+\varrho_n}{2})\right\|\right)\\ &\leq\|\varrho_{n-1}-\varrho^*\|^p-\|\varrho_n-\varrho^*\|^p+2M^p\frac{\delta}{1-\delta}(k_n^p-1).\end{aligned}$$

Thus

$$\begin{aligned}&4\delta^{p+1}\sum_{j=1}^{m}g\left(\left\|\frac{\varrho_{n-1}+\varrho_n}{2}-T^n(\frac{\varrho_{n-1}+\varrho_n}{2})\right\|\right)\\ &\leq\sum_{j=1}^{m}\left(\|\varrho_{j-1}-\varrho^*\|^p-\|\varrho_j-\varrho^*\|^p+2M^p\frac{\delta}{1-\delta}(k_n^p-1)\right).\end{aligned}$$

Hence

$$\sum_{j=1}^{\infty} g\left(\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - T^n(\frac{\varrho_{n-1}+\varrho_n}{2})\right\|\right) < +\infty.$$

It implies that

$$\lim_{n\to\infty}\left\|\frac{\varrho_{n-1}+\varrho_n}{2} - T^n\left(\frac{\varrho_{n-1}+\varrho_n}{2}\right)\right\| = 0.$$

From this, it can be easily see that

$$\lim_{n\to\infty}\|\varrho_{n-1}-\varrho_n\| = 0.$$

This completes the proof. □

Lemma 3.10. *Let E and T as in Lemma 3.9. Let $\{\varrho_n\}$ be the sequence generated by Algorithm 3.1, where $\{t_n\} \in (0,1)$ satisfying $\{t_n\} \in [\delta, 1-\delta]$. Then* $\lim_{n\to\infty}\|\varrho_n - T\varrho_n\| = 0$.

Theorem 3.11. *Let E and T as in Lemma 3.9. For arbitrary $\varrho_0 \in K$, generate the sequence $\{\varrho_n\}$ by the Algorithm 3.1. If T is completely continuos, then $\{\varrho_n\}$ converges strongly to some fixed point of T in E.*

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Hesitant fuzzy filters in lattice implication algebras

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Abstract. The notion of hesitant fuzzy filters in lattice implication algebras is introduced, and several properties are investigated. Characterizations of hesitant fuzzy filters are discussed.

1. **Introduction**

In the field of many-valued logic, lattice-valued logic plays an important role for two aspects: One is that it extends the chain-type truth-value field of some well-known presented logic [1] to some relatively general lattices. The other is that the incompletely comparable property of truth value characterized by general lattice can more efficiently reflect the uncertainty of people's thinking, judging and decision. Hence, lattice-valued logic is becoming a research field which strongly influences the development of Algebraic Logic, Computer Science and Artificial Intelligence Technology. Therefore Goguen, Novak and Pavelka researched on this lattice-valued logic formal systems (see [2, 10, 11]). In order to research the logical system whose propositional value is given in a lattice, Xu [12] proposed the concept of lattice implication algebras, and discussed their some properties. For the general development of lattice implication algebras, filter theory and its fuzzification play an important role. Xu and Qin [14] introduced the notion of (implicative) filters in a lattice implication algebra, and investigated their properties. Jun (together with Xu and Qin) [3, 9] discussed positive implicative and associative filters of a lattice implication algebra, and Jun [4] considered the fuzzification of positive implicative and associative filters of a lattice implication algebra. In [13], Xu and Qin considered the fuzzification of (implicative) filters.

Torra [16] introduced the hesitant fuzzy set which is a useful generalization of the fuzzy set that is designed for situations in which it is difficult to determine the membership of an element to a set owing to ambiguity between a few different values. The hesitant fuzzy set permits the

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membership degree of an element to a set to be represented by a set of possible values between 0 and 1 (see [16] and [17]). Jun et al. applied the notion of hesitant fuzzy sets to semigroups, MTL-algebras and EQ-algebras (see [5, 6, 7, 8]).

In this paper, we apply the notion of hesitant fuzzy sets to the filter theory in lattice implication algebras. We introduce the concept of hesitant fuzzy filters in lattice implication algebras, and investigate several properties. We discuss characterizations of hesitant fuzzy filters.

2. Preliminaries

By a *lattice implication algebra* we mean a bounded lattice $L := (L, \vee, \wedge, 0, 1)$ with order-reversing involution " $'$ " and a binary operation " $\to$ " satisfying the following axioms:

(I1) $x \to (y \to z) = y \to (x \to z)$,
(I2) $x \to x = 1$,
(I3) $x \to y = y' \to x'$,
(I4) $x \to y = y \to x = 1 \Rightarrow x = y$,
(I5) $(x \to y) \to y = (y \to x) \to x$,
(L1) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$,
(L2) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$,

for all $x, y, z \in L$. We define a relation $\leq$ on a lattice implication algebra L by $x \leq y$ if and only if $x \to y = 1$.

In a lattice implication algebra L, the following hold (see [12]):

(a1) $0 \to x = 1$, $1 \to x = x$ and $x \to 1 = 1$.
(a2) $x \to y \leq (y \to z) \to (x \to z)$.
(a3) $x \leq y$ implies $y \to z \leq x \to z$ and $z \to x \leq z \to y$.
(a4) $x' = x \to 0$.
(a5) $x \vee y = (x \to y) \to y$.
(a6) $((y \to x) \to y')' = x \wedge y = ((x \to y) \to x')'$.
(a7) $x \leq (x \to y) \to y$

where $x \leq y$ means $x \to y = 1$.

A subset F of a lattice implication algebra L is called a *filter* of L (see [14]) if it satisfies:

(F1) $1 \in F$,
(F2) $x \in F$ and $x \to y \in F$ imply $y \in F$

for all $x, y \in L$.

Let L be a reference set. Then we define hesitant fuzzy set on L in terms of a function $\mathcal{H}$ that when applied to X returns a subset of $[0, 1]$ (see [16]).

For a hesitant fuzzy set $\mathcal{H}$ on L and $x, y, z \in L$, we use the notations $\mathcal{H}_x := \mathcal{H}(x)$, $\mathcal{H}_x^y := \mathcal{H}(x) \cap \mathcal{H}(y)$, $\mathcal{H}_x(\varepsilon) := \mathcal{H}(x) \cap \varepsilon$ and $\mathcal{H}_x^y(\varepsilon) := \mathcal{H}(x) \cap \mathcal{H}(y) \cap \varepsilon$ where $\varepsilon \in \mathscr{P}([0,1])$. It is clear that $\mathcal{H}_x^y = \mathcal{H}_y^x$, $\mathcal{H}_x^y(\varepsilon) \subseteq \mathcal{H}_x(\varepsilon)$ and

$$\mathcal{H}_x = \mathcal{H}_y \;\Leftrightarrow\; \mathcal{H}_x \subseteq \mathcal{H}_y,\ \mathcal{H}_y \subseteq \mathcal{H}_x$$

for all $x, y \in L$.

For a hesitant fuzzy set $\mathcal{H}$ on L and a subset ε of $[0,1]$, the set

$$L(\mathcal{H};\varepsilon) := \{x \in L \mid \varepsilon \subseteq \mathcal{H}_x\},$$

is called the hesitant level set of $\mathcal{H}$.

3. Hesitant fuzzy filters

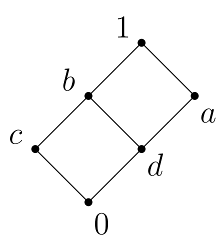
In what follows, we take a lattice implication algebra L as a reference set unless otherwise specified.

Definition 3.1. A hesitant fuzzy set $\mathcal{H}$ on L is a hesitant fuzzy filter of L if it satisfies the following assertions.

$$(\forall x \in L)\,(\mathcal{H}_1 \supseteq \mathcal{H}_x)\,, \tag{3.1}$$

$$(\forall x, y \in L)\,\big(\mathcal{H}_y \supseteq \mathcal{H}_{x\to y)}^x\big)\,. \tag{3.2}$$

Example 3.2. Let $L = \{0, a, b, c, d, 1\}$ be a set with the following Hasse diagram and Cayley tables:



x	x'
0	1
a	c
b	d
c	a
d	b
1	0

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

Then L is a lattice implication algebra (see [15]). Let $\mathcal{H}$ be a hesitant fuzzy set on L which is given as follows:

$$\mathcal{H} : L \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} [0.2, 0.8] & \text{if } x \in \{a, 1\}, \\ [0.3, 0.7] & \text{otherwise.} \end{cases}$$

Then $\mathcal{H}$ is a hesitant fuzzy filter of L.

Theorem 3.3. *A hesitant fuzzy set $\mathcal{H}$ on L is a hesitant fuzzy filter of L if and only if the hesitant level set $L(\mathcal{H};\varepsilon)$ of $\mathcal{H}$ is a filter of L for all $\varepsilon \in \mathscr{P}([0,1])$ with $L(\mathcal{H};\varepsilon) \neq \emptyset$.*

Proof. Assume that $\mathcal{H}$ is a hesitant fuzzy filter of L. Let $\varepsilon \in \mathscr{P}([0,1])$ be such that $L(\mathcal{H};\varepsilon) \neq \emptyset$. Then there exists $a \in L(\mathcal{H};\varepsilon)$, and so $\mathcal{H}_a \supseteq \varepsilon$. It follows from (3.1) that $\mathcal{H}_1 \supseteq \mathcal{H}_a \supseteq \varepsilon$ and so that $1 \in L(\mathcal{H};\varepsilon)$. Let $x, y \in L$ be such that $x \in L(\mathcal{H};\varepsilon)$ and $x \to y \in L(\mathcal{H};\varepsilon)$. Then $\varepsilon \subseteq \mathcal{H}_x$ and $\varepsilon \subseteq \mathcal{H}_{x\to y}$. Using (3.2), we get $\mathcal{H}_y \supseteq \mathcal{H}^x_{x\to y} \supseteq \varepsilon$. Thus $y \in L(\mathcal{H};\varepsilon)$, and hence $L(\mathcal{H};\varepsilon)$ is a filter of L for all $\varepsilon \in \mathscr{P}([0,1])$ with $L(\mathcal{H};\varepsilon) \neq \emptyset$.

Conversely, suppose that the nonempty hesitant level set $L(\mathcal{H};\varepsilon)$ of $\mathcal{H}$ is a filter of L for all $\varepsilon \in \mathscr{P}([0,1])$. For any $x \in L$, let $\mathcal{H}_x = \varepsilon_x$. Then $x \in L(\mathcal{H};\varepsilon_x)$, and so $L(\mathcal{H};\varepsilon_x) \neq \emptyset$. Hence $1 \in L(\mathcal{H};\varepsilon_x)$, and thus $\mathcal{H}_1 \supseteq \varepsilon_x = \mathcal{H}_x$ for all $x \in L$. For any $x, y \in L$, let $\mathcal{H}^x_{x\to y} = \delta$. Then $\mathcal{H}_x \supseteq \delta$ and $\mathcal{H}_{x\to y} \supseteq \delta$, that is, $x \in L(\mathcal{H};\delta)$ and $x \to y \in L(\mathcal{H};\delta)$. It follows from (F2) that $y \in L(\mathcal{H};\delta)$ and so that $\mathcal{H}_y \supseteq \delta = \mathcal{H}^x_{x\to y}$ for all $x, y \in L$. Therefore $\mathcal{H}$ is a hesitant fuzzy filter of L. □

Proposition 3.4. *Every hesitant fuzzy filter $\mathcal{H}$ of L satisfies:*

$$(\forall x, y \in L)\,(x \leq y \;\Rightarrow\; \mathcal{H}_x \subseteq \mathcal{H}_y)\,. \tag{3.3}$$

Proof. Let $x, y \in L$ satisfy $x \leq y$. Then $x \to y = 1$, and so

$$\mathcal{H}_y \supseteq \mathcal{H}^x_{x\to y} = \mathcal{H}^x_1 = \mathcal{H}_x$$

by (3.2) and (3.1). □

Theorem 3.5. *A hesitant fuzzy set $\mathcal{H}$ on L is a hesitant fuzzy filter of L if and only if it satisfies* (3.1) *and*

$$(\forall x, y, z \in L)\left(\mathcal{H}_{x\to z} \supseteq \mathcal{H}^{x\to y}_{y\to z}\right). \tag{3.4}$$

Proof. Assume that $\mathcal{H}$ is a hesitant fuzzy filter of L. Since $x \to y \leq (y \to z) \to (x \to z)$ for all $x, y, z \in L$, it follows from (3.3) that $\mathcal{H}_{x\to y} \subseteq \mathcal{H}_{(y\to z)\to(x\to z)}$ and so from (3.2) that

$$\mathcal{H}_{x\to z} \supseteq \mathcal{H}^{y\to z}_{(y\to z)\to(x\to z)} \supseteq \mathcal{H}^{y\to z}_{x\to y}$$

for all $x, y, z \in L$.

Conversely, let $\mathcal{H}$ satisfy (3.1) and (3.4). Taking $x = 1$ in (3.4) and using (a1), we have

$$\mathcal{H}_z = \mathcal{H}_{1\to z} \supseteq \mathcal{H}^{1\to y}_{y\to z} = \mathcal{H}^y_{y\to z}$$

for all $y, z \in L$. Therefore $\mathcal{H}$ is a hesitant fuzzy filter of L. □

Theorem 3.6. *For any hesitant fuzzy set $\mathcal{H}$ on L, the following assertions are equivalent.*

(1) *$\mathcal{H}$ is a hesitant fuzzy filter of L.*
(2) $(\forall x, y, z \in L)\left(x \leq y \to z \;\Rightarrow\; \mathcal{H}_z \supseteq \mathcal{H}^x_y\right)$.

Proof. Suppose that $\mathcal{H}$ is a hesitant fuzzy filter of L. Let $x, y, z \in L$ satisfy $x \leq y \to z$. Using (3.2) and (3.3) implies that $\mathcal{H}_z \supseteq \mathcal{H}^y_{y\to z} \supseteq \mathcal{H}^y_x$.

Assume that the second condition is valid. Since $x \leq x \to 1$ for all $x \in L$, we have $\mathcal{H}_1 \supseteq \mathcal{H}^x_x = \mathcal{H}_x$ for all $x \in L$. Note that $y \leq (y \to x) \to x$ for all $x, y \in L$. Hence $\mathcal{H}_x \supseteq \mathcal{H}^y_{y\to x}$ for all $x, y \in L$. Therefore $\mathcal{H}$ is a hesitant fuzzy filter of L. □

Theorem 3.7. *A hesitant fuzzy set $\mathcal{H}$ on L is a hesitant fuzzy filter of L if and only if it satisfies* (3.1)*,* (3.3) *and*

$$(\forall x, y \in L)\left(\mathcal{H}_{(x\to y')'} \supseteq \mathcal{H}^x_y\right). \tag{3.5}$$

Proof. Assume that $\mathcal{H}$ is a hesitant fuzzy filter of L. Then the conditions (3.1) and (3.3) are valid by Definition 3.1 and Proposition 3.4. Using (3.1), (3.2) and (I2), we have

$$\begin{aligned}\mathcal{H}_{(x\to y')'} &\supseteq \mathcal{H}^y_{y\to(x\to y')'} \supseteq \mathcal{H}^y_x(x \to (y \to (x \to y')'))\\ &= \mathcal{H}^y_x((x \to y')' \to (x \to y')')\\ &= \mathcal{H}^x_y(1) = \mathcal{H}^x_y\end{aligned}$$

for all $x, y \in L$. Hence (3.5) is valid.

Conversely, let $\mathcal{H}$ satisfy conditions (3.1), (3.3) and (3.5). Note that

$$(x \to (x \to y)')' \leq y$$

for all $x, y \in L$. It follows from (3.3) and (3.5) that

$$\mathcal{H}_y \supseteq \mathcal{H}_{(x\to(x\to y)')'} \supseteq \mathcal{H}^x_{x\to y}$$

for all $x, y \in L$. Therefore $\mathcal{H}$ is a hesitant fuzzy filter of L by Theorem 3.3. □

Theorem 3.8. *A hesitant fuzzy set $\mathcal{H}$ on L is a hesitant fuzzy filter of L if and only if it satisfies* (3.1) *and*

$$(\forall x, y, z \in L)\left(\mathcal{H}_{z\to x} \supseteq \mathcal{H}^y_{(z\to y)\to x}\right). \tag{3.6}$$

Proof. Suppose that $\mathcal{H}$ is a hesitant fuzzy filter of L. Let $x, y, z \in L$. Since $x \leq z \to x$ and $y \leq z \to y$, we have

$$(z \to y) \to x \leq (z \to y) \to (z \to x) \leq y \to (z \to x).$$

It follows from (3.2) and (3.3) that

$$\mathcal{H}_{z\to x} \supseteq \mathcal{H}^y_{y\to(z\to x)} \supseteq \mathcal{H}^y_{(z\to y)\to x}.$$

Hence (3.6) is valid.

Conversely, let $\mathcal{H}$ satisfy conditions (3.1) and (3.6). If we take $z = 1$ in (3.6) and use (a1), then

$$\mathcal{H}_x = \mathcal{H}_{1\to x} \supseteq \mathcal{H}^y_{(1\to y)\to x} = \mathcal{H}^y_{y\to x}$$

for all $x, y \in L$. Therefore $\mathcal{H}$ is a hesitant fuzzy filter of L. □

Let $\mathcal{H}$ be a hesitant fuzzy set on L and $a \in L$. We consider the set

$$\mathcal{H}^{\to}_a := \{x \in L \mid \mathcal{H}_a \subseteq \mathcal{H}_x\}.$$

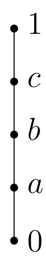
Obviously, $a \in \mathcal{H}^{\to}_a$. If $\mathcal{H}$ is a hesitant fuzzy filter of L , then $1 \in \mathcal{H}^{\to}_a$ since $\mathcal{H}_1 \supseteq \mathcal{H}_x$ for all $x \in L$.

Let $\mathcal{H}$ satisfy the condition (3.1). Then there exists $a \in L$ such that $\mathcal{H}^{\to}_a$ is not a filter of L as seen in the following example.

Example 3.9. Consider the set $L = \{a_i \mid i = 1, 2, \cdots, n\}$. For any $1 \leq j, k \leq n$, define

$$\begin{aligned} a_j \vee a_k &= a_{\max\{j,k\}},\\ a_j \wedge a_k &= a_{\min\{j,k\}},\\ (a_j)' &= a_{n-j+1},\\ a_j \to a_k &= a_{\min\{n-j+k,n\}}. \end{aligned}$$

Then $(L, \vee, \wedge,', \to)$ is a lattice implication algebra which is called the Łukasiewicz implication algebra (of order n) (see [15]). The Łukasiewicz implication algebra $L = \{0, a, b, c, 1\}$ of order 5 is represented by



x	x'
0	1
a	c
b	b
c	a
1	0

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	c	1	1	1
c	a	b	c	1	1
1	0	a	b	c	1

Let $\mathcal{H}$ be a hesitant fuzzy set on L defined by

$$\mathcal{H} : L \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} (0.2, 0.3) \cup (0.6, 0.8] & \text{if } x \in \{0, c\},\\ [0.1, 0.3) \cup (0.5, 0.9) & \text{if } x = a,\\ [0.2, 0.3) \cup [0.6, 0.9) & \text{if } x = b,\\ [0.1, 0.3] \cup [0.5, 0.9] & \text{if } x = 1. \end{cases}$$

Then $\mathcal{H}^{\to}_b = \{a, b, 1\}$ is not a filter of L since $a \to c = 1 \in \mathcal{H}^{\to}_b$ and $a \in \mathcal{H}^{\to}_b$, but $c \notin \mathcal{H}^{\to}_b$.

We provide conditions for the set $\mathcal{H}^{\to}_a$ to be a filter of L for $a \in L$.

Theorem 3.10. *Let $a \in L$. If $\mathcal{H}$ is a hesitant fuzzy filter of L , then $\mathcal{H}^{\to}_a$ is a filter of L.*

Proof. Obviously $1 \in \mathcal{H}_a^{\rightarrow}$ by (3.1). Let $x, y \in L$ satisfy $x \rightarrow y \in \mathcal{H}_a^{\rightarrow}$ and $x \in \mathcal{H}_a^{\rightarrow}$. Then $\mathcal{H}_{x\rightarrow y} \supseteq \mathcal{H}_a$ and $\mathcal{H}_x \supseteq \mathcal{H}_a$. It follows from (3.2) that

$$\mathcal{H}_y \supseteq \mathcal{H}_{x\rightarrow y}^x \supseteq \mathcal{H}_a.$$

Thus $y \in \mathcal{H}_a^{\rightarrow}$ and $\mathcal{H}_a^{\rightarrow}$ is a filter of L. $\square$

Theorem 3.11. *For any $a \in L$ and a hesitant fuzzy set $\mathcal{H}$ on L, we have the following assertions:*

(1) *If $\mathcal{H}_a^{\rightarrow}$ is a filter of L, then $\mathcal{H}$ satisfies the following implication.*

$$(\forall x, y \in L) \left(\mathcal{H}_a \subseteq \mathcal{H}_{x\rightarrow y}^x \;\Rightarrow\; \mathcal{H}_a \subseteq \mathcal{H}_y \right). \tag{3.7}$$

(2) *If $\mathcal{H}$ satisfies* (3.1) *and* (3.7)*, then $\mathcal{H}_a^{\rightarrow}$ is a filter of L.*

Proof. (1) Assume that $\mathcal{H}_a^{\rightarrow}$ is a filter of L for $a \in L$. Let $x, y \in L$ be such that

$$\mathcal{H}_a \subseteq \mathcal{H}_{x\rightarrow y}^x.$$

Then $x \rightarrow y \in \mathcal{H}_a^{\rightarrow}$ and $x \in \mathcal{H}_a^{\rightarrow}$. Since $\mathcal{H}_a^{\rightarrow}$ is a filter of L, it follows that $y \in \mathcal{H}_a^{\rightarrow}$, that is, $\mathcal{H}_a \subseteq \mathcal{H}_y$.

(2) Suppose that $\mathcal{H}$ satisfies (3.1) and (3.7). Let $x, y \in L$ be such that $x \rightarrow y \in \mathcal{H}_a^{\rightarrow}$ and $x \in \mathcal{H}_a^{\rightarrow}$. Then $\mathcal{H}_a \subseteq \mathcal{H}_{x\rightarrow y}$ and $\mathcal{H}_a \subseteq \mathcal{H}_x$, which implies that $\mathcal{H}_a \subseteq \mathcal{H}_{x\rightarrow y}^x$. It follows from (3.7) that $\mathcal{H}_a \subseteq \mathcal{H}_y$, i.e., $y \in \mathcal{H}_a^{\rightarrow}$. Since $\mathcal{H}$ satisfies (3.1), we have $1 \in \mathcal{H}_a^{\rightarrow}$. Therefore $\mathcal{H}_a^{\rightarrow}$ is a filter of L. $\square$

For a fixed element $a \in L$ and a hesitant fuzzy set $\mathcal{H}$ on L, let $[a\mathcal{H}]$ be a hesitant fuzzy set on L given as follows:

$$[a\mathcal{H}] : L \rightarrow \mathscr{P}([0,1]),\; x \mapsto \begin{cases} \varepsilon_1 & \text{if } a \leq x, \\ \varepsilon_2 & \text{otherwise} \end{cases}$$

where $\varepsilon_1, \varepsilon_2 \in \mathscr{P}([0,1])$ with $\varepsilon_1 \supsetneq \varepsilon_2$.

Let $L = \{0, a, b, c, 1\}$ be the lattice implication algebra in Example 3.9. For $b \in L$, the hesitant fuzzy set $[b\mathcal{H}]$ on L which is given by

$$[b\mathcal{H}] : L \rightarrow \mathscr{P}([0,1]),\; x \mapsto \begin{cases} [0.2, 0.7] & \text{if } b \leq x, \\ [0.3, 0.6] & \text{otherwise} \end{cases}$$

is not a hesitant fuzzy filter of L since $[b\mathcal{H}]_a = [0.3, 0.6] \not\supseteq [0.2, 0.7] = [b\mathcal{H}]_{c\rightarrow a}^c$.

Given $a \in L$, we provide conditions for the hesitant fuzzy set $[a\mathcal{H}]$ to be a hesitant fuzzy filter of L.

Theorem 3.12. *Given $a \in L$, the hesitant fuzzy set $[a\mathcal{H}]$ is a hesitant fuzzy filter of L if and only if the following assertion is valid.*

$$(\forall x, y \in L) \left(a \leq y \rightarrow x,\; a \leq y \;\Rightarrow\; a \leq x \right). \tag{3.8}$$

Proof. Suppose that $[a\mathcal{H}]$ is a hesitant fuzzy filter of L and let $x, y \in L$ satisfy $a \leq y \to x$ and $a \leq y$. Then $[a\mathcal{H}]_{y\to x} = \varepsilon_1 = [a\mathcal{H}]_y$, and so $[a\mathcal{H}]_x \supseteq [a\mathcal{H}]^y_{y\to x} = \varepsilon_1$. Thus $a \leq x$, which satisfies the condition (3.8).

Conversely, assume that the condition (3.8) is valid. Note that

$$L([a\mathcal{H}];\varepsilon) = \begin{cases} L & \text{if } \varepsilon \subseteq \varepsilon_2, \\ \{x \in L \mid a \leq x\} & \text{if } \varepsilon_2 \subsetneq \varepsilon \subseteq \varepsilon_1, \\ \emptyset & \text{otherwise} \end{cases}$$

For the case of $\varepsilon_2 \subsetneq \varepsilon \subseteq \varepsilon_1$, obviously $1 \in L([a\mathcal{H}];\varepsilon)$. Let $x, y \in L$ be such that $x \in L([a\mathcal{H}];\varepsilon)$ and $x \to y \in L([a\mathcal{H}];\varepsilon)$. Then $a \leq x$ and $a \leq x \to y$, which imply from the hypothesis that $a \leq y$, that is, $y \in L([a\mathcal{H}];\varepsilon)$. Hence $L([a\mathcal{H}];\varepsilon)$ is a filter of L whenever it is nonempty. Therefore $[a\mathcal{H}]$ is a hesitant fuzzy filter of L. □

Theorem 3.13. *For a subset J of L, let $\mathcal{G}$ be a hesitant fuzzy set on L given as follows:*

$$\mathcal{G} : L \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} \varepsilon_1 & \text{if } x \in J, \\ \varepsilon_2 & \text{otherwise} \end{cases}$$

where $\varepsilon_1, \varepsilon_2 \in \mathscr{P}([0,1])$ with $\varepsilon_1 \supsetneq \varepsilon_2$. Then $\mathcal{G}$ is a hesitant fuzzy filter of L if and only if the following assertion is valid.

$$(\forall x, y \in J)(\forall z \in L)\left(x, y \in J,\ y \leq x \to z \ \Rightarrow\ z \in J\right). \tag{3.9}$$

Proof. Note that

$$L(\mathcal{G};\varepsilon) = \begin{cases} L & \text{if } \varepsilon \subseteq \varepsilon_2, \\ J & \text{if } \varepsilon_2 \subsetneq \varepsilon \subseteq \varepsilon_1, \\ \emptyset & \text{otherwise} \end{cases}$$

Assume that $\mathcal{G}$ is a hesitant fuzzy filter of L. Then $J = L(\mathcal{G};\varepsilon)$ for $\varepsilon_2 \subsetneq \varepsilon \subseteq \varepsilon_1$, and J is a filter of L. Let $x, y, z \in L$ be such that $x, y \in J$ and $y \leq x \to z$. Then $y \to (x \to z) = 1 \in J$, and so $z \in J$.

Conversely, let $\mathcal{G}$ be a hesitant fuzzy set on L and suppose that (3.9) is valid. Since $y \leq 1 = x \to 1$ for all $x, y \in L$, we have $1 \in J$ by (3.9), and so $1 \in L(\mathcal{G};\varepsilon)$ for $\varepsilon_2 \subsetneq \varepsilon \subseteq \varepsilon_1$. Let $x, y \in L$ be such that $y \in J = L(\mathcal{G};\varepsilon)$ and $y \to x \in J = L(\mathcal{G};\varepsilon)$ for $\varepsilon_2 \subsetneq \varepsilon \subseteq \varepsilon_1$. Since $y \leq (y \to x) \to x$, it follows from (3.9) that $x \in J = L(\mathcal{G};\varepsilon)$. Hence $L(\mathcal{G};\varepsilon)$ is a filter of L for all $\varepsilon \in \mathscr{P}([0,1])$ with $L(\mathcal{G};\varepsilon) \neq \emptyset$. Therefore $\mathcal{G}$ is a hesitant fuzzy filter of L. □

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3D Green's Function and Its Finite Element Error Estimates

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In our previous article, we introduced the definition of the 3D Green's function, and gave some estimates for this function. In this article, we will give the finite element approximation to the 3D Green's function. Moreover, some error estimates between 3D Green's function and its finite element approximation are derived, which will be used to the local superconvergence analysis.

1 Introduction

Superconvergence study is still an important topic in the finite element method, and the Green's function plays very important roles in the study of the superconvergence (especially, pointwise superconvergence) of the finite element method (see [1–9]). As for the global superconvergence, we know that the discrete Green's function and the discrete derivative Green's function are usually used. However, as for the local superconvergence, we need to use the Green's function which is independent of the mesh-size h. In our recent articles, we have introduced the definition of the 3D Green's function and its some estimates. This article will focus on the finite element approximation to the 3D Green's function.

we shall use the symbol C to denote a generic constant, which is independent of the mesh-size h and which may not be the same in each occurrence and also use the standard notations for the Sobolev spaces and their norms.

In this article, we consider the following Poisson equation:

$$\mathcal{L}u \equiv -\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where $\Omega \subset \mathcal{R}^3$ is a bounded polytopic domain. The weak formulation of the above equation reads,

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ satisfying} \\ a(u\,,\,v) = (f\,,\,v) \text{ for all } v \in H_0^1(\Omega), \end{cases} \tag{1.1}$$

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where

$$a(u\,,\,v) \equiv \int_\Omega \nabla u \cdot \nabla v \, dX, \;\; (f\,,\,v) \equiv \int_\Omega f v \, dX.$$

Let $\{\mathcal{T}^h\}$ be a regular family of partitions of $\bar{\Omega}$. Denote by $S^h(\Omega)$ a continuous finite elements space of degree $m(m \geq 1)$ regarding this kind of partitions and let $S_0^h(\Omega) = S^h(\Omega) \cap H_0^1(\Omega)$.

For every $Z \in \bar{\Omega}$, we define the discrete δ function $\delta_Z^h \in S_0^h(\Omega)$, the discrete derivative δ function $\partial_{Z,\ell}\delta_Z^h \in S_0^h(\Omega)$, the regularized Green's function $G_Z^* \in H^2(\Omega) \cap H_0^1(\Omega)$, the regularized derivative Green's function $\partial_{Z,\ell} G_Z^* \in H^2(\Omega) \cap H_0^1(\Omega)$, the discrete Green's function $G_Z^h \in S_0^h(\Omega)$, the discrete derivative Green's function $\partial_{Z,\ell} G_Z^h \in S_0^h(\Omega)$, and the L^2-projection $P_h u \in S_0^h(\Omega)$ such that (see [9])

$$(v,\, \delta_Z^h) = v(Z) \;\; \forall\, v \in S_0^h(\Omega), \tag{1.2}$$

$$(v,\, \partial_{Z,\ell}\delta_Z^h) = \partial_\ell v(Z) \;\; \forall\, v \in S_0^h(\Omega), \tag{1.3}$$

$$a(G_Z^*,\, v) = (\delta_Z^h,\, v) \;\; \forall\, v \in H_0^1(\Omega), \tag{1.4}$$

$$a(\partial_{Z,\ell} G_Z^*,\, v) = (\partial_{Z,\ell}\delta_Z^h,\, v) \;\; \forall\, v \in H_0^1(\Omega), \tag{1.5}$$

$$a(G_Z^h,\, v) = v(Z) \;\; \forall\, v \in S_0^h(\Omega), \tag{1.6}$$

$$a(\partial_{Z,\ell} G_Z^h,\, v) = \partial_\ell v(Z) \;\; \forall\, v \in S_0^h(\Omega), \tag{1.7}$$

$$(u - P_h u,\, v) = 0 \;\; \forall\, v \in S_0^h(\Omega). \tag{1.8}$$

Here, for any direction $\ell \in R^3$, $|\ell| = 1$, $\partial_{Z,\ell}\delta_Z^h$, $\partial_{Z,\ell} G_Z^h$, and $\partial_\ell v(Z)$ stand for the following onesided directional derivatives, respectively.

$$\partial_{Z,\ell}\delta_Z^h = \lim_{|\Delta Z| \to 0} \frac{\delta_{Z+\Delta Z}^h - \delta_Z^h}{|\Delta Z|},\; \partial_{Z,\ell} G_Z^h = \lim_{|\Delta Z| \to 0} \frac{G_{Z+\Delta Z}^h - G_Z^h}{|\Delta Z|},$$

$$\partial_\ell v(Z) = \lim_{|\Delta Z| \to 0} \frac{v(Z + \Delta Z) - v(Z)}{|\Delta Z|},\; \Delta Z = |\Delta Z| \ell.$$

As for G_Z^*, $\partial_{Z,\ell} G_Z^*$, G_Z^h, and $\partial_{Z,\ell} G_Z^h$, we have obtained some optimal estimates (see [4–6]), which will be used in next section. From (1.4)–(1.7), we easily find G_Z^h and $\partial_{Z,\ell} G_Z^h$ are the finite element approximations to G_Z^* and $\partial_{Z,\ell} G_Z^*$, respectively.

For the L^2-projection operator P_h, we have (see [4])

Lemma 1.1. *For $P_h w$ the L^2-projection of $w \in L^p(\Omega)$, we have the following stability estimate:*

$$\|P_h w\|_{0,\,p,\,\Omega} \leq C^t \|w\|_{0,\,p,\,\Omega}, \tag{1.9}$$

where $t = \left|1 - \frac{2}{p}\right|$, and $1 \leq p \leq \infty$.

Further, by Lemma 1.1, we easily obtain the following result:

$$\|w - P_h w\|_{0,\,p,\,\Omega} \leq (1 + C^t) \inf_{v \in S_0^h \Omega} \|w - v\|_{0,\,p,\,\Omega}, \tag{1.10}$$

where $1 \le p \le \infty$. Using the result (1.10), we easily obtain

$$\|P_h w\|_{1,\,p,\,\Omega} \le C\|w\|_{1,\,p,\,\Omega}, \text{ for } 3 < p \le \infty. \tag{1.11}$$

In addition, we also assume the following a priori estimate holds.

Lemma 1.2. *For the true solution u of (1.1), there exists a $q_0 (1 < q_0 \le \infty)$ such that for every $1 < q < q_0$,*

$$\|u\|_{2,\,q,\,\Omega} \le C(q)\|\mathcal{L}u\|_{0,\,q,\,\Omega}. \tag{1.12}$$

2 Regularized Green's Function and Its Finite Element Approximation

We introduce two weight functions defined by

$$\phi = \left(|X-Z|^2+\theta^2\right)^{-\frac{3}{2}} \text{ and } \tau = |X-Z|^{-3} \;\; \forall\, X \in \bar{\Omega},$$

where $Z \in \bar{\Omega}$ is a fixed point, $\theta = \gamma h$, and $\gamma \in [3, +\infty)$ is a suitable real number. They will be used in this section and next section.

In [4], we derived the following Lemma 2.1 (see (2.62) and (2.63) in [4]).

Lemma 2.1. *Suppose $q_0 > 3$. For G_Z^* and G_Z^h defined by (1.4) and (1.6), respectively, we have*

$$\left\|G_Z^* - G_Z^h\right\|_{1,\phi^{-1}} \le Ch\left|\nabla^2 G_Z^*\right|_{\phi^{-1}} \le Ch\,|\ln h|^{\frac{1}{6}}\,. \tag{2.1}$$

Lemma 2.2. *For G_Z^* and G_Z^h defined by (1.4) and (1.6), respectively, we have*

$$\left\|G_Z^* - G_Z^h\right\|_{1,\phi^{-\alpha}} \le C(\alpha)h \begin{cases} \forall\, 1<\alpha<\frac{5}{3}-\frac{2}{q_0} \text{ when } 3<q_0<6, \\ \forall\, 1<\alpha<\frac{4}{3} \text{ when } q_0 \ge 6. \end{cases} \tag{2.2}$$

Proof. Similar to the proof of the result (2.43) in [4], we have

$$\left\|G_Z^* - G_Z^h\right\|^2_{1,\,\phi^{-\alpha}} \le Ch^2\left\|\nabla^2 G_Z^*\right\|^2_{\phi^{-\alpha}} + C\left\|G_Z^* - G_Z^h\right\|^2_{\phi^{-\alpha+\frac{2}{3}}}\,. \tag{2.3}$$

We easily obtain

$$\begin{aligned} & \left\|G_Z^* - G_Z^h\right\|^2_{\phi^{-\alpha+\frac{2}{3}}} = \left(\phi^{-\alpha+\frac{2}{3}}(G_Z^* - G_Z^h),\, G_Z^* - G_Z^h\right) \\ = \;& a(v,\, G_Z^* - G_Z^h) = a(v-\Pi v,\, G_Z^* - G_Z^h) \\ \le \;& \left|G_Z^* - G_Z^h\right|_{1,\,\phi^{-\alpha}} \cdot |v-\Pi v|_{1,\,\phi^{\alpha}} \\ \le \;& \varepsilon\left|G_Z^* - G_Z^h\right|^2_{1,\,\phi^{-\alpha}} + C(\varepsilon)\,|v-\Pi v|^2_{1,\,\phi^{\alpha}} \\ \le \;& \varepsilon\left|G_Z^* - G_Z^h\right|^2_{1,\,\phi^{-\alpha}} + C(\varepsilon)h^2\left|\nabla^2 v\right|^2_{\phi^{\alpha}} \\ \le \;& \varepsilon\left|G_Z^* - G_Z^h\right|^2_{1,\,\phi^{-\alpha}} + C(\varepsilon)h^2\theta^{-2}\left|\nabla(\phi^{-\alpha+\frac{2}{3}}(G_Z^* - G_Z^h))\right|^2_{\phi^{\alpha-\frac{4}{3}}}, \end{aligned} \tag{2.4}$$

where $\mathcal{L}v = \phi^{-\alpha+\frac{2}{3}}(G_Z^* - G_Z^h)$.

Note that the result $\left|\nabla^2 v\right|^2_{\phi^\alpha} \le C\theta^{-2}\left|\nabla(\phi^{-\alpha+\frac{2}{3}}(G^*_Z - G^h_Z))\right|^2_{\phi^{\alpha-\frac{4}{3}}}$ in (2.4) should satisfy one of the following two conditions: (1) $1 < \alpha < \frac{5}{3} - \frac{2}{q_0}$ when $3 < q_0 < 6$; (2) $1 < \alpha < \frac{4}{3}$ when $q_0 \ge 6$. In addition,

$$\begin{aligned}
&\left|\nabla(\phi^{-\alpha+\frac{2}{3}}(G^*_Z - G^h_Z))\right|^2_{\phi^{\alpha-\frac{4}{3}}} \\
= &\int_\Omega \phi^{\alpha-\frac{4}{3}}\left|\nabla\phi^{-\alpha+\frac{2}{3}}\cdot(G^*_Z - G^h_Z) + \phi^{-\alpha+\frac{2}{3}}\cdot\nabla(G^*_Z - G^h_Z)\right|^2 dX \\
\le\; & C\int_\Omega \phi^{\alpha-\frac{4}{3}}\left(|\nabla\phi^{-\alpha+\frac{2}{3}}|^2|G^*_Z - G^h_Z|^2 + (\phi^{-\alpha+\frac{2}{3}})^2|\nabla(G^*_Z - G^h_Z)|^2\right) dX \\
\le\; & C\left(\left|G^*_Z - G^h_Z\right|^2_{1,\,\phi^{-\alpha}} + \left\|G^*_Z - G^h_Z\right\|^2_{\phi^{-\alpha+\frac{2}{3}}}\right).
\end{aligned}$$

Combining (2.4) and the above result, we have

$$\begin{aligned}
\left\|G^*_Z - G^h_Z\right\|^2_{\phi^{-\alpha+\frac{2}{3}}} \le\; & \varepsilon\left|G^*_Z - G^h_Z\right|^2_{1,\,\phi^{-\alpha}} \\
& + C(\varepsilon)h^2\theta^{-2}\left(\left|G^*_Z - G^h_Z\right|^2_{1,\,\phi^{-\alpha}} + \left\|G^*_Z - G^h_Z\right\|^2_{\phi^{-\alpha+\frac{2}{3}}}\right) \\
=\; & \varepsilon\left|G^*_Z - G^h_Z\right|^2_{1,\,\phi^{-\alpha}} \\
& + C(\varepsilon)\gamma^{-2}\left(\left|G^*_Z - G^h_Z\right|^2_{1,\,\phi^{-\alpha}} + \left\|G^*_Z - G^h_Z\right\|^2_{\phi^{-\alpha+\frac{2}{3}}}\right).
\end{aligned} \tag{2.5}$$

Choosing $\gamma \in [3, +\infty)$ in (2.5) such that $0 < C(\varepsilon)\gamma^{-2} < \min(\varepsilon, \frac{1}{2})$, we have

$$\left\|G^*_Z - G^h_Z\right\|^2_{\phi^{-\alpha+\frac{2}{3}}} \le 4\varepsilon\left|G^*_Z - G^h_Z\right|^2_{1,\,\phi^{-\alpha}}. \tag{2.6}$$

Taking a suitable $\varepsilon \in (0, +\infty)$, from (2.3) and (2.6), we obtain

$$\left\|G^*_Z - G^h_Z\right\|_{1,\,\phi^{-\alpha}} \le Ch\left\|\nabla^2 G^*_Z\right\|_{\phi^{-\alpha}}. \tag{2.7}$$

We can prove

$$\left\|\nabla^2 G^*_Z\right\|_{\phi^{-\alpha}} \le C\left\|\delta^h_Z\right\|_{\phi^{-\alpha}} + C\left\|G^*_Z\right\|_{\phi^{-\alpha+\frac{4}{3}}} \le Ch^{\frac{3(\alpha-1)}{2}} + C\left\|G^*_Z\right\|_{\phi^{-\alpha+\frac{4}{3}}}. \tag{2.8}$$

Further, from (1.4), (1.8), (1.9), (1.12), and the Sobolev Embedding Theorem [10], we have

$$\begin{aligned}
&\|G^*_Z\|^2_{\phi^{-\alpha+\frac{4}{3}}} = (G^*_Z, \phi^{-\alpha+\frac{4}{3}}G^*_Z) = a(G^*_Z, w) \\
= \;& P_h w(Z) \le \|P_h w\|_{0,\infty} \le C\|w\|_{0,\infty} \le C\|w\|_{2,p} \le C\left\|\phi^{-\alpha+\frac{4}{3}}G^*_Z\right\|_{0,p} \\
= \;& C\left(\int_\Omega \phi^{(\frac{4}{3}-\alpha)p}|G^*_Z|^p\, dX\right)^{\frac{1}{p}} \le C\left(\int_\Omega \phi^{\frac{(\frac{4}{3}-\alpha)p}{2-p}}\, dX\right)^{\frac{2-p}{2p}} \|G^*_Z\|_{\phi^{-\alpha+\frac{4}{3}}}.
\end{aligned}$$

Here we choose p such that $\frac{3}{2} < p < \frac{6}{7-3\alpha} < 2$ and $0 < \frac{(\frac{4}{3}-\alpha)p}{2-p} < 1$. It is easy to prove

$$\int_\Omega \phi^{\frac{(\frac{4}{3}-\alpha)p}{2-p}}\, dX \le C(\alpha).$$

Thus we have

$$\|G_Z^*\|_{\phi^{-\alpha+\frac{4}{3}}} \le C(\alpha). \tag{2.9}$$

From (2.7)–(2.9), the result (2.2) is obtained.

Lemma 2.3. *For $\partial_{Z,\ell}G_Z^*$ and $\partial_{Z,\ell}G_Z^h$ defined by (1.5) and (1.7), respectively, we have*

$$\left\|\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h\right\|_{1,\phi^{-\alpha}} \le Ch^{\frac{3(\alpha-1)}{2}} |\ln h|^{\frac{4-3\alpha}{6}}, \tag{2.10}$$

where $1<\alpha<\frac{5}{3}-\frac{2}{q_0}$ when $3<q_0<6$ and $1<\alpha<\frac{4}{3}$ when $q_0 \ge 6$.

Proof. Similar to the result (2.7), we have

$$\left\|\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h\right\|_{1,\,\phi^{-\alpha}} \le Ch\left\|\nabla^2\partial_{Z,\ell}G_Z^*\right\|_{\phi^{-\alpha}}. \tag{2.11}$$

In addition

$$\begin{aligned}\left\|\nabla^2\partial_{Z,\ell}G_Z^*\right\|_{\phi^{-\alpha}} &\le C\left\|\partial_{Z,\ell}\delta_Z^h\right\|_{\phi^{-\alpha}} + C\left\|\partial_{Z,\ell}G_Z^*\right\|_{\phi^{-\alpha+\frac{4}{3}}}\\ &\le Ch^{\frac{3\alpha-5}{2}} + C\left\|\partial_{Z,\ell}G_Z^*\right\|_{\phi^{-\alpha+\frac{4}{3}}}.\end{aligned} \tag{2.12}$$

Further, from (1.5), (1.8), (1.11), (1.12), the inverse inequality, the Sobolev Embedding Theorem [10], and the Hölder inequality, we have

$$\begin{aligned}&\|\partial_{Z,\ell}G_Z^*\|^2_{\phi^{-\alpha+\frac{4}{3}}} = (\partial_{Z,\ell}G_Z^*, \phi^{-\alpha+\frac{4}{3}}\partial_{Z,\ell}G_Z^*) = a(\partial_{Z,\ell}G_Z^*, w) = \partial_{Z,\ell}P_h w(Z)\\ \le\;& |P_h w|_{1,\infty} \le Ch^{-\frac{3}{q}}|P_h w|_{1,q} \le Ch^{-\frac{3}{q}}|w|_{1,q} \le Ch^{-\frac{3}{q}}\|w\|_{2,s}\\ \le\;& Ch^{-\frac{3}{q}}\left\|\phi^{\frac{4}{3}-\alpha}\partial_{Z,\ell}G_Z^*\right\|_{0,s} = Ch^{-\frac{3}{q}}\left(\int_\Omega \phi^{(\frac{4}{3}-\alpha)s}|\partial_{Z,\ell}G_Z^*|^s\,dX\right)^{\frac{1}{s}}\\ \le\;& Ch^{-\frac{3}{q}}\left(\int_\Omega \phi^{\frac{(\frac{4}{3}-\alpha)s}{2-s}}\,dX\right)^{\frac{2-s}{2s}}\|\partial_{Z,\ell}G_Z^*\|_{\phi^{-\alpha+\frac{4}{3}}}.\end{aligned}$$

Here we choose $s=\frac{6}{7-3\alpha}$ and $\frac{1}{q}=\frac{1}{s}-\frac{1}{3}$. Obviously,

(A) $\frac{3}{2}<s<\frac{3q_0}{3+q_0}$ and $3<q<q_0$ when $3<q_0<6$.

(B) $\frac{3}{2}<s<2$ and $3<q<6$ when $q_0\ge 6$.

In the meantime, we have $\frac{(\frac{4}{3}-\alpha)s}{2-s}=1$. By the result (2.14) in [4], we then get

$$\left(\int_\Omega \phi^{\frac{(\frac{4}{3}-\alpha)s}{2-s}}\,dX\right)^{\frac{2-s}{2s}} \le C|\ln h|^{\frac{4-3\alpha}{6}}.$$

Thus we have

$$\|\partial_{Z,\ell}G_Z^*\|_{\phi^{-\alpha+\frac{4}{3}}} \le Ch^{\frac{3\alpha-5}{2}}|\ln h|^{\frac{4-3\alpha}{6}}. \tag{2.13}$$

From (2.11)–(2.13), $\left\|\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h\right\|_{1,\,\phi^{-\alpha}} \le Ch^{\frac{3(\alpha-1)}{2}}|\ln h|^{\frac{4-3\alpha}{6}}$. The proof of the result (2.10) is completed.

Lemma 2.4. *For G_Z^* and G_Z^h defined by (1.4) and (1.6), respectively, we have*

$$\left\|G_Z^* - G_Z^h\right\|_{1,p} \le \begin{cases} Ch^{\frac{3-2p}{p}}|\ln h|^{\frac{1}{6}}, & 1<p<\frac{3}{2},\\ Ch|\ln h|^{\frac{2}{3}}, & p=1.\end{cases} \tag{2.14}$$

Proof. When $p = 1$, the result can be seen in [4]. Thus we only need to prove the case of $1 < p < \frac{3}{2}$. By the Hölder inequality, we have

$$\left\|G_Z^* - G_Z^h\right\|_{1,p} \leq \left(\int_\Omega \phi^{\frac{p}{2-p}}\, dX\right)^{\frac{2-p}{2p}} \left\|G_Z^* - G_Z^h\right\|_{1,\phi^{-1}}. \tag{2.15}$$

From (2.13) in [4],

$$\int_\Omega \phi^{\frac{p}{2-p}}\, dX \leq Ch^{\frac{6-6p}{2-p}}. \tag{2.16}$$

Combining (2.1), (2.15), and (2.16) yields $\left\|G_Z^* - G_Z^h\right\|_{1,p} \leq Ch^{\frac{3-2p}{p}} \left|\ln h\right|^{\frac{1}{6}}$. The proof of the result (2.14) is completed.

3 Finite Element Approximation to the 3D Green's Function

In this section, we discuss the 3D Green's function and its finite element approximation. We call G_Z Green's function which satisfies the following Theorem 3.1.

Theorem 3.1. *There exists a unique* $G_Z \in W_0^{1,p}(\Omega)$ $(1 \leq p < \frac{3}{2})$ *such that*

$$a(G_Z, v) = v(Z) \ \ \forall\, v \in W_0^{1,p'}(\Omega),\ \frac{1}{p} + \frac{1}{p'} = 1. \tag{3.1}$$

Proof. We first prove the uniqueness of G_Z. Suppose there exists another Green's function $G'_Z \in W_0^{1,p}(\Omega)$ satisfying (3.1). Set $E_Z = G_Z - G'_Z$, thus

$$a(E_Z, v) = 0 \ \ \forall\, v \in W_0^{1,p'}(\Omega). \tag{3.2}$$

When $1 < p < \frac{3}{2}$, for each $\varphi \in L^{p'}(\Omega)$, there exists a $w \in W^{2,p'} \cap W_0^{1,p'}(\Omega)$ such that $\mathcal{L}w = \varphi$. Obviously, $\mathrm{sgn}E_Z|E_Z|^{p-1} \in L^{p'}(\Omega)$, thus we can find $w \in W^{2,p'} \cap W_0^{1,p'}(\Omega)$ such that $\mathcal{L}w = v$. Then we have

$$\|E_Z\|_{0,p}^p = (E_Z, \mathrm{sgn}E_Z|E_Z|^{p-1}) = a(E_Z, w), \tag{3.3}$$

From (3.2) and (3.3), $\|E_Z\|_{0,p} = 0$, i.e., $G_Z = G'_Z$. Similarly, when $p = 1$, we can also prove $G_Z = G'_Z$. Thus we have completed the proof of the uniqueness.

Next, we prove the existence of G_Z. We give a series of finite element spaces $S_0^{h_i}(\Omega)$, $i = 0, 1, 2, \cdots$ satisfying $S_0^{h_i}(\Omega) \subset S_0^{h_j}(\Omega)$ when $i < j$, where $h_0 \equiv h$ and $\frac{1}{4}h_{i-1} \leq h_i \leq \frac{1}{2}h_{i-1}$. Let $G_{Z,i}^*$ be the regularized Green's function for the finite element space $S_0^{h_i}(\Omega)$, and $G_Z^{h_i}$ the discrete Green's function. Their definitions can be seen in Section 1. Obviously, we have $a(G_{Z,i+1}^* - G_Z^{h_i}, v) = 0 \ \forall\, v \in S_0^{h_i}(\Omega)$. Similar to the proof of the result (2.14), we have for $1 < p < \frac{3}{2}$

$$\left\|G_{Z,i+1}^* - G_Z^{h_i}\right\|_{1,p} \leq Ch_i^{\frac{3-2p}{p}} \left|\ln h_i\right|^{\frac{1}{6}},$$

which combined with (2.14), we get

$$\left\|G^*_{Z,i+1} - G^*_{Z,i}\right\|_{1,p} \le Ch_i^{\frac{3-2p}{p}} \left|\ln h_i\right|^{\frac{1}{6}}. \tag{3.4}$$

Thus

$$\sum_{i=0}^{\infty} \left\|G^*_{Z,i+1} - G^*_{Z,i}\right\|_{1,p} \le C\sum_{i=0}^{\infty} \left(\frac{h}{2^i}\right)^{\frac{3-2p}{p}} \left|\ln \frac{h}{2^i}\right|^{\frac{1}{6}} \le Ch^{\frac{3-2p}{p}} \left|\ln h\right|^{\frac{1}{6}}. \tag{3.5}$$

Set

$$G_Z \equiv G^*_Z + \sum_{i=0}^{\infty}(G^*_{Z,i+1} - G^*_{Z,i}).$$

Thus we have $G_Z \in W^{1,p}_0(\Omega)$. From (3.5),

$$\|G_Z - G^*_Z\|_{1,p} \le Ch^{\frac{3-2p}{p}} \left|\ln h\right|^{\frac{1}{6}}. \tag{3.6}$$

Similarly, when $p = 1$, we have

$$\|G_Z - G^*_Z\|_{1,1} \le Ch \left|\ln h\right|^{\frac{2}{3}}. \tag{3.7}$$

Therefore, for $1 \le p < \frac{3}{2}$, we have $G^*_{Z,i} \longrightarrow G_Z$ in $W^{1,p}(\Omega)$ when $i \to \infty$. Using (1.10) and the interpolation error estimate, we obtain

$$\|v - P_h v\|_{0,\,\infty,\,\Omega} \le C\|v - \Pi v\|_{0,\,\infty,\,\Omega} \le Ch^{1-\frac{3}{p'}}\|v\|_{1,p',\,\Omega}, \tag{3.8}$$

where $3 < p' \le \infty$. Thus, for every $v \in W^{1,p'}_0(\Omega)$, we have by (3.6)–(3.8)

$$a(G_Z, v) = \lim_{i\to\infty} a(G^*_{Z,i}, v) = \lim_{i\to\infty} P_{h_i} v(Z) = v(Z).$$

The proof of Theorem 3.1 is completed. Now we show G_Z is independent of h. Suppose there exists a Green's function $\tilde{G}_Z$ for the mesh-size $\tilde{h}$. In addition, $\frac{1}{4}\tilde{h}_{i-1} \le \tilde{h}_i \le \frac{1}{2}\tilde{h}_{i-1}$ and $\tilde{h}_0 = \tilde{h}$. Thus, for every $f \in L^{p'}(\Omega)$, we choose $v \in W^{2,p'}(\Omega) \cap W^{1,p'}_0(\Omega)$ such that $\mathcal{L}v = f$. Then we get $(G_Z, f) = a(G_Z, v) = v(Z)$ and $(\tilde{G}_Z, f) = a(\tilde{G}_Z, v) = v(Z)$. Thus, $(G_Z, f) = (\tilde{G}_Z, f)$, i.e., $(G_Z - \tilde{G}_Z, f) = 0$. So we get $G_Z = \tilde{G}_Z$. Namely, G_Z is independent of h.

In addition, we find

$$a(G_Z,\, v) = v(Z) \quad \forall\, v \in S^h_0(\Omega) \subset W^{1,p'}(\Omega). \tag{3.9}$$

Combining (1.6) and (3.9), we have $a(G_Z - G^h_Z,\, v) = 0 \;\forall\, v \in S^h_0(\Omega)$. Thus G^h_Z is the finite element approximation to G_Z. Further, we have the following error estimates.

Theorem 3.2. *For G_Z and G^h_Z defined by (3.1) and (1.6), respectively, we have*

$$\left\|G_Z - G^h_Z\right\|_{1,p} \le \begin{cases} Ch^{\frac{3-2p}{p}} \left|\ln h\right|^{\frac{1}{6}}, & 1 < p < \frac{3}{2}, \\ Ch \left|\ln h\right|^{\frac{2}{3}}, & p = 1, \end{cases} \tag{3.10}$$

where C is independent of h and Z.
Proof. From (2.14), (3.6), (3.7), and the triangular inequality, we immediately obtain the result (3.10).
Theorem 3.3. *Suppose $q_0 = \infty$, for G_Z and G_Z^h defined by (3.1) and (1.6), respectively, we have*

$$\left\|G_Z - G_Z^h\right\|_{0,1} \leq Ch^2 \left|\ln h\right|^{\frac{5}{3}}, \tag{3.11}$$

where C is independent of h and Z.
Proof. For every $\varphi \in L^\infty(\Omega)$, there exists a unique $v \in W^{2,\infty}(\Omega) \cap H_0^1(\Omega)$ such that $\mathcal{L}v = \varphi$ and

$$(G_Z - G_Z^h, \varphi) = a(G_Z - G_Z^h, v) = a(G_Z, v - v_h) = v(Z) - v_h(Z), \tag{3.12}$$

where v_h is the finite element approximation to v. From (1.10),

$$|v(Z) - P_h v(Z)| \leq \|v - P_h v\|_{0,\infty} \leq C\|v - \Pi v\|_{0,\infty} \leq Ch^{2-\frac{3}{q}} \|v\|_{2,q}, \tag{3.13}$$

where $1 < q < q_0$. In addition, by (2.14), the Hölder inequality, and the interpolation error estimate, we have

$$\begin{aligned} & |P_h v(Z) - v_h(Z)| = |a(v - v_h, G_Z^*)| = |a(v - v_h, G_Z^* - G_Z^h)| \\ = \ & |a(v - \Pi v, G_Z^* - G_Z^h)| \leq C\left\|G_Z^* - G_Z^h\right\|_{1,1} \|v - \Pi v\|_{1,\infty} \\ \leq \ & Ch^{2-\frac{3}{q}} \left|\ln h\right|^{\frac{2}{3}} \|v\|_{2,q}. \end{aligned} \tag{3.14}$$

From (3.12)–(3.14), and the triangular inequality,

$$|(G_Z - G_Z^h, \varphi)| = |v(Z) - v_h(Z)| \leq Ch^{2-\frac{3}{q}} \left|\ln h\right|^{\frac{2}{3}} \|v\|_{2,q}.$$

From (1.12),

$$|(G_Z - G_Z^h, \varphi)| \leq C(q)h^{2-\frac{3}{q}} \left|\ln h\right|^{\frac{2}{3}} \|\varphi\|_{0,q}. \tag{3.15}$$

Because of $q_0 = \infty$, we can take $q = |\ln h| < q_0$ in (3.15), and we have $C(q) \leq Cq$. Thus,

$$|(G_Z - G_Z^h, \varphi)| \leq Ch^2 \left|\ln h\right|^{\frac{5}{3}} \|\varphi\|_{0,\infty}. \tag{3.16}$$

From (3.16), we know the result (3.11) holds. So, the proof of the result (3.11) is completed.
Theorem 3.4. *For G_Z and G_Z^h defined by (3.1) and (1.6), respectively, we have*

$$\left\|G_Z - G_Z^h\right\|_{1,\tau^{-1}} \leq Ch\left|\ln h\right|^{\frac{1}{6}}, \tag{3.17}$$

$$\left\|G_Z - G_Z^h\right\|_{1,\tau^{-\alpha}} \leq C(\alpha)h \begin{cases} \forall\, 1 < \alpha < \frac{5}{3} - \frac{2}{q_0} & \text{when } 3 < q_0 < 6, \\ \forall\, 1 < \alpha < \frac{4}{3} & \text{when } q_0 \geq 6, \end{cases} \tag{3.18}$$

where C is independent of h and Z.
Proof. Obviously, $\tau^{-k} < \phi^{-k}$ when $k > 0$. Thus from (2.1) and (2.2),

$$\left\|G_Z^* - G_Z^h\right\|_{1,\tau^{-1}} \leq Ch\left|\ln h\right|^{\frac{1}{6}}, \tag{3.19}$$

$$\left\|G_Z^* - G_Z^h\right\|_{1,\tau^{-\alpha}} \leq C(\alpha)h \begin{cases} \forall\, 1 < \alpha < \frac{5}{3} - \frac{2}{q_0} \text{ when } 3 < q_0 < 6, \\ \forall\, 1 < \alpha < \frac{4}{3} \text{ when } q_0 \geq 6, \end{cases} \tag{3.20}$$

Similar to the arguments of Theorem 3.1, we can obtain the results (3.17) and (3.18). Obviously,

$$\left\|\partial_{Z,\ell} G_Z^* - \partial_{Z,\ell} G_Z^h\right\|_{1,\tau^{-\alpha}} \leq \left\|\partial_{Z,\ell} G_Z^* - \partial_{Z,\ell} G_Z^h\right\|_{1,\phi^{-\alpha}} \leq C h^{\frac{3(\alpha-1)}{2}} \left|\ln h\right|^{\frac{4-3\alpha}{6}}, \tag{3.21}$$

where $1 < \alpha < \frac{5}{3} - \frac{2}{q_0}$ when $3 < q_0 < 6$ and $1 < \alpha < \frac{4}{3}$ when $q_0 \geq 6$. Adopting the techniques in the proof of Theorem 3.1, we can derive by (3.21)

$$\sum_{i=0}^{\infty} \left\|\partial_{Z,\ell} G_{Z,i+1}^* - \partial_{Z,\ell} G_{Z,i}^*\right\|_{1,\tau^{-\alpha}} \leq C h^{\frac{3(\alpha-1)}{2}} \left|\ln h\right|^{\frac{4-3\alpha}{6}}.$$

Set

$$F \equiv \partial_{Z,\ell} G_Z^* + \sum_{i=0}^{\infty} (\partial_{Z,\ell} G_{Z,i+1}^* - \partial_{Z,\ell} G_{Z,i}^*).$$

Here, $\|F\|_{1,\tau^{-\alpha}} < \infty$ and $\partial_{Z,\ell} G_{Z,i}^* = \lim_{|\Delta Z| \to 0} \frac{G_{Z+\Delta Z,i}^* - G_{Z,i}^*}{|\Delta Z|}$, $\Delta Z = |\Delta Z|\ell$. By the arguments of Theorem 3.1,

$$\begin{aligned} G_{Z+\Delta Z} &\equiv G_{Z+\Delta Z}^* + \sum_{i=0}^{\infty} (G_{Z+\Delta Z,i+1}^* - G_{Z+\Delta Z,i}^*), \\ G_Z &\equiv G_Z^* + \sum_{i=0}^{\infty} (G_{Z,i+1}^* - G_{Z,i}^*). \end{aligned}$$

Thus we have $F = \lim_{|\Delta Z| \to 0} \frac{G_{Z+\Delta Z} - G_Z}{|\Delta Z|} = \partial_{Z,\ell} G_Z$. Namely,

$$\partial_{Z,\ell} G_Z = \partial_{Z,\ell} G_Z^* + \sum_{i=0}^{\infty} (\partial_{Z,\ell} G_{Z,i+1}^* - \partial_{Z,\ell} G_{Z,i}^*),\ \|\partial_{Z,\ell} G_Z\|_{1,\tau^{-\alpha}} < \infty. \tag{3.22}$$

We write $W_\beta(\Omega) = \{v : v|_{\partial\Omega} = 0, \|v\|_{1,\tau^\beta} < \infty\}$. From (3.22), $\partial_{Z,\ell} G_Z \in W_{-\alpha}(\Omega)$. Further, we can obtain the following Theorem 3.5.

Theorem 3.5. *There exists a unique* $\partial_{Z,\ell} G_Z \in W_{-\alpha}(\Omega)$ *such that*

$$a(\partial_{Z,\ell} G_Z, v) = \partial_\ell v(Z)\ \ \forall\, v \in W_\alpha(\Omega) \cap C_0^\infty(\Omega), \tag{3.23}$$

where $1 < \alpha < \frac{5}{3} - \frac{2}{q_0}$ *when* $3 < q_0 < 6$ *and* $1 < \alpha < \frac{4}{3}$ *when* $q_0 \geq 6$.

Proof. From (3.22),

$$\|\partial_{Z,\ell} G_Z - \partial_{Z,\ell} G_Z^*\|_{1,\tau^{-\alpha}} \leq C h^{\frac{3(\alpha-1)}{2}} \left|\ln h\right|^{\frac{4-3\alpha}{6}}. \tag{3.24}$$

Namely, $\partial_{Z,\ell} G_Z^* \longrightarrow \partial_{Z,\ell} G_Z$ in $W_{-\alpha}(\Omega)$ when $h \to 0$. Then we have by (1.3), (1.5), and (1.8)

$$a(\partial_{Z,\ell} G_Z, v) = \lim_{h \to 0} a(\partial_{Z,\ell} G_Z^*, v) = \lim_{h \to 0} \partial_\ell P_h v(Z). \tag{3.25}$$

From (1.11), $\|v - P_h v\|_{1,\infty} \le C\,\|v - \Pi v\|_{1,\infty} \le Ch\,\|v\|_{2,\infty}$. That is

$$\|v - P_h v\|_{1,\infty} \longrightarrow 0 \text{ when } h \to 0. \tag{3.26}$$

Combining (3.25) and (3.26) yields

$$a(\partial_{Z,\ell} G_Z, v) = \partial_\ell v(Z). \tag{3.27}$$

The uniqueness of $\partial_{Z,\ell} G_Z$ satisfying (3.27) can be similarly proved as that of G_Z in (3.1).

By (3.21), (3.24), and the triangular inequality, we immediately obtain the following result (3.28).

Theorem 3.6. *For $\partial_{Z,\ell} G_Z$ and $\partial_{Z,\ell} G_Z^h$ defined by (3.23) and (1.7), respectively, we have*

$$\left\|\partial_{Z,\ell} G_Z - \partial_{Z,\ell} G_Z^h\right\|_{1,\tau^{-\alpha}} \le Ch^{\frac{3(\alpha-1)}{2}} \left|\ln h\right|^{\frac{4-3\alpha}{6}}, \tag{3.28}$$

where $1 < \alpha < \frac{5}{3} - \frac{2}{q_0}$ when $3 < q_0 < 6$ and $1 < \alpha < \frac{4}{3}$ when $q_0 \ge 6$.

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Hermite–Hadamard Type Inequalities for s-Convex Functions via Riemann–Liouville Fractional Integrals

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Abstract

In the paper, by establishing a Riemann–Liouville fractional integral identity involving an n-times differentiable function, the authors present some Hermite–Hadamard type inequalities involving Riemann–Liouville fractional integrals for s-convex functions.

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1 Introduction

Throughout this paper, let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_0 = [0, \infty)$, use $I \subseteq \mathbb{R}$ and I° to denote an interval and the interior of I respectively, and utilize $\mathbb{N}$ to denote the set of all positive integers.

The following definition is well known in the literature.

Definition 1.1. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$$

holds for $x, y \in I$ and $\lambda \in [0, 1]$. If this inequality reverses, then f is said to be concave on I.

The most important inequality in the theory of convex functions, Hermite–Hadamard's inequality, may be stated as follows. If f is a convex function on $[a, b]$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

1

If f is concave on $[a,b]$, then the inequality (1.1) is reversed. See [6], for example.

The inequality (1.1) has been generalized in many articles. Some of them may be recited as follows.

Theorem 1.1 ([2, Theorem 2.2]). *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $a,b \in I^\circ$ with $a<b$. If $|f'(x)|$ is convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{(b-a)(|f'(a)|+|f'(b)|)}{8}.$$

Theorem 1.2 ([7, Theorem 1]). *If f is differentiable on $[a,b]$ such that $|f'(x)|^q$ is a convex function on $[a,b]$ for $q\ge1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q}.$$

Theorem 1.3 ([5, Theorem 2.3]). *Let $f : I \to \mathbb{R}$ be differentiable on I°, $a,b\in I^\circ$ with $a<b$, and $p>1$. If $|f'(x)|^{p/(p-1)}$ is convex on $[a,b]$, then*

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|&\le\frac{b-a}{16}\left(\frac{4}{p+1}\right)^{1/p}\\&\times\left\{\left[|f'(a)|^{p/(p-1)}+3|f'(b)|^{p/(p-1)}\right]^{1-1/p}+\left[3|f'(a)|^{p/(p-1)}+|f'(b)|^{p/(p-1)}\right]^{1-1/p}\right\}.\end{aligned}$$

For more information, please refer to [2, 5, 6, 7] and references therein.

In addition to the classical convex functions, the class of functions which are s-convex has been introduced in [4] as follows.

Definition 1.2 ([4, p. 100]). A function $f:\mathbb{R}_0\to\mathbb{R}$ is said to be s-convex for some fixed $s\in(0,1]$ if $f(tx+(1-t)y)\le t^s f(x)+(1-t)^s f(y)$ holds for all $x,y\in\mathbb{R}_0$ and $t\in[0,1]$.

It is obvious that when $s=1$, the so-called s-convexity reduces to the ordinary convexity of functions defined on $\mathbb{R}_0$.

Some inequalities of Hermite–Hadamard type for s-convex functions may be narrated as follows.

Theorem 1.4 ([3]). *Suppose that $f:\mathbb{R}_0\to\mathbb{R}_0$ is a s-convex function for $s\in(0,1)$ and let $a,b\in\mathbb{R}_0$ and $a<b$. If $f'\in L_1([a,b])$, then*

$$2^{s-1}f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\le\frac{f(a)+f(b)}{s+1}. \tag{1.2}$$

The constant $\frac{1}{s+1}$ is the best possible in the right hand side inequality in (1.2).

Theorem 1.5 ([1]). *Let $f:I\subseteq\mathbb{R}_0\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L_1([a,b])$, where $a,b\in I$ and $a<b$. If $|f'|^q$ is s-convex on $[a,b]$ for some fixed $s\in(0,1]$, $q>1$, and $p=\frac{q}{q-1}$, and if $|f'(x)|\le M$, then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}\,t\right|\le\frac{M}{(1+p)^{1/p}}\left(\frac{2}{s+1}\right)^{1/q}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right],\quad x\in[a,b].$$

For more results about s-convex functions, one can see [1, 3, 4, 8] and references therein.

Definition 1.3 ([9])**.** Let $f \in L_1([a,b])$. The Riemann–Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $b > a > 0$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) \,\mathrm{d}\, t \quad \text{and} \quad J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) \,\mathrm{d}\, t$$

for $x \in (a,b)$ respectively, where Γ is the classical Euler gamma function defined for $\mathfrak{Re}(z) > 0$ by $\Gamma(z) = \int_0^\infty e^{-u} u^{z-1} \,\mathrm{d}\, u$. Moreover, define $J_{b-}^{0} f(x) = J_{a+}^{0} f(x) = f(x)$.

In the case $\alpha = 1$, the fractional integral reduces to the classical and usual integral.

Very recently, Hermite–Hadamard's inequality was extended in [9] to the case of Riemann–Liouville fractional integrals.

Theorem 1.6 ([9, Theorem 2])**.** *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $x \in [a,b]$. If f is a convex function on $[a,b]$, then*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \le \frac{f(a)+f(b)}{2}, \quad \alpha > 0.$$

Theorem 1.7 ([9, Theorem 3])**.** *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) and $a < b$. If $|f'|$ is convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right]\right| \le \frac{b-a}{2(\alpha+1)} \left(1 - \frac{1}{2^{\alpha}}\right) \left[|f'(a)| + |f'(b)|\right], \quad \alpha > 0.$$

Theorem 1.8 ([10, Theorem 7])**.** *Let $f : [a,b] \subseteq \mathbb{R}_0 \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$ such that $f' \in L_1([a,b])$. If $|f'|$ is s-convex on $[a,b]$ for some fixed $s \in (0,1]$ and $|f'(x)| \le M$, then*

$$\begin{aligned} &\left|\frac{(x-a)^{\alpha} + (b-x)^{\alpha}}{b-a} f(x) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J_{x+}^{\alpha} f(b) + J_{x-}^{\alpha} f(a)\right]\right| \\ &\qquad \le \frac{M}{b-a} \left[1 + \frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\right] \frac{(x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{\alpha+s+1}, \quad \alpha > 0, \quad x \in [a,b]. \end{aligned}$$

For recent development on fractional calculus, one can see the monographs [9, 10, 11] and the references therein.

Motivated by the above results, we establish a Riemann–Liouville fractional integral identity involving a n-times differentiable mapping and give some new Hermite–Hadamard type inequalities involving Riemann–Liouville fractional integrals for s-convex functions.

2 A lemma

In order to obtain our main results, we need the following lemma.

Lemma 2.1. *For $n \in \mathbb{N}$ and $a < b$, let $f : [a,b] \subseteq \mathbb{R}_0 \to \mathbb{R}$ be an n-times differentiable mapping on (a,b) and $\alpha > 0$. If $f^{(n)} \in L_1([a,b])$, then*

$$\frac{\Gamma(\alpha+n)}{2(b-a)^\alpha}\big[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\big]=\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\big[f^{(k)}(a)+(-1)^k f^{(k)}(b)\big]$$
$$-\frac{(b-a)^n}{2}\int_0^1\big[(-1)^{n-1}(1-t)^{\alpha+n-1}-t^{\alpha+n-1}\big]f^{(n)}(ta+(1-t)b)\,\mathrm{d}\,t.$$

Proof. When $n=1$, by integrating by part in the right-hand side of (2.1), we have

$$\frac{b-a}{2}\int_0^1\big[(1-t)^\alpha-t^\alpha\big]f'(ta+(1-t)b)\,\mathrm{d}\,t$$
$$=\frac{f(a)+f(b)}{2}-\frac{\alpha}{2}\int_0^1\big[(1-t)^{\alpha-1}+t^{\alpha-1}\big]f(ta+(1-t)b)\,\mathrm{d}\,t, \quad (2.1)$$

where

$$\alpha\int_0^1(1-t)^{\alpha-1}f(ta+(1-t)b)\,\mathrm{d}\,t=\frac{\alpha}{b-a}\int_a^b\left(\frac{x-a}{b-a}\right)^{\alpha-1}f(x)\,\mathrm{d}\,x=\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}J_{b-}^\alpha f(a) \quad (2.2)$$

and

$$\alpha\int_0^1 t^{\alpha-1}f(ta+(1-t)b)\,\mathrm{d}\,t=\frac{\alpha}{b-a}\int_a^b\left(\frac{b-x}{b-a}\right)^{\alpha-1}f(x)\,\mathrm{d}\,x=\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}J_{a+}^\alpha f(b). \quad (2.3)$$

Substituting (2.2) and (2.3) into (2.1) yields the identity (2.1) for $n=1$.

When $n=m-1$ and $m\ge2$, suppose that the identity (2.1) is valid. When $n=m$, by the hypothesis, we have

$$\frac{(b-a)^m}{2}\int_0^1\big[(-1)^{m-1}(1-t)^{\alpha+m-1}-t^{\alpha+m-1}\big]f^{(m)}(ta+(1-t)b)\,\mathrm{d}\,t$$
$$=\frac{(b-a)^{m-1}}{2}\bigg\{\big[f^{(m-1)}(a)+(-1)^{m-1}f^{(m-1)}(b)\big]$$
$$+(\alpha+m-1)\int_0^1\big[(-1)^{m-2}(1-t)^{\alpha+m-2}-t^{\alpha+m-2}\big]f^{(m-1)}(ta+(1-t)b)\,\mathrm{d}\,t\bigg\}$$
$$=\frac{(b-a)^{m-1}}{2}\big[f^{(m-1)}(a)+(-1)^{m-1}f^{(m-1)}(b)\big]$$
$$+\frac{(\alpha+m-1)(b-a)^{m-1}}{2}\int_0^1\big[(-1)^{m-2}(1-t)^{\alpha+m-2}-t^{\alpha+m-2}\big]f^{(m-1)}(ta+(1-t)b)\,\mathrm{d}\,t$$
$$=\frac{(b-a)^{m-1}}{2}\big[f^{(m-1)}(a)+(-1)^{m-1}f^{(m-1)}(b)\big]$$
$$+\sum_{k=0}^{m-2}\frac{(\alpha+m-1)\Gamma(\alpha+m-1)(b-a)^k}{2\Gamma(\alpha+k+1)}\big[f^{(k)}(a)+(-1)^k f^{(k)}(b)\big]$$
$$-\frac{(\alpha+m-1)\Gamma(\alpha+m-1)}{2(b-a)^\alpha}\big[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\big]$$
$$=\sum_{k=0}^{m-1}\frac{\Gamma(\alpha+m)(b-a)^k}{2\Gamma(\alpha+k+1)}\big[f^{(k)}(a)+(-1)^k f^{(k)}(b)\big]-\frac{\Gamma(\alpha+m)}{2(b-a)^\alpha}\big[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\big].$$

Therefore, when $n = m$, the identity (2.1) holds. By induction, the proof of Lemma 2.1 is complete. $\square$

Remark 2.1. When $n = 1$ in (2.1), we obtain the identity

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right] = \frac{b-a}{2}\int_0^1 \left[(1-t)^\alpha - t^\alpha\right] f'(ta+(1-t)b)\,\mathrm{d}\,t,$$

which is the identity established in [9].

3 Hermite–Hadamard type inequalities involving Riemann–Liouville fractional integrals

Now we start out to establish some new Hermite–Hadamard type inequalities involving Riemann–Liouville fractional integrals for s-convex functions.

Theorem 3.1. *For $n \in \mathbb{N}$ and $a, b \in \mathbb{R}_0$ with $a < b$, let $f : \mathbb{R}_0 \to \mathbb{R}$ be an n-times differentiable function on $\mathbb{R}_0$ such that $f^{(n)} \in L_1([a,b])$. If $|f^{(n)}|^q$ is s-convex on $[a,b]$ for $q \ge 1$ and some fixed $s \in (0,1]$, then*

$$\begin{aligned}\Bigg|\frac{\Gamma(\alpha+n)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right] - \sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left[f^{(k)}(a) + (-1)^k f^{(k)}(b)\right]\Bigg| \\ \le \frac{(b-a)^n}{2(\alpha+n)^{1-1/q}}\Bigg\{\bigg[B(s+1,\alpha+n)\left|f^{(n)}(a)\right|^q + \frac{1}{\alpha+n+s}\left|f^{(n)}(b)\right|^q\bigg]^{1/q} \\ + \bigg[\frac{1}{\alpha+n+s}\left|f^{(n)}(a)\right|^q + B(s+1,\alpha+n)\left|f^{(n)}(b)\right|^q\bigg]^{1/q}\Bigg\},\end{aligned}$$

where $\alpha > 0$ and B is the classical Beta function which may be defined for $\mathfrak{Re}(x) > 0$ and $\mathfrak{Re}(y) > 0$ by $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}\,t$.

Proof. By Lemma 2.1, s-convexity of $|f^{(n)}|^q$, and Hölder's inequality, we obtain

$$\begin{aligned}&\Bigg|\frac{\Gamma(\alpha+n)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right] - \sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left[f^{(k)}(a) + (-1)^k f^{(k)}(b)\right]\Bigg| \\ &\le \frac{(b-a)^n}{2}\left[\int_0^1 (1-t)^{\alpha+n-1}\left|f^{(n)}(ta+(1-t)b)\right|\mathrm{d}\,t + \int_0^1 t^{\alpha+n-1}\left|f^{(n)}(ta+(1-t)b)\right|\mathrm{d}\,t\right] \\ &\le \frac{(b-a)^n}{2}\Bigg\{\left[\int_0^1 (1-t)^{\alpha+n-1}\,\mathrm{d}\,t\right]^{1-1/q}\left[\int_0^1 (1-t)^{\alpha+n-1}\left|f^{(n)}(ta+(1-t)b)\right|^q\mathrm{d}\,t\right]^{1/q} \\ &\quad + \left[\int_0^1 t^{\alpha+n-1}\,\mathrm{d}\,t\right]^{1-1/q}\left[\int_0^1 t^{\alpha+n-1}\left|f^{(n)}(ta+(1-t)b)\right|^q\mathrm{d}\,t\right]^{1/q}\Bigg\} \\ &\le \frac{(b-a)^n}{2(\alpha+n)^{1-1/q}}\Bigg\{\left[\int_0^1\Big((1-t)^{\alpha+n-1}t^s\left|f^{(n)}(a)\right|^q + (1-t)^{\alpha+n+s-1}\left|f^{(n)}(b)\right|^q\Big)\mathrm{d}\,t\right]^{1/q}\end{aligned}$$

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$$
\begin{aligned}
&+\left[\int_0^1 \left(t^{\alpha+n+s-1}\big|f^{(n)}(a)\big|^q + t^{\alpha+n-1}(1-t)^s\big|f^{(n)}(b)\big|^q\right)\mathrm{d}\,t\right]^{1/q}\Bigg\}\\
&=\frac{(b-a)^n}{2(\alpha+n)^{1-1/q}}\left\{\left[B(s+1,\alpha+n)\big|f^{(n)}(a)\big|^q + \frac{1}{\alpha+n+s}\big|f^{(n)}(b)\big|^q\right]^{1/q}\right.\\
&\left.+\left[\frac{1}{\alpha+n+s}\big|f^{(n)}(a)\big|^q + B(s+1,\alpha+n)\big|f^{(n)}(b)\big|^q\right]^{1/q}\right\}.
\end{aligned}
$$

Theorem 3.1 is proved. □

Corollary 3.1.1. *Under the assumptions of Theorem 3.1,*

1. *when $s=1$, we have*

$$
\begin{aligned}
&\left|\frac{\Gamma(\alpha+n)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right|\\
&\quad\le\frac{(b-a)^n}{2(\alpha+n)(\alpha+n+1)^{1/q}}\left\{\left[\big|f^{(n)}(a)\big|^q+(\alpha+n)\big|f^{(n)}(b)\big|^q\right]^{1/q}\right.\\
&\qquad\left.+\left[(\alpha+n)\big|f^{(n)}(a)\big|^q+\big|f^{(n)}(b)\big|^q\right]^{1/q}\right\};
\end{aligned}
$$

2. *when $n=1$, we have*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\right|\le\frac{b-a}{2(\alpha+1)^{1-1/q}}\left\{\left[B(s+1,\alpha+1)\big|f'(a)\big|^q\right.\right.\\
&\quad\left.\left.+\frac{1}{\alpha+s+1}\big|f'(b)\big|^q\right]^{1/q}+\left[\frac{1}{\alpha+s+1}\big|f'(a)\big|^q+B(s+1,\alpha+1)\big|f'(b)\big|^q\right]^{1/q}\right\};
\end{aligned}
$$

3. *when $q=1$, we have*

$$
\begin{aligned}
&\left|\frac{\Gamma(\alpha+n)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right|\\
&\quad\le\frac{(b-a)^n}{2}\left[B(s+1,\alpha+n)+\frac{1}{\alpha+n+s}\right]\left[\big|f^{(n)}(a)\big|+\big|f^{(n)}(b)\big|\right];
\end{aligned}
$$

4. *when $s=n=q=1$, we have*

$$
\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\right|\le\frac{b-a}{2(\alpha+1)}\left[\big|f'(a)\big|+\big|f'(b)\big|\right].
$$

Theorem 3.2. *For $n\in\mathbb{N}$ and $a,b\in\mathbb{R}_0$ with $a<b$, let $f:\mathbb{R}_0\to\mathbb{R}$ be an n-times differentiable function on $\mathbb{R}_0$ such that $f^{(n)}\in L_1([a,b])$. If $|f^{(n)}|^q$ is s-convex on $[a,b]$ for $q>1$ and some fixed*

$s \in (0,1]$, *then*

$$\begin{aligned}&\left|\frac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left[f^{(k)}(a)+(-1)^kf^{(k)}(b)\right]\right|\\&\quad\le\frac{(b-a)^n}{2}\left[\frac{q-1}{q(\alpha+n)-r-1}\right]^{1-1/q}\left\{\left[B(s+1,r+1)\left|f^{(n)}(a)\right|^q+\frac{1}{r+s+1}\left|f^{(n)}(b)\right|^q\right]^{1/q}\right.\\&\qquad\left.+\left[\frac{1}{r+s+1}\left|f^{(n)}(a)\right|^q+B(s+1,r+1)\left|f^{(n)}(b)\right|^q\right]^{1/q}\right\}\end{aligned}$$

for $\alpha>0$ *and* $0\le r\le q(\alpha+n-1)$.

Proof. From Lemma 2.1, s-convexity of $|f^{(n)}|^q$, and the Hölder's inequality, it follows that

$$\begin{aligned}&\left|\frac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left[f^{(k)}(a)+(-1)^kf^{(k)}(b)\right]\right|\\&\le\frac{(b-a)^n}{2}\left[\int_0^1(1-t)^{\alpha+n-1}\left|f^{(n)}(ta+(1-t)b)\right|\,\mathrm{d}\,t+\int_0^1t^{\alpha+n-1}\left|f^{(n)}(ta+(1-t)b)\right|\,\mathrm{d}\,t\right]\\&\le\frac{(b-a)^n}{2}\left\{\left[\int_0^1(1-t)^{[q(\alpha+n-1)-r]/(q-1)}\,\mathrm{d}\,t\right]^{1-1/q}\left[\int_0^1(1-t)^r\left|f^{(n)}(ta+(1-t)b)\right|^q\,\mathrm{d}\,t\right]^{1/q}\right.\\&\quad\left.+\left[\int_0^1t^{[q(\alpha+n-1)-r]/(q-1)}\,\mathrm{d}\,t\right]^{1-1/q}\left[\int_0^1t^r\left|f^{(n)}(ta+(1-t)b)\right|^q\,\mathrm{d}\,t\right]^{1/q}\right\}\\&\le\frac{(b-a)^n}{2}\left[\frac{q-1}{q(\alpha+n)-r-1}\right]^{1-1/q}\left\{\left[\int_0^1\Big((1-t)^rt^s\left|f^{(n)}(a)\right|^q+(1-t)^{r+s}\left|f^{(n)}(b)\right|^q\Big)\,\mathrm{d}\,t\right]^{1/q}\right.\\&\quad\left.+\left[\int_0^1\Big(t^{r+s}\left|f^{(n)}(a)\right|^q+t^r(1-t)^s\left|f^{(n)}(b)\right|^q\Big)\,\mathrm{d}\,t\right]^{1/q}\right\}\\&=\frac{(b-a)^n}{2}\left[\frac{q-1}{q(\alpha+n)-r-1}\right]^{1-1/q}\left\{\left[B(s+1,r+1)\left|f^{(n)}(a)\right|^q+\frac{1}{r+s+1}\left|f^{(n)}(b)\right|^q\right]^{1/q}\right.\\&\quad\left.+\left[B(1,r+s+1)\left|f^{(n)}(a)\right|^q+\frac{1}{r+s+1}\left|f^{(n)}(b)\right|^q\right]^{1/q}\right\}.\end{aligned}$$

Theorem 3.2 is proved. □

Corollary 3.2.1. *Under the assumptions of Theorem 3.2,*

1. *if* $s=1$, *then*

$$\begin{aligned}&\left|\frac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left[f^{(k)}(a)+(-1)^kf^{(k)}(b)\right]\right|\\&\qquad\le\frac{(b-a)^n}{2\big((r+1)(r+2)\big)^{1/q}}\left[\frac{q-1}{q(\alpha+n)-r-1}\right]^{1-1/q}\\&\qquad\quad\times\left\{\left[\left|f^{(n)}(a)\right|^q+(r+1)\left|f^{(n)}(b)\right|^q\right]^{1/q}+\left[(r+1)\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(b)\right|^q\right]^{1/q}\right\};\end{aligned}$$

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2. if $n=1$, *then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\right|\le\frac{b-a}{2}\left[\frac{q-1}{q(\alpha+1)-r-1}\right]^{1-1/q}$$
$$\times\left\{\left[B(s+1,r+1)\left|f'(a)\right|^q+\frac{1}{r+s+1}\left|f'(b)\right|^q\right]^{1/q}\right.$$
$$\left.+\left[\frac{1}{r+s+1}\left|f'(a)\right|^q+B(s+1,r+1)\left|f'(b)\right|^q\right]^{1/q}\right\};$$

3. is $s=n=1$, *then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\right|\le\frac{b-a}{2\left[(r+1)(r+2)\right]^{1/q}}$$
$$\times\left[\frac{q-1}{q(\alpha+1)-r-1}\right]^{1-1/q}\left\{\left[\left|f'(a)\right|^q+(r+1)\left|f'(b)\right|^q\right]^{1/q}+\left[(r+1)\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]^{1/q}\right\}.$$

Corollary 3.2.2. *Under the assumptions of Theorem 3.2,*

1. when $r=0$, *we have*

$$\left|\frac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left[f^{(k)}(a)+(-1)^kf^{(k)}(b)\right]\right|$$
$$\le\frac{(b-a)^n}{(s+1)^{1/q}}\left[\frac{q-1}{q(\alpha+n)-1}\right]^{1-1/q}\left[\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(b)\right|^q\right]^{1/q};$$

2. when $r=0$ *and* $s=n=1$, *we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\right|$$
$$\le(b-a)\left[\frac{q-1}{q(\alpha+1)-1}\right]^{1-1/q}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{1/q};$$

3. when $r=q$, *we have*

$$\left|\frac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left[f^{(k)}(a)+(-1)^kf^{(k)}(b)\right]\right|$$
$$\le\frac{(b-a)^n}{2}\left[\frac{q-1}{q(\alpha+n-1)-1}\right]^{1-1/q}\left\{\left[B(s+1,q+1)\left|f^{(n)}(a)\right|^q+\frac{1}{q+s+1}\left|f^{(n)}(b)\right|^q\right]^{1/q}\right.$$
$$\left.+\left[\frac{1}{q+s+1}\left|f^{(n)}(a)\right|^q+B(s+1,q+1)\left|f^{(n)}(b)\right|^q\right]^{1/q}\right\};$$

4. *when $r = q$ and $s = n = 1$, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\right|\le \frac{b-a}{2\left[(q+1)(q+2)\right]^{1/q}}\left(\frac{q-1}{q\alpha-1}\right)^{1-1/q}$$
$$\times\left\{\left[\left|f'(a)\right|^q+(q+1)\left|f'(b)\right|^q\right]^{1/q}+\left[(q+1)\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]^{1/q}\right\};$$

5. *when $r = q(\alpha + n - 1)$, we have*

$$\left|\frac{\Gamma(\alpha+n)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right|$$
$$\le\frac{(b-a)^n}{2}\left\{\left[B(s+1,q(\alpha+n-1)+1)\left|f^{(n)}(a)\right|^q+\frac{1}{q(\alpha+n-1)+s+1}\left|f^{(n)}(b)\right|^q\right]^{1/q}\right.$$
$$\left.+\left[\frac{1}{q(\alpha+n-1)+s+1}\left|f^{(n)}(a)\right|^q+B(s+1,q(\alpha+n-1)+1)\left|f^{(n)}(b)\right|^q\right]^{1/q}\right\};$$

6. *when $r = q(\alpha + n - 1)$ and $s = n = 1$, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\right|\le \frac{b-a}{2\left[(q\alpha+1)(q\alpha+2)\right]^{1/q}}$$
$$\times\left\{\left[\left|f'(a)\right|^q+(q\alpha+1)\left|f'(b)\right|^q\right]^{1/q}+\left[(q\alpha+1)\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]^{1/q}\right\}.$$

Theorem 3.3. *For $n \in \mathbb{N}$ and $a, b \in \mathbb{R}_0$ with $a < b$, let $f : \mathbb{R}_0 \to \mathbb{R}$ be an n-times differentiable function on $\mathbb{R}_0$ such that $f^{(n)} \in L_1([a, b])$. If $|f^{(n)}|^q$ is s-concave on $[a, b]$ for $q > 1$ and some fixed $s \in (0, 1]$, then*

$$\left|\frac{\Gamma(\alpha+n)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right|$$
$$\le\frac{(b-a)^n}{2^{(1-s)/q}}\left[\frac{q-1}{q(\alpha+n)-1}\right]^{1-1/q}\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|,\quad \alpha>0.$$

Proof. Using Lemma 2.1 and the well-known Hölder's inequality yields

$$\left|\frac{\Gamma(\alpha+n)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right|$$
$$\le\frac{(b-a)^n}{2}\left[\int_0^1(1-t)^{\alpha+n-1}\left|f^{(n)}(ta+(1-t)b)\right|\,\mathrm{d}\,t+\int_0^1 t^{\alpha+n-1}\left|f^{(n)}(ta+(1-t)b)\right|\,\mathrm{d}\,t\right]$$
$$\le\frac{(b-a)^n}{2}\left\{\left[\int_0^1(1-t)^{q(\alpha+n-1)/(q-1)}\,\mathrm{d}\,t\right]^{1-1/q}\left[\int_0^1\left|f^{(n)}(ta+(1-t)b)\right|^q\,\mathrm{d}\,t\right]^{1/q}\right.$$

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$$
\begin{aligned}
&+\left[\int_0^1 t^{q(\alpha+n-1)/(q-1)}\,\mathrm{d}\,t\right]^{1-1/q}\left[\int_0^1\left|f^{(n)}(ta+(1-t)b)\right|^q\mathrm{d}\,t\right]^{1/q}\Bigg\}\\
&=(b-a)^n\left[\frac{q-1}{q(\alpha+n)-1}\right]^{1-1/q}\left[\int_0^1\left|f^{(n)}(ta+(1-t)b)\right|^q\mathrm{d}\,t\right]^{1/q}.
\end{aligned}
$$

Since $|f^{(n)}|^q$ is s-concave, we have

$$
\int_0^1\left|f^{(n)}(ta+(1-t)b)\right|^q\mathrm{d}\,t\le 2^{s-1}\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q.
$$

Combining the above two inequalities yields (3.3). The proof of Theorem 3.3 is complete. □

Corollary 3.3.1. *Under the assumptions of Theorem 3.3,*

1. *if $s=1$, then*

$$
\begin{aligned}
&\left|\frac{\Gamma(\alpha+n)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right|\\
&\qquad\le(b-a)^n\left[\frac{q-1}{q(\alpha+n)-1}\right]^{1-1/q}\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|;
\end{aligned}
$$

2. *if $n=1$, then*

$$
\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\right|\le\frac{b-a}{2^{(1-s)/q}}\left[\frac{q-1}{q(\alpha+1)-1}\right]^{1-1/q}\left|f'\left(\frac{a+b}{2}\right)\right|;
$$

3. *if $s=n=1$, then*

$$
\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\right|\le(b-a)\left[\frac{q-1}{q(\alpha+1)-1}\right]^{1-1/q}\left|f'\left(\frac{a+b}{2}\right)\right|.
$$

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A Monotone Hybrid Projection Algorithm for Solving Fixed Point and Equilibrium Problems in a Banach Space

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Abstract. In this paper, an uncountable infinite family of nonlinear mappings are investigated. Strong convergence theorems of common solutions are established in a strictly convex and uniformly smooth Banach space which also has the Kadec-Klee property. The results obtained in this paper unify and improve many corresponding results announced recently.

Keywords: quasi-ϕ-nonexpansive mapping; equilibrium problem; fixed point; projection.

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1 Introduction

Recently, common solution problems have been intensively investigated based on iterative methods. The so called common solution problems which capture lots of applications in multi-disciplines such as image restoration, and radiation therapy treatment planning are to find a special point in the intersection of a family of convex sets, which are usually considered as solution sets of nonlinear problems; see [1]-[15] and the references therein. Mean-valued iterative processes, in particular, Mann iterative process and Ishikawa iterative process, are efficient and powerful for studying fixed points of Lipschitz continuous nonlinear operators. However, in the framework of infinite-dimensional Hilbert spaces,

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they are only weakly convergent; see [16], [17] and the references therein. In many modern disciplines, including image recovery, economics, control theory, and quantum physics, problems arises in the framework of infinite dimension spaces. In such nonlinear problems, strong convergence is often much more desirable than the weak convergence; see [18] and the references therein. To guarantee the strong convergence of mean-valued iteration processes, many authors use different regularization methods. The projection method which was first introduced by Haugazeau [19] has been considered for the approximation of fixed points of nonexpansive mappings. The advantage of projection methods is that strong convergence of iterative sequences can be guaranteed without compact restrictions imposed on operators.

In this paper, we study a common solution problem via projection methods. Strong convergence theorems of common solutions are established with the aid of a generalized projection in a Banach space. The results obtained in this paper mainly unify and improve the corresponding results in [20]-[30].

2 Preliminaries

Let E be a real Banach space and let E^* be the dual space of E. Let B_E be the unit sphere of E. Recall that E is said to be a strictly convex space if for all $x, y \in B_E$ and $x \neq y$, $\|x + y\| < 2$. It is said to be uniformly convex if for any $\epsilon \in (0, 2]$ there exists $\delta > 0$ such that for any $x, y \in B_E$,

$$\|x - y\| \geq \epsilon \quad \text{implies} \quad \|x + y\| \leq 2 - 2\delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex; see [31] and the references therein.

Recall that E is said to have a Gâteaux differentiable norm if for all $x, y \in B_E$. $\lim_{t\to 0}(\|\frac{x}{t} + y\| - \|\frac{x}{t}\|)$. In this case, we also say that E is a smooth space. E is said to have a uniformly Gâteaux differentiable norm if for each $y \in B_E$, the limit is attained uniformly for all $x \in B_E$. E is also said to have a uniformly Fréchet differentiable norm if the above limit is attained uniformly for $x, y \in B_E$. In this case, we say that E is uniformly smooth. It is known that a uniformly smooth Banach space is reflexive and smooth.

Recall that normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx = \{y \in E^* : \|x\|^2 = \langle x, y\rangle = \|y\|^2\}.$$

It is known if E is uniformly smooth, then J is uniformly norm-to-norm continuous on every bounded subset of E; if E is a strictly convex Banach space, then

J is strictly monotone; if E is a smooth Banach space, then J is single-valued and demicontinuous, i.e.,continuous from the strong topology of E to the weak star topology of E; if E is a reflexive and strictly convex Banach space with a strictly convex dual E^* and $J^* : E^* \to E$ is the normalized duality mapping in E^*, then $J^{-1} = J^*$; if E is a smooth, strictly convex and reflexive Banach space, then J is single-valued, one-to-one and onto.

Recall that E has the Kadec-Klee Property (KKP) if $\lim_{m\to\infty} \|x_m - x\| = 0$, for any sequence $\{x_m\} \subset E$, and $x \in E$ with $\{x_n\}$ converges weakly to x, and $\{\|x_n\|\}$ converges strongly to $\|x\|$. It is known that every uniformly convex Banach space has the KKP; see [31] and the references therein.

Let C be a nonempty closed and convex subset of E and let $B : C \times C \to \mathbb{R}$ be a function. Recall that the following equilibrium problem in the terminology of Blum and Oettli [32]. Find $\bar{x} \in C$ such that $B(\bar{x}\, y) \geq 0$, $\forall y \in C$. We use $Sol(B)$ to denote the solution set of the equilibrium problem. That is, $Sol(B) = \{x \in C : B(x, y) \geq 0, \forall y \in C\}$.

The following restrictions are essential for solving the equilibrium problem in this paper.

(R-1) $B(a, a) \equiv 0, \forall a \in C$;

(R-2) $B(b, a) + B(a, b) \leq 0, \forall a, b \in C$;

(R-3) $B(a, b) \geq \limsup_{t\downarrow 0} B(tc + (1-t)a, b)$, $\forall a, b, c \in C$;

(R-4) $b \mapsto B(a, b)$ is convex and weakly lower semi-continuous, $\forall a \in C$.

Let T be a self mapping on C. T is said to be closed if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = \bar{x}$ and $\lim_{m\to\infty} Tx_n = \bar{y}$, then $\bar{y} = T\bar{x}$. Let B be a bounded subset of C. Recall that T is said to be uniformly asymptotically regular on C if and only if $\limsup_{n\to\infty} \sup_{x\in B}\{\|T^n x - T^{n+1}x\|\} = 0$. From now on, we use $\to$ and $\rightharpoonup$ to stand for the strong convergence and weak convergence, respectively. and use $Fix(T)$ to denote the fixed point set of mapping T.

Recall that a point p is said to be an asymptotic fixed point of mapping T if and only if subset C contains a sequence $\{x_m\}$ which converges weakly to p such that $\lim_{m\to\infty} \|Tx_m - x_m\| = 0$. We use $\widetilde{Fix}(T)$ to stand for the asymptotic fixed point set in this paper.

Next, we assume that E is a smooth Banach space which means duality mapping J is single-valued. Study the functional

$$\phi(x, y) := \|x\|^2 + \|y\|^2 - 2\langle x, Jy\rangle, \quad \forall x, y \in E.$$

In [33], Alber studied a generalized projection $Proj_C : E \to C$, which is a mapping assigning to an arbitrary point $x \in E$ the minimum point of $\phi(x, y)$, which implies from the definition of ϕ $\phi(x, y)+2\|x\|\|y\| \geq \|x\|^2+\|y\|^2, \forall x, y \in E$.

T is said to be relatively nonexpansive iff

$$\phi(p, x) \geq \phi(p, Tx), \quad \forall x \in C, \forall p \in \widetilde{Fix}(T) = Fix(T) \neq \emptyset.$$

T is said to be relatively asymptotically nonexpansive iff

$$\phi(p, x) + \xi_n\phi(p, x) \geq \phi(p, T^n x), \quad \forall x \in C, \forall p \in Fix(T) = \widetilde{Fix}(T) \neq \emptyset, \forall n \geq 1,$$

where $\{\xi_n\} \subset [0, \infty)$ is a sequence such that $\mu_n \to 0$ as $n \to \infty$.

Remark 2.1. The class of relatively asymptotically nonexpansive mappings, which was first considered in [34], covers the class of relatively nonexpansive mappings [35].

T is said to be quasi-ϕ-nonexpansive iff

$$\phi(p, x) \geq \phi(p, Tx), \quad \forall x \in C, \forall p \in Fix(T) \neq \emptyset.$$

T is said to be asymptotically quasi-ϕ-nonexpansive if and only if there exists a sequence $\{\xi_n\} \subset [0, \infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$\phi(p, x) + \xi_n\phi(p, x) \geq \phi(p, T^n x), \quad \forall x \in C, \forall p \in Fix(T) \neq \emptyset, \forall n \geq 1.$$

Remark 2.2. The class of quasi-ϕ-nonexpansive mappings [26] and the class of asymptotically quasi-ϕ-nonexpansive mappings [27] cover the class of relatively nonexpansive mappings and the class of relatively asymptotically nonexpansive mappings. Quasi-ϕ-nonexpansive mappings and asymptotically quasi-ϕ-nonexpansive mappings do not require the strong restriction that the fixed point set equals the asymptotic fixed point set.

Remark 2.3. The class of quasi-ϕ-nonexpansive mappings and the class of asymptotically quasi-ϕ-nonexpansive mappings are generalizations of the class of quasi-nonexpansive mappings and the class of asymptotically quasi-nonexpansive mappings in Banach spaces because of $\sqrt{\phi(x, y)} = \|x - y\|$.

The following lemmas also play an important role in this paper.

Lemma 2.4. [33] *Let E be a strictly convex, reflexive, and smooth Banach space and let C be a nonempty, closed, and convex subset of E. Let $x \in E$. Then*

$$\phi(y, \Pi_C x) \leq \phi(y, x) - \phi(\Pi_C x, x), \quad \forall y \in C,$$

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$\langle y - x_0, Jx - Jx_0 \rangle \le 0$, $\forall y \in C$ *if and only if* $x_0 = \Pi_C x$.

Lemma 2.5. ([26], [32]) *Let E be a strictly convex, smooth, and reflexive Banach space and let C be a closed convex subset of E. Let B be a function with the restrictions (R-1), (R-2), (R-3) and (R-4), from $C \times C$ to $\mathbb{R}$. Let $x \in E$ and let $r > 0$. Then there exists $z \in C$ such that $rB(z, y) + \langle z - y, Jz - Jx \rangle \le 0$, $\forall y \in C$ Define a mapping $K^{B,r}$ by*

$$K^{B,r}x = \{z \in C : rB(z, y) + \langle y - z, Jz - Jx \rangle \ge 0, \quad \forall y \in C\}.$$

The following conclusions hold:

(1) *$K^{B,r}$ is single-valued quasi-ϕ-nonexpansive;*

(2) *$Sol(B) = Fix(K^{B,r})$ is closed and convex.*

Lemma 2.6 [36] *Let E be a strictly convex and uniformly smooth Banach space which also has the KKP. Let C be a convex and closed subset of E and let T be an asymptotically quasi-ϕ-nonexpansive mapping on C. $Fix(T)$ is convex.*

Lemma 2.7 [37] *Let r be a positive real number and let E be uniformly convex. Then there exists a convex, strictly increasing and continuous function cof : $[0, 2r] \to \mathbb{R}$ such that $cof(0) = 0$ and*

$$t\|a\|^2 + (1-t)\|b\|^2 \ge \|(1-t)b + ta\|^2 + t(1-t)cof(\|b - a\|)$$

for all $t \in [0, 1]$ and $a, b \in B^r := \{a \in E : \|a\| \le r\}$.

3 Main results

Theorem 3.1. *Let E be a strictly convex and uniformly smooth Banach space which also has the KKP. Let C be a convex and closed subset of E and let Λ be an arbitrary index set. Let B_i be a bifunction with (R-1), (R-2), (R-3) and (R-4). Let T_i be an asymptotically quasi-ϕ-nonexpansive mapping on C for every $i \in \Lambda$. Assume that T_i is uniformly asymptotically regular and closed for every $i \in \Lambda$ and $\cap_{i\in\Lambda} Sol(B_i) \bigcap \cap_{i\in\Lambda} Fix(T_i)$ is nonempty and bounded. Let $\{x_j\}$ be a*

sequence generated by

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_{(1,i)} = C, \forall i \in \Lambda, \\ C_1 = \cap_{i\in\Lambda} C_{(1,i)}, x_1 = Proj_{C_1} x_0, \\ Jy_{(j,i)} = \alpha_{(j,i)} JT_i^j x_j + (1-\alpha_{(j,i)}) Ju_{(j,i)}, \\ C_{(j+1,i)} = \{z \in C_{(j,i)} : \phi(z, y_{(j,i)}) - \phi(z, x_j) \le \alpha_{(j,i)} \xi_{(j,i)} D_{(j,i)}\}, \\ C_{j+1} = \cap_{i\in\Lambda} C_{(j+1,i)}, x_{j+1} = Proj_{C_{j+1}} x_1, \end{cases}$$

where $u_{(j,i)}$ *is such that* $r_{(j,i)} B_i(u_{(j,i)}, \mu) \ge \langle u_{(j,i)} - \mu, Ju_{(j,i)} - Jx_j \rangle$, $\forall \mu \in C_j$, $D_{(j,i)} = \sup\{\phi(z, x_j) : z \in \cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i)\}$, $\{\alpha_{(j,i)}\}$ *is a real sequence in* $(0,1)$ *such that* $\liminf_{j\to\infty} \alpha_{(j,i)}(1-\alpha_{(j,i)}) > 0$ *and* $\{r_{(j,i)}\} \subset [r, \infty)$ *is a real sequence, where* r *is some positive real number. Then* $\{x_j\}$ *converges strongly to* $Proj_{\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i)} x_1$.

Proof. First, we prove $\cap_{i\in\Lambda} Sol(B_i) \bigcap \cap_{i\in\Lambda} Fix(T_i)$ is convex and closed. Using Lemma 2.5 and 2.6, we find that $Sol(B_i)$ is convex and closed and $Fix(T_i)$ is convex for every $i \in \Lambda$. Since T_i is closed, we find that $Fix(T_i)$ is also closed. So, $Proj_{\cap_{i\in\Lambda} Sol(B_i) \bigcap \cap_{i\in\Lambda} Fix(T_i)} x$ is well defined, for any element x in E.

Next, we prove that C_j is convex and closed. It is obvious that $C_{(1,i)} = C$ is convex and closed. Assume that $C_{(m,i)}$ is convex and closed for some $m \ge 1$. Let $p_1, p_2 \in C_{(m+1,i)}$. It follows that $p = sp_1 + (1-s)p_2 \in C_{(m,i)}$, where $s \in (0,1)$. Notice that $\phi(p_1, y_{(m,i)}) - \phi(p_1, x_m) \le \alpha_{(m,i)} \xi_{(m,i)} D_{(m,i)}$, and $\phi(p_2, y_{(m,i)}) - \phi(p_2, x_m) \le \alpha_{(m,i)} \xi_{(m,i)} D_{(m,i)}$. Hence, one has

$$2\langle p_1, Jx_m - Jy_{(m,i)} \rangle - \|x_m\|^2 + \|y_{(m,i)}\|^2 \le \alpha_{(m,i)} \xi_{(m,i)} D_{(m,i)},$$

and

$$2\langle p_2, Jx_m - Jy_{(m,i)} \rangle - \|x_m\|^2 + \|y_{(m,i)}\|^2 \le \alpha_{(m,i)} \xi_{(m,i)} D_{(m,i)}.$$

Using the above two inequalities, one has $\phi(p, y_{(m,i)}) - \phi(p, x_m) \le \alpha_{(m,i)} \xi_{(m,i)} D_{(m,i)}$. This shows that $C_{(m+1,i)}$ is closed and convex. Hence, $C_j = \cap_{i\in\Lambda} C_{(j,i)}$ is a convex and closed set. This proves that $Proj_{C_{j+1}} x_1$ is well defined.

On the other hand, we find that $\cap_{i\in\Lambda} Sol(B_i) \bigcap \cap_{i\in\Lambda} Fix(T_i) \subset C_1 = C$ is clear. Suppose that $\cap_{i\in\Lambda} Sol(B_i) \bigcap \cap_{i\in\Lambda} Fix(T_i) \subset C_{(m,i)}$ for some positive

integer m. For any $w \in \cap_{i\in\Lambda} Sol(B_i) \bigcap \cap_{i\in\Lambda} Fix(T_i) \subset C_{(m,i)}$, we see that

$$\begin{aligned}\phi(z, y_{(m,i)}) &= \|z\|^2 + \|\alpha_{(m,i)} JT_i^m x_m + (1-\alpha_{(m,i)})Ju_{(m,i)}\|^2 \\ &\quad - 2\langle z, \alpha_{(m,i)} JT_i^m x_m + (1-\alpha_{(m,i)})Ju_{(m,i)}\rangle \\ &\leq \|z\|^2 + \alpha_{(m,i)}\|T_i^m x_m\|^2 + (1-\alpha_{(m,i)})\|u_{(m,i)}\|^2 \\ &\quad - 2\alpha_{(m,i)}\langle z, JT_i^m x_m\rangle - 2(1-\alpha_{(m,i)})\langle z, Ju_{(m,i)}\rangle \\ &\leq \phi(z, x_m) + \alpha_{(m,i)}\xi_{(m,i)} D_{(m,i)},\end{aligned}$$

where $D_{(m,i)} = \sup\{\phi(z, x_m) : z \in \cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i)\}$. This shows that $z \in C_{(m+1,i)}$. This implies that $\cap_{i\in\Lambda} Sol(B_i) \bigcap \cap_{i\in\Lambda} Fix(T_i) \subset \cap_{i\in\Lambda} C_{(j,i)} = C_j$. Using Lemma 2.4, one has $\langle z - x_j, Jx_1 - Jx_j\rangle \leq 0$, for any $z \in C_j$. It follows that

$$\langle z - x_j, Jx_1 - Jx_j\rangle \leq 0, \quad \forall z \in \cap_{i\in\Lambda} Sol(B_i) \bigcap \cap_{i\in\Lambda} Fix(T_i) \subset C_j. \tag{3.1}$$

Using Lemma 2.4 yields that

$$\begin{aligned}\phi(x_j, x_1) &\leq \phi(Proj_{\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i)} x_1, x_1) \\ &\quad - \phi(Proj_{\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i)} x_1, x_j),\end{aligned}$$

which shows that $\{\phi(x_j, x_1)\}$ is bounded. Hence, $\{x_j\}$ is also bounded. Without loss of generality, we assume $x_j \rightharpoonup \bar{x} \in C_j$. Hence $\phi(x_j, x_1) \leq \phi(\bar{x}, x_1)$. This implies that

$$\phi(\bar{x}, x_1) \leq \liminf_{j\to\infty}(\|x_j\|^2 + \|x_1\|^2 - 2\langle x_j, Jx_1\rangle) = \limsup_{j\to\infty} \phi(x_j, x_1) \leq \phi(\bar{x}, x_1).$$

It follows that $\lim_{j\to\infty} \phi(x_j, x_1) = \phi(\bar{x}, x_1)$. Hence, we have $\lim_{j\to\infty} \|x_j\| = \|\bar{x}\|$. Using the KKP, one obtains that $\{x_j\}$ converges strongly to $\bar{x}$ as $j \to \infty$. On the other hand, we find that $\phi(x_{j+1}, x_1) \geq \phi(x_j, x_1)$, which shows that $\{\phi(x_j, x_1)\}$ is nondecreasing. It follows that $\lim_{j\to\infty} \phi(x_j, x_1)$ exists. Since $\phi(x_{j+1}, x_1) - \phi(x_j, x_1) \geq \phi(x_{j+1}, x_j)$, one has $\lim_{j\to\infty} \phi(x_{j+1}, x_j) = 0$. Since $x_{j+1} \in C_{j+1}$, one sees that $\phi(x_{j+1}, y_{(j,i)}) - \phi(x_{j+1}, x_j) \leq \alpha_{(j,i)}\xi_{(j,i)} D_{(j,i)}$. It follows that $\lim_{j\to\infty} \phi(x_{j+1}, y_{(j,i)}) = 0$. Hence, one has $\lim_{j\to\infty}(\|y_{(j,i)}\| - \|x_{j+1}\|) = 0$. This implies that $\lim_{j\to\infty} \|Jy_{(j,i)}\| = \lim_{j\to\infty} \|y_{(j,i)}\| = \|\bar{x}\| = \|J\bar{x}\|$. This implies that $\{Jy_{(j,i)}\}$ is bounded. Without loss of generality, we assume that $\{Jy_{(j,i)}\}$ converges weakly to $y^{(*,i)} \in E^*$. In view of the reflexivity of E, we see that $J(E) = E^*$. This shows that there exists an element $y^i \in E$ such that $Jy^i = y^{(*,i)}$. It follows that $\phi(x_{j+1}, y_{(j,i)}) + 2\langle x_{j+1}, Jy_{(j,i)}\rangle = \|x_{j+1}\|^2 + \|Jy_{(j,i)}\|^2$. Taking $\liminf_{j\to\infty}$, one has $0 \geq \|\bar{x}\|^2 - 2\langle \bar{x}, y^{(*,i)}\rangle + \|y^{(*,i)}\|^2 = \|\bar{x}\|^2 + \|Jy^i\|^2 - 2\langle \bar{x}, Jy^i\rangle = \phi(\bar{x}, y^i) \geq 0$. That is, $\bar{x} = y^i$, which in turn implies that $J\bar{x} = y^{(*,i)}$.

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Hence, $Jy_{(j,i)} \rightharpoonup J\bar{x} \in E^*$. Since E^* is uniformly convex. Hence, it has the KKP, we obtain $\lim_{i\to\infty} Jy_{(j,i)} = J\bar{x}$. Since $J^{-1} : E^* \to E$ is demi-continuous and E has the KKP, one gets that $y_{(j,i)} \to \bar{x}$, as $j \to \infty$. Using the fact

$$\phi(z, x_j) - \phi(z, y_{(j,i)}) \le (\|x_j\| + \|y_{(j,i)}\|)\|y_{(j,i)} - x_j\| + 2\langle z, Jy_{(j,i)} - Jx_j\rangle,$$

we find

$$\lim_{j\to\infty} \big(\phi(z, x_j) - \phi(z, y_{(j,i)})\big) = 0. \tag{3.2}$$

On the other hand, one sees from Lemma 2.7

$$\begin{aligned}
\phi(z, y_{(j,i)}) &= \|z\|^2 + \|\alpha_{(j,i)} JT_i^j x_j + (1 - \alpha_{(j,i)})Ju_{(j,i)}\|^2 \\
&\quad - 2\langle z, \alpha_{(j,i)} JT_i^j x_j + (1 - \alpha_{(j,i)})Ju_{(j,i)}\rangle \\
&\le \|z\|^2 + \alpha_{(j,i)}\|T_i^j x_j\|^2 + (1 - \alpha_{(j,i)})\|u_{(j,i)}\|^2 \\
&\quad - \alpha_{(j,i)}(1 - \alpha_{(j,i)})cof(\||Ju_{(j,i)} - JT_i^j x_j\|) \\
&\quad - 2\alpha_{(j,i)}\langle z, JT_i^j x_j\rangle - 2(1 - \alpha_{(j,i)})\langle z, Ju_{(j,i)}\rangle \\
&\le \phi(z, x_j) + \alpha_{(j,i)}\xi_{(j,i)}D_{(j,i)} - \alpha_{(j,i)}(1 - \alpha_{(j,i)})cof(\||Ju_{(j,i)} - JT_i^j x_j\|).
\end{aligned}$$

This implies

$$\begin{aligned}
&\alpha_{(j,i)}(1 - \alpha_{(j,i)})cof(\||Ju_{(j,i)} - JT_i^j x_j\|) \\
&\le \phi(z, x_j) - \phi(z, y_{(j,i)}) + \alpha_{(j,i)}\xi_{(j,i)}D_{(j,i)}.
\end{aligned}$$

Using the restriction imposed on the sequence $\{\alpha_{(j,i)}\}$ and (3.2), one has

$$\lim_{j\to\infty} \||Ju_{(j,i)} - JT_i^j x_j\| = 0.$$

It follows that $JT_i^j x_j \to J\bar{x}$ as $j \to \infty$. Since $J^{-1} : E^* \to E$ is demi-continuous, one has $T_i^j x_j \rightharpoonup \bar{x}$. Using the fact $\|\|T_i^j x_j\| - \|\bar{x}\|\| = \|\|JT_i^j x_j\| - \|J\bar{x}\|\| \le \|JT_i^j x_j - J\bar{x}\|$, one has $\|T_i^j x_j\| \to \|\bar{x}\|$ as $j \to \infty$. Since E has the KKP, one has $\lim_{j\to\infty} \||\bar{x} - T_i^j x_j\| = 0$. Since T_i is also uniformly asymptotically regular, one has $\lim_{j\to\infty} \|\bar{x} - T_i^{j+1} x_j\| = 0$. That is, $T_i(T_i^j x_j) \to \bar{x}$. Using the closedness of T_i, we find $T_i\bar{x} = \bar{x}$. This proves $\bar{x} \in Fix(T_i)$, that is, $\bar{x} \in \cap_{i\in\Lambda} Fix(T_i)$.

Next, we show that $\bar{x} \in \cap_{i\in\Lambda} Sol(B_i)$. Since B_i is monotone, we find that

$$r_{(j,i)} B_i(\mu, u_{(j,i)}) \le \|\mu - u_{(j,i)}\|\|Ju_{(j,i)} - Jx_j\|.$$

Therefore, one sees $B_i(\mu, \bar{x}) \le 0$. For $0 < t_i < 1$, define $\mu_{(t,i)} = (1 - t_i)\bar{x} + t_i\mu$. This implies that $0 \ge B_i(\mu_{(t,i)}, \bar{x})$. Hence, we have $0 = B_i(\mu_{(t,i)}, \mu_{(t,i)}) \le t_i B_i(\mu_{(t,i)}, \mu)$. It follows that $B_i(\bar{x}, \mu) \ge 0$, $\forall \mu \in C$. This implies that $\bar{x} \in Sol(B_i)$ for every $i \in \Lambda$.

Finally, we prove $\bar{x} = Proj_{\cap_{i\in\Lambda}(Fix(T_i)\cap Sol(B_i))}x_1$. Using (3.1), one has $\langle \bar{x} - z, Jx_1 - J\bar{x}\rangle \geq 0$ $z \in \cap_{i\in\Lambda}(Fix(T_i) \cap Sol(B_i))$. Using Lemma 2.4, we find that $\bar{x} = Proj_{\cap_{i\in\Lambda}(Fix(T_i)\cap Sol(B_i))}x_1$. This completes the proof.

For the class of quasi-ϕ-nonexpansive mappings, the boundedness of the common solution set is not required. Indeed, we have the following result.

Corollary 3.2. *Let E be a strictly convex and uniformly smooth Banach space which also has the KKP. Let C be a convex and closed subset of E and let Λ be an arbitrary index set. Let B_i be a bifunction with (R-1), (R-2), (R-3) and (R-4). Let T_i be a quasi-ϕ-nonexpansive mapping on C for every $i \in \Lambda$. Assume that T_i is closed for every $i \in \Lambda$ and $\cap_{i\in\Lambda}Sol(B_i) \bigcap \cap_{i\in\Lambda}Fix(T_i)$ is nonempty. Let $\{x_j\}$ be a sequence generated by*

$$\begin{cases} x_0 \in E \textit{ chosen arbitrarily}, \\ C_{(1,i)} = C, \forall i \in \Lambda, \\ C_1 = \cap_{i\in\Lambda}C_{(1,i)}, x_1 = Proj_{C_1}x_0, \\ Jy_{(j,i)} = \alpha_{(j,i)}JT_ix_j + (1 - \alpha_{(j,i)})Ju_{(j,i)}, \\ C_{(j+1,i)} = \{z \in C_{(j,i)} : \phi(z, y_{(j,i)}) \leq \phi(z, x_j)\}, \\ C_{j+1} = \cap_{i\in\Lambda}C_{(j+1,i)}, x_{j+1} = Proj_{C_{j+1}}x_1, \end{cases}$$

where $u_{(j,i)}$ is such that $r_{(j,i)}B_i(u_{(j,i)}, \mu) \geq \langle u_{(j,i)} - \mu, Ju_{(j,i)} - Jx_j\rangle$, $\forall \mu \in C_j$, $D_{(j,i)} = \sup\{\phi(z, x_j) : z \in \cap_{i\in\Lambda}Fix(T_i) \bigcap \cap_{i\in\Lambda}Sol(B_i)\}$, $\{\alpha_{(j,i)}\}$ is a real sequence in $(0, 1)$ such that $\liminf_{j\to\infty} \alpha_{(j,i)}(1-\alpha_{(j,i)}) > 0$ and $\{r_{(j,i)}\} \subset [r, \infty)$ is a real sequence, where r is some positive real number. Then $\{x_j\}$ converges strongly to $Proj_{\cap_{i\in\Lambda}Fix(T_i)\bigcap\cap_{i\in\Lambda}Sol(B_i)}x_1$.

From Theorem 3.1, we also have the following result.

Corollary 3.3. *Let E be a strictly convex and uniformly smooth Banach space which also has the KKP. Let C be a convex and closed subset of E and let B be a bifunction with (R-1), (R-2), (R-3) and (R-4). Let T be an asymptotically quasi-ϕ-nonexpansive mapping on C. Assume that T is uniformly asymptotically regular and closed and $Sol(B) \cap Fix(T)$ is nonempty and bounded. Let $\{x_j\}$ be*

a sequence generated by

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, x_1 = Proj_{C_1} x_0, \\ Jy_j = \alpha_j JT^j x_j + (1-\alpha_j) Ju_j, \\ C_{j+1} = \{z \in C_j : \phi(z, y_j) - \phi(z, x_j) \le \alpha_j \xi_j D_j\}, \\ x_{j+1} = Proj_{C_{j+1}} x_1, \end{cases}$$

where u_j *is such that* $r_j B(u_j, \mu) \ge \langle u_j - \mu, Ju_j - Jx_j\rangle$, $\forall \mu \in C_j$, $D_j = \sup\{\phi(z, x_j) : z \in Fix(T) \cap Sol(B)\}$, $\{\alpha_j\}$ *is a real sequence in* $(0,1)$ *such that* $\liminf_{j\to\infty} \alpha_j(1-\alpha_j) > 0$ *and* $\{r_j\} \subset [r, \infty)$ *is a real sequence, where* r *is some positive real number. Then* $\{x_j\}$ *converges strongly to* $Proj_{Fix(T)\cap Sol(B)} x_1$.

4 Applications

In this section, we consider common solutions of a family of variational inequalities in the framework Banach spaces. we give some deduced results of our main results in the framework of Hilbert spaces.

Let $A : C \to E^*$ be a single valued monotone operator which is continuous along each line segment in C with respect to the weak* topology of E^* (hemicontinuous). Recall the the following variational inequality. Finding a point $x \in C$ such that $\langle x - y, Ax\rangle \le 0, \forall y \in C$. The symbol $Nc(x)$ stand for the normal cone for C at a point $x \in C$; that is, $Nc(x) = \{x^* \in E^* : \langle x - y, x^*\rangle \ge 0, \ \forall y \in C\}$. From now on, we use $VI(C, A)$ to denote the solution set of the variational inequality.

Theorem 4.1. *Let* E *be a strictly convex and uniformly smooth Banach space which also has the KKP. Let* C *be a convex and closed subset of* E. *Let* Λ *be an index set and let* $A_i : C \to E^*$ *be a single valued, monotone and hemicontinuous operator. Let* B_i *be a bifunction with (R-1), (R-2), (R-3) and (R-4). Assume that* $\cap_{i\in\Lambda} VI(C, A_i)$ *is not empty. Let* $\{x_n\}$ *be a sequence generated in the*

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following process.

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_{(1,i)} = C, \forall i \in \Lambda, \\ C_1 = \cap_{i\in\Lambda} C_{(1,i)}, x_1 = Proj_{C_1} x_0, \\ u_{(n,i)} = VI(C, A_i + \frac{1}{r_i}(J - Jx_n)), \\ Jy_{(j,i)} = \alpha_{(j,i)} Jx_j + (1-\alpha_{(j,i)}) Ju_{(j,i)}, \\ C_{(j+1,i)} = \{z \in C_{(j,i)} : \phi(z, y_{(j,i)}) \le \phi(z, x_j)\}, \\ C_{j+1} = \cap_{i\in\Lambda} C_{(j+1,i)}, x_{j+1} = Proj_{C_{j+1}} x_1, \end{cases}$$

where $\{\alpha_{(j,i)}\}$ *is a real sequence in* $(0,1)$ *such that* $\liminf_{j\to\infty} \alpha_{(j,i)}(1-\alpha_{(j,i)}) > 0$. *Then* $\{x_j\}$ *converges strongly to* $Proj_{\cap_{i\in\Lambda} VI(C,A_i)} x_1$.

Proof. Define a new operator M_i by $M_i x = A_i x + Nc(x)$, $x \in C$, $M_i x = \emptyset$, $x \notin C$. Hence, M_i is maximal monotone and $M_i^{-1}(0) = VI(C, A_i)$, where $M_i^{-1}(0)$ stand for the zero point set of M_i. For each $r_i > 0$, and $x \in E$, we see that there exists a unique x_{r_i} in the domain of M_i such that $Jx \in Jx_{r_i} + r_i M_i(x_{r_i})$, where $x_{r_i} = (J + r_i M_i)^{-1} Jx$. Notice that $u_{j,i} = VI(C, \frac{1}{r_i}(J - Jx_j) + A_i)$, which is equivalent to $\langle u_{j,i} - y, A_i z_{j,i} + \frac{1}{r_i}(Jz_{j,i} - Jx_j)\rangle \le 0$, $\forall y \in C$, that is, $\frac{1}{r_i}\big(Jx_j - Ju_{j,i}\big) \in Nc(u_{j,i}) + A_i z_{j,i}$. This implies that $u_{j,i} = (J + r_i M_i)^{-1} Jx_j$. From [26], we find that $(J + r_i M_i)^{-1} J$ is closed quasi-ϕ-nonexpansive with $Fix((J + r_i M_i)^{-1} J) = M_i^{-1}(0)$. Using Theorem 3.1, we find the desired conclusion immediately.

Theorem 4.2. *Let E be a Hilbert. Let C be a convex and closed subset of E and let Λ be an arbitrary index set. Let B_i be a function with (R-1), (R-2), (R-3) and (R-4). Let T_i be an asymptotically quasi-nonexpansive mapping on C for every $i \in \Lambda$. Assume that T_i is uniformly asymptotically regular and closed for every $i \in \Lambda$ and $\cap_{i\in\Lambda} Sol(B_i) \bigcap \cap_{i\in\Lambda} Fix(T_i)$ is nonempty and bounded. Let $\{x_j\}$ be a sequence generated by*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_{(1,i)} = C, \forall i \in \Lambda, \\ C_1 = \cap_{i\in\Lambda} C_{(1,i)}, x_1 = P_{C_1} x_0, \\ y_{(j,i)} = \alpha_{(j,i)} T_i^j x_j + (1-\alpha_{(j,i)}) u_{(j,i)}, \\ C_{(j+1,i)} = \{z \in C_{(j,i)} : \|z - y_{(j,i)}\|^2 - \|z - x_j\|^2 \le \alpha_{(j,i)} \xi_{(j,i)} D_{(j,i)}\}, \\ C_{j+1} = \cap_{i\in\Lambda} C_{(j+1,i)}, x_{j+1} = P_{C_{j+1}} x_1, \end{cases}$$

where $u_{(j,i)}$ *is such that* $r_{(j,i)} B_i(u_{(j,i)}, \mu) \ge \langle u_{(j,i)} - \mu, u_{(j,i)} - x_j\rangle$, $\forall \mu \in C_j$, $D_{(j,i)} = \sup\{\|z - x_j\|^2 : z \in \cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(B_i)\}$, $\{\alpha_{(j,i)}\}$ *is a real*

sequence in $(0,1)$ *such that* $\liminf_{j\to\infty}\alpha_{(j,i)}(1-\alpha_{(j,i)})>0$ *and* $\{r_{(j,i)}\}\subset[r,\infty)$ *is a real sequence, where* r *is some positive real number. Then* $\{x_j\}$ *converges strongly to* $P_{\cap_{i\in\Lambda}Fix(T_i)\bigcap\cap_{i\in\Lambda}Sol(B_i)}x_1$.

Proof. In the framework of Hilbert spaces, we see that $\sqrt{\phi(x,y)}=\|x-y\|$, $\forall x,y\in E$. The generalized projection is reduced to the metric projection and the asymptotically-ϕ-nonexpansive mapping is reduced to the asymptotically quasi-nonexpansive mapping. Using Theorem 3.1, we find the desired conclusion immediately.

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Inner-outer factorization on Besov-type spaces

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Abstract. In this paper, motivated by some results of Dyakonov, we give an inner-outer factorization on Besov-type spaces.

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1 Introduction

We denote the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ by $\mathbb{D}$ and its boundary by $\partial\mathbb{D}$. Let $H(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$. For $0 < p < \infty$, the Hardy space H^p is the set of $f \in H(\mathbb{D})$ for which

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

As usual, H^∞ is the set of $f \in H(\mathbb{D})$ with $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)| < \infty$ (see [5]).

For $0 < p, q < \infty$ and $0 < s < 1$, the Besov-type space, denoted by B^s_{pq}, is the set of functions $f \in L^p(\partial\mathbb{D})$ such that

$$\int_0^\infty \frac{\omega_p(t, f)^q dt}{|t|^{sq+1}} < \infty,$$

where

$$\omega_p(t, f)^p := \sup_{-t\le h\le t} \int_{\partial\mathbb{D}} |f(e^{ih}\zeta) - f(\zeta)|^p dm(\zeta), \quad 0 \le t \le \pi$$

and

$$\omega_p(t, f) := \omega_p(\pi, f) \quad \text{when} \quad \pi < t < \infty.$$

Here dm is the normalized Lebesgue measure on $\partial\mathbb{D}$.

The analytic Besov space, denoted by $AB^s_{pq} = B^s_{pq} \cap H^p$, is the space of functions $f \in H^p$ such that

$$\int_0^1 (1-r)^{(1-s)q-1} \left(\int_{\partial\mathbb{D}} |f'(r\zeta)|^p dm(\zeta) \right)^{\frac{q}{p}} dr < \infty.$$

We refer the reader to [2], [3], [4] and [10]. For the simplicity of notation, we denote B^s_{pp} and AB^s_{pp} by B^s_p and AB^s_p, respectively.

Let $0 < p, s < \infty$, $-2 < q < \infty$. An $f \in H(\mathbb{D})$ is said to belong to $F(p, q, s)$ if (see [24])

$$\|f\|^p_{p,q,s} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z) < \infty,$$

where $g(z, a) = \log \frac{1}{|\varphi_a(z)|}$, $\quad z, a \in \mathbb{D}, z \neq a$, $\varphi_a(z) = \frac{a-z}{1-\overline{a}z}$, $dA(z) = \frac{1}{\pi}dxdy$. $F(p, q, s)$ is called general function space because it can get many function spaces if it takes special parameters of p, q, s. For example, when $s > 1$, $F(p, q, s) = \mathcal{B}^{\frac{q+2}{p}}$, which is called the Bloch-type space; $F(2, 0, s) = Q_s$ (see [23]); $F(2, 0, 1) = BMOA$, the space of analytic functions in the Hardy s-pace $H^1(\mathbb{D})$ whose boundary functions have bounded mean oscillation (see [13, 14, 19]). It is easy to see that $F(p, p-2, s)$ is a Möbius invariant Besov-type space. In fact, from [17], we know that $f \in F(p, p - 2, s)$ if and only if

$$\sup_{a\in\mathbb{D}} \|f \circ \varphi_a - f(a)\|_{AB_p^{\frac{1-s}{p}}} < \infty$$

when $0 < p, s < \infty$ and $F(p, p-2, s) \subseteq BMOA$ when $1 \le p < \infty$ and $0 < s < 1$.

For a sequence $\{z_n\}$ in $\mathbb{D}$ with $\sum_{n=1}^{\infty}(1 - |z_n|) < \infty$, the Blaschke product is defined by

$$B(z) = \prod_{n=1}^{\infty} \frac{-\overline{z_n}}{|z_n|} \frac{z - \overline{z_n}}{1 - z\overline{z_n}}.$$

If for every bounded sequence of complex numbers $\{a_n\}$, there exists an $f \in H^\infty$ such that $f(z_n) = a_n$ for every n, then both the sequence $\{z_n\}$ and the Blaschke product B are called interpolating. A Blaschke product B is called Carleson-Newman if B is a product of finitely many interpolating Blaschke products. Products of finitely many interpolating Blaschke products is an important tool in the study of H^∞, see [13].

An $f \in H(\mathbb{D})$ is called an inner function if it is bounded and has boundary values of modulus 1 almost everywhere on $\partial\mathbb{D}$. It is obvious that every Blaschke product is an inner function. For an inner function θ and $\epsilon \in (0, 1)$, define the level set of order ϵ of θ as

$$\Omega(\theta, \epsilon) = \{z \in \mathbb{D} : |\theta(z)| < \epsilon\}.$$

We refer to [1, 12, 15, 16, 20] for more information about inner function.

A function $g \in H(\mathbb{D})$ is said to be an outer function if there exists a positive function h with $\log h \in L^1(\partial\mathbb{D})$ and a complex number C with $|C| = 1$ such that

$$g(z) := C \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log h(e^{it}) \frac{e^{it} + z}{e^{it} - z} dt\right).$$

Moreover, for almost all $\zeta \in \partial\mathbb{D}$, $h(\zeta) = |g(\zeta)|$.

It is well known that each $f \in H^p$ has a unique factorization θg, where θ is an inner function and g is an outer function. Hence if we fix a function $f \in H^p$, there must have some relationship between θ and g. Dyakonov obtained many results on inner-outer factorization and characterized the moduli of analytic functions in $\mathbb{D}$ whose boundary values belong to certain smoothness classes. For many nice results about this topic, we refer to [6, 7, 9, 11, 22]. The following result can be found in [7, Theorem 1].

Theorem A. *If $f \in BMOA$ and θ is an inner function, then the following conditions are equivalent:*

(1) $f\theta \in BMOA$;

(2) $\sup_{z\in\mathbb{D}} |f(z)|^2(1-|\theta(z)|^2) < \infty$;

(3) $\sup_{z\in\Omega(\theta,\ \epsilon)} |f(z)| < \infty$, *for every* ϵ, $0<\epsilon<1$;

(4) $\sup_{z\in\Omega(\theta,\ \epsilon)} |f(z)| < \infty$, *for some* ϵ, $0<\epsilon<1$.

In this paper, we extend Theorem A from $BMOA$ to a more general spaces $F(p, p-2, s)$ and give the similar theorem as Theorem A.

Theorem 1. *Let $1 \le p < \infty$ and $0 < s < 1$. If $f \in F(p, p-2, s)$ and $\theta \in F(p, p-2, s)$ is an inner function, then the following statements are equivalent:*

(1) $f\theta \in F(p, p-2, s)$;

(2) $\sup_{z\in\mathbb{D}} |f(z)|^2(1-|\theta(z)|^2) < \infty$;

(3) $\sup_{z\in\Omega(\theta,\ \epsilon)} |f(z)| < \infty$, *for every* ϵ, $0<\epsilon<1$;

(4) $\sup_{z\in\Omega(\theta,\ \epsilon)} |f(z)| < \infty$, *for some* ϵ, $0<\epsilon<1$.

For more general Besov space, we have the following result.

Theorem 2. *Suppose that $2 \le p < \infty$, $0 < q < \infty$ and $0 < s < \frac{1}{2}$. If $f \in AB^s_{pq} \cap BMOA$ and $\theta \in AB^s_{pq}$ is an inner function, then the following statements are equivalent:*

(1) $f\theta \in AB^s_{pq} \cap BMOA$;

(2) $\sup_{z\in\mathbb{D}} |f(z)|^2(1-|\theta(z)|^2) < \infty$;

(3) $\sup_{z\in\Omega(\theta,\ \epsilon)} |f(z)| < \infty$, *for every* ϵ, $0<\epsilon<1$;

(4) $\sup_{z\in\Omega(\theta,\ \epsilon)} |f(z)| < \infty$, *for some* ϵ, $0<\epsilon<1$.

Throughout this paper, for two functions f and g, $f \asymp g$ means that $g \lesssim f \lesssim g$, that is, there are positive constants C_1 and C_2, such that $C_1 g \le f \le C_2 g$.

2 Proof of main results

In this section, we will give the proof of main results in this paper. To prove Theorem 1, we need the following lemmas.

Lemma 1. ([21, Theorem 1.4]) *Let $0 < s < 1$. Then an inner function belongs to the Möbius invariant Besov-type space $F(p, p-2, s)$ for all $p > \max\{s, 1-s\}$ if and only if it is the Blaschke product associated with a sequence $\{a_k\}_{k=1}^{\infty}$ which satisfies*

$$\sup_{a\in\mathbb{D}} \sum_{k=1}^{\infty} (1 - |\varphi_a(a_k)|^2)^s < \infty.$$

Lemma 2. ([18, Lemma 21]) *Let $\{a_k\}_{k=1}^{\infty}$ be a sequence in $\mathbb{D}$. Then the measure $d\mu_{a_k} = \sum_{k=1}^{\infty}(1-|a_k|^2)\delta_{a_k}$ is a Carleson measure, i.e.*

$$\sup_{a\in\mathbb{D}} \sum_{k=1}^{\infty} (1 - |\varphi_a(a_k)|^2) < \infty,$$

if and only if $\{a_k\}_{k=1}^{\infty}$ is a finite union of uniformly separated sequences.

Lemma 3. *Let $1 \leq p < \infty$, $0 < s < 1$, $f \in F(p, p-2, s)$ and B be a Carleson-Newman Blaschke product with a sequence of zeros $\{a_k\}_{k=1}^{\infty}$. Then $fB \in F(p, p-2, s)$ if and only if*

$$\sup_{a\in\mathbb{D}} \sum_{k=1}^{\infty} |f(a_k)|^p (1 - |\varphi_a(a_k)|^2)^s < \infty.$$

Proof. Necessity. The proof is similar to the proof of [17, Lemma 2.6].

Sufficiency. Let B be a Carleson-Newman Blaschke products with zeros $\{a_k\}_{k=1}^{\infty}$. Suppose that $B = \prod_{i=1}^{n} B_i$, B_i is an interpolating Blaschke products with zeros $\{a_{i,k}\}_{k=1}^{\infty}$ and

$$\{a_k\}_{k=1}^{\infty} = \bigcup_{i=1}^{n} \{a_{i,k}\}_{k=1}^{\infty}.$$

It is easy to see that

$$\sup_{a\in\mathbb{D}} \sum_{k=1}^{\infty} |f(a_{i,k})|^p (1 - |\varphi_a(a_{i,k})|^2)^s \leq \sup_{a\in\mathbb{D}} \sum_{k=1}^{\infty} |f(a_k)|^p (1 - |\varphi_a(a_k)|^2)^s < \infty.$$

Since $f \in F(p, p-2, s)$, $\rho(w, z) = \rho(\varphi_a(w), \varphi_a(z))$, $B_i \circ \varphi_a$ is an interpolating Blachke products with zeros $\{\varphi_a(a_{i,k})\}_{k=1}^{\infty}$. By [8, Theorem 8] and its

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remark (1), we have

$$\begin{aligned}\sup_{a\in\mathbb{D}}\|P_-\Big((f\circ\varphi_a)\cdot\overline{B_i\circ\varphi_a}\Big)\|^p_{B_p^{\frac{1-s}{p}}} &\lesssim \sup_{a\in\mathbb{D}}\sum\frac{|f\circ\varphi_a(\varphi_a(a_{i,k}))|^p}{(1-|\varphi_a(a_{i,k})|^2)^{\frac{1-s}{p}p-1}}\\ &=\sup_{a\in\mathbb{D}}\sum_{k=1}^{\infty}|f(a_{i,k})|^p(1-|\varphi_a(a_{i,k})|^2)^s.\end{aligned}$$

Combine with [20, Theorem 5], we get

$$\begin{aligned}&\sup_{a\in\mathbb{D}}\|f\circ\varphi_a-f(a)\|^p_{AB_p^{\frac{1-s}{p}}}+\sup_{a\in\mathbb{D}}\|(fB_i)\circ\varphi_a-f(a)B_i(a)\|^p_{AB_p^{\frac{1-s}{p}}}\\ \approx&\sup_{a\in\mathbb{D}}\|f\circ\varphi_a-f(a)\|^p_{B_p^{\frac{1-s}{p}}}+\sup_{a\in\mathbb{D}}\|(fB_i)\circ\varphi_a-f(a)B_i(a)\|^p_{B_p^{\frac{1-s}{p}}}\\ \approx&\sup_{a\in\mathbb{D}}\|f\circ\varphi_a-f(a)\|^p_{B_p^{\frac{1-s}{p}}}+\sup_{a\in\mathbb{D}}\|P_-\Big((f\circ\varphi_a)\cdot\overline{B_i\circ\varphi_a}\Big)\|^p_{B_p^{\frac{1-s}{p}}}\\ \lesssim&\sup_{a\in\mathbb{D}}\|f\circ\varphi_a-f(a)\|^p_{B_p^{\frac{1-s}{p}}}+\sup_{a\in\mathbb{D}}\sum_{k=1}^{\infty}|f(a_{i,k})|^p(1-|\varphi_a(a_{i,k})|^2)^s\\ \approx&\sup_{a\in\mathbb{D}}\|f\circ\varphi_a-f(a)\|^p_{AB_p^{\frac{1-s}{p}}}+\sup_{a\in\mathbb{D}}\sum_{k=1}^{\infty}|f(a_{i,k})|^p(1-|\varphi_a(a_{i,k})|^2)^s.\end{aligned}$$

Thus,

$$\begin{aligned}&\sup_{a\in\mathbb{D}}\|(fB_i)\circ\varphi_a-f(a)B_i(a)\|^p_{AB_p^{\frac{1-s}{p}}}\\ \lesssim&\sup_{a\in\mathbb{D}}\sum_{k=1}^{\infty}|f(a_{i,k})|^p(1-|\varphi_a(a_{i,k})|^2)^s+\sup_{a\in\mathbb{D}}\|f\circ\varphi_a-f(a)\|^p_{AB_p^{\frac{1-s}{p}}}.\end{aligned}$$

Since $f\in F(p,p-2,s)$, by Lemma 2.1 in [17], we have

$$fB_i\in F(p,p-2,s),\qquad i=1,...,n.$$

By inductive, we have

$$(fB)'(z)=\sum_{j=1}^{n}(fB_j)'(z)\prod_{i=1,i\neq j}^{n}B_i(z)-(n-1)f'(z)\prod_{i=1}^{n}B_i(z).$$

Hence,

$$|(fB)'(z)|\le\sum_{j=1}^{n}|(fB_j)'(z)|+(n-1)|f'(z)|,\qquad z\in\mathbb{D}.$$

Notice that $f\in F(p,p-2,s)$, $fB_i\in F(p,p-2,s)$, combine with p-inequality, we obtain $fB\in F(p,p-2,s)$. The proof is complete.

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Proof of Theorem 1. (1) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (2). Since $f \in F(p,p-2,s) \subseteq BMOA$, $f\theta \in F(p,p-2,s) \subseteq BMOA$. From Theorem A, we easily get our result.

(2) $\Rightarrow$ (1). Assume that (2) holds. Since $\theta \in F(p,p-2,s)$, by Lemma 1, we see that θ is a Blaschke product with zeros $\{a_k\}_{k=1}^{\infty}$, and

$$\sup_{a\in\mathbb{D}}\sum_{k=1}^{\infty}(1-|\varphi_a(a_k)|^2)^s<\infty, \quad 0<s<1,$$

which implies that

$$\sup_{a\in\mathbb{D}}\sum_{k=1}^{\infty}(1-|\varphi_a(a_k)|^2)<\infty.$$

From Lemma 2, we get that θ is a Carleson-Newman Blaschke product. Since $f\in F(p,p-2,s)\subseteq BMOA$, by the assumption that $\sup_{z\in\mathbb{D}}|f(z)|^2(1-|\theta(z)|^2)<\infty$ and Theorem A, we see that $f\theta\in BMOA$. Theorem A gives

$$\sup_{z\in\Omega(\theta,\ \epsilon)}|f(z)|<\infty, \quad 0<\epsilon<1,$$

which implies that $\sup_k |f(a_k)|<\infty$. Thus,

$$\begin{aligned}
&\sup_{a\in\mathbb{D}}\sum_{k=1}^{\infty}|f(a_k)|^p(1-|\varphi_a(a_k)|^2)^s\\
\leq\ & \sup_k |f(a_k)|^p \sup_{a\in\mathbb{D}}\sum_{k=1}^{\infty}(1-|\varphi_a(a_k)|^2)^s<\infty.
\end{aligned}$$

Applying Lemma 3, we see that $f\theta\in F(p,p-2,s)$. The proof is complete.

Proof of Theorem 2. (1) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (2). The proof is similar to Theorem 1 and hence we omit the details.

(2) $\Rightarrow$ (1). Suppose that $f\in AB_{pq}^s\cap BMOA$ and $\theta\in AB_{pq}^s$. Since θ is bounded, if we want to prove $f\theta\in AB_{pq}^s$, we only need to prove

$$\int_0^1(1-r)^{(1-s)q-1}\left(\int_{\partial\mathbb{D}}|f(r\zeta)\theta'(r\zeta)|^p dm(\zeta)\right)^{\frac{q}{p}}dr<\infty.$$

Using the well known Schwarz's Lemma, we have

$$|\theta'(z)|\leq\frac{1-|\theta(z)|^2}{1-|z|^2}.$$

Therefore

$$\begin{aligned}
&\int_0^1(1-r)^{(1-s)q-1}\left(\int_{\partial\mathbb{D}}|f(r\zeta)\theta'(r\zeta)|^p dm(\zeta)\right)^{\frac{q}{p}}dr\\
\lesssim&\int_0^1(1-r)^{(1-s)q-1}\left(\int_{\partial\mathbb{D}}|f(r\zeta)|^p\left|\frac{1-|\theta(r\zeta)|^2}{1-r^2}\right|^p dm(\zeta)\right)^{\frac{q}{p}}dr.
\end{aligned}$$

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From [10, Theorem 3.2], we known that $\theta \in AB^s_{pq}$ if and only if

$$\int_0^1 \left(\int_{\partial\mathbb{D}} (1 - |\theta(r\zeta)|)^{\frac{p}{2}} dm(\zeta) \right)^{\frac{q}{p}} \frac{dr}{(1-r)^{sq+1}} < \infty.$$

Thus, combine with the assumption that $\sup_{z\in\mathbb{D}} |f(z)|^2(1 - |\theta(z)|^2) < \infty$, we deduce that

$$\int_0^1 (1-r)^{(1-s)q-1} \left(\int_{\partial\mathbb{D}} |f(r\zeta)\theta'(r\zeta)|^p dm(\zeta) \right)^{\frac{q}{p}} dr < \infty,$$

which implies that $f\theta \in AB^s_{pq}$. In addition, by Theorem A, we see that $f\theta \in BMOA$. Hence $f\theta \in AB^s_{pq} \cap BMOA$. The proof is complete.

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GENERALIZED RATIONAL CONTRACTIONS ENDOWED WITH A GRAPH AND AN APPLICATION TO A SYSTEM OF INTEGRAL EQUATIONS

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Abstract. In the present paper, we introduce the notion of generalized rational contraction including admissible mappings and establish coincidence point and common fixed point results for this class of mappings defined on ordinary as well as ordered metric spaces. Our results extend, generalize and unify comparable results in the existing literature. Applying these results, we deduce fixed point results on metric spaces endowed with graph. An example and application to obtain the existence of common solution for a system of integral equations are also given in order to illustrate the effectiveness of the offered results.

1. Introduction and Preliminaries

Fixed point theory is one of the most powerful and effective tools in mathematics which has enormous applications within as well as outside mathematics. One of the most fundamental fixed point theorems is the Banach contraction principle [8] which gives an answer on the existence and uniqueness of a solution of an operator equation $Fx = x$. Since then, there is a great number of generalizations of this fundamental principle (for example, see [1]-[7], [9]-[29]).

Recently, Samet et al. [28] first introduced α-admissible mappings and then α-ψ-contractive type mappings to obtain some interesting generalizations of Banach contraction principle. For more results in this direction, we refer to [3, 5, 6, 11, 15, 17, 21, 23, 25, 27, 22] and references mentioned therein.

Definition 1 ([28])**.** *Let X be a nonempty set and $\alpha : X \times X \longrightarrow [0, +\infty)$. A self-mapping T on X is called α-admissible mapping if*

$$x, y \in X, \qquad \alpha(x, y) \geq 1 \text{ implies } \alpha(Tx, Ty) \geq 1.$$

Afterward, Patel et al. [25] extended the definition of α-admissible mapping to a pair of two mappings to obtain common fixed point results as follows:

Definition 2 ([25])**.** *Let f, g, S and T be four self-mappings of a non-empty set X, and let $\alpha : S(X) \cup T(X) \times S(X) \cup T(X) \to [0, +\infty)$. Then the pair (f, g) is called α-admissible with respect to S and T (in short, α_{ST}-admissible) if for all $x, y \in X$,*

$$\alpha(Sx, Ty) \geq 1 \text{ or } \alpha(Tx, Sy) \geq 1 \Longrightarrow \alpha(fx, gy) \geq 1 \text{ and } \alpha(gx, fy) \geq 1.$$

If we take $S = T = I_X$ (identity mapping on X) in above definition, then we have:

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Definition 3 ([3]). *Let f and g be self-mappings of a non-empty set X and $\alpha : X \times X \to [0,+\infty)$. Then the pair (f,g) is called α-admissible if for all $x,y \in X$,*

$$\alpha(x,y) \geq 1 \Longrightarrow \alpha(fx,gy) \geq 1 \quad \textit{and} \quad \alpha(gx,fy) \geq 1.$$

Definition 4 ([19]). *A pair (f,T) of self-mappings on a set X is said to be* weakly compatible *if f and T commute at their coincidence point (i.e. $fTx = Tfx$, $x \in X$ whenever $fx = Tx$).*

A point $y \in X$ is called a *point of coincidence* of two self-mappings f and T on X if there exists a point $x \in X$ such that $y = fx = Tx$. Also, $x \in X$ is called a *common fixed point* of mappings f and T if $x = fx = Tx$.

The notations $\mathcal{F}(f,T)$ and $\mathcal{C}(f,T)$ stand for the set of all common fixed point and the set of all coincidence points of f and T, respectively. In the sequel, we will indicate the set of all real numbers, the set of all non-negative real numbers and the set of all natural numbers by the letters $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$, respectively.

On the other side, Khan et al. [20] introduced and employed the notion of altering distance function to obtain some interesting fixed point results in metric spaces. Note that altering distance functions are continuous whereas Su [29] defined generalized altering distance function, not necessarily continuous, as follows:

Definition 5 ([29]). *A mapping $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called generalized altering distance function if*

(*a*) *ψ is non-decreasing,*
(*b*) *$\psi(t) = 0$ iff $t = 0$.*

We set $\Psi = \{\psi : \mathbb{R}^+ \to \mathbb{R}^+ : \psi$ is a generalized altering distance function$\}$ and $\Phi = \{\varphi : \mathbb{R}^+ \to \mathbb{R}^+ : \varphi$ is a nondecreasing and right upper semi-continuous function and we have $\psi(t) > \varphi(t)$ for all $t > 0$ where $\psi \in \Psi\}$.

We now introduce generalized rational contraction mappings as follows:

Definition 6. *Let f, g, S and T be selfmaps of a metric space (X,d), and (f,g) be an α_{ST}-admissible pair. We say that (f,g) is a generalized $(\alpha,\psi,\varphi)_{(S,T)}$-rational contraction if*

$$\alpha(Sx,Ty) \geq 1 \textit{ implies } \psi(d(fx,gy)) \leq \varphi(M(x,y)) \tag{1.1}$$

for all $x,y \in X$, where $\psi \in \Psi$, $\varphi \in \Phi$ and

$$\begin{aligned} M(x,y) \;=\; \max\Bigg(& d(Sx,Ty), d(Sx,fx), d(Ty,gy), \frac{d(Sx,gy)+d(fx,Ty)}{2}, \\ & \frac{d(Ty,gy)[1+d(Sx,fx)]}{1+d(Sx,Ty)}, \frac{d(fx,Ty)[1+d(Sx,gy)]}{1+d(Sx,Ty)} \Bigg). \end{aligned}$$

In this paper, we prove some common fixed point results of generalized $(\alpha,\psi,\varphi)_{(S,T)}$-rational contractions for a quadruple of self-mappings defined on ordinary as well as ordered metric spaces. Our results extend, generalize and unify comparable results in the existing literature. Applying these results, we deduce fixed point results on metric spaces endowed with graph. An example is presented to support the results obtained herein. As an application of offered results, the existence of the common solution for a system of integral equations are also investigated.

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2. Main Results

We start with the following first result.

Theorem 1. *Let f, g, S and T be selfmaps of a complete metric space (X, d) with $f(X) \subset T(X)$, $g(X) \subset S(X)$ and (f, g) be a generalized $(\alpha, \psi, \varphi)_{(S,T)}$-rational contraction pair. Suppose that:*

- (a) *there exists $x_0 \in X$ such that $\alpha(Sx_0, fx_0) \geq 1$;*
- (b) *$\alpha(Sx_n, Tx_{n+1}) \geq 1$ for all n even implies that $\alpha(Sx_n, Tx_j) \geq 1$ for all n even and $j > n$ odd;*
- (c) *$\alpha(Sx_n, Tx_{n+1}) \geq 1$ for all n even and, Sx_n and Tx_{n+1} converge to an $x \in X$ as $n \to \infty$ implies that $\alpha(Sx_n, x) \geq 1$ and $\alpha(x, Tx_{n+1}) \geq 1$ for all n even.*

Then the pairs (f, S) and (g, T) have a point of coincidence in X. Moreover, if

- (i) *$\{f, S\}$ and $\{g, T\}$ are weakly compatible,*
- (ii) *$\alpha(Su, Tv) \geq 1$ whenever $u \in \mathcal{C}(f, S)$ and $v \in \mathcal{C}(g, T)$.*

Then f, g, S and T have a common fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(Sx_0, fx_0) \geq 1$. Since $fX \subset TX$, there exists an $x_1 \in X$ such that $fx_0 = Tx_1$. Again since $gX \subset SX$, there exists an $x_2 \in X$ such that $gx_1 = Tx_2$. Continuing this process, we can construct the sequences $\{x_n\}$ and $\{y_n\}$ in X defined by

$$y_{2n} = fx_{2n} = Tx_{2n+1}, \quad y_{2n+1} = gx_{2n+1} = Sx_{2n+2}, \quad n \in \mathbb{N}_0, \tag{2.1}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. As (f, g) is an α_{ST}-admissible pair and $\alpha(Sx_0, fx_0) = \alpha(Sx_0, Tx_1) \geq 1$, we have $\alpha(fx_0, gx_1) \geq 1$ and $\alpha(gx_0, fx_1) \geq 1$ which implies that $\alpha(Tx_1, Sx_2) \geq 1$. Again, since $\alpha(Tx_1, Sx_2) \geq 1$, we have $\alpha(fx_1, gx_2) \geq 1$ and $\alpha(gx_1, fx_2) \geq 1$ which gives that $\alpha(Sx_2, Tx_3) \geq 1$. Continuing this way, we obtain

$$\alpha(Sx_{2n}, Tx_{2n+1}) \geq 1 \quad \text{and} \quad \alpha(Tx_{2n+1}, Sx_{2n+2}) \geq 1 \quad \text{for all } n \in \mathbb{N}_0. \tag{2.2}$$

Suppose that $y_{2n} \neq y_{2n+1}$ for all $n \in \mathbb{N}_0$. Now we show that

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0. \tag{2.3}$$

Putting $x = x_{2n}$ and $y = x_{2n+1}$ in (1.1) and using (2.1) and (2.2), we get

$$\begin{aligned} \psi(d(y_{2n}, y_{2n+1})) &= \psi(d(fx_{2n}, gx_{2n+1})) \\ &\leq \varphi(M(x_{2n}, x_{2n+1})), \end{aligned} \tag{2.4}$$

where

$$\begin{aligned}
M(x_{2n},x_{2n+1}) &= \max\Bigg(d(Sx_{2n},Tx_{2n+1}), d(Sx_{2n},fx_{2n}), d(Tx_{2n+1},gx_{2n+1}),\\
&\quad \frac{d(Sx_{2n},gx_{2n+1})+d(fx_{2n},Tx_{2n+1})}{2},\\
&\quad \frac{d(Tx_{2n+1},gx_{2n+1})[1+d(Sx_{2n},fx_{2n})]}{1+d(Sx_{2n},Tx_{2n+1})},\\
&\quad \frac{d(fx_{2n},Tx_{2n+1})[1+d(Sx_{2n},gx_{2n+1})]}{1+d(Sx_{2n},Tx_{2n+1})}\Bigg)\\
&= \max\Bigg(d(y_{2n-1},y_{2n}), d(y_{2n-1},y_{2n}), d(y_{2n},y_{2n+1}),\\
&\quad \frac{d(y_{2n-1},y_{2n+1})+d(y_{2n},y_{2n})}{2},\\
&\quad \frac{d(y_{2n},y_{2n+1})[1+d(y_{2n-1},y_{2n})]}{1+d(y_{2n-1},y_{2n})},\\
&\quad \frac{d(y_{2n},y_{2n})[1+d(y_{2n-1},y_{2n+1})]}{1+d(y_{2n-1},y_{2n})}\Bigg)\\
&\le \max\left(d(y_{2n-1},y_{2n}), d(y_{2n},y_{2n+1}), \frac{d(y_{2n-1},y_{2n})+d(y_{2n},y_{2n+1})}{2}\right)\\
&= \max(d(y_{2n-1},y_{2n}), d(y_{2n},y_{2n+1})).
\end{aligned}$$

If $d(y_{2n-1},y_{2n}) \le d(y_{2n},y_{2n+1})$ for some $n\in\mathbb{N}$, then by (2.4), we have

$$\psi(d(y_{2n},y_{2n+1})) \le \varphi(d(y_{2n},y_{2n+1})),$$

a contradiction to the fact that $y_{2n}\ne y_{2n+1}$. So for all $n\in\mathbb{N}$, we have $d(y_{2n},y_{2n+1}) < d(y_{2n-1},y_{2n})$.

From (2.4), we also obtain

$$\psi(d(y_{2n},y_{2n+1})) \le \varphi(d(y_{2n-1},y_{2n})). \tag{2.5}$$

Again, putting $x=x_{2n-1}$ and $y=x_{2n}$ in (1.1) and following arguing similar to those given above, we get

$$\psi(d(y_{2n-1},y_{2n})) \le \varphi(d(y_{2n-2},y_{2n-1})). \tag{2.6}$$

From (2.5) and (2.6), we conclude

$$\psi(d(y_n,y_{n+1})) \le \varphi(d(y_{n-1},y_n)). \tag{2.7}$$

It follows that the sequence $\{d(y_n,y_{n+1})\}$ is decreasing and bounded below. Hence, there exists $r\ge 0$ such that $\lim_{n\to\infty} d(y_n,y_{n+1}) = r$. If $r>0$, then taking limit as $n\to\infty$ on both sides of (2.7), we have

$$\begin{aligned}
\psi(r) &\le \lim_{n\to\infty}\psi(d(y_n,y_{n+1}))\\
&\le \lim_{n\to\infty}\varphi(d(y_{n-1},y_n)) \le \varphi(r),
\end{aligned}$$

a contradiction and hence $r=0$, that is, the equation (2.3) holds.

Now, we prove that $\{y_n\}$ is a Cauchy sequence. To this end, it is sufficient to verify that $\{y_{2n}\}$ is a Cauchy sequence. Suppose, to the contrary, that $\{y_{2n}\}$ is not a Cauchy sequence. Then, there exists an $\varepsilon>0$ for which we can find two

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subsequences $\{y_{2m_k}\}$ and $\{y_{2n_k}\}$ of $\{y_{2n}\}$ such that m_k is the smallest index for which $m_k > n_k > k$ and

$$d\left(y_{2m_k}, y_{2n_k}\right) \geq \varepsilon \quad \text{and} \quad d\left(y_{2m_k-1}, y_{2n_k}\right) < \varepsilon. \tag{2.8}$$

Using the triangular inequality and (2.8), we have

$$\begin{aligned} \varepsilon &\leq d\left(y_{2m_k}, y_{2n_k}\right) \leq d\left(y_{2m_k}, y_{2m_k-1}\right) + d\left(y_{2m_k-1}, y_{2n_k}\right) \\ &< d\left(y_{2m_k}, y_{2m_k-1}\right) + \varepsilon. \end{aligned}$$

Taking $k \to \infty$ on both sides of above inequality and using (2.3), we obtain

$$\lim_{k\to\infty} d\left(y_{2m_k}, y_{2n_k}\right) = \varepsilon. \tag{2.9}$$

Again, using the triangular inequality, we get

$$\left|d\left(y_{2n_k}, y_{2m_k+1}\right) - d\left(y_{2n_k}, y_{2m_k}\right)\right| \leq d\left(y_{2m_k}, y_{2m_k+1}\right).$$

Letting $k \to \infty$ in the above inequality and using (2.3) and (2.9), we have

$$\lim_{k\to\infty} d\left(y_{2n_k}, y_{2m_k+1}\right) = \varepsilon. \tag{2.10}$$

Similarly, one can easily show that

$$\lim_{k\to\infty} d\left(y_{2n_k-1}, y_{2m_k}\right) = \lim_{k\to\infty} d\left(y_{2n_k-1}, y_{2m_k+1}\right) = \varepsilon. \tag{2.11}$$

Since $\alpha\left(Sx_{2n_k}, Tx_{2m_k+1}\right) \geq 1$ from (2.2) and the hypothesis (b), putting $x = x_{2n_k}$ and $y = x_{2m_k+1}$ in (1.1), we get

$$\begin{aligned} \psi\left(d\left(y_{2n_k}, y_{2m_k+1}\right)\right) &= \psi\left(d\left(fx_{2n_k}, gx_{2m_k+1}\right)\right) \\ &\leq \varphi\left(M\left(x_{2n_k}, x_{2m_k+1}\right)\right), \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} M\left(x_{2n_k}, x_{2m_k+1}\right) &= \max\Bigg(d\left(Sx_{2n_k}, Tx_{2m_k+1}\right), d\left(Sx_{2n_k}, fx_{2n_k}\right), d\left(Tx_{2m_k+1}, gx_{2m_k+1}\right), \\ &\quad \frac{d\left(Sx_{2n_k}, gx_{2m_k+1}\right) + d\left(fx_{2n_k}, Tx_{2m_k+1}\right)}{2}, \\ &\quad \frac{d\left(Tx_{2m_k+1}, gx_{2m_k+1}\right)\left[1 + d\left(Sx_{2n_k}, fx_{2n_k}\right)\right]}{1 + d\left(Sx_{2n_k}, Tx_{2m_k+1}\right)}, \\ &\quad \frac{d\left(fx_{2n_k}, Tx_{2m_k+1}\right)\left[1 + d\left(Sx_{2n_k}, gx_{2m_k+1}\right)\right]}{1 + d\left(Sx_{2n_k}, Tx_{2m_k+1}\right)}\Bigg) \\ &= \max\Bigg(d\left(y_{2n_k-1}, y_{2m_k}\right), d\left(y_{2n_k-1}, y_{2n_k}\right), d\left(y_{2m_k}, y_{2m_k+1}\right), \\ &\quad \frac{d\left(y_{2n_k-1}, y_{2m_k+1}\right) + d\left(y_{2n_k}, y_{2m_k}\right)}{2}, \\ &\quad \frac{d\left(y_{2m_k}, y_{2m_k+1}\right)\left[1 + d\left(y_{2n_k-1}, y_{2n_k}\right)\right]}{1 + d\left(y_{2n_k-1}, y_{2m_k}\right)}, \\ &\quad \frac{d\left(y_{2n_k}, y_{2m_k}\right)\left[1 + d\left(y_{2n_k-1}, y_{2m_k+1}\right)\right]}{1 + d\left(y_{2n_k-1}, y_{2m_k}\right)}\Bigg). \end{aligned}$$

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Now, from the properties of ψ and φ and using (2.3), (2.9), (2.10) and (2.11) as $k \to \infty$ in (2.12), we obtain

$$\begin{aligned}\psi(\varepsilon) &\leq \lim_{k\to\infty} \psi(d(y_{2n_k}, y_{2m_k+1})) \\ &\leq \lim_{k\to\infty} \varphi(M(x_{2n_k}, x_{2m_k+1})) \\ &\leq \varphi(\max(\varepsilon, 0, 0, \varepsilon, 0, \varepsilon)) = \varphi(\varepsilon),\end{aligned}$$

which implies that $\varepsilon = 0$, a contradiction with $\varepsilon > 0$. Thus $\{y_{2n}\}$ is a Cauchy sequence in X and hence $\{y_n\}$ is a Cauchy sequence. From the completeness of (X, d), there exists $z \in X$ such that

$$\lim_{n\to\infty} y_n = z. \tag{2.13}$$

From (2.1) and (2.13), we get

$$fx_{2n} \to z, \qquad Tx_{2n+1} \to z, \qquad gx_{2n+1} \to z, \qquad Sx_{2n+2} \to z \quad \text{as } n \to \infty. \tag{2.14}$$

Now we shall prove that z is a common fixed point of f, g, S and T.

Since $g(X) \subset S(X)$, we can choose a point u in X such that $z = Su$. Suppose that $d(z, fu) \neq 0$.

By (2.2), (2.14) and the condition (c), we have $\alpha(Su, Tx_{2n+1}) \geq 1$. Then, substituting $x = u$ and $y = x_{2n+1}$ in (1.1), we deduce

$$\psi(d(fu, gx_{2n+1})) \leq \varphi(M(u, x_{2n+1})), \tag{2.15}$$

where

$$\begin{aligned}M(u, x_{2n+1}) = \max\Bigg(& d(Su, Tx_{2n+1}), d(Su, fu), d(Tx_{2n+1}, gx_{2n+1}), \\ & \frac{d(Su, gx_{2n+1}) + d(fu, Tx_{2n+1})}{2}, \\ & \frac{d(Tx_{2n+1}, gx_{2n+1})[1 + d(Su, fu)]}{1 + d(Su, Tx_{2n+1})}, \\ & \frac{d(fu, Tx_{2n+1})[1 + d(Su, gx_{2n+1})]}{1 + d(Su, Tx_{2n+1})}\Bigg).\end{aligned}$$

Letting $k \to \infty$ in (2.15), we have

$$\begin{aligned}\psi(d(fu, z)) &\leq \lim_{n\to\infty} \psi(d(fu, gx_{2n+1})) \\ &\leq \lim_{n\to\infty} \varphi(M(u, x_{2n+1})) \\ &\leq \varphi\left(\max\left(0, d(z, fu), 0, \frac{d(fu, z)}{2}, 0, d(fu, z)\right)\right) \\ &= \varphi(d(fu, z)),\end{aligned}$$

a contradiction and hence $d(fu, z) = 0$, that is $fu = z$, and so $u \in \mathcal{C}(f, S)$.

Similarly, since $f(X) \subset T(X)$, we can choose a point v in X such that $z = Tv$. Suppose that $d(z, gv) \neq 0$.

By (2.2), (2.14) and the condition (c), we have $\alpha(Sx_{2n}, Tv) \geq 1$. Then, putting $x = x_{2n}$ and $y = v$ in (1.1), we obtain

$$\psi(d(fx_{2n}, gv)) \leq \varphi(M(x_{2n}, v)), \tag{2.16}$$

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where

$$\begin{aligned} M\left(x_{2n},v\right) &= \max\Bigg(d\left(Sx_{2n},Tv\right),d\left(Sx_{2n},fx_{2n}\right),d\left(Tv,gv\right), \\ &\quad \frac{d\left(Sx_{2n},gv\right)+d\left(fx_{2n},Tv\right)}{2}, \\ &\quad \frac{d\left(Tv,gv\right)\left[1+d\left(Sx_{2n},fx_{2n}\right)\right]}{1+d\left(Sx_{2n},Tv\right)}, \\ &\quad \frac{d\left(fx_{2n},Tv\right)\left[1+d\left(Sx_{2n},gv\right)\right]}{1+d\left(Sx_{2n},Tv\right)}\Bigg). \end{aligned}$$

Taking limit on (2.16), we get

$$\begin{aligned} \psi\left(d\left(z,gv\right)\right) &\leq \lim_{n\to\infty}\psi\left(d\left(fx_{2n},gv\right)\right) \\ &\leq \lim_{n\to\infty}\varphi\left(M\left(x_{2n},v\right)\right) \\ &\leq \varphi\left(\max\left(0,0,d\left(z,gv\right),\frac{d\left(z,gv\right)}{2},d\left(z,gv\right),0\right)\right) \\ &= \varphi\left(d\left(z,gv\right)\right), \end{aligned}$$

a contradiction and hence $d\left(z,gv\right)=0$, that is $z=gv$, and so $v\in\mathcal{C}\left(g,T\right)$.

Thus, $z=fu=Su=gv=Tv$. By the weak compatibility of the pairs (f,S) and (g,T), we deduce that $fz=Sz$ and $gz=Tz$.

Since $z\in\mathcal{C}\left(f,S\right)$ and $v\in\mathcal{C}\left(g,T\right)$, by (ii), we have $\alpha\left(Sz,Tv\right)\geq 1$ and so, from (1.1)

$$\psi\left(d\left(fz,z\right)\right)=\psi\left(d\left(fz,gv\right)\right)\leq\varphi\left(M\left(z,v\right)\right), \tag{2.17}$$

where

$$\begin{aligned} M\left(z,v\right) &= \max\Bigg(d\left(Sz,Tv\right),d\left(Sz,fz\right),d\left(Tv,gv\right), \\ &\quad \frac{d\left(Sz,gv\right)+d\left(fz,Tv\right)}{2},\frac{d\left(Tv,gv\right)\left[1+d\left(Sz,fz\right)\right]}{1+d\left(Sz,Tv\right)}, \\ &\quad \frac{d\left(fz,Tv\right)\left[1+d\left(Sz,gv\right)\right]}{1+d\left(Sz,Tv\right)}\Bigg) \\ &= \max\left(d\left(fz,z\right),0,0,d\left(fz,z\right),0,d\left(fz,z\right)\right)=d\left(fz,z\right) \end{aligned}$$

By (2.17), we get

$$\psi\left(d\left(fz,z\right)\right)\leq\varphi\left(d\left(fz,z\right)\right),$$

which implies that $z=fz$, and so $z=fz=Sz$. Similarly, it can be shown that $z=gz=Tz$. This completes the proof. □

Corollary 1. *Let f,g,S and T be selfmaps of a complete metric space (X,d) with $f(X)\subset T(X)$, $g(X)\subset S(X)$ and (f,g) be an α_{ST}-admissible pair such that*

$$\alpha\left(Sx,Ty\right)\psi\left(d\left(fx,gy\right)\right)\leq\varphi\left(M\left(x,y\right)\right), \tag{2.18}$$

for all $x,y\in X$, where $\psi\in\Psi$ and $\varphi\in\Phi$. Assume that the following conditions are satisfied:

- (a) *there exists $x_0\in X$ such that $\alpha\left(Sx_0,fx_0\right)\geq 1$;*
- (b) *$\alpha\left(Sx_n,Tx_{n+1}\right)\geq 1$ for all n even implies that $\alpha\left(Sx_n,Tx_j\right)\geq 1$ for all n even and $j>n$ odd;*

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(c) $\alpha(Sx_n, Tx_{n+1}) \geq 1$ *for all* n *even and,* Sx_n *and* Tx_{n+1} *converge to an* $x \in X$ *as* $n \to \infty$ *implies that* $\alpha(Sx_n, x) \geq 1$ *and* $\alpha(x, Tx_{n+1}) \geq 1$ *for all* n *even.*

Then the pairs (f, S) *and* (g, T) *have a point of coincidence in* X*. Moreover, if*

(i) $\{f, S\}$ *and* $\{g, T\}$ *are weakly compatible,*

(ii) $\alpha(Su, Tv) \geq 1$ *whenever* $u \in \mathcal{C}(f, S)$ *and* $v \in \mathcal{C}(g, T)$.

Then f, g, S *and* T *have a common fixed point.*

Proof. Let $\alpha(Sx, Ty) \geq 1$ for $x, y \in X$. Then by (2.18), we have

$$\psi(d(fx, gy)) \leq \varphi(M(x, y)).$$

This implies that the inequality (1.1) holds. Therefore, the proof follows from Theorem 1. □

If we take $\alpha(Sx, Ty) = 1$ in Corollary 1, we have a generalized version of Theorem 2.3 in [29]:

Theorem 2. *Let* f, g, S *and* T *be selfmaps of a complete metric space* (X, d) *with* $f(X) \subset T(X)$ *and* $g(X) \subset S(X)$*. Suppose that*

$$\psi(d(fx, gy)) \leq \varphi(M(x, y)), \tag{2.19}$$

for all $x, y \in X$*, where* $\psi \in \Psi$ *and* $\varphi \in \Phi$*. Then the pairs* (f, S) *and* (g, T) *have a point of coincidence in* X*. Moreover, if* $\{f, S\}$ *and* $\{g, T\}$ *are weakly compatible, then* f, g, S *and* T *have a common fixed point.*

If we take $\psi(t) = t$ in Corollary 1, we have a generalized version of Theorem 2.2 in [28]:

Theorem 3. *Let* f, g, S *and* T *be selfmaps of a complete metric space* (X, d) *with* $f(X) \subset T(X)$*,* $g(X) \subset S(X)$ *and* (f, g) *be an* α_{ST}*-admissible pair such that*

$$\alpha(Sx, Ty)\, d(fx, gy) \leq \varphi(M(x, y)), \tag{2.20}$$

for all $x, y \in X$*, where* $\varphi \in \Phi$*. Assume that the following conditions are satisfied:*

(a) *there exists* $x_0 \in X$ *such that* $\alpha(Sx_0, fx_0) \geq 1$*;*

(b) $\alpha(Sx_n, Tx_{n+1}) \geq 1$ *for all* n *even implies that* $\alpha(Sx_n, Tx_j) \geq 1$ *for all* n *even and* $j > n$ *odd;*

(c) $\alpha(Sx_n, Tx_{n+1}) \geq 1$ *for all* n *even and,* Sx_n *and* Tx_{n+1} *converge to an* $x \in X$ *as* $n \to \infty$ *implies that* $\alpha(Sx_n, x) \geq 1$ *and* $\alpha(x, Tx_{n+1}) \geq 1$ *for all* n *even.*

Then the pairs (f, S) *and* (g, T) *have a point of coincidence in* X*. Moreover, if*

(i) $\{f, S\}$ *and* $\{g, T\}$ *are weakly compatible,*

(ii) $\alpha(Su, Tv) \geq 1$ *whenever* $u \in \mathcal{C}(f, S)$ *and* $v \in \mathcal{C}(g, T)$.

Then f, g, S *and* T *have a common fixed point.*

If we take $\varphi(t) = \psi(t) - \phi(t)$ in Corollary 1, we have the following result.

Corollary 2. *Let* f, g, S *and* T *be selfmaps of a complete metric space* (X, d) *with* $f(X) \subset T(X)$*,* $g(X) \subset S(X)$ *and* (f, g) *be an* α_{ST}*-admissible pair such that*

$$\alpha(Sx, Ty)\, \psi(d(fx, gy)) \leq \psi(M(x, y)) - \phi(M(x, y)), \tag{2.21}$$

for all $x, y \in X$*, where* $\psi \in \Psi$ *and* $\phi \in \Phi$*. Assume that the following conditions are satisfied:*

(a) *there exists* $x_0 \in X$ *such that* $\alpha(Sx_0, fx_0) \geq 1$;

(b) $\alpha(Sx_n, Tx_{n+1}) \geq 1$ *for all* n *even implies that* $\alpha(Sx_n, Tx_j) \geq 1$ *for all* n *even and* $j > n$ *odd;*

(c) $\alpha(Sx_n, Tx_{n+1}) \geq 1$ *for all* n *even and,* Sx_n *and* Tx_{n+1} *converge to an* $x \in X$ *as* $n \to \infty$ *implies that* $\alpha(Sx_n, x) \geq 1$ *and* $\alpha(x, Tx_{n+1}) \geq 1$ *for all* n *even.*

Then the pairs (f, S) *and* (g, T) *have a point of coincidence in* X. *Moreover, if*

(i) $\{f, S\}$ *and* $\{g, T\}$ *are weakly compatible,*

(ii) $\alpha(Su, Tv) \geq 1$ *whenever* $u \in \mathcal{C}(f, S)$ *and* $v \in \mathcal{C}(g, T)$.

Then f, g, S *and* T *have a common fixed point.*

Let us give the following hypothesis for the uniqueness of the common fixed point in Theorem 1.

(H) For all $x, y \in \mathcal{F}(f, g, S, T)$, we have $\alpha(Sx, Ty) \geq 1$.

Theorem 4. *Adding condition* (H) *to the hypotheses of Theorem 1, we obtain the uniqueness of the common fixed point of* f, g, S *and* T.

Proof. Suppose that $x = fx = gx = Sx = Tx$ and $y = fy = gy = Sy = Ty$. Then, from (H), we have $\alpha(Sx, Ty) \geq 1$. Then, applying (1.1), we obtain

$$\psi(d(x, y)) = \psi(d(fx, gy)) \leq \varphi(M(x, y)), \tag{2.22}$$

where

$$\begin{aligned} M(x, y) &= \max\Bigg(d(Sx, Ty), d(Sx, fx), d(Ty, gy), \\ &\qquad \frac{d(Sx, gy) + d(fx, Ty)}{2}, \frac{d(Ty, gy)[1 + d(Sx, fx)]}{1 + d(Sx, Ty)}, \\ &\qquad \frac{d(fx, Ty)[1 + d(Sx, gy)]}{1 + d(Sx, Ty)}\Bigg) \\ &= \max(d(x, y), 0, 0, d(x, y), 0, d(x, y)) = d(x, y). \end{aligned}$$

From (2.22), we have

$$\psi(d(x, y)) \leq \varphi(d(x, y)),$$

which implies that $d(x, y) = 0$, that is, $x = y$. $\square$

Remark 1. *Adding condition* (H) *to the hypotheses of Corollaries 1 and 2, we obtain the uniqueness of the common fixed point.*

If we choose $S = T = I_X$ in Corollary 1, we have the following corollary.

Corollary 3. *Let* f *and* g *be selfmaps of a complete metric space* (X, d) *and* (f, g) *be an* α*-admissible pair such that*

$$\alpha(x, y)\psi(d(fx, gy)) \leq \varphi(M_{fg}(x, y)), \tag{2.23}$$

for all $x, y \in X$, *where* $\psi \in \Psi$, $\varphi \in \Phi$ *and*

$$\begin{aligned} M_{fg}(x, y) &= \max\Bigg(d(x, y), d(x, fx), d(y, gy), \frac{d(x, gy) + d(fx, y)}{2}, \\ &\qquad \frac{d(y, gy)[1 + d(x, fx)]}{1 + d(x, y)}, \frac{d(fx, y)[1 + d(x, gy)]}{1 + d(x, y)}\Bigg). \end{aligned}$$

Assume that the following conditions are satisfied:

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(a) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, fx_0) \geq 1$*;*
(b) $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *implies that* $\alpha(x_n, x_j) \geq 1$ *for all* $j > n$*;*
(c) $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and,* $x_n \to x \in X$ *as* $n \to \infty$ *implies that* $\alpha(x_n, x) \geq 1$ *for all* n*.*

Then f *and* g *have a common fixed point. Moreover, if* $\alpha(x, y) \geq 1$ *whenever* $x, y \in \mathcal{F}(f, g)$*, then* f *and* g *have a unique common fixed point.*

Now, we furnish the following example which illustrates Theorem 1 as well as Theorem 4.

Example 1. *Let* $X = \mathbb{R}^+$ *with the usual metric* $d(x, y) = |x - y|$ *for all* $x, y \in X$ *and* $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *be defined by* $\psi(t) = t$ *and* $\varphi(t) = \frac{t}{2}$*. Define the mappings* f, g, S *and* T *on* X *by*

$$fx = \begin{cases} \frac{x}{6} & \text{if } x \in [0,1], \\ 3x & \text{if } x > 1, \end{cases} \quad \text{and} \quad gx = \begin{cases} \frac{x}{4} & \text{if } x \in [0,1], \\ 6x & \text{if } x > 1, \end{cases}$$

$$Sx = \begin{cases} \frac{x}{2} & \text{if } x \in [0,1], \\ 3x & \text{if } x > 1, \end{cases} \quad \text{and} \quad Tx = \begin{cases} \frac{x}{3} & \text{if } x \in [0,1], \\ 2x & \text{if } x > 1. \end{cases}$$

Note that $f(X) \subset T(X)$ *and* $g(X) \subset S(X)$*,* $\{f, S\}$ *and* $\{g, T\}$ *are weakly compatible.*

Also, we define the mapping $\alpha : S(X) \cup T(X) \times S(X) \cup T(X) \to \mathbb{R}^+$ *by*

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in \left[0, \frac{1}{2}\right], \\ 0 & \text{otherwise.} \end{cases}$$

Now, let $x, y \in X$ *such that* $\alpha(Sx, Ty) \geq 1$*. Then* $Sx, Ty \in \left[0, \frac{1}{2}\right]$ *and this implies that* $x, y \in [0, 1]$*. By the definitions of* f, g *and* α*, we have* $fx, gy \in \left[0, \frac{1}{2}\right]$ *and* $gx, fy \in \left[0, \frac{1}{2}\right]$ *which implies that* $\alpha(fx, gy) \geq 1$ *and* $\alpha(gx, fy) \geq 1$*.*

In case of $\alpha(Tx, Sy) \geq 1$*, analogously to the above proof, one can easily obtain that* $\alpha(fx, gy) \geq 1$ *and* $\alpha(gx, fy) \geq 1$*.*

Then (f, g) *is* α_{ST}*-admissible. Moreover, the condition* $\alpha(Sx_0, fx_0) \geq 1$ *is satisfied with* $x_0 = 0$*.*

Let $\{x_n\}$ *be a sequence in* X *such that* $\alpha(Sx_n, Tx_{n+1}) \geq 1$ *for all* n *even. Then, by the definition of* α*, we get* $x_n \in [0, 1]$ *for all* n *even. Thus,* $x_j \in [0, 1]$ *for all* $j > n$ *odd, and so* $\alpha(Sx_n, Tx_j) \geq 1$*.*

Similarly, if $\{x_n\}$ *is any sequence in* X *such that* $\alpha(Sx_n, Tx_{n+1}) \geq 1$ *for all* n *even and,* Sx_n *and* Tx_{n+1} *converge to an* $x \in X$ *as* $n \to \infty$*, then by the definition of* α*, we have* $Sx_n \in \left[0, \frac{1}{2}\right]$ *and* $Tx_{n+1} \in \left[0, \frac{1}{2}\right]$ *for all* n *even and so* $x \in \left[0, \frac{1}{2}\right]$ *which implies that* $\alpha(Sx_n, x) \geq 1$ *and* $\alpha(x, Tx_{n+1}) \geq 1$*.*

Now, we prove that (f, g) *is a generalized* $(\alpha, \psi, \varphi)_{(S,T)}$*-rational contraction. Let* $\alpha(Sx, Ty) \geq 1$*. Then,* $x, y \in [0, 1]$*, and so*

$$\begin{aligned} \psi(d(fx, gy)) &= |fx - gy| = \left|\frac{x}{6} - \frac{y}{4}\right| \\ &\leq \frac{x}{6} = \frac{1}{2}|Sx - fx| \\ &\leq \frac{1}{2} M(x, y) = \varphi(M(x, y)). \end{aligned}$$

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Obviously, assumption (ii) *of Theorem 1 and condition* (H) *are satisfied. Consequently, by Theorems 1 and 4,* f, g, S *and* T *have a unique common fixed point which is* 0.

3. Fixed Point Results on Partially Ordered Metric Spaces

The existence of fixed points of nonlinear contraction mappings in metric spaces endowed with a partial ordering has been considered recently by Ran and Reurings [26] in order to obtain a solution of a matrix equation in 2004. Nieto and Lopez [24] extended the results in [26] by removing the continuity condition of the mapping. They applied their result to get a solution of a boundary value problem (see also [4, 13, 14] and references mentioned therein).

Let X be a non-empty set. If d is a complete metric on X and $\preceq$ is a partial order on the set X, then $(X, d, \preceq)$ is called complete partially ordered metric space. Let $(X, \preceq)$ be a partially ordered set and f, g, S and T be self-mappings on X. Then, (f, g) is called a (S, T)-nondecreasing mapping pair if $fx \preceq gy$ and $gx \preceq fy$ whenever $Sx \preceq Ty$ or $Tx \preceq Sy$ for all $x, y \in X$.

From Theorem 1, in the setting of complete partially ordered metric spaces, we obtain the following theorem.

Theorem 5. *Let* $(X, d, \preceq)$ *be a complete partially ordered metric space and let* f, g, S *and* T *be self-mappings on* X *such that* $f(X) \subset T(X)$, $g(X) \subset S(X)$. *Let* (f, g) *be a* (S, T)*-nondecreasing pair such that*

$$\psi\left(d\left(fx, gy\right)\right) \leq \varphi\left(M\left(x, y\right)\right), \tag{3.1}$$

for all $x, y \in X$ *such that* $Sx \preceq Ty$, *where* $\psi \in \psi$ *and* $\varphi \in \Phi$.

Assume that the following conditions are satisfied:

- (a) *there exists* $x_0 \in X$ *such that* $Sx_0 \preceq fx_0$;
- (b) $Sx_n \preceq Tx_{n+1}$ *for all* n *even implies that* $Sx_n \preceq Tx_j$ *for all* n *even and* $j > n$ *odd;*
- (c) $Sx_n \preceq Tx_{n+1}$ *for all* n *even and,* Sx_n *and* Tx_{n+1} *converge to an* $x \in X$ *as* $n \to \infty$ *implies that* $Sx_n \preceq x$ *and* $x \preceq Tx_{n+1}$ *for all* n *even.*

Then the pairs (f, S) *and* (g, T) *have point of coincidence in* X. *Moreover, if*

- (i) $\{f, S\}$ *and* $\{g, T\}$ *are weakly compatible,*
- (ii) $Su \preceq Tv$ *whenever* $u \in \mathcal{C}(f, S)$ *and* $v \in \mathcal{C}(g, T)$.

Then f, g, S *and* T *have common fixed point. Moreover, if* $Sx \preceq Ty$ *whenever* $x, y \in \mathcal{F}(f, g, S, T)$, *then* f, g, S *and* T *have a unique common fixed point.*

Proof. Define the function $\alpha : X \times X \to \mathbb{R}^+$ by

$$\alpha\left(x, y\right) = \begin{cases} 1 & \text{if } x \preceq y, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\alpha\left(Sx, Ty\right) \geq 1$. Then

$$Sx \preceq Ty. \tag{3.2}$$

From (3.1), we obtain that

$$\psi\left(d\left(fx, gy\right)\right) \leq \varphi\left(M\left(x, y\right)\right).$$

Also, since (f, g) is (S, T)-nondecreasing, by (3.2) we have $fx \preceq gy$ and $gx \preceq fy$, which gives us that $\alpha(fx, gy) \geq 1$ and $\alpha(gx, fy) \geq 1$. Then (f, g) is α_{ST}-admissible.

On the other hand, one can easily show that the hypotheses $(a), (b), (c)$ and (ii) imply the conditions $(a), (b), (c)$ and (ii) of Theorem 1.

Now, let $x, y \in \mathcal{F}(f, g, S, T)$. Then, $Sx \preceq Ty$ and so $\alpha(Sx, Ty) \geq 1$. Therefore, the uniqueness of the common fixed point follows from condition (H). □

If we take $\varphi(t) = \psi(t) - \eta(t)$ in Theorem 5, we have the following result.

Corollary 4. *Let $(X, d, \preceq)$ be a complete partially ordered metric space and let f, g, S and T be self-mappings on X such that $f(X) \subset T(X)$, $g(X) \subset S(X)$. Let (f, g) be a (S, T)-nondecreasing pair such that*

$$\psi(d(fx, gy)) \leq \psi(M(x, y)) - \eta(M(x, y)), \tag{3.3}$$

for all $x, y \in X$ such that $Sx \preceq Ty$, where $\psi \in \psi$ and $\varphi \in \Phi$.

Assume that the following conditions are satisfied:

- (a) *there exists $x_0 \in X$ such that $Sx_0 \preceq fx_0$;*
- (b) *$Sx_n \preceq Tx_{n+1}$ for all n even implies that $Sx_n \preceq Tx_j$ for all n even and $j > n$ odd;*
- (c) *$Sx_n \preceq Tx_{n+1}$ for all n even and, Sx_n and Tx_{n+1} converge to an $x \in X$ as $n \to \infty$ implies that $Sx_n \preceq x$ and $x \preceq Tx_{n+1}$ for all n even.*

Then the pairs (f, S) and (g, T) have point of coincidence in X. Moreover, if

- (i) *$\{f, S\}$ and $\{g, T\}$ are weakly compatible,*
- (ii) *$Su \preceq Tv$ whenever $u \in \mathcal{C}(f, S)$ and $v \in \mathcal{C}(g, T)$.*

Then f, g, S and T have common fixed point. Moreover, if $Sx \preceq Ty$ whenever $x, y \in \mathcal{F}(f, g, S, T)$, then f, g, S and T have a unique common fixed point.

If we take $\psi(t) = t$ and $\eta(t) = (1 - k)t$ in Corollary 4, we have the following result.

Corollary 5. *Let $(X, d, \preceq)$ be a complete partially ordered metric space and let f, g, S and T be self-mappings on X such that $f(X) \subset T(X)$, $g(X) \subset S(X)$. Let (f, g) be a (S, T)-nondecreasing pair such that*

$$d(fx, gy) \leq kM(x, y), \tag{3.4}$$

for all $x, y \in X$ such that $Sx \preceq Ty$, where $k \in [0, 1)$.

Assume that the following conditions are satisfied:

- (a) *there exists $x_0 \in X$ such that $Sx_0 \preceq fx_0$;*
- (b) *$Sx_n \preceq Tx_{n+1}$ for all n even implies that $Sx_n \preceq Tx_j$ for all n even and $j > n$ odd;*
- (c) *$Sx_n \preceq Tx_{n+1}$ for all n even and, Sx_n and Tx_{n+1} converge to an $x \in X$ as $n \to \infty$ implies that $Sx_n \preceq x$ and $x \preceq Tx_{n+1}$ for all n even.*

Then the pairs (f, S) and (g, T) have point of coincidence in X. Moreover, if

- (i) *$\{f, S\}$ and $\{g, T\}$ are weakly compatible,*
- (ii) *$Su \preceq Tv$ whenever $u \in \mathcal{C}(f, S)$ and $v \in \mathcal{C}(g, T)$.*

Then f, g, S and T have common fixed point. Moreover, if $Sx \preceq Ty$ whenever $x, y \in \mathcal{F}(f, g, S, T)$, then f, g, S and T have a unique common fixed point.

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4. Some Results for Graphic Contractions

Consistent with Jachymski [18], let (X,d) be a metric space and let $\Delta := \{(x,x) : x \in X\}$ be a diagonal of the Cartesian product $X \times X$. Consider a graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops; that is, $E(G) \supseteq \Delta$. We assume G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph by assigning to each edge the distance between its vertices. If x and y are vertices in a graph G, then a path in G from x to y of length N ($N \in \mathbb{N}$) is a sequence $\{x_i\}_{i=0}^{N}$ of $N+1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, \ldots, N$. A graph G is connected if there is a path between any two vertices. G is weakly connected if $\widetilde{G}$ is connected (see for more details [2, 9, 10]).

In this section, we give the existence and uniqueness of fixed point theorems on a metric space endowed with graph. Before presenting our results, we give the following notions and definitions.

Definition 7 ([18])**.** *Let (X,d) be a metric space endowed with a graph G and $T : X \to X$ be a mapping. One says that T preserves edges of G if*

$$\forall x, y \in X, \qquad (x,y) \in E(G) \Rightarrow (Tx, Ty) \in E(G). \tag{4.1}$$

Definition 8. *Let f, g, S and T be selfmaps of a metric space (X,d) endowed with a graph G. One says that (f,g) preserves edges of G with respect to (S,T) if for all $x, y \in X$,*

$$(Sx, Ty) \in E(G) \Rightarrow (fx, gy) \in E(G) \text{ and } (gx, fy) \in E(G). \tag{4.2}$$

Definition 9. *Let (X,d) be a metric space endowed with a graph G and f, g, S and T be selfmaps on X such that (f,g) preserves edges of G with respect to (S,T). We say that (f,g) is a generalized $(\alpha,\psi,\varphi)_{(S,T)}$-graphic contraction involving rational expressions if*

$$\psi(d(fx, gy)) \le \varphi(M(x,y)), \tag{4.3}$$

for all $x, y \in X$ for which $(Sx, Ty) \in E(G)$, where $\psi \in \Psi$, $\varphi \in \Phi$ and

$$\begin{aligned} M(x,y) &= \max\Bigg(d(Sx,Ty), d(Sx,fx), d(Ty,gy), \frac{d(Sx,gy)+d(fx,Ty)}{2}, \\ &\qquad \frac{d(Ty,gy)[1+d(Sx,fx)]}{1+d(Sx,Ty)}, \frac{d(fx,Ty)[1+d(Sx,gy)]}{1+d(Sx,Ty)}\Bigg). \end{aligned}$$

Theorem 6. *Let f, g, S and T be selfmaps of a metric space (X,d) endowed with a graph G, and $f(X) \subset T(X)$, $g(X) \subset S(X)$ and (f,g) be a generalized $(\alpha,\psi,\varphi)_{(S,T)}$-graphic contraction involving rational expressions. Assume that the following conditions are satisfied:*

- *(a) there exists $x_0 \in X$ such that $(Sx_0, fx_0) \in E(G)$;*
- *(b) $(Sx_n, Tx_{n+1}) \in E(G)$ for all n even implies that $(Sx_n, Tx_j) \in E(G)$ for all n even and $j > n$ odd;*
- *(c) $(Sx_n, Tx_{n+1}) \in E(G)$ for all n even and, Sx_n and Tx_{n+1} converge to an $x \in X$ as $n \to \infty$ implies that $(Sx_n, x) \in E(G)$ and $(x, Tx_{n+1}) \in E(G)$ for all n even.*

Then the pairs (f,S) and (g,T) have a point of coincidence in X. Moreover, if

- *(i) $\{f,S\}$ and $\{g,T\}$ are weakly compatible,*

(*ii*) $(Su, Tv) \in E(G)$ *whenever* $u \in \mathcal{C}(f, S)$ *and* $v \in \mathcal{C}(g, T)$.

Then f, g, S *and* T *have common fixed point. Moreover, if* $(Sx, Ty) \in E(G)$ *whenever* $x, y \in \mathcal{F}(f, g, S, T)$*, then* f, g, S *and* T *have a unique common fixed point.*

Proof. Define the function $\alpha : X \times X \to \mathbb{R}^+$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } (x, y) \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\alpha(Sx, Ty) \geq 1$. Then

$$(Sx, Ty) \in E(G). \tag{4.4}$$

From (4.3), we obtain that

$$\psi(d(fx, gy)) \leq \varphi(M(x, y)).$$

Also, since (f, g) preserves edges of G with respect to (S, T), by (4.4) we have $(fx, gy) \in E(G)$ and $(gx, fy) \in E(G)$, which gives us that $\alpha(fx, gy) \geq 1$ and $\alpha(gx, fy) \geq 1$. Then (f, g) is α_{ST}-admissible.

On the other hand, it is easy to see that the hypotheses $(a), (b), (c)$ and (ii) imply the conditions $(a), (b), (c)$ and (ii) of Theorem 1.

Now, let $x, y \in \mathcal{F}(f, g, S, T)$. Then, $(Sx, Ty) \in E(G)$ and so $\alpha(Sx, Ty) \geq 1$. Therefore, the uniqueness of the common fixed point follows from condition (H). □

If we take $\varphi(t) = \psi(t) - \phi(t)$ in Theorem 6, we have the following result.

Corollary 6. *Let* f, g, S *and* T *be selfmaps of a metric space* (X, d) *endowed with a graph* G*, and* $f(X) \subset T(X)$*,* $g(X) \subset S(X)$*. Assume that* (f, g) *preserves edges of* G *with respect to* (S, T) *such that*

$$\psi(d(fx, gy)) \leq \psi(M(x, y)) - \phi(M(x, y)), \tag{4.5}$$

for all $x, y \in X$ *for which* $(Sx, Ty) \in E(G)$*, where* $\psi \in \Psi$ *and* $\phi \in \Phi$.

Suppose also that the following conditions are satisfied:

(*a*) *there exists* $x_0 \in X$ *such that* $(Sx_0, fx_0) \in E(G)$;

(*b*) $(Sx_n, Tx_{n+1}) \in E(G)$ *for all* n *even implies that* $(Sx_n, Tx_j) \in E(G)$ *for all* n *even and* $j > n$ *odd;*

(*c*) $(Sx_n, Tx_{n+1}) \in E(G)$ *for all* n *even and,* Sx_n *and* Tx_{n+1} *converge to an* $x \in X$ *as* $n \to \infty$ *implies that* $(Sx_n, x) \in E(G)$ *and* $(x, Tx_{n+1}) \in E(G)$ *for all* n *even.*

Then the pairs (f, S) *and* (g, T) *have a point of coincidence in* X*. Moreover, if*

(*i*) $\{f, S\}$ *and* $\{g, T\}$ *are weakly compatible and,*

(*ii*) $(Su, Tv) \in E(G)$ *whenever* $u \in \mathcal{C}(f, S)$ *and* $v \in \mathcal{C}(g, T)$.

Then f, g, S *and* T *have common fixed point. Moreover, if* $(Sx, Ty) \in E(G)$ *whenever* $x, y \in \mathcal{F}(f, g, S, T)$*, then* f, g, S *and* T *have a unique common fixed point.*

If we take $\psi(t) = t$ and $\phi(t) = (1 - k)t$ in Corollary 6, we have the following result.

Corollary 7. *Let* f, g, S *and* T *be selfmaps of a metric space* (X, d) *endowed with a graph* G*, and* $f(X) \subset T(X)$*,* $g(X) \subset S(X)$*. Assume that* (f, g) *preserves edges of* G *with respect to* (S, T) *such that*

$$d(fx, gy) \leq kM(x, y), \tag{4.6}$$

for all $x, y \in X$ for which $(Sx, Ty) \in E(G)$, where $\psi \in \Psi$ and $\phi \in \Phi$.

Suppose also that the following conditions are satisfied:

- (a) *there exists $x_0 \in X$ such that $(Sx_0, fx_0) \in E(G)$;*
- (b) *$(Sx_n, Tx_{n+1}) \in E(G)$ for all n even implies that $(Sx_n, Tx_j) \in E(G)$ for all n even and $j > n$ odd;*
- (c) *$(Sx_n, Tx_{n+1}) \in E(G)$ for all n even and, Sx_n and Tx_{n+1} converge to an $x \in X$ as $n \to \infty$ implies that $(Sx_n, x) \in E(G)$ and $(x, Tx_{n+1}) \in E(G)$ for all n even.*

Then the pairs (f, S) and (g, T) have a point of coincidence in X. Moreover, if

- (i) *$\{f, S\}$ and $\{g, T\}$ are weakly compatible and,*
- (ii) *$(Su, Tv) \in E(G)$ whenever $u \in \mathcal{C}(f, S)$ and $v \in \mathcal{C}(g, T)$.*

Then f, g, S and T have common fixed point. Moreover, if $(Sx, Ty) \in E(G)$ whenever $x, y \in \mathcal{F}(f, g, S, T)$, then f, g, S and T have a unique common fixed point.

5. An Application

Consider the following integral equations:

$$x(s) = \int_a^b H_1(s, r, x(r))\, dr, \tag{5.1}$$

and

$$x(s) = \int_a^b H_2(s, r, x(r))\, dr, \tag{5.2}$$

where $s, r \in I = [a, b]$, $H_1, H_2 : I \times I \times \mathbb{R} \to \mathbb{R}$ and $b > a \geq 0$.

In this section, we present an existence and uniqueness theorem for a common solution to (5.1) and (5.2) that belongs to $X := C(I, \mathbb{R})$ (the set of continuous functions defined on I) by using the obtained result in Corollary 3.

We consider the operators $f, g : X \to X$ given by for all $x \in X$

$$fx(s) = \int_a^b H_1(s, r, x(r))\, dr, \quad s \in I,$$

and

$$gx(s) = \int_a^b H_2(s, r, x(r))\, dr, \quad s \in I.$$

Then the existence of a common solution to (5.1) and (5.2) are equivalent to the existence of a common fixed point of f and g.

Meanwhile, X endowed with the metric d defined by

$$d(x, y) = \sup_{s \in I} |x(s) - y(s)|$$

for all $x, y \in X$, is a complete metric space.

Suppose that the following conditions hold.

- $(A1)$ $H_1, H_2 : I \times I \times \mathbb{R} \to \mathbb{R}$ are continuous;
- $(A2)$ there exist $\xi : X \times X \to \mathbb{R}$ such that if $\xi(x, y) \geq 0$ for all $x, y \in X$, then for every $s, r \in I$, we have

$$|H_1(s, r, x(r)) - H_2(s, r, y(r))|^2 \leq \gamma(s, r) \ln\left(1 + |x(r) - y(r)|^2\right)$$

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where $\gamma : I \times I \to \mathbb{R}^+$ is a continuous function satisfying $\sup_{s\in I} \int_a^b \gamma(s,r) \leq 1/(b-a)$;

$(A3)$ for every $s \in I$ there exist $x_0 \in X$ such that $\xi(x_0(s), fx_0(s)) \geq 0$;

$(A4)$ for all $s \in I$ and $x, y \in X$,

$$\xi(x(s), y(s)) \geq 0 \;\Rightarrow\; \xi(fx(s), gy(s)) \geq 0 \text{ and } \xi(gx(s), fy(s)) \geq 0,$$

$(A5)$ $\xi(x_n(s), x_{n+1}(s)) \geq 0$ for all n and $s \in I$ implies that $\xi(x_n(s), x_j(s)) \geq 0$ for all $j > n$;

$(A6)$ $\xi(x_n(s), x_{n+1}(s)) \geq 0$ for all n and $s \in I$ and, $x_n \to x \in X$ as $n \to \infty$ implies that $\xi(x_n(s), x(s)) \geq 0$ for all n.

Theorem 7. *Assume that the conditions $(A1)-(A6)$ are satisfied. Then, integral equations (5.1) and (5.2) have a common solution in X.*

Proof. Let $x, y \in X$ such that $\xi(x,y) \geq 0$. Then, by $(A2)$, for all $s, r \in I$, we deduce

$$\begin{aligned}
|fx(s) - gy(s)|^2 &\leq \left(\int_a^b |H_1(s,r,x(r)) - H_2(s,r,y(r))|\, dr\right)^2 \\
&\leq \int_a^b 1^2 dr \int_a^b |H_1(s,r,x(r)) - H_2(s,r,y(r))|^2\, dr \\
&\leq (b-a)\int_a^b \gamma(s,r)\ln\left(1 + |x(r) - y(r)|^2\right) dr \\
&\leq (b-a)\int_a^b \gamma(s,r)\ln\left(1 + d(x,y)^2\right) dr \\
&= (b-a)\left(\int_a^b \gamma(s,r)\, dr\right)\ln\left(1 + d(x,y)^2\right) \\
&\leq \ln\left(1 + d(x,y)^2\right) \leq \ln\left(1 + M_{fg}(x,y)^2\right),
\end{aligned}$$

where

$$\begin{aligned}
M_{fg}(x,y) = \max\Bigg(& d(x(s), y(s)), d(x(s), fx(s)), d(y(s), gy(s)), \\
& \frac{d(x(s), gy(s)) + d(fx(s), y(s))}{2}, \\
& \frac{d(y(s), gy(s))\,[1 + d(x(s), fx(s))]}{1 + d(x(s), y(s))}, \\
& \frac{d(fx(s), y(s))\,[1 + d(x(s), gy(s))]}{1 + d(x(s), y(s))}\Bigg).
\end{aligned}$$

Therefore, we obtain

$$\left(\sup_{s\in I} |fx(s) - gy(s)|\right)^2 \leq \ln\left(1 + M_{fg}(x,y)^2\right).$$

Now, define $\alpha : X \times X \to \mathbb{R}^+$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } \xi(x,y) \geq 0 \text{ where } x, y \in X, \\ 0 & \text{otherwise.} \end{cases}$$

Also, define $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\psi(t) = t^2$ and $\varphi(t) = \ln\left(1 + t^2\right)$. Therefore, using the last inequality, we have

$$\alpha(x, y)\,\psi(d(fx, gy)) \leq \varphi(M_{fg}(x, y)).$$

It easily shows that all the hypotheses of Corollary 3 are satisfied. Therefore f and g have a common fixed point, that is, integral equations (5.1) and (5.2) have a common solution. □

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ON QUADRATIC ρ-FUNCTIONAL INEQUALITIES IN FUZZY NORMED SPACES

CHOONKIL PARK, SUN YOUNG JANG, AND SUNGSIK YUN*

Abstract. In this paper, we solve the following quadratic ρ-functional inequalities

$$N\left(f(x+y)+f(x-y)-2f(x)-2f(y)-\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right),t\right) \geq \frac{t}{t+\varphi(x,y)}, \tag{0.1}$$

where ρ is a fixed real number with $\rho \neq 2$, and

$$N\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)-\rho\left(f(x+y)+f(x-y)-2f(x)-2f(y)\right),t\right) \geq \frac{t}{t+\varphi(x,y)}, \tag{0.2}$$

where ρ is a fixed real number with $\rho \neq \frac{1}{2}$.

Using the direct method, we prove the Hyers-Ulam stability of the quadratic ρ-functional inequalities (0.1) and (0.2) in fuzzy Banach spaces.

1. Introduction and preliminaries

Katsaras [14] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [9, 16, 37]. In particular, Bag and Samanta [2], following Cheng and Mordeson [6], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [15]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [3].

We use the definition of fuzzy normed spaces given in [2, 19, 20] to investigate the Hyers-Ulam stability of quadratic ρ-functional inequalities in fuzzy Banach spaces.

Definition 1.1. [2, 19, 20, 21] Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N_1) $N(x,t) = 0$ for $t \leq 0$;
(N_2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
(N_3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N_4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;
(N_5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.
(N_6) for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a *fuzzy normed vector space.*

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [18, 19].

Definition 1.2. [2, 19, 20, 21] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$

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for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by $N\text{-}\lim_{n\to\infty} x_n = x$.

Definition 1.3. [2, 19, 20, 21] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [3]).

The stability problem of functional equations originated from a question of Ulam [36] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [27] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [10] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

The functional equation $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [35] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [7] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. Czerwik [8] proved the Hyers-Ulam stability of the quadratic functional equation. The functional equation $f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$ is called a *Jensen type quadratic equation*. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [5, 12, 13, 17, 24, 25, 26, 28, 29, **?**, 30, 31, 32, 33, 34]).

Park [22, 23] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces.

In Section 2, we solve the quadratic ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.1) in fuzzy Banach spaces by using the direct method.

In Section 3, we solve the quadratic ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.2) in fuzzy Banach spaces by using the direct method.

Throughout this paper, assume that X is a real vector space and (Y, N) is a fuzzy Banach space.

2. Quadratic ρ-functional inequality (0.1)

In this section, we prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.1) in fuzzy Banach spaces. Let ρ be a real number with $\rho \neq 2$. We need the following lemma to prove the main results.

Lemma 2.1. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$f(x+y) + f(x-y) - 2f(x) - 2f(y) = \rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right) \quad (2.1)$$

for all $x, y \in X$. Then $f : X \to Y$ is quadratic.

Proof. Replacing y by x in (2.1), we get $f(2x) - 4f(x) = 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$\begin{aligned} f(x+y) + f(x-y) - 2f(x) - 2f(y) &= \rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right) \\ &= \frac{\rho}{2}(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \end{aligned}$$

and so $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in X$, as desired. $\square$

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\Phi(x,y) := \sum_{j=1}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \quad (2.2)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\begin{aligned} &N(f(x+y) + f(x-y) - 2f(x) - 2f(y) \quad (2.3) \\ &-\rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right), t) \geq \frac{t}{t + \varphi(x,y)} \end{aligned}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{t}{t + \frac{1}{4}\Phi(x,x)} \quad (2.4)$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $y = x$ in (2.3), we get

$$N(f(2x) - 4f(x), t) \geq \frac{t}{t + \varphi(x,x)} \quad (2.5)$$

and so $N\left(f(x) - 4f\left(\frac{x}{2}\right), t\right) \geq \frac{t}{t+\varphi\left(\frac{x}{2},\frac{x}{2}\right)}$ for all $x \in X$. Hence

$$\begin{aligned} &N\left(4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right), t\right) \quad (2.6) \\ &\geq \min\left\{N\left(4^l f\left(\frac{x}{2^l}\right) - 4^{l+1} f\left(\frac{x}{2^{l+1}}\right), t\right), \cdots, N\left(4^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 4^m f\left(\frac{x}{2^m}\right), t\right)\right\} \\ &= \min\left\{N\left(f\left(\frac{x}{2^l}\right) - 4f\left(\frac{x}{4^{l+1}}\right), \frac{t}{4^l}\right), \cdots, N\left(f\left(\frac{x}{2^{m-1}}\right) - 4f\left(\frac{x}{2^m}\right), \frac{t}{4^{m-1}}\right)\right\} \\ &\geq \min\left\{\frac{\frac{t}{4^l}}{\frac{t}{4^l} + \varphi\left(\frac{x}{2^{l+1}}, \frac{x}{2^{l+1}}\right)}, \cdots, \frac{\frac{t}{4^{m-1}}}{\frac{t}{4^{m-1}} + \varphi\left(\frac{x}{2^m}, \frac{x}{2^m}\right)}\right\} \\ &= \min\left\{\frac{t}{t + 4^l\varphi\left(\frac{x}{2^{l+1}}, \frac{x}{2^{l+1}}\right)}, \cdots, \frac{t}{t + 4^{m-1}\varphi\left(\frac{x}{2^m}, \frac{x}{2^m}\right)}\right\} \\ &\geq \frac{t}{t + \frac{1}{4}\sum_{j=l+1}^{m} 4^j \varphi\left(\frac{x}{2^j}, \frac{x}{2^j}\right)} \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$ and all $t > 0$. It follows from (2.2) and (2.6) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.6), we get (2.4).

By (2.3),

$$\begin{aligned}&N\left(4^n\left(f\left(\frac{x+y}{2^n}\right)+f\left(\frac{x-y}{2^n}\right)-2f\left(\frac{x}{2^n}\right)-2f\left(\frac{y}{2^n}\right)\right)\right.\\&\quad\left.-\rho\left(4^n\left(2f\left(\frac{x+y}{2^{n+1}}\right)+2f\left(\frac{x-y}{2^{n+1}}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right)\right),4^n t\right)\geq\frac{t}{t+\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}\end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\begin{aligned}&N\left(4^n\left(f\left(\frac{x+y}{2^n}\right)+f\left(\frac{x-y}{2^n}\right)-2f\left(\frac{x}{2^n}\right)-2f\left(\frac{y}{2^n}\right)\right)\right.\\&\quad\left.-\rho\left(4^n\left(2f\left(\frac{x+y}{2^{n+1}}\right)+2f\left(\frac{x-y}{2^{n+1}}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right)\right),t\right)\\&\quad\geq\frac{\frac{t}{4^n}}{\frac{t}{4^n}+\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}=\frac{t}{t+4^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}\end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{t}{t+4^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$Q(x+y)+Q(x-y)-2Q(x)-2Q(y)=\rho\left(2Q\left(\frac{x+y}{2}\right)+2Q\left(\frac{x-y}{2}\right)-Q(x)-Q(y)\right)$$

for all $x, y \in X$. By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic, as desired. □

Corollary 2.3. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned}&N\left(f(x+y)+f(x-y)-2f(x)-2f(y)-\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right),t\right)\\&\quad\geq\frac{t}{t+\theta(\|x\|^p+\|y\|^p)}\end{aligned}\tag{2.7}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x)-Q(x),t\right)\geq\frac{(2^p-4)t}{(2^p-4)t+2\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$, as desired. □

Theorem 2.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\Phi(x,y):=\sum_{j=0}^{\infty}\frac{1}{4^j}\varphi\left(2^jx,2^jy\right)<\infty$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.3). Then $Q(x) := N$-$\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{1}{t + \frac{1}{4}\Phi(x, x)}$$

for all $x \in X$ and all $t > 0$.

Proof. It follows from (2.5) that $N\left(f(x) - \frac{1}{4}f(2x), \frac{1}{4}t\right) \geq \frac{t}{t+\varphi(x,x)}$ and so

$$N\left(f(x) - \frac{1}{4}f(2x), t\right) \geq \frac{4t}{4t + \varphi(x, x)} = \frac{t}{t + \frac{1}{4}\varphi(x, x)}$$

for all $x \in X$ and all $t > 0$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.7). Then $Q(x) := N$-$\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N(f(x) - Q(x), t) \geq \frac{(4 - 2^p)t}{(4 - 2^p)t + 2\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$, as desired. □

3. Quadratic ρ-functional inequality (0.2)

In this section, we prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.2) in fuzzy Banach spaces. Let ρ be a real number with $\rho \neq \frac{1}{2}$. We need the following lemma to prove the main results.

Lemma 3.1. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$2f\left(\frac{x + y}{2}\right) + 2f\left(\frac{x - y}{2}\right) - f(x) - f(y) = \rho\left(f(x + y) + f(x - y) - 2f(x) - 2f(y)\right) \quad (3.1)$$

for all $x, y \in X$. Then $f : X \to Y$ is quadratic.

Proof. Letting $y = 0$ in (3.1), we get $4f\left(\frac{x}{2}\right) - f(x) = 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$\begin{aligned}\frac{1}{2}f(x + y) - \frac{1}{2}f(x - y) - f(x) - f(y) &= 2f\left(\frac{x + y}{2}\right) + 2f\left(\frac{x - y}{2}\right) - f(x) - f(y) \\ &= \rho(f(x + y) + f(x - y) - 2f(x) - 2f(y))\end{aligned}$$

and so $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ for all $x, y \in X$, as desired. □

Theorem 3.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\Phi(x, y) := \sum_{j=0}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \quad (3.2)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\begin{aligned} N&\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)-\rho\left(f(x+y)+f(x-y)-2f(x)-2f(y)\right),t\right) \\ &\geq \frac{t}{t+\varphi(x,y)} \end{aligned} \tag{3.3}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x)-Q(x),t\right) \geq \frac{t}{t+\Phi(x,0)} \tag{3.4}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $y = 0$ in (3.3), we get

$$N\left(f(x)-4f\left(\frac{x}{2}\right),t\right) = N\left(4f\left(\frac{x}{2}\right)-f(x),t\right) \geq \frac{t}{t+\varphi(x,0)} \tag{3.5}$$

for all $x \in X$. Hence

$$\begin{aligned} N&\left(4^l f\left(\frac{x}{2^l}\right)-4^m f\left(\frac{x}{2^m}\right),t\right) \\ &\geq \min\left\{N\left(4^l f\left(\frac{x}{2^l}\right)-4^{l+1} f\left(\frac{x}{2^{l+1}}\right),t\right),\cdots,N\left(4^{m-1} f\left(\frac{x}{2^{m-1}}\right)-4^m f\left(\frac{x}{2^m}\right),t\right)\right\} \\ &= \min\left\{N\left(f\left(\frac{x}{2^l}\right)-4f\left(\frac{x}{2^{l+1}}\right),\frac{t}{4^l}\right),\cdots,N\left(f\left(\frac{x}{2^{m-1}}\right)-4f\left(\frac{x}{2^m}\right),\frac{t}{4^{m-1}}\right)\right\} \\ &\geq \min\left\{\frac{\frac{t}{4^l}}{\frac{t}{4^l}+\varphi\left(\frac{x}{2^l},0\right)},\cdots,\frac{\frac{t}{4^{m-1}}}{\frac{t}{4^{m-1}}+\varphi\left(\frac{x}{2^{m-1}},0\right)}\right\} \\ &= \min\left\{\frac{t}{t+4^l\varphi\left(\frac{x}{2^l},0\right)},\cdots,\frac{t}{t+4^{m-1}\varphi\left(\frac{x}{2^{m-1}},0\right)}\right\} \\ &\geq \frac{t}{t+\sum_{j=l}^{m-1} 4^j\varphi\left(\frac{x}{2^j},0\right)} \end{aligned} \tag{3.6}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$ and all $t > 0$. It follows from (3.2) and (3.6) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.6), we get (3.4).

By (3.2),

$$\begin{aligned} N&\left(4^n\left(2f\left(\frac{x+y}{2^{n+1}}\right)+2f\left(\frac{x-y}{2^{n+1}}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right)\right. \\ &\left.-\rho\left(4^n\left(f\left(\frac{x+y}{2^n}\right)+f\left(\frac{x-y}{2^n}\right)-2f\left(\frac{x}{2^n}\right)-2f\left(\frac{y}{2^n}\right)\right)\right),4^n t\right) \geq \frac{t}{t+\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\begin{aligned}
&N\left(4^n\left(2f\left(\frac{x+y}{2^{n+1}}\right)+2f\left(\frac{x-y}{2^{n+1}}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right)\right.\\
&\left.-\rho\left(4^n\left(f\left(\frac{x+y}{2^n}\right)+f\left(\frac{x-y}{2^n}\right)-2f\left(\frac{x}{2^n}\right)-2f\left(\frac{y}{2^n}\right)\right)\right),t\right)\\
&\geq \frac{\frac{t}{4^n}}{\frac{t}{4^n}+\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}=\frac{t}{t+4^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}
\end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{t}{t+4^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$2Q\left(\frac{x+y}{2}\right)+2\left(\frac{x-y}{2}\right)-Q(x)-Q(y)=\rho\left(Q\left(x+y\right)+Q\left(x-y\right)-2Q(x)-2Q(y)\right)$$

for all $x, y \in X$=. By Lemma 3.1, the mapping $Q : X \to Y$ is quadratic, as desired. □

Corollary 3.3. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned}
&N(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y) \qquad (3.7)\\
&-\rho\left(f(x+y)+f(x-y)-2f(x)-2f(y)\right),t)\geq \frac{t}{t+\theta(\|x\|^p+\|y\|^p)}
\end{aligned}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x)-Q(x),t\right)\geq \frac{(2^p-4)t}{(2^p-4)t+2^p\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$, as desired. □

Theorem 3.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\Phi(x,y):=\sum_{j=1}^{\infty}\frac{1}{4^j}\varphi\left(2^jx,2^jy\right)<\infty$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.3). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x)-Q(x),t\right)\geq \frac{t}{t+\Phi(x,0)}$$

for all $x \in X$ and all $t > 0$.

Proof. It follows from (3.5) that $N\left(f(x) - \frac{1}{4}f(2x), \frac{t}{4}\right) \geq \frac{t}{t+\varphi(2x,0)}$ and so

$$N\left(f(x)-\frac{1}{4}f(2x),t\right)\geq \frac{4t}{4t+\varphi(2x,0)}=\frac{t}{t+\frac{1}{4}\varphi(2x,0)}$$

for all $x \in X$ and all $t > 0$.

The rest of the proof is similar to the proof of Theorem 3.2. □

Corollary 3.5. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.7). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N(f(x) - Q(x), t) \geq \frac{(4 - 2^p)t}{(4 - 2^p)t + 2^p\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$, as desired. □

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ON A DOUBLE INTEGRAL EQUATION INCLUDING A SET OF TWO VARIABLES POLYNOMIALS SUGGESTED BY LAGUERRE POLYNOMIALS

M. ALİ ÖZARSLAN AND CEMALİYE KÜRT

ABSTRACT. In this paper, we introduce general classes of bivariate and Mittag-Leffler functions $E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$ and Laguerre polynomials $L_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$. We investigate double fractional integrals and derivative properties of the above mentioned classes. We further obtain linear generating function for $L_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$ in terms of $E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$. Finally, we calculate double Laplace transforms of the above mentioned classes and then we consider a general singular integral equation with $L_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$ in the kernel and obtain the solution in terms of $E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$.

1. Introduction

The special function of the form [7]

$$E_{\alpha}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)} \tag{1.1}$$

$$(\alpha\in\mathbb{C},\mathrm{Re}(\alpha)>0,z\in\mathbb{C})$$

and more general function [12] of (1.1)

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)} \tag{1.2}$$

$$(\alpha,\beta\in\mathbb{C},\mathrm{Re}(\alpha),\mathrm{Re}(\beta)>0,z\in\mathbb{C})$$

are known as Mittag-Leffler functions the first of which was introduced by Swedish mathematician G. Mittag-Leffler and the second one by Wiman.

Setting $\alpha=\beta=1$, the equation (1.2) becomes the exponential function e^z. When $0<\alpha<1$, it bridges an interpolation between the pure exponential function e^z and a geometric function

$$\frac{1}{1-z}=\sum_{n=0}^{\infty}z^n.$$

$$(|z|<1)$$

Key words and phrases. Double fractional integrals and derivatives, Bivariate Mittag-Leffler function, Bivariate Laguerre polynomials, Double generating functions, Singular double integral equation, Double Laplace integral.

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A further generalization of (1.2) was introduced by Prabhakar (see [9]) as

$$E_{\alpha,\beta}^{\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!} \tag{1.3}$$
$$(\alpha,\beta,\gamma\in\mathbb{C},\mathrm{Re}(\alpha),\mathrm{Re}(\beta)>0)$$

where the Pochammer symbol [11], $(\gamma)_n$, is defined as

$$(\gamma)_n=\frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}=\begin{cases}1 & ;n=0,\gamma\neq 0\\ \gamma(\gamma+1)\cdots(\gamma+n-1) & ;n=1,2,\cdots\end{cases}$$

In the special case, we have the polynomials $Z_n^{\alpha}(x;k)$ (see [6],[10]) which were defined by

$$Z_n^{\alpha}(x;k)=\frac{\Gamma(kn+\alpha+1)}{n!}E_{k,\alpha+1}^{-n}(x^k)\quad .$$
$$(\mathrm{Re}(\alpha)>0,k\in\mathbb{Z}_{0^+})$$

Note that, in [6] and [10], generating functions, integrals and recurrence relations were developed for the polynomials $Z_n^{\alpha}(x;k)$ of degree n in x^k, which form one set of the biorthogonal pair corresponding to the weight function $e^{-x}x^{\alpha}$ over the interval $(0,\infty)$.

For $k=1$, we have $Z_n^{\alpha}(x;1)=L_n^{\alpha}(x)$ where $L_n^{\alpha}(x)$ is the usual Laguerre polynomial which were given as follows

$$L_n^{\alpha}(x)=\frac{(1+\alpha)_n}{n!}\ {}_1F_1(-n;1+\alpha;x)$$

where

$${}_1F_1(-n;1+\alpha;x)=\sum_{k=0}^{n}\frac{(-n)_k}{(1+\alpha)_k}\frac{x^k}{k!}.$$

Very recently, a class of polynomials $Z_{n_1,\cdots,n_j}^{(\alpha)}\left(x_1,\cdots,x_j;\rho_1,\cdots,\rho_j\right)$ (see [8]) suggested by the multivariate Laguerre polynomials were defined by

(1.4)

$$Z_{n_1,\cdots,n_j}^{(\alpha)}\left(x_1,\cdots,x_j;\rho_1,\cdots,\rho_j\right)$$
$$=\frac{\Gamma\left(\rho_1 n_1+\cdots+\rho_j n_j+\alpha+1\right)}{n_1!\cdots n_j!}\sum_{k_1,\cdots,k_j=0}^{n_1,\cdots,n_j}\frac{(-n_1)_{k_1}\cdots(-n_j)_{k_j}\,x_1^{\rho_1 k_1}\cdots x_j^{\rho_j k_j}}{\Gamma\left(\rho_1 k_1+\cdots+\rho_j k_j+\alpha+1\right)k_1!\cdots k_j!}.$$

$(\alpha,\rho_1,\cdots,\rho_j\in\mathbb{C},\ \mathrm{Re}(\rho_i)>0\ (i=1,\cdots,j))$

Obviously $Z_{n_1,\cdots,n_j}^{(\alpha)}\left(x_1,\cdots,x_j;\rho_1,\cdots,\rho_j\right)$ gives $L_{n_1,\cdots,n_j}^{(\alpha)}(x_1,\cdots,x_j)$ when $\rho_1=\cdots=\rho_j=1$, where $L_{n_1,\cdots,n_j}^{(\alpha)}(x_1,\cdots,x_j)$ is the multivariable Laguerre polynomial [2] given by

$$L_{n_1,\cdots,n_j}^{(\alpha)}(x_1,\cdots,x_j)=\frac{\Gamma\left(n_1+\cdots+n_j+\alpha+1\right)}{n_1!\cdots n_j!}\sum_{k_1,\cdots,k_j=0}^{n_1,\cdots,n_j}\frac{(-n_1)_{k_1}\cdots(-n_j)_{k_j}\,x_1^{k_1}\cdots x_j^{k_j}}{\Gamma\left(k_1+\cdots+k_j+\alpha+1\right)k_1!\cdots k_j!}.$$

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It is known that the multivariate Mittag-Leffler functions were defined by the multiple series as [13]

$$E_{\rho_1,\cdots,\rho_j,\lambda}^{(\gamma_1,\cdots,\gamma_j)}(x_1,\cdots,x_j)=\sum_{k_1,\cdots,k_j=0}^{\infty}\frac{(\gamma_1)_{k_1}\cdots(\gamma_j)_{k_j}\,x_1^{k_1}\cdots x_j^{k_j}}{\Gamma\left(\rho_1k_1+\cdots+\rho_jk_j+\lambda\right)k_1!\cdots k_j!}. \tag{1.5}$$

$$(\lambda,\rho_1,\cdots,\rho_j,\gamma_1,\cdots,\gamma_j\in\mathbb{C},\ \mathrm{Re}(\rho_i)>0\ (i=1,\cdots,j))$$

Note that the function in (1.5) is a special case of the generalized Lauricella series in several variables introduced and investigated by Srivastava and Daoust [16] (see also see [14],[15]). Also, when $j=1,\rho_1=\alpha,\lambda=\beta,\gamma_1=\gamma,x_1=z$, the function (1.5) reduces to (1.3).

The polynomials $Z_{n_1,\cdots,n_j}^{(\alpha)}\left(x_1,\cdots,x_j;\rho_1,\cdots,\rho_j\right)$ can be represented in terms of the multivariate Mittag-Leffler functions as follows (see [8]):

$$\begin{aligned} &Z_{n_1,\cdots,n_j}^{(\alpha)}\left(x_1,\cdots,x_j;\rho_1,\cdots,\rho_j\right) \\ =\ &\frac{\Gamma\left(\rho_1n_1+\cdots+\rho_jn_j+\alpha+1\right)}{n_1!\cdots n_j!}E_{\rho_1,\cdots,\rho_j,\alpha+1}^{(-n_1,\cdots,-n_j)}(x_1^{\rho_1},\cdots,x_j^{\rho_j}).\end{aligned} \tag{1.6}$$

Clearly, setting $\rho_1=\rho_2=\cdots=\rho_j=1$ in (1.6) gives

$$L_{n_1,\cdots,n_j}^{(\alpha)}(x_1,\cdots,x_j)=\frac{\Gamma(n_1+\cdots+n_j+\alpha+1)}{n_1!\cdots n_j!}E_{1,\cdots,1,\alpha+1}^{(-n_1,\cdots,-n_j)}(x_1,\cdots,x_j).$$

Very recently, a slight motivated form of the multivariate Mittag-Leffler functions were introduced and investigated in [3].

On the other hand, a nontrivial two variables Mittag-Leffler functions were defined in [4] by

$$\begin{aligned} E_1(x,y)&=E_1\left(\begin{array}{c|c}\gamma_1,\alpha_1;\gamma_2,\beta_1 & x\\ \delta_1,\alpha_2,\beta_2;\delta_2,\alpha_3;\delta_3,\beta_3 & y\end{array}\right)\\ &=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\gamma_1)_{\alpha_1m}(\gamma_2)_{\beta_1n}}{\Gamma(\delta_1+\alpha_2m+\beta_2n)}\frac{x^m}{\Gamma(\delta_2+\alpha_3m)}\frac{y^n}{\Gamma(\delta_3+\beta_3n)}.\end{aligned}$$

$$(\gamma_1,\gamma_2,\delta_1,\delta_2,\delta_3,x,y\in\mathbb{C},\min\{\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3\}>0\})$$

Motivated essentially by the above definitions and investigations, in this paper, we introduce a class of bivariate Mittag-Leffler function

$$E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(\gamma_1)_{k_1}(\gamma_2)_{k_2}x^{k_1}y^{k_2}}{\Gamma(\alpha k_1+\beta k_2+\lambda)\Gamma(\eta k_2+\xi)k_1!k_2!} \tag{1.7}$$

where $\gamma_1,\gamma_2,\alpha,\beta,\lambda,\eta,\xi\in\mathbb{C},\mathrm{Re}(\alpha+\eta)>0$ and $\mathrm{Re}(\beta)>0$.

According to the convergence conditions investigated by Srivastava and Daoust ([15], p. 155) for the generalized Lauricella series in two variables, the series in (1.7) converges absolutely for $\mathrm{Re}(\alpha+\eta)>0$ and $\mathrm{Re}(\beta)>0$.

We also introduce a general class of bivariate Laguerre polynomials

$$\begin{aligned} &L_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)\\ =\ &\frac{\Gamma(\alpha n+\beta m+\gamma+1)}{\Gamma(\xi+\eta m)}\sum_{k_1=0}^{n}\sum_{k_2=0}^{m}\frac{(-n)_{k_1}(-m)_{k_2}x^{\alpha k_1}y^{\beta k_2}}{\Gamma(\alpha k_1+\beta k_2+\gamma+1)\Gamma(\eta k_2+\xi)k_1!k_2!}\end{aligned} \tag{1.8}$$

where $\alpha,\beta,\gamma,\eta,\xi\in\mathbb{C},\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\eta),\mathrm{Re}(\xi)>0,\mathrm{Re}(\gamma)>-1$.

Comparing (1.7) and (1.8), we see that

$$L_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)=\frac{\Gamma(\alpha n+\beta m+\gamma+1)}{\Gamma(\xi+\eta m)}E_{-n,-m}^{(\alpha,\beta,\eta,\xi,\lambda)}(x^{\alpha},y^{\beta}). \tag{1.9}$$

This paper is organized as follows. In section 2, we calculate the double fractional integrals and derivatives of the above mentioned classes (1.7) and (1.8). Linear generating functions for $L_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$ are given in terms of $E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$ in Section 3. In the last section, we first investigate double Laplace transforms of the above mentioned classes and then we consider a general singular integral equation with $L_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$ in the kernel and obtain the solution by means of $E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$.

2. Fractional integrals and derivatives

This section aims to provide the fractional integral formulas of the functions $E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$ and $L_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$. Throughout this section, we assume that $\mathrm{Re}(\alpha),\mathrm{Re}(\beta)>0,\mathrm{Re}(\mu),\mathrm{Re}(\lambda)>0,\mathrm{Re}(\gamma)>-1$.

Definition 2.1. ([1],[8]) *Let $\Omega=[a,b]$ be a finite interval of the real axis. The Riemann-Liouville fractional integral of order $\mu\in\mathbb{C}$ $(\mathrm{Re}(\mu)>0)$ is defined by*

$${}_xI_{a^+}^{\mu}[f]=\frac{1}{\Gamma(\mu)}\int_a^x\frac{f(t)\;dt}{(x-t)^{1-\mu}}\;.\quad(x>a,\mathrm{Re}(\mu)>0)$$

Similarly, the partial fractional integrals of a function $f(x,t)$, where $(x,t)\in\mathbb{R}\times\mathbb{R}$ is defined as follows:

$${}_xI_{a^+}^{\mu}f(x,t)=\frac{1}{\Gamma(\mu)}\int_a^x(x-\xi)^{\mu-1}f(\xi,t)d\xi\;,\quad(x>a,\mathrm{Re}(\mu)>0)$$

$${}_tI_{a^+}^{\lambda}f(x,t)=\frac{1}{\Gamma(\lambda)}\int_b^t(t-\tau)^{\lambda-1}f(x,\tau)d\tau,\quad(t>b,\mathrm{Re}(\lambda)>0)$$

$$\begin{aligned}&{}_tI_{b^+}^{\lambda}{}_xI_{a^+}^{\mu}f(x,t)\\=&\frac{1}{\Gamma(\mu)\Gamma(\lambda)}\int_b^t\int_a^x(t-\tau)^{\lambda-1}(x-\xi)^{\mu-1}f(\xi,\tau)d\xi d\tau\;.\quad(x>a,y>b,\mathrm{Re}(\lambda)>0,\mathrm{Re}(\mu)>0)\end{aligned}$$

Definition 2.2. ([1],[8]) *The Riemann-Liouville fractional derivative of order $\mu\in\mathbb{C}$ $(\mathrm{Re}(\mu)\geq0)$ is defined by*

$${}_xD_{a^+}^{\mu}[f]=\left(\frac{d}{dx}\right)^n\frac{1}{\Gamma(n-\alpha)}\int_a^x(x-\xi)^{\alpha-n-1}f(\xi)d\xi\;,\;(n=[\mathrm{Re}(\mu)]+1,\;x>a)$$

where, as usual, $[\mathrm{Re}(\mu)]$ means the integral part of $\mathrm{Re}(\mu)$.

Similarly, the partial fractional derivatives of a function $f(x,t)$, where $(x,t) \in \mathbb{R} \times \mathbb{R}$ is defined as follows:

$$
\begin{aligned}
{}_xD^{\mu}_{a+}f(x,t) &= \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\mu)} \int_a^x (x-\xi)^{n-\mu-1} f(\xi,t)d\xi \ , \ (n = [\mathrm{Re}(\mu)]+1, \ x > a) \\
{}_tD^{\lambda}_{b+}f(x,t) &= \left(\frac{d}{dt}\right)^m \frac{1}{\Gamma(m-\lambda)} \int_b^t (t-\tau)^{m-\lambda-1} f(x,\tau)d\tau \ , \ (m = [\mathrm{Re}(\lambda)]+1, \ t > b)
\end{aligned}
$$

$$
\begin{aligned}
&{}_tD^{\lambda}_{b+}{}_xD^{\mu}_{a+}f(x,t) \\
&= \left(\frac{d}{dt}\right)^m \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\mu)} \frac{1}{\Gamma(m-\lambda)} \int_b^t \int_a^x (t-\tau)^{m-\lambda-1}(x-\xi)^{n-\mu-1} f(\xi,\tau)d\xi d\tau. \\
&(n = [\mathrm{Re}(\mu)]+1, m = [\mathrm{Re}(\lambda)]+1, \ t > b, \ x > a)
\end{aligned}
$$

Theorem 2.1. *We have for* $\mathrm{Re}(\alpha+\eta) > 0$ *and* $\mathrm{Re}(\alpha) > 0$ *and* $(\beta) > 0$, *that*

$$
{}_yI^{\alpha}_{0+}{}_xI^{\beta}_{0+}\left[x^{\lambda-1}y^{\xi-1}E^{(\alpha,\beta,\eta,\xi,\lambda)}_{\gamma_1,\gamma_2}\left(x^{\alpha}, x^{\beta}y^{\eta}\right)\right] = x^{\beta+\lambda-1}y^{\alpha+\xi-1}E^{(\alpha,\beta,\eta,\xi+\alpha,\lambda+\beta)}_{\gamma_1,\gamma_2}\left(x^{\alpha}, x^{\beta}y^{\eta}\right)
$$

Proof. Because of the hypothesis of the Theorem, we have a right to interchange of the order of series and fractional integral operators, which yields

$$
\begin{aligned}
&{}_yI^{\alpha}_{0+}{}_xI^{\beta}_{0+}\left[x^{\lambda-1}y^{\xi-1}E^{(\alpha,\beta,\eta,\xi,\lambda)}_{\gamma_1,\gamma_2}\left(x^{\alpha}, x^{\beta}y^{\eta}\right)\right] \\
&= \int_0^y \int_0^x \frac{(y-\tau)^{\alpha-1}(x-t)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} t^{\lambda-1}\tau^{\xi-1}E^{(\alpha,\beta,\eta,\xi,\lambda)}_{\gamma_1,\gamma_2}\left(t^{\alpha}, t^{\beta}\tau^{\eta}\right)dt d\tau \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{(\gamma_1)_{k_1}(\gamma_2)_{k_2}}{\Gamma(\alpha k_1 + \beta k_2 + \lambda)\Gamma(\eta k_2 + \xi)k_1!k_2!} \\
&\times \int_0^y (y-\tau)^{\alpha-1}\tau^{\eta k_2+\xi-1}d\tau \int_0^x (x-t)^{\beta-1}t^{\alpha k_1+\beta k_2+\lambda-1}dt \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{(\gamma_1)_{k_1}(\gamma_2)_{k_2}x^{\alpha k_1+\beta k_2+\beta+\lambda-1}y^{\eta k_2+\xi+\alpha-1}}{\Gamma(\alpha k_1 + \beta k_2 + \lambda + \beta)\Gamma(\eta k_2 + \xi + \alpha)k_1!k_2!} \\
&= x^{\beta+\lambda-1}y^{\alpha+\xi-1}E^{(\alpha,\beta,\eta,\xi+\alpha,\lambda+\beta)}_{\gamma_1,\gamma_2}\left(x^{\alpha}, x^{\beta}y^{\eta}\right)
\end{aligned}
$$

□

In a similar manner, we have the following corollary:

Corollary 2.2. *For* $\mathrm{Re}(\alpha) > 0$ *and* $\mathrm{Re}(\beta) > 0$, *that*

$$
\begin{aligned}
&{}_yI^{\alpha}_{0+}{}_xI^{\beta}_{0+}\left[x^{\gamma}y^{\xi-1}L^{(\alpha,\beta,\gamma,\eta,\xi)}_{n,m}\left(x, xy^{\frac{\eta}{\beta}}\right)\right] \\
&= \frac{\Gamma(\alpha n + \beta m + \gamma + 1)}{\Gamma(\xi + \eta m)} \frac{\Gamma(\alpha + \xi + \eta m)}{\Gamma(\alpha n + \beta m + \gamma + \beta + 1)} x^{\beta+\gamma}y^{\alpha+\xi-1}L^{(\alpha,\beta,\gamma+\beta,\eta,\alpha+\xi)}_{n,m}\left(x, xy^{\frac{\eta}{\beta}}\right).
\end{aligned}
$$

Theorem 2.3. *For* $\mathrm{Re}(\alpha+\eta) > 0$,$\mathrm{Re}(\alpha) \geqslant 0$ *and* $(\beta) > 0$, *that*

$$
{}_yD^{\alpha}_{0+}{}_xD^{\beta}_{0+}\left[x^{\lambda-1}y^{\xi-1}E^{(\alpha,\beta,\eta,\xi,\lambda)}_{\gamma_1,\gamma_2}\left(x^{\alpha}, x^{\beta}y^{\eta}\right)\right] = x^{\lambda-\beta-1}y^{\xi-\alpha-1}E^{(\alpha,\beta,\eta,\xi-\alpha,\lambda-\beta)}_{\gamma_1,\gamma_2}\left(x^{\alpha}, x^{\beta}y^{\eta}\right).
$$

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Proof. Because of the hypothesis of the Theorem, we have a right to interchange of the order of series and fractional derivate operators, which yields

$$
\begin{aligned}
& {}_yD^{\alpha}_{0+\,x}D^{\beta}_{0+}\left[x^{\lambda-1}y^{\xi-1}E^{(\alpha,\beta,\eta,\xi,\lambda)}_{\gamma_1,\gamma_2}\left(x^{\alpha},x^{\beta}y^{\eta}\right)\right] \\
&= {}_yD^{\alpha}_{0+\,x}D^{\beta}_{0+}\left[x^{\lambda-1}y^{\xi-1}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(\gamma_1)_{k_1}(\gamma_2)_{k_2}x^{\alpha k_1}x^{\beta k_2}y^{\eta k_2}}{\Gamma(\alpha k_1+\beta k_2+\lambda)\Gamma(\eta k_2+\xi)k_1!k_2!}\right] \\
&= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(\gamma_1)_{k_1}(\gamma_2)_{k_2\,x}D^{\alpha}_{0+\,y}D^{\beta}_{0+}[x^{\alpha k_1+\beta k_2+\lambda-1}y^{\eta k_2+\xi-1}]}{\Gamma(\alpha k_1+\beta k_2+\lambda)\Gamma(\eta k_2+\xi)k_1!k_2!} \\
&= (\frac{d}{dy})^m(\frac{d}{dx})^n\frac{1}{\Gamma(n-\beta)}\frac{1}{\Gamma(m-\alpha)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(\gamma_1)_{k_1}(\gamma_2)_{k_2}}{\Gamma(\alpha k_1+\beta k_2+\lambda)\Gamma(\eta k_2+\xi)k_1!k_2!} \\
&\times\int_0^y(y-\tau)^{m-\alpha-1}\tau^{\eta k_2+\xi-1}d\tau\int_0^x(x-\xi)^{n-\beta-1}\xi^{\alpha k_1+\beta k_2+\lambda-1}d\xi \\
&= x^{\lambda-\beta-1}y^{\xi-\alpha-1}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(\gamma_1)_{k_1}(\gamma_2)_{k_2}x^{\alpha k_1+\beta k_2}y^{\eta k_2}}{\Gamma(\alpha k_1+\beta k_2+\lambda-\beta)\Gamma(\eta k_2+\xi-\alpha)k_1!k_2!} \\
&= x^{\lambda-\beta-1}y^{\xi-\alpha-1}E^{(\alpha,\beta,\eta,\xi-\alpha,\lambda-\beta)}_{\gamma_1,\gamma_2}\left(x^{\alpha},x^{\beta}y^{\eta}\right).
\end{aligned}
$$

□

In a similar manner, we have the following corollary:

Corollary 2.4. *For* $\mathrm{Re}(\alpha+\eta)>0$ *and* $\mathrm{Re}(\beta)>0$*, that*

$$
\begin{aligned}
& {}_yD^{\alpha}_{0+\,x}D^{\beta}_{0+}[x^{\gamma}y^{\xi-1}L^{(\alpha,\beta,\gamma,\eta,\xi)}_{n,m}\left(x,xy^{\frac{\eta}{\beta}}\right)] \\
&= \frac{\Gamma(\alpha n+\beta m+\gamma+1)}{\Gamma(\xi+\eta m)}\frac{\Gamma(\xi-\alpha+\eta m)}{\Gamma(\alpha n+\beta m+\gamma-\beta+1)}x^{\gamma-\beta}y^{\xi-\alpha-1}L^{(\alpha,\beta,\gamma-\beta,\eta,\xi-\alpha)}_{n,m}\left(x,xy^{\frac{\eta}{\beta}}\right).
\end{aligned}
$$

3. Linear generating function

In this section, we provide a linear generating function for the polynomials $L^{(\alpha,\beta,\gamma,\eta,\xi)}_{n,m}(x,y)$ by means of two variables analogue of Mittag-Leffler functions defined in (1.7).

Theorem 3.1. *For* $|t_1|<1$ *and* $|t_2|<1$*,* $\gamma_1,\gamma_2\in\mathbb{C}$ *and* $\alpha,\beta,\gamma,\xi,\eta\in\mathbb{C}$*, we have*

$$
\begin{aligned}
& \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\gamma_1)_n(\gamma_2)_mL^{(\alpha,\beta,\gamma,\eta,\xi)}_{n,m}(x,y)\,\Gamma(\xi+\eta m)}{\Gamma(\alpha n+\beta m+\gamma+1)n!m!}t_1^nt_2^m \\
&= (1-t_1)^{-\gamma_1}(1-t_2)^{-\gamma_2}E^{(\alpha,\beta,\eta,\xi,\gamma+1)}_{\gamma_1,\gamma_2}\left(\frac{-x^{\alpha}t_1}{1-t_1},\frac{-y^{\beta}t_2}{1-t_2}\right).
\end{aligned}
$$

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Proof. Direct calculations yield that

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\gamma_1)_n(\gamma_2)_m L_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)\,\Gamma(\xi+\eta m)}{\Gamma(\alpha n+\beta m+\gamma+1)n!m!}t_1^n t_2^m$$
$$=\sum_{n,m=0}^{\infty}\sum_{k_1,k_2=0}^{n,m}\frac{(\gamma_1)_n(\gamma_2)_m(-n)_{k_1}(-m)_{k_2}x^{\alpha k_1}y^{\beta k_2}}{\Gamma(\alpha k_1+\beta k_2+\gamma+1)\Gamma(\xi+\eta k_2)k_1!k_2!n!m!}t_1^n t_2^m$$
$$=\sum_{n,m=0}^{\infty}\sum_{k_1,k_2=0}^{n,m}\frac{(-1)^{k_1+k_2}(\gamma_1)_n(\gamma_2)_m x^{\alpha k_1}y^{\beta k_2}}{\Gamma(\alpha k_1+\beta k_2+\gamma+1)\Gamma(\xi+\eta k_2)k_1!k_2!(n-k_1)!(m-k_2)!}t_1^n t_2^m.$$

Letting $n\to n+k_1$ and $m\to m+k_2$, we get

$$\sum_{n,m=0}^{\infty}\sum_{k_1,k_2=0}^{\infty}\frac{(-1)^{k_1+k_2}(\gamma_1)_{n+k_1}(\gamma_2)_{m+k_2}x^{\alpha k_1}y^{\beta k_2}}{\Gamma(\alpha k_1+\beta k_2+\gamma+1)\Gamma(\xi+\eta k_2)k_1!k_2!(n)!(m)!}t_1^{n+k_1}t_2^{m+k_2}.$$

Since $(\gamma_1)_{n+k_1}=(\gamma_1+k_1)_n(\gamma_1)_{k_1}$ and $(\gamma_2)_{m+k_2}=(\gamma_2+k_2)_m(\gamma_2)_{k2}$, we have

$$\sum_{k_1,k_2=0}^{\infty}\frac{(\gamma_1)_{k_1}(\gamma_2)_{k_2}(-x^{\alpha}t_1)^{k_1}(-y^{\beta}t_2)^{k_2}}{\Gamma(\alpha k_1+\beta k_2+\gamma+1)\Gamma(\xi+\eta k_2)k_1!k_2!}\sum_{n,m=0}^{\infty}\frac{(\gamma_1+k_1)_n(\gamma_2+k_2)_m}{(n)!(m)!}t_1^n t_2^m$$
$$=(1-t_1)^{-\gamma_1}(1-t_2)^{-\gamma_2}E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\gamma+1)}\left(\frac{-x^{\alpha}t_1}{1-t_1},\frac{-y^{\beta}t_2}{1-t_2}\right).$$

Note that, because of the uniform converge of the series under the conditions $|t_1|<1$ and $|t_2|<1$, we have interchanged the order of summations. □

4. Singular double integral equation

In this section, we first obtain the double Laplace transform of the functions $E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$ and $L_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$. Then, we compute the double integral involving the product of two $E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$ functions in the integrand. Finally, we solve a double integral equation with $L_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}(x,y)$ in the kernel, in terms of the two $E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}(x,y)$ functions.

As usual [5],

$$\mathbb{L}_2[f(x,t)]=\int_0^{\infty}e^{-px}\int_0^{\infty}e^{-st}f(x,t)dtdx \qquad (4.1)$$
$$(x,t>0,\quad p,s\in\mathbb{C})$$

denotes the double Laplace transform of f.

Lemma 4.1. *For* $\mathrm{Re}(\lambda_1),\mathrm{Re}(\lambda_2),\mathrm{Re}(\alpha+\eta)>0,\mathrm{Re}(\beta)>0,\mathrm{Re}(s_1),\mathrm{Re}(s_2)>0$ *and* $\left|\frac{\lambda_1^{\alpha}}{s_1^{\alpha}}\right|,\left|\frac{\lambda_2^{\beta}}{s_1^{\beta}s_2^{\eta}}\right|<1$*, we have*

$$\mathbb{L}_2[x^{\lambda-1}y^{\xi-1}E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}((\lambda_1 x)^{\alpha},(\lambda_2^{\beta}x^{\beta}y^{\eta}))](s_1,s_2)=\frac{1}{s_1^{\lambda}}\frac{1}{s_2^{\xi}}(1-\frac{\lambda_1^{\alpha}}{s_1^{\alpha}})^{-\gamma_1}(1-\frac{\lambda_2^{\beta}}{s_1^{\beta}s_2^{\eta}})^{-\gamma_2}.$$

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Proof. Using definition (4.1) and taking into account that $\left|\frac{\lambda_1^\alpha}{s_1^\alpha}\right| < 1$ and $\left|\frac{\lambda_2^\beta}{s_1^\beta s_2^\eta}\right| < 1$, we get

$$\begin{aligned}
&\mathbb{L}_2[x^{\lambda-1}y^{\xi-1}E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}((\lambda_1 x)^\alpha,(\lambda_2^\beta x^\beta y^\eta))](s_1,s_2)\\
&=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(\gamma_1)_{k_1}(\gamma_2)_{k_2}\lambda_1^{\alpha k_1}\lambda_2^{\beta k_2}}{\Gamma(\alpha k_1+\beta k_2+\lambda)\Gamma(\eta k_2+\xi)k_1!k_2!}\\
&\times\int_0^\infty x^{\alpha k_1+\beta k_2+\lambda-1}e^{-s_1x}dx\int_0^\infty y^{\eta k_2+\xi-1}e^{-s_2y}dy\\
&=\frac{1}{s_1^\lambda}\frac{1}{s_2^\xi}\sum_{k_1=0}^{\infty}\frac{(\gamma_1)_{k_1}}{k_1!}(\frac{\lambda_1^\alpha}{s_1^\alpha})^{k_1}\sum_{k_2=0}^{\infty}\frac{(\gamma_2)_{k_2}}{k_2!}(\frac{\lambda_2^\beta}{s_1^\beta s_2^\eta})^{k_2}=\frac{1}{s_1^\lambda}\frac{1}{s_2^\xi}(1-\frac{\lambda_1^\alpha}{s_1^\alpha})^{-\gamma_1}(1-\frac{\lambda_2^\beta}{s_1^\beta s_2^\eta})^{-\gamma_2}.
\end{aligned}$$

□

We deduce the following result from *Lemma* 4.1 by setting $\lambda-1=\gamma$ and using equation (1.9).

Corollary 4.2. *For* $\mathrm{Re}(\lambda_1),\mathrm{Re}(\lambda_2),\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\lambda),\mathrm{Re}(s_1),\mathrm{Re}(s_2)>0$ *and* $\left|\frac{\lambda_1^\alpha}{s_1^\alpha}\right|,\left|\frac{\lambda_2^\beta}{s_1^\beta s_2^\eta}\right|<1$, *we have*

$$\begin{aligned}
&\mathbb{L}_2[t^\gamma\tau^{\xi-1}L_{n,m}^{(\alpha,\beta,\gamma,\eta,\xi)}((\lambda_1 t),(\lambda_2 t\tau^{\frac{\eta}{\beta}}))](s_1,s_2)\\
=\;&\frac{1}{s_1^{\gamma+1}}\frac{1}{s_2^\xi}\frac{\Gamma(\alpha n+\beta m+\gamma+1)}{\Gamma(\eta m+\xi)}(1-\frac{\lambda_1^\alpha}{s_1^\alpha})^n(1-\frac{\lambda_2^\beta}{s_1^\beta s_2^\eta})^m.
\end{aligned}$$

Theorem 4.3. *Let* $\lambda_1,\lambda_2\in\mathbb{C},\mathrm{Re}(\alpha+\eta)>0$ *and* $\mathrm{Re}(\beta)>0$. *Then*

$$\begin{aligned}
&\int_0^y\int_0^x\Big[(x-t)^{\lambda-1}(y-\tau)^{\xi-1}E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}(\lambda_1^\alpha(x-t)^\alpha,\lambda_2^\beta(x-t)^\beta(y-\tau)^\eta)\\
&\times t^{\gamma-1}\tau^{\zeta-1}E_{\gamma_3,\gamma_4}^{(\alpha,\beta,\eta,\zeta,\gamma)}(\lambda_1^\alpha t^\alpha,\lambda_2^\beta t^\beta\tau^\eta)dtd\tau\Big]\\
&=x^{\lambda+\gamma}y^{\xi+\zeta}E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}(\lambda_1^\alpha x^\alpha,\lambda_2^\beta x^\beta y^\eta)E_{\gamma_3,\gamma_4}^{(\alpha,\beta,\eta,\xi,\lambda)}(\lambda_1^\alpha x^\alpha,\lambda_2^\beta x^\beta y^\eta).
\end{aligned}$$

Proof. Using the convolution theorem for the Laplace transform we have,

$$\begin{aligned}
&\mathbb{L}_2\left[\int_0^y\int_0^x(x-t)^{\lambda-1}(y-\tau)^{\xi-1}E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}(\lambda_1^\alpha(x-t)^\alpha,\lambda_2^\beta(x-t)^\beta(y-\tau)^\eta)t^{\gamma-1}\tau^{\zeta-1}\right.\\
&\left.\times E_{\gamma_3,\gamma_4}^{(\alpha,\beta,\eta,\zeta,\gamma)}(\lambda_1^\alpha t^\alpha,\lambda_2^\beta t^\beta\tau^\eta)dtd\tau\right](s_1,s_2)\\
&=\mathbb{L}_2[x^{\lambda-1}y^{\xi-1}E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}(\lambda_1^\alpha x^\alpha,\lambda_2^\beta x^\beta y^\eta)]\mathbb{L}_2[x^{\gamma-1}y^{\zeta-1}E_{\gamma_3,\gamma_4}^{(\alpha,\beta,\eta,\zeta,\gamma)}(\lambda_1^\alpha t^\alpha,\lambda_2^\beta t^\beta\tau^\eta)](s_1,s_2)\\
&=\frac{1}{s_1^\lambda}\frac{1}{s_2^\xi}\left(1-\frac{\lambda_1^\alpha}{s_1^\alpha}\right)^{-\gamma_1}\left(1-\frac{\lambda_2^\beta}{s_1^\beta s_2^\eta}\right)^{-\gamma_2}\frac{1}{s_1^\gamma}\frac{1}{s_2^\zeta}\left(1-\frac{\lambda_1^\alpha}{s_1^\alpha}\right)^{-\gamma_3}\left(1-\frac{\lambda_2^\beta}{s_1^\beta s_2^\eta}\right)^{-\gamma_4}
\end{aligned}$$

We have for $\mathrm{Re}(s_1), \mathrm{Re}(s_2) > 0$

$$\begin{aligned}(4.2)\quad & \mathbb{L}_2\left[\int_0^y\int_0^x (x-t)^{\lambda-1}(y-\tau)^{\xi-1}E^{(\alpha,\beta,\eta,\xi,\lambda)}_{\gamma_1,\gamma_2}(\lambda_1^{\alpha}(x-t)^{\alpha},\lambda_2^{\beta}(x-t)^{\beta}(y-\tau)^{\eta})\right.\\ & \left.\times t^{\gamma-1}\tau^{\zeta-1}E^{(\alpha,\beta,\eta,\zeta,\gamma)}_{\gamma_1,\gamma_2}(\lambda_1^{\alpha}t^{\alpha},\lambda_2^{\beta}t^{\beta}\tau^{\eta})dtd\tau\right](s_1,s_2)\\ = & \frac{1}{s_1^{\lambda+\gamma}}\frac{1}{s_2^{\xi+\zeta}}\left(1-\frac{\lambda_1^{\alpha}}{s_1^{\alpha}}\right)^{-\gamma_1}\left(1-\frac{\lambda_2^{\beta}}{s_1^{\beta}s_2^{\eta}}\right)^{-\gamma_2}\left(1-\frac{\lambda_1^{\alpha}}{s_1^{\alpha}}\right)^{-\gamma_3}\left(1-\frac{\lambda_2^{\beta}}{s_1^{\beta}s_2^{\eta}}\right)^{-\gamma_4}\\ = & \mathbb{L}_2\left[x^{\lambda+\gamma}y^{\xi+\zeta}E^{(\alpha,\beta,\eta,\xi,\lambda)}_{\gamma_1,\gamma_2}(\lambda_1^{\alpha}x^{\alpha},\lambda_2^{\beta}x^{\beta}y^{\eta})E^{(\alpha,\beta,\eta,\xi,\lambda)}_{\gamma_3,\gamma_4}(\lambda_1^{\alpha}x^{\alpha},\lambda_2^{\beta}x^{\beta}y^{\eta})\right](s_1,s_2).\end{aligned}$$

Taking inverse Laplace transform on both sides of (4.2), the result follows. □

The next assertion follows from *Theorem* 4.5 by letting $\lambda - 1 = \gamma$ and taking into account (1.9).

Corollary 4.4. *Let* $\lambda_1, \lambda_2 \in \mathbb{C}, \mathrm{Re}(\lambda), \mathrm{Re}(\xi), \mathrm{Re}(\gamma), \mathrm{Re}(\zeta) > 0$. *Then*

$$\begin{aligned}&\int_0^y\int_0^x (x-t)^{\gamma}(y-\tau)^{\xi-1}L^{(\alpha,\beta,\gamma_1,\eta,\xi_1)}_{n_1,m_1}(\lambda_1(x-t),\lambda_2(x-t)(y-\tau)^{\frac{\eta}{\beta}})\\ &\times t^{\gamma}\tau^{\xi-1}L^{(\alpha,\beta,\gamma_2,\eta,\xi_2)}_{n_2,m_2}(\lambda_1 t,\lambda_2 t\tau^{\frac{\eta}{\beta}})dtd\tau\\ &= x^{\gamma_1+\gamma_2+1}y^{\xi_1+\xi_2-1}L^{(\alpha,\beta,\gamma_1,\eta,\xi_1)}_{n_1,m_1}(\lambda_1 x,\lambda_2 xy^{\frac{\eta}{\beta}})L^{(\alpha,\beta,\gamma_2,\eta,\xi_2)}_{n_2,m_2}(\lambda_1 x,\lambda_2 xy^{\frac{\eta}{\beta}}).\end{aligned}$$

Now, we consider the following double convolution equation:

(4.3)
$$\int_0^y\int_0^x (x-t)^{\gamma-1}(y-\tau)^{\xi-1}L^{(\alpha,\beta,\gamma,\eta,\xi)}_{n,m}((\lambda_1 x)^{\alpha},(\lambda_2^{\beta}x^{\beta}y^{\eta}))\Phi(t,\tau)dtd\tau = \Psi(x,y)$$

where $\mathrm{Re}(\gamma) > -1$.

For the solution of the integral equation (4.3), we have the following theorem:

Theorem 4.5. *The singular double integral equation (4.3) admits a locally integrable solution*

$$\Phi(t,\tau) = \frac{\Gamma(\eta m+\xi)}{\Gamma(\alpha n+\beta m+\gamma+1)}$$

$$\times\int_0^y\int_0^x (x-t)^{\alpha_1-\gamma-2}(y-\tau)^{\alpha_2-\xi-1}E^{(\alpha,\beta,\eta,\xi,\lambda)}_{\gamma_1,\gamma_2}((\lambda_1 x)^{\alpha},(\lambda_2^{\beta}x^{\beta}y^{\eta}))[I_{0^+}^{-\alpha_1}I_{0^+}^{-\alpha_2}\Psi(t,\tau)]dtd\tau.$$

Proof. Applying double Laplace transform on both sides of (4.3), then using double convolution theorem, we get

$$\frac{1}{s_1^{\gamma+1}}\frac{1}{s_2^{\xi}}\frac{\Gamma(\alpha n+\beta m+\gamma+1)}{\Gamma(\eta m+\xi)}(1-\frac{\lambda_1^{\alpha}}{s_1^{\alpha}})^{n}(1-\frac{\lambda_2^{\beta}}{s_1^{\beta}s_2^{\eta}})^{m}\mathbb{L}_2[\Phi(t,\tau)](s_1,s_2) = \mathbb{L}_2[\Psi(t,\tau)](s_1,s_2)$$

Therefore, we have,

$$\mathbb{L}_2[\Phi(t,\tau)](s_1,s_2) = \frac{\Gamma(\eta m+\xi)}{\Gamma(\alpha n+\beta m+\gamma+1)}$$

$$\times(s_1)^{\gamma-\alpha_1+1}(s_2)^{\xi-\alpha_2}(1-\frac{\lambda_1^{\alpha}}{s_1^{\alpha}})^{-n}(1-\frac{\lambda_2^{\beta}}{s_1^{\beta}s_2^{\eta}})^{-m}\{s_1^{\alpha_1}s_2^{\alpha_2}\mathbb{L}_2[\Psi(t,\tau)](s_1,s_2)\}$$

Finally taking inverse Laplace transform on both sides and using *Lemma* 3.2 of [1] and *Lemma* 4.1, we get

$$\Phi(t,\tau)=\frac{\Gamma(\eta m+\xi)}{\Gamma(\alpha n+\beta m+\gamma+1)}$$
$$\times\int_0^y\int_0^x(x-t)^{\alpha_1-\gamma-2}(y-\tau)^{\alpha_2-\xi-1}E_{\gamma_1,\gamma_2}^{(\alpha,\beta,\eta,\xi,\lambda)}((\lambda_1x)^{\alpha},(\lambda_2^{\beta}x^{\beta}y^{\eta}))[I_{0^+}^{-\alpha_1}I_{0^+}^{-\alpha_2}\Psi(t,\tau)]dtd\tau$$

and the proof is completed. □

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Generalized Inequalities of the type of Hermite-Hadamard-Fejer with Quasi-Convex Functions by way of k-Fractional Derivatives

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Abstract: In this article, Hermite-Hadamard-Fejer type inequalities are discussed with quasi-convex functions and obtained the generalized results of the type using k-fractional derivatives. And proposed some new bounds in terms of some special means.

Keywords: Hermite-Hadamard inequality, Hermite-Hadamard-Fejer inequality, quasi convex functions, k-Riemann-Liouville fractional derivatives, Hölder's integral inequality, Power mean inequality.

1. Introduction

The function $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is said to be convex on I if for every $x, y \in I$ and $t \in [0, 1]$, we get

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y).$$

Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, f satisfies the following well-known Hermite-Hadamard type inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(a)+f(b)}{2}.$$

Definition 1. *The function $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is said to be quasi-convex if*

$$f(tx + (1-t)y) \le max\{f(x), f(y)\},$$

for every $x, y \in I$ and $t \in [0, 1]$ (see [4]*).*

In [3] Mubeen and Habibullah introduced the following class of fractional derivatives.

Definition 2. *Let $f \in L[a,b]$, then k-Riemann-Liouville fractional derivatives ${}_kJ^{\alpha}_{a^+}f(x)$ and ${}_kJ^{\alpha}_{b^-}f(x)$ of order $\alpha > 0$ are defined by*

$$ {}_kJ^{\alpha}_{a^+}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_a^x (x-t)^{\frac{\alpha}{k}-1}f(t)dt, \qquad (0 \le a < x < b) $$

and

$$ {}_kJ^{\alpha}_{b^-}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_x^b (t-x)^{\frac{\alpha}{k}-1}f(t)dt, \qquad (0 \le a < x < b) $$

respectively, where $k > 0$ and $\Gamma_k(\alpha)$ is the k-gamma function given as $\Gamma_k(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{\frac{-t^k}{k}}dt$. Furthermore $\Gamma_k(\alpha+k) = \alpha\Gamma_k(\alpha)$ and ${}_kJ^0_{a^+}f(x) = {}_kJ^0_{b^-}f(x) = f(x)$.

In [1] Fejér established the following inequality.

Lemma 1. *Let $f : [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function, the inequality*

$$ f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)dx $$

holds, where $g : [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ is non-negative integrable and symmetric to $\frac{a+b}{2}$. This inequality is called Hermite-Hadamard-Fejer *inequality.*

Lemma 2. *(*[7]*) For $0 < t \le 1$ and $0 \le a < b$, we get*

$$ |a^t - b^t| \le (b-a)^t. $$

E. Set et al. established the following Lemma in [6].

Lemma 3. *Let $f : [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable mapping on (a,b) and $g : [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}$. If $f', g \in L[a,b]$, the following identity for fractional derivatives holds*

$$ f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(b)\right] - \left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}(fg)(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}(fg)(b)\right] = \frac{1}{\Gamma(\alpha)}\int_a^b m(t)f'(t)dt \tag{1.1} $$

where

$$ m(t) = \begin{cases} \int_a^t (s-a)^{\alpha-1}g(s)ds & t \in \left[a, \frac{a+b}{2}\right) \\ -\int_t^b (b-s)^{\alpha-1}g(s)ds & t \in \left[\frac{a+b}{2}, b\right]. \end{cases} $$

Iscan obtained the following lemma in [2].

Lemma 4. *Let $f : [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable mapping on (a,b) and $f^{'} \in L[a,b]$. If $g : [a,b] \longrightarrow R$ is integrable and symmetric to $\frac{a+b}{2}$, the following identity for fractional derivatives holds*

$$\begin{aligned} &\frac{f(a)+f(b)}{2}\left[J_{a^+}^{\alpha}g(b)+J_{b^-}^{\alpha}g(a)\right]-\left[J_{a^+}^{\alpha}(fg)(b)+J_{b^-}^{\alpha}(fg)(a)\right] \\ &= \frac{1}{\Gamma(\alpha)}\int_a^b\left(\int_a^t(b-s)^{\alpha-1}g(s)ds-\int_t^b(s-a)^{\alpha-1}g(s)ds\right)f^{'}(t)dt \qquad (1.2) \end{aligned}$$

where $\alpha > 0$.

In the present paper motivated by the recent results given in [5] we established some Hermite-Hadamard-Fejér type inequalities for quasi-convex functions via k-fractional derivatives.

2. Main Findings

Throughout this paper, let I be an interval on R and let $||g||_{[a,b],\infty} = \sup_{t\in[a,b]} g(t)$ for continuous function $g : [a,b]\mathbb{R} \longrightarrow \mathbb{R}$.

The following identity is the generalization of identity (1.1) in Lemma 3 for k-fractional derivatives.

Lemma 5. *Let $f : [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable mapping on (a,b) and $g : [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}$. If $f^{'}, g \in L[a,b]$, the following identity for k-fractional derivatives holds*

$$\begin{aligned} &f\left(\frac{a+b}{2}\right)\left[{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(a)+{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(b)\right]-\left[{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^-}(fg)(a)+{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^+}(fg)(b)\right] \\ &= \frac{1}{\Gamma_k(\alpha)}\int_a^b m(t)f^{'}(t)dt \end{aligned}$$

where

$$m(t) = \begin{cases} \int_a^t (s-a)^{\frac{\alpha}{k}-1}g(s)ds & t \in \left[a, \frac{a+b}{2}\right) \\ -\int_t^b (b-s)^{\frac{\alpha}{k}-1}g(s)ds & t \in \left[\frac{a+b}{2}, b\right]. \end{cases}$$

Here the identity (1.2) of Lemma 4 is also generalized for k-fractional derivatives.

Lemma 6. *Let $f : [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable mapping on (a,b) and $f^{'} \in L[a,b]$. If $g : [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ is integrable and symmetric to $\frac{a+b}{2}$, the following for k-fractional*

derivatives holds

$$\frac{f(a)+f(b)}{2}\left[{}_kJ^{\alpha}_{a^+}g(b)+{}_kJ^{\alpha}_{b^-}g(a)\right]-\left[{}_kJ^{\alpha}_{a^+}(fg)(b)+{}_kJ^{\alpha}_{b^-}(fg)(a)\right]$$
$$=\frac{1}{\Gamma_k(\alpha)}\int_a^b\left(\int_a^t(b-s)^{\frac{\alpha}{k}-1}g(s)ds-\int_t^b(s-a)^{\frac{\alpha}{k}-1}g(s)ds\right)f^{'}(t)dt$$

where $\frac{\alpha}{k}>0$.

Theorem 1. *Let* $f:I\subset\mathbb{R}\longrightarrow\mathbb{R}$ *be a differentiable mapping on* I^o *and* $f^{'}\in L[a,b]$ *and* $g:[a,b]\subset\mathbb{R}\longrightarrow\mathbb{R}$ *is continuous. If* $|f^{'}|^q$ *is quasi-convex function on* $[a,b], q\geq 1$*, the following inequality for* k*-fractional derivatives holds*

$$\left|f\left(\frac{a+b}{2}\right)\left[{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(a)+{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(b)\right]-\left[{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^-}(fg)(a)+{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^+}(fg)(b)\right]\right|$$
$$\leq\frac{(b-a)^{\frac{\alpha}{k}+1}||g||_{[a,b],\infty}}{2^{\frac{\alpha}{k}}\left(\frac{\alpha}{k}+1\right)\Gamma_k(\alpha+k)}\left(max\left\{|f^{'}(a)|^q,|f^{'}(b)|^q\right\}\right)^{\frac{1}{q}}$$

where $\frac{\alpha}{k}>0$.

Proof. Since $|f^{'}|^q$ is quasi-convex on $[a,b]$, we know that for $t\in[a,b]$

$$|f^{'}(t)|^q=\left|f^{'}\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right|^q\leq\max\left\{|f^{'}(a)|^q,|f^{'}(b)|^q\right\}.$$

Using lemma 5, power mean inequality and the fact that $|f^{'}|^q$ is quasi-convex function on $[a,b]$, it follows that

$$\left|f\left(\frac{a+b}{2}\right)\left[{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(a)+{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(b)\right]-\left[{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^-}(fg)(a)+{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^+}(fg)(b)\right]\right|$$
$$\leq\frac{1}{\Gamma_k(\alpha)}\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t(s-a)^{\frac{\alpha}{k}-1}g(s)ds\right|dt\right)^{1-\frac{1}{q}}\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t(s-a)^{\frac{\alpha}{k}-1}g(s)ds\right||f^{'}(t)|^qdt\right)^{\frac{1}{q}}$$
$$+\frac{1}{\Gamma_k(\alpha)}\left(\int_{\frac{a+b}{2}}^b\left|\int_t^b(b-s)^{\frac{\alpha}{k}-1}g(s)ds\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{a+b}{2}}^b\left|\int_t^b(b-s)^{\frac{\alpha}{k}-1}g(s)ds\right||f^{'}(t)|^qdt\right)^{\frac{1}{q}}$$

$$\leq \frac{||g||_{\left[a,\frac{a+b}{2}\right],\infty}}{\Gamma_k(\alpha)}\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\frac{\alpha}{k}-1}ds\right|dt\right)^{1-\frac{1}{q}}\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\frac{\alpha}{k}-1}ds\right||f^{'}(t)|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{||g||_{\left[\frac{a+b}{2},b\right],\infty}}{\Gamma_k(\alpha)}\left(\int_{\frac{a+b}{2}}^{b}\left|\int_t^b (b-s)^{\frac{\alpha}{k}-1}ds\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{a+b}{2}}^{b}\left|\int_t^b (b-s)^{\frac{\alpha}{k}-1}ds\right||f^{'}(t)|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{\Gamma_k(\alpha+k)}\left(\frac{(b-a)^{\frac{\alpha}{k}+1}}{2^{\frac{\alpha}{k}+1}\left(\frac{\alpha}{k}+1\right)}\right)^{1-\frac{1}{q}}\left(\frac{(b-a)^{\frac{\alpha}{k}+1}}{2^{\frac{\alpha}{k}+1}\left(\frac{\alpha}{k}+1\right)}\right)^{\frac{1}{q}}\left(||g||_{\left[a,\frac{a+b}{2}\right],\infty}+||g||_{\left[\frac{a+b}{2},b\right],\infty}\right)$$

$$\left(\max\left\{|f^{'}(a)|^q,|f^{'}(b)|^q\right\}\right)^{\frac{1}{q}}$$

$$\leq \frac{(b-a)^{\frac{\alpha}{k}+1}||g||_{[a,b],\infty}}{2^{\frac{\alpha}{k}}\left(\frac{\alpha}{k}+1\right)\Gamma_k(\alpha+k)}\left(\max\left\{|f^{'}(a)|^q,|f^{'}(b)|^q\right\}\right)^{\frac{1}{q}}$$

where

$$\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\frac{\alpha}{k}-1}ds\right|dt=\int_{\frac{a+b}{2}}^{b}\left|\int_t^b (b-s)^{\frac{\alpha}{k}-1}ds\right|dt=\frac{(b-a)^{\frac{\alpha}{k}+1}}{2^{\frac{\alpha}{k}+1}\left(\frac{\alpha}{k}+1\right)\frac{\alpha}{k}}$$

Which completes the proof. □

Corollary 1. *If we choose $g(x)=1$ and $\alpha=k$ in Theorem 1, we get*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|\leq\frac{b-a}{4}\left(\mathit{max}\left\{|f^{'}(a)|^q,|f^{'}(b)|^q\right\}\right)^{\frac{1}{q}}.$$

Theorem 2. *Let $f:I\subset\mathbb{R}\longrightarrow\mathbb{R}$ be a differentiable mapping on I^o and $f^{'}\in L[a,b]$ and $g:[a,b]\subset\mathbb{R}\longrightarrow\mathbb{R}$ is continuous. If $|f^{'}|^q$ is quasi-convex function on $[a,b], q>1$, the following inequality for k-fractional derivatives holds*

$$\left|f\left(\frac{a+b}{2}\right)\left[{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(a)+{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(b)\right]-\left[{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^-}(fg)(a)+{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^+}(fg)(b)\right]\right|$$
$$\leq\frac{(b-a)^{\frac{\alpha}{k}+1}||g||_{\infty}}{2^{\frac{\alpha}{k}}\left(\frac{\alpha}{k}p+1\right)^{\frac{1}{p}}\Gamma_k(\alpha+k)}\left(\mathit{max}\left\{|f^{'}(a)|^q,|f^{'}(b)|^q\right\}\right)^{\frac{1}{q}}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 5, Hölder's inequality and the fact that $|f^{'}|^q$ is quasi-convex function on $[a,b]$, it follows that

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right)\left[{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^-} g(a) + {}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^+} g(b)\right] - \left[{}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^-}(fg)(a) + {}_kJ^{\alpha}_{\left(\frac{a+b}{2}\right)^+}(fg)(b)\right]\right| \\
&\leq \frac{1}{\Gamma_k(\alpha)}\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\frac{\alpha}{k}-1} g(s)ds\right| |f^{'}(t)|dt\right. \\
&\quad \left. + \int_{\frac{a+b}{2}}^b \left|\int_t^b (b-s)^{\frac{\alpha}{k}-1} g(s)ds\right| |f^{'}(t)|dt\right) \\
&\leq \frac{1}{\Gamma_k(\alpha)}\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\frac{\alpha}{k}-1} g(s)ds\right|^p dt\right)^{\frac{1}{p}} \left(\int_a^{\frac{a+b}{2}} |f^{'}(t)|^q dt\right)^{\frac{1}{q}} \\
&\quad + \frac{1}{\Gamma_k(\alpha)}\left(\int_{\frac{a+b}{2}}^b \left|\int_t^b (b-s)^{\frac{\alpha}{k}-1} g(s)ds\right|^p dt\right)^{\frac{1}{p}} \left(\int_{\frac{a+b}{2}}^b |f^{'}(t)|^q dt\right)^{\frac{1}{q}} \\
&= \frac{1}{\Gamma_k(\alpha)}\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\frac{\alpha}{k}-1} g(s)ds\right|^p dt\right)^{\frac{1}{p}} \\
&\quad \left[\left(\int_a^{\frac{a+b}{2}} |f^{'}(t)|^q dt\right)^{\frac{1}{q}} + \left(\int_{\frac{a+b}{2}}^b |f^{'}(t)|^q dt\right)^{\frac{1}{q}}\right] \\
&\leq \frac{||g||_\infty}{\Gamma_k(\alpha)}\left(\frac{(b-a)^{\frac{\alpha}{k}p+1}}{2^{\frac{\alpha}{k}p+1}\left(\frac{\alpha}{k}p+1\right)\left(\frac{\alpha}{k}\right)^p}\right)^{\frac{1}{p}} \\
&\quad \left[\left(\int_a^{\frac{a+b}{2}} \max\left\{|f^{'}(a)|^q, |f^{'}(b)|^q\right\} dt\right)^{\frac{1}{q}} + \left(\int_{\frac{a+b}{2}}^b \max\left\{|f^{'}(a)|^q, |f^{'}(b)|^q\right\} dt\right)^{\frac{1}{q}}\right] \\
&= \frac{(b-a)^{\frac{\alpha}{k}+1}||g||_\infty}{2^{\frac{\alpha}{k}}\left(\frac{\alpha}{k}p+1\right)^{\frac{1}{p}}\Gamma_k(\alpha+k)}\left(\max\left\{|f^{'}(a)|^q, |f^{'}(b)|^q\right\}\right)^{\frac{1}{q}}.
\end{aligned}$$

Where

$$\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\frac{\alpha}{k}-1} ds\right|^p dt = \frac{(b-a)^{\frac{\alpha}{k}p+1}}{2^{\frac{\alpha}{k}p+1}\left(\frac{\alpha}{k}p+1\right)\left(\frac{\alpha}{k}\right)^p}.$$

□

Corollary 2. *If we choose $g(x) = 1$ and $\alpha = k$ in Theorem 2, then we get*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left(max\left\{|f^{'}(a)|^q, |f^{'}(b)|^q\right\}\right)^{\frac{1}{q}}.$$

Theorem 3. *Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable mapping on I^o and $f^{'} \in L[a,b]$. If $|f^{'}|$ is quasi-convex function on $[a,b]$ and $g : [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and symmetric to $\frac{a+b}{2}$, the following inequality for k-fractional derivatives holds*

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$$\left|\frac{f(a)+f(b)}{2}\left[{}_kJ^{\alpha}_{a^+}g(b)+{}_kJ^{\alpha}_{b^-}g(a)\right]-\left[{}_kJ^{\alpha}_{a^+}(fg)(b)+{}_kJ^{\alpha}_{b^-}(fg)(a)\right]\right|$$
$$\leq \frac{2(b-a)^{\frac{\alpha}{k}+1}||g||_{\infty}}{\left(\frac{\alpha}{k}+1\right)\Gamma_k(\alpha+k)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right) max\left\{|f^{'}(a)|,|f^{'}(b)|\right\}$$

where $\frac{\alpha}{k}>0$.

Proof. From Lemma 6, we get

$$\left|\frac{f(a)+f(b)}{2}\left[{}_kJ^{\alpha}_{a^+}g(b)+{}_kJ^{\alpha}_{b^-}g(a)\right]-\left[{}_kJ^{\alpha}_{a^+}(fg)(b)+{}_kJ^{\alpha}_{b^-}(fg)(a)\right]\right|$$
$$\leq \frac{1}{\Gamma_k(\alpha)}\int_a^b\left|\int_a^t(b-s)^{\frac{\alpha}{k}-1}g(s)ds-\int_t^b(s-a)^{\frac{\alpha}{k}-1}g(s)ds\right||f^{'}(t)|dt.$$

Since $|f^{'}|$ is quasi-convex on $[a,b]$, we know that for $t\in[a,b]$

$$|f^{'}(t)|=\left|f^{'}\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right|\leq \max\left\{|f^{'}(a)|,|f^{'}(b)|\right\}$$

and since $g:[a,b]\subset\mathbb{R}\longrightarrow\mathbb{R}$ is continuous and symmetric to $\frac{a+b}{2}$ we can write

$$\int_t^b(s-a)^{\frac{\alpha}{k}-1}g(s)ds=\int_a^{a+b-t}(b-s)^{\frac{\alpha}{k}-1}g(a+b-s)ds$$
$$=\int_a^{a+b-t}(b-s)^{\frac{\alpha}{k}-1}g(s)ds$$

therefore we get

$$\left|\int_a^t(b-s)^{\frac{\alpha}{k}-1}g(s)ds-\int_t^b(s-a)^{\frac{\alpha}{k}-1}g(s)ds\right|$$
$$=\int_t^{a+b-t}(b-s)^{\frac{\alpha}{k}-1}g(s)ds$$

$$\leq\begin{cases}\int_t^{a+b-t}\left|(b-s)^{\frac{\alpha}{k}-1}g(s)\right|ds, & t\in\left[a,\frac{a+b}{2}\right]\\ \int_{a+b-t}^{t}\left|(b-s)^{\frac{\alpha}{k}-1}g(s)\right|ds, & t\in\left[\frac{a+b}{2},b\right].\end{cases} \tag{2.3}$$

Therefore we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}\left[{}_kJ^{\alpha}_{a^+}g(b)+{}_kJ^{\alpha}_{b^-}g(a)\right]-\left[{}_kJ^{\alpha}_{a^+}(fg)(b)+{}_kJ^{\alpha}_{b^-}(fg)(a)\right]\right| \\
&\leq \frac{1}{\Gamma_k(\alpha)}\left[\int_a^{\frac{a+b}{2}}\int_t^{a+b-t}\left|(b-s)^{\frac{\alpha}{k}-1}g(s)\right|dsdt+\int_{\frac{a+b}{2}}^{b}\int_{a+b-t}^{t}\left|(b-s)^{\frac{\alpha}{k}-1}g(s)\right|dsdt\right] \\
&\qquad \left(\max\left\{|f^{'}(a)|,|f^{'}(b)|\right\}\right) \\
&\leq \frac{||g||_{\infty}}{\Gamma_k(\alpha+k)}\left(\int_a^{\frac{a+b}{2}}\left[(b-t)^{\frac{\alpha}{k}}-(t-a)^{\frac{\alpha}{k}}\right]dt+\int_{\frac{a+b}{2}}^{b}\left[(t-a)^{\frac{\alpha}{k}}-(b-t)^{\frac{\alpha}{k}}\right]dt\right) \\
&\qquad \left(\max\left\{|f^{'}(a)|,|f^{'}(b)|\right\}\right) \\
&= \frac{2(b-a)^{\frac{\alpha}{k}+1}||g||_{\infty}}{\left(\frac{\alpha}{k}+1\right)\Gamma_k(\alpha+k)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\left(\max\left\{|f^{'}(a)|,|f^{'}(b)|\right\}\right)
\end{aligned}$$

since

$$\int_a^{\frac{a+b}{2}}(b-t)^{\frac{\alpha}{k}}dt=\int_{\frac{a+b}{2}}^{b}(t-a)^{\frac{\alpha}{k}}dt=\frac{(b-a)^{\frac{\alpha}{k}+1}\left(2^{\frac{\alpha}{k}+1}-1\right)}{2^{\frac{\alpha}{k}+1}\left(\frac{\alpha}{k}+1\right)}$$

and

$$\int_a^{\frac{a+b}{2}}(t-a)^{\frac{\alpha}{k}}dt=\int_{\frac{a+b}{2}}^{b}(b-t)^{\frac{\alpha}{k}}dt=\frac{(b-a)^{\frac{\alpha}{k}+1}}{2^{\frac{\alpha}{k}+1}\left(\frac{\alpha}{k}+1\right)}.$$

□

Corollary 3. *In Theorem 3, if we take $g(x)=1$, we get the inequality*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[{}_kJ^{\alpha}_{a^+}f(b)+{}_kJ^{\alpha}_{b^-}f(a)\right]\right| \\
&\qquad\qquad \leq \frac{b-a}{\left(\frac{\alpha}{k}+1\right)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\mathit{max}\left\{|f^{'}(a)|,|f^{'}(b)|\right\}.
\end{aligned}$$

.

Theorem 4. *Let $f: I\subset\mathbb{R}\longrightarrow\mathbb{R}$ be a differentiable mapping on I^o and $f^{'}\in L[a,b]$. If $|f^{'}|^q, q\geq 1$ is quasi-convex function on $[a,b]$ and $g:[a,b]\subset\mathbb{R}\longrightarrow\mathbb{R}$ is continuous and symmetric to $\frac{a+b}{2}$, the following inequality for k-fractional derivatives holds*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}\left[{}_kJ^{\alpha}_{a^+}g(b)+{}_kJ^{\alpha}_{b^-}g(a)\right]-\left[{}_kJ^{\alpha}_{a^+}(fg)(b)+{}_kJ^{\alpha}_{b^-}(fg)(a)\right]\right| \\
&\qquad \leq \frac{2(b-a)^{\frac{\alpha}{k}+1}||g||_{\infty}}{\left(\frac{\alpha}{k}+1\right)\Gamma_k(\alpha+k)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\left(\mathit{max}\left\{|f^{'}(a)|^q,|f^{'}(b)|^q\right\}\right)^{\frac{1}{q}}
\end{aligned}$$

where $\frac{\alpha}{k}>0$.

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Proof. From Lemma 6, power mean inequality, inequality (2.3) and the quasi-convexity of $|f^{'}|^q$, we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}\left[{}_kJ^{\alpha}_{a^+}g(b)+{}_kJ^{\alpha}_{b^-}g(a)\right]-\left[{}_kJ^{\alpha}_{a^+}(fg)(b)+{}_kJ^{\alpha}_{b^-}(fg)(a)\right]\right| \\
&\leq \frac{1}{\Gamma_k(\alpha)}\left(\int_a^b\left|\int_t^{a+b-t}(b-s)^{\frac{\alpha}{k}-1}g(s)ds\right|dt\right)^{1-\frac{1}{q}}\left(\int_a^b\left|\int_t^{a+b-t}(b-s)^{\frac{\alpha}{k}-1}g(s)ds\right||f^{'}(t)|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{1}{\Gamma_k(\alpha)}\left(\int_a^{\frac{a+b}{2}}\left[\int_t^{a+b-t}\left|(b-s)^{\frac{\alpha}{k}-1}g(s)\right|ds\right]dt+\int_{\frac{a+b}{2}}^b\left[\int_{a+b-t}^t\left|(b-s)^{\frac{\alpha}{k}-1}g(s)\right|ds\right]dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_a^{\frac{a+b}{2}}\left[\int_t^{a+b-t}|(b-s)^{\frac{\alpha}{k}-1}g(s)|ds\right]|f^{'}(t)|^q dt+\int_{\frac{a+b}{2}}^b\left[\int_{a+b-t}^t|(b-s)^{\frac{\alpha}{k}-1}g(s)|ds\right]|f^{'}(t)|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{2(b-a)^{\frac{\alpha}{k}+1}||g||_\infty}{\left(\frac{\alpha}{k}+1\right)\Gamma_k(\alpha+k)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\left(\max\left\{|f^{'}(a)|^q,|f^{'}(b)|^q\right\}\right)^{\frac{1}{q}}
\end{aligned}$$

where

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}}\left[\int_t^{a+b-t}|(b-s)^{\frac{\alpha}{k}-1}g(s)|ds\right]dt+\int_{\frac{a+b}{2}}^b\left[\int_{a+b-t}^t|(b-s)^{\frac{\alpha}{k}-1}g(s)|ds\right]dt \\
&=\frac{2(b-a)^{\frac{\alpha}{k}+1}}{\frac{\alpha}{k}\left(\frac{\alpha}{k}+1\right)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right).
\end{aligned}$$

□

Theorem 5. *Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable mapping on I^o and $f^{'} \in L[a,b]$. If $|f^{'}|^q, q > 1$ is quasi-convex function on $[a,b]$, and $g : [a,b] \longrightarrow R$ is continuous and symmetric to $\frac{a+b}{2}$, the following inequality for k-fractional derivatives holds*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}\left[{}_kJ^{\alpha}_{a^+}g(b)+{}_kJ^{\alpha}_{b^-}g(a)\right]-\left[{}_kJ^{\alpha}_{a^+}(fg)(b)+{}_kJ^{\alpha}_{b^-}(fg)(a)\right]\right| \\
&\leq \frac{2^{\frac{1}{p}}(b-a)^{\frac{\alpha}{k}+1}||g||_\infty}{\left(\frac{\alpha}{k}p+1\right)^{\frac{1}{p}}\Gamma_k(\alpha+k)}\left(1-\frac{1}{2^{\frac{\alpha}{k}p}}\right)^{\frac{1}{p}}\left(\mathit{max}\left\{|f^{'}(a)|^q,|f^{'}(b)|^q\right\}\right)^{\frac{1}{q}},
\end{aligned}$$

where $\frac{\alpha}{k} > 0$.

Proof. From Lemma 6, Hölder's inequality, inequality (2.3) and the quasi-convexity of $|f^{'}|^q$, we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}[{}_kJ^{\alpha}_{a^+}g(b)+{}_kJ^{\alpha}_{b^-}g(a)]-[{}_kJ^{\alpha}_{a^+}(fg)(b)+{}_kJ^{\alpha}_{b^-}(fg)(a)]\right| \\
&\leq \frac{1}{\Gamma_k(\alpha)}\left(\int_a^b\left|\int_t^{a+b-t}(b-s)^{\frac{\alpha}{k}-1}g(s)ds\right|^p dt\right)^{\frac{1}{p}}\left(\int_a^b|f^{'}(t)|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{||g||_\infty}{\Gamma_k(\alpha+k)}\left(\int_a^{\frac{a+b}{2}}\left[(b-t)^{\frac{\alpha}{k}}-(t-a)^{\frac{\alpha}{k}}\right]^p dt+\int_{\frac{a+b}{2}}^b\left[(t-a)^{\frac{\alpha}{k}}-(b-t)^{\frac{\alpha}{k}}\right]^p dt\right)^{\frac{1}{p}} \\
&\qquad \left(\int_a^b \max\left\{|f^{'}(a)|^q,|f^{'}(b)|^q\right\}dt\right)^{\frac{1}{q}} \\
&\leq \frac{||g||_\infty(b-a)^{\frac{\alpha}{k}+1}}{\Gamma_k(\alpha+k)}\left(\int_0^{\frac{1}{2}}\left[(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right]^p dt+\int_{\frac{1}{2}}^1\left[t^{\frac{\alpha}{k}}-(1-t)^{\frac{\alpha}{k}}\right]^p dt\right)^{\frac{1}{p}} \\
&\qquad \left(\max\left\{|f^{'}(a)|^q,|f^{'}(b)|^q\right\}\right)^{\frac{1}{q}} \\
&\leq \frac{||g||_\infty(b-a)^{\frac{\alpha}{k}+1}}{\Gamma_k(\alpha+k)}\left(\int_0^{\frac{1}{2}}\left[(1-t)^{\frac{\alpha}{k}p}-t^{\frac{\alpha}{k}p}\right] dt+\int_{\frac{1}{2}}^1\left[t^{\frac{\alpha}{k}p}-(1-t)^{\frac{\alpha}{k}p}\right] dt\right)^{\frac{1}{p}} \\
&\qquad \left(\max\left\{|f^{'}(a)|^q,|f^{'}(b)|^q\right\}\right)^{\frac{1}{q}} \\
&\leq \frac{2^{\frac{1}{p}}(b-a)^{\frac{\alpha}{k}+1}||g||_\infty}{\left(\frac{\alpha}{k}p+1\right)^{\frac{1}{p}}\Gamma_k(\alpha+k)}\left(1-\frac{1}{2^{\frac{\alpha}{k}p}}\right)^{\frac{1}{p}}\left(\max\left\{|f^{'}(a)|^q,|f^{'}(b)|^q\right\}\right)^{\frac{1}{q}}.
\end{aligned}$$

Where

$$\left[(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right]^p \leq (1-t)^{\frac{\alpha}{k}p}-t^{\frac{\alpha}{k}p}, \qquad \text{for } t\in\left[0,\frac{1}{2}\right]$$

and

$$\left[t^{\frac{\alpha}{k}}-(1-t)^{\frac{\alpha}{k}}\right]^p \leq t^{\frac{\alpha}{k}p}-(1-t)^{\frac{\alpha}{k}p}, \qquad \text{for } t\in\left[\frac{1}{2},1\right]$$

which follows from $(A-B)^q \leq A^q - B^q$, for any $A > B \geq 0$ and $q \geq 1$. Hence the proof is complete. □

Theorem 6. *Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable mapping on I^o and $f^{'} \in L[a,b]$. If $|f^{'}|^q, q > 1$ is quasi-convex function on $[a,b]$, and $g : [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and symmetric to $\frac{a+b}{2}$, the following inequality for k-fractional derivatives holds*

$$\left|\frac{f(a)+f(b)}{2}[{}_kJ^{\alpha}_{a^+}g(b)+{}_kJ^{\alpha}_{b^-}g(a)]-[{}_kJ^{\alpha}_{a^+}(fg)(b)+{}_kJ^{\alpha}_{b^-}(fg)(a)]\right|$$
$$\leq \frac{(b-a)^{\frac{\alpha}{k}+1}||g||_{\infty}}{\left(\frac{\alpha}{k}p+1\right)^{\frac{1}{p}}\Gamma_k(\alpha+k)}\left(max\left\{|f^{'}(a)|^q,|f^{'}(b)|^q\right\}\right)^{\frac{1}{q}}$$

where $0<\frac{\alpha}{k}\leq 1, \frac{1}{p}+\frac{1}{q}=1.$

Proof. The inequality can also be proved by using Lemma 6, Hölder's inequality, inequality (2.3), the quasi-convexity of $|f^{'}|^q$ and Lemma 2. ☐

3. Applications to Special Means

We now consider the means of arbitrary real numbers ξ,η $(\xi\neq\eta)$. We take

Arithmetic mean

$$A(\xi,\eta)=\frac{\xi+\eta}{2},\ \xi,\eta\in R.$$

Logarithmic mean

$$L(\xi,\eta)=\frac{\xi-\eta}{ln|\xi|-ln|\eta|},\ \xi,\eta\in R,\ \xi\neq\eta,\ |\xi|\neq|\eta|,\ \xi\eta\neq 0.$$

Genralised log-mean

$$L_n(\xi,\eta)=\left[\frac{\xi^{n+1}-\eta^{n+1}}{(n+1)(\xi-\eta)}\right]^{\frac{1}{n}},\ n\in Z\setminus\{-1,0\},\ \xi,\eta\in R,\ \xi\neq\eta.$$

Proposition 1. *Let* $a,b\in R\setminus\{0\}, a<b,$ *and* $n\in Z\setminus\{-1,0\}$, *then we have*

$$|A^n(a,b)A(a,b)-L_n^n(a,b)|\leq\frac{kb\,(b-a)}{4}\left(max\left\{|na^{n-1}|^q,|nb^{n-1}|^q\right\}\right)^{\frac{1}{q}}.$$

$$|A^n(a,b)A(a,b)-L_n^n(a,b)|\leq\frac{kb\,(b-a)}{2(p+1)^{\frac{1}{p}}}\left(max\left\{|na^{n-1}|^q,|nb^{n-1}|^q\right\}\right)^{\frac{1}{q}}.$$

Proof. The assertion follows from Theorem 1 and Theorem 2, applied to $f(x)=x^n, x\in R$, $g(x)=x$ and $\alpha=k$. ☐

Proposition 2. *Let* $a,b\in R, a<b,$ *and* $n\in Z\setminus\{-1,0\}$ *is odd, then for every* $q\geq 1$,*we have*

$$\left|A(a^n,b^n)[(a-A(a,b))^2+(b-A(a,b))^2]-\frac{2}{n+1}[(a-A(a,b))^{n+1}+(b-A(a,b))^{n+1}]\right|$$
$$\leq\frac{k|b-a|^3}{4}max\left\{|na^{n-1}|,|nb^{n-1}|\right\}$$

$$\left|A(a^n,b^n)[(a-A(a,b))^2+(b-A(a,b))^2]-\frac{2}{n+1}[(a-A(a,b))^{n+1}+(b-A(a,b))^{n+1}]\right|$$
$$\leq \frac{k|b-a|^3}{4}\left(max\left\{|na^{n-1}|^q,|nb^{n-1}|^q\right\}\right)^{\frac{1}{q}}$$

$$\left|A(a^n,b^n)[(a-A(a,b))^2+(b-A(a,b))^2]-\frac{2}{n+1}[(a-A(a,b))^{n+1}+(b-A(a,b))^{n+1}]\right|$$
$$\leq \frac{2^{\frac{1}{p}-1}k|b-a|^3}{(p+1)^{\frac{1}{p}}}\left(1-\frac{1}{2^p}\right)^{\frac{1}{p}}\left(max\left\{|na^{n-1}|^q,|nb^{n-1}|^q\right\}\right)^{\frac{1}{q}}$$

$$\left|A(a^n,b^n)[(a-A(a,b))^2+(b-A(a,b))^2]-\frac{2}{n+1}[(a-A(a,b))^{n+1}+(b-A(a,b))^{n+1}]\right|$$
$$\leq \frac{k|b-a|^3}{(p+1)^{\frac{1}{p}}}\left(max\left\{|na^{n-1}|^q,|nb^{n-1}|^q\right\}\right)^{\frac{1}{q}}$$

Proof. The assertion follows from Theorems 3, 4, 5 and 6 respectively, applied to $f(x)=x^n, x\in R$, $g(x)=|x-\frac{a+b}{2}|$ and $\alpha=k$. □

Note: If $n\in Z\setminus\{-1,0\}$ is even in Proposition 2, then the term $(a-A(a,b))^{n+1}$ in the left hand side of each of above inequalities will bear negative sign instead of positive.

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NONLINEAR DIFFERENTIAL POLYNOMIALS OF MEROMORPHIC FUNCTIONS WITH REGARD TO MULTIPLICITY SHARING A SMALL FUNCTION

JIANREN LONG

Abstract. We study the uniqueness problem of nonlinear differential polynomials of meromorphic functions that share one small function. A uniqueness result which related to multiplicity of meromorphic function is proved in this paper.

1. Introduction and main results

Let f be a nonconstant meromorphic function in the complex plane $\mathbb{C}$. We will assume that the reader is familiar with the standard notation of the Nevanlinna's theory of meromorphic functions, such as $T(r,f), m(r,f)$, $\overline{N}(r,f)$ and $N(r,f)$, see [9, 14, 16] for more details. The notation $S(r,f)$ is defined to be any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty, r \notin E$, where E is a set of positive real number of finite linear measure, not necessarily the same at each occurrence. The notations $T(r)$ and $S(r)$ are defined respectively by

$$T(r) = \max\{T(r,f), T(r,g)\}, \quad S(r) = o\left(T(r)\right) \text{ as } r \to \infty, r \notin E,$$

for any two nonconstant meromorphic functions f and g. A meromorphic function h is called a small function with respect to f, proved that $T(r,a) = S(r,f)$. Moreover, $\mathrm{GCD}(n_1, n_2, \ldots, n_k)$ denotes the greatest common divisor of positive integers $n_1, n_2, \ldots, n_k$.

Let f and g be two nonconstant meromorphic functions, and let $a \in \mathbb{C}$. We say that f and g share the value a CM (counting multiplicities), provided that $f - a$ and $g - a$ have the same zeros with the same multiplicities. If $f - a$ and $g - a$ have the same zeros, then we say that f and g share a IM (ignoring multiplicities). Similarly, we immediately get the definitions of f and g share h IM (or CM), where h is a small function of f and g. In addition, we also need the following notation, for any $a \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$,

$$\Theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,a;f)}{T(r,f)}.$$

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Hayman [10] and Clunie [5] proved the following result.

Theorem 1.1. *Let f be a transcendental entire function, $n \geq 1$ be a positive integer. Then $f^n f' = 1$ has infinitely many zeros.*

Remark 1. The similar result of Theorem 1.1 in which entire function is replaced with meromorphic function is proved in [2] and [4].

Fang and Hua [8], Yang and Hua [15] obtained a uniqueness theorem corresponding to Theorem 1.1.

Theorem 1.2. *Let f and g be two nonconstant entire (meromorphic) functions, and let $n \geq 6$ $(n \geq 11)$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$, or $f = tg$ for a constant t such that $t^{n+1} = 1$.*

Fang [7] considered the case of the kth derivative, and proved the following result.

Theorem 1.3. *Let f and g be two nonconstant entire functions, and let n, k be two positive integers with $n \geq 2k + 8$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share 1 CM, then $f = g$.*

Zhang et al. [18] considered some general differential polynomials. They proved the following results.

Theorem 1.4. *Let f and g be two nonconstant entire functions. Let n, k and m be three positive integers with $n \geq 3m+2k+5$ and let $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$, where $a_0 \neq 0, a_1, \ldots, a_{m-1}, a_m \neq 0$ are complex constants. If $(f^n P(f))^{(k)}$ and $(g^n P(g))^{(k)}$ share 1 CM, then either $f = tg$ for a constant t such that $t^d = 1$, where $d = \mathrm{GCD}(n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$, or f and g satisfying the algebraic function equation $R(f,g) = 0$, where $R(w_1, w_2) = w_1^n(a_m w_1^m + a_{m-1} w_1^{m-1} + \cdots + a_0) - w_2^n(a_m w_2^m + a_{m-1} w_2^{m-1} + \cdots + a_0)$.*

Theorem 1.5. *Let f and g be two nonconstant meromorphic functions, and $h(\not\equiv 0, \infty)$ be a small function with respect to f and g. Let n, k and m be three positive integers with $n > 3k + m + 8$ and $P(z)$ be defined as in Theorem 1.4. If $(f^n P(f))^{(k)}$ and $(g^n P(g))^{(k)}$ share $h(z)$ CM, then one of the following three cases holds:*

(i) *$f = tg$ for a constant t such that $t^d = 1$, where $d = \mathrm{GCD}(n + m, \ldots, n + m - i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$;*

(ii) *f and g satisfying the algebraic function equation $R(f,g)=0$, where $R(w_1,w_2)=w_1^n(a_mw_1^m+a_{m-1}w_1^{m-1}+\cdots+a_0)-w_2^n(a_mw_2^m+a_{m-1}w_2^{m-1}+\cdots+a_0)$;*

(iii) $(f^nP(f))^{(k)}(g^nP(g))^{(k)}=h^2$.

In 2011 Dyavanal [6] considered the uniqueness problem of meromorphic function related to the value sharing of two nonlinear differential polynomials in which the multiplicities of zeros and poles of f and g are taken into account. In 2013, Bhoosnurmath and Kabbur [3] proved the following uniqueness theorem by using the idea from Dyavanal [6].

Theorem 1.6. *Let f and g be two nonconstant meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer. Let n and m be two positive integers with $(n-m-1)s\geq \max\{10,2m+3\}$, and let $P(z)$ be defined as in Theorem 1.4. If $f^nP(f)f'$ and $g^nP(g)g'$ share 1 CM, then either $f=tg$ for a constant t such that $t^d=1$, where $d=\mathrm{GCD}(n+m+1,\cdots,n+m+1-i,\cdots,n+1)$, $a_{m-i}\neq 0$ for some $i=0,1,\cdots,m$ or f and g satisfy the algebraic function equation $R(f,g)=0$, where $R(x,y)=x^{n+1}(\frac{a_m}{n+m+1}x^m+\frac{a_{m-1}}{n+m}x^{m-1}+\cdots+\frac{a_0}{n+1})-y^{n+1}(\frac{a_m}{n+m+1}y^m+\frac{a_{m-1}}{n+m}y^{m-1}+\cdots+\frac{a_0}{n+1})$.*

Similar Theorem 1.5 in which a small function and kth derivative are considered, what can we say when the condition sharing 1 and the first derivative in Theorem 1.6 are replaced with sharing a small function and kth derivative respectively? In this paper, we will study the problem and establish the following uniqueness theorem.

Theorem 1.7. *Let f and g be two transcendental meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer. Let n and m be two positive integers with $n-m>\max\{2+\frac{2m}{s},\frac{(n+2)(k+4)}{ns}\}$, $\Theta(\infty,f)+\Theta(\infty,g)>\frac{4}{n}$ and let $P(z)$ be defined as in Theorem 1.4. If $(f^nP(f))^{(k)}$ and $(g^nP(g))^{(k)}$ share $h(z)$ CM, where $h(z)(\not\equiv 0,\infty)$ is a small function of f and g, then one of the following three cases hold:*

(i) $(f^nP(f))^{(k)}(g^nP(g))^{(k)}=h^2$;

(ii) *$f=tg$ for a constant t such that $t^d=1$, where $d=\mathrm{GCD}(n+m,n+m-1,\dots,n+m-i,\dots,n+1,n)$, $a_{m-i}\neq 0$ for $i=0,1,\dots,m$;*

(iii) *f and g satisfy the algebraic equation $R(f,g)=0$, where $R(f,g)=f^nP(f)-g^nP(g)$.*

The possibility $(f^nP(f))^{(k)}(g^nP(g))^{(k)}=h^2$ does not occur for $k=1$.

2. Auxiliary results

For the proof of our result, we need the following lemmas and definitions.

Definition 2.1. [11] Let $a \in \overline{\mathbb{C}}$. We use $N(r, a; f| = 1)$ to denote the counting function of simple $a-$points of f. For a positive integer p we denote by $N(r, a; f| \leq p)$ the counting function of those $a-$points of f (counted with proper multiplicities) whose multiplicities are not greater than p. By $\overline{N}(r, a; f| \leq p)$ we denote the corresponding reduced counting function. Similarly, we can define $N(r, a; f| \geq p)$ and $\overline{N}(r, a; f| \geq p)$.

Definition 2.2. [1] Let $a \in \overline{\mathbb{C}}$, and let k be a nonnegative integer. We denote by $N_k(r, \frac{1}{f-a})$ the counting function of $a-$points of f, where an $a-$point of multiplicity m is counted m times if $m \leq k$ and k times if $m > k$. Then

$$N_k(r, \frac{1}{f-a}) = \overline{N}(r, \frac{1}{f-a}) + \overline{N}(r, a; f| \geq 2) + \cdots + \overline{N}(r, a; f| \geq k). \tag{2.1}$$

Obviously $N_1(r, \frac{1}{f-a}) = \overline{N}(r, \frac{1}{f-a})$.

Lemma 2.1. [15] *Let f and g be two nonconstant meromorphic functions that share 1 CM. Then one of the following cases hold:*

(i) $T(r) \leq N_2(r, \frac{1}{f}) + N_2(r, \frac{1}{g}) + N_2(r, f) + N_2(r, g) + S(r)$;
(ii) $f = g$;
(iii) $fg = 1$.

Lemma 2.2. [17] *Let f be a nonconstant meromorphic function, and p, k be positive integers. Then*

$$N_p(r, \frac{1}{f^{(k)}}) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, \frac{1}{f}) + S(r, f), \tag{2.2}$$

$$N_p(r, \frac{1}{f^{(k)}}) \leq k\overline{N}(r, f) + N_{p+k}(r, \frac{1}{f}) + S(r, f). \tag{2.3}$$

Lemma 2.3. [13] *Let f be a nonconstant meromorphic function, and let $P_n(f) = \sum_{j=0}^{n} a_j f^j$ be a polynomial in f, where $a_n \neq 0$, $a_{n-1}, \cdots, a_1, a_0$ satisfying $T(r, a_j) = S(r, f)$. Then*

$$T(r, P_n) = nT(r, f) + S(r, f). \tag{2.4}$$

Lemma 2.4. *Let f and g be two nonconstant meromorphic functions such that*

$$\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n}$$

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for all integer $n \geq 3$. *Then*

$$f^n(af+b) = g^n(ag+b)$$

implies $f = g$, *where* a *and* b *are two finite nonzero complex constants.*

Proof. By using similar way in [12], we can obtain the lemma. □

Lemma 2.5. *Let* f *and* g *be two nonconstant meromorphic functions, whose zeros and poles are of multiplicities at least* s, *where* s *is a positive integer and let* n, k *be two positive integers. Let* $F = (f^nP(f))^{(k)}$ *and* $G = (g^nP(g))^{(k)}$, *where* $P(z)$ *be defined as in Theorem 1.4. If there exist two nonzero constants* b_1 *and* b_2 *such that* $\overline{N}(r, \frac{1}{F}) = \overline{N}(r, \frac{1}{G-b_1})$ *and* $\overline{N}(r, \frac{1}{G}) = \overline{N}(r, \frac{1}{F-b_2})$, *then* $n - m \leq \frac{(k+1)(n+2)}{ns}$.

Proof. By the second fundamental theorem of Nevanlinna's theory,

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,\frac{1}{F}) + \overline{N}(r,F) + \overline{N}(r,\frac{1}{F-b_2}) + S(r,F) \\ &\leq \overline{N}(r,\frac{1}{F}) + \overline{N}(r,F) + \overline{N}(r,\frac{1}{G}) + S(r,F). \end{aligned} \tag{2.5}$$

Combining (2.2), (2.3), (2.5) and Lemma 2.3, we get

$$\begin{aligned} (n+m)T(r,f) &\leq T(r,F) - \overline{N}(r,\frac{1}{F}) + N_{k+1}(r,\frac{1}{f^nP(f)}) + S(r,f) \\ &\leq \overline{N}(r,\frac{1}{G}) + \overline{N}(r,f) + N_{k+1}(r,\frac{1}{f^nP(f)}) + S(r,f) \\ &\leq N_{k+1}(r,\frac{1}{f^nP(f)}) + N_{k+1}(r,\frac{1}{g^nP(g)}) + \overline{N}(r,f) \\ &+ k\overline{N}(r,g) + S(r,f) + S(r,g) \\ &\leq (\frac{k+1+n}{ns} + m)T(r,f) + (\frac{k+1+nk}{ns} + m)T(r,g) \\ &+ S(r,f) + S(r,g) \\ &\leq (\frac{(k+1)(n+2)}{ns} + 2m)T(r) + S(r). \end{aligned} \tag{2.6}$$

Similarly, for the case of g,

$$(n+m)T(r,g) \leq (\frac{(k+1)(n+2)}{ns} + 2m)T(r) + S(r). \tag{2.7}$$

It follows from (2.6) and (2.7) that

$$(n - \frac{(k+1)(n+2)}{ns} - m)T(r) \leq S(r), \tag{2.8}$$

which gives $n - m \leq \frac{(k+1)(n+2)}{ns}$. This completes the proof. □

Lemma 2.6. *Let* f *and* g *be two transcendental meromorphic functions, whose zeros and poles are of multiplicities at least* s, *where* s *is a positive*

integer. Let $P(z)$ be defined as in Theorem 1.4, and n, m, k be three positive integers, and $h(\not\equiv 0, \infty)$ be small function of f and g. Then

$$(f^nP(f))^{(k)}\,(g^nP(g))^{(k)} \not\equiv h^2 \tag{2.9}$$

holds for $k = 1$ and $(n+m-2)p > 2m(1+\frac{1}{s})$, where p is the number of distinct roots of $P(z) = 0$.

Proof. If (2.9) is possible for $k = 1$, i.e.,

$$(f^nP(f))'(g^nP(g))' = h^2.$$

Then

$$f^{n-1}Q(f)f'g^{n-1}Q(g)g' = h^2, \tag{2.10}$$

where $Q(z) = \sum_{j=0}^{m} b_jz^j$, $b_j = (n+j)a_j$, $j = 0, 1, \ldots, m$. Denote $Q(z)$ as

$$Q(z) = b_m(z-d_1)^{l_1}(z-d_2)^{l_2}\cdots(z-d_p)^{l_p},$$

where $\sum_{i=1}^{p} l_i = m$, $1 \le p \le m$, $d_i \ne d_j$, $i \ne j$, $1 \le i, j \le p$, d_i are nonzero constants and l_i are positive integers, $i = 1, 2, \ldots p$.

Suppose that $z_1 \notin S_0$ is a zero of f with multiplicity $s_1(\ge s)$, where S_0 is a set defined as

$$S_0 = \{z : h(z) = 0\} \cup \{z : h(z) = \infty\}.$$

Then z_1 is a pole of g with multiplicity $q_1(\ge s)$. We deduce from (2.10) that

$$ns_1 - 1 = (n+m)q_1 + 1$$

and so

$$mq_1 + 2 = n(s_1 - q_1). \tag{2.11}$$

From (2.11) we get $q_1 \ge \frac{n-2}{m}$, so

$$s_1 \ge \frac{n+m-2}{m}.$$

Hence,

$$\overline{N}(r, \frac{1}{f}) \le \frac{m}{n+m-2}N(r, \frac{1}{f}) + S(r, f). \tag{2.12}$$

Suppose that $z_2 \notin S_0$ is a zero of $Q(f)$ with multiplicity s_2 and is a zero of $f - d_i$ of order q_i, $i = 1, 2, \ldots p$. Then $s_2 = l_iq_i$, $i = 1, 2, \cdots p$. Then z_2 is a pole of g with multiplicity $q(\ge s)$. It follows from (2.10) that

$$q_il_i + q_i - 1 = (n+m)q + 1 \ge (n+m)s + 1.$$

So

$$q_i \geq \frac{(n+m)s+2}{l_i+1}, \quad i = 1, 2, \ldots p.$$

Hence,

$$\overline{N}(r, \frac{1}{f-d_i}) \leq \frac{l_i+1}{(n+m)s+2} N(r, d_i, f) + S(r, f), \quad i = 1, 2, \ldots p.$$

By this and the first fundamental theorem of Nevanlinna's theory, we have

$$\sum_{i=1}^{p} \overline{N}(r, \frac{1}{f-d_i}) \leq \frac{m+p}{(n+m)s+2} T(r, f) + S(r, f). \tag{2.13}$$

Suppose that $z_3 \notin S_0$ is a pole of f. Then we know that z_3 is either a zero of $g^{n-1}Q(g)$ or a zero of g' by (2.10). Therefore,

$$\begin{aligned} \overline{N}(r, f) &\leq \overline{N}(r, \frac{1}{g}) + \sum_{i=1}^{p} \overline{N}(r, \frac{1}{g-d_i}) + \overline{N_0}(r, \frac{1}{g'}) + S(r, f) + S(r, g) \\ &\leq (\frac{m}{n+m-2} + \frac{m+p}{(n+m)s+2}) T(r, g) + \overline{N_0}(r, \frac{1}{g'}) \\ &+ S(r, f) + S(r, g), \end{aligned} \tag{2.14}$$

where $\overline{N_0}(r, \frac{1}{g'})$ denote the reduce counting function of those zeros of g' which are not the zeros of $gQ(g)$.

By (2.12)-(2.14), and the second fundamental theorem of Nevanlinna's theory,

$$\begin{aligned} pT(r, f) &\leq \overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) + \sum_{i=1}^{p} \overline{N}(r, \frac{1}{f-d_i}) - \overline{N_0}(r, \frac{1}{f'}) + S(r, f) \\ &\leq (\frac{m}{n+m-2} + \frac{m+p}{(n+m)s+2})(T(r, f) + T(r, g)) + \overline{N_0}(r, \frac{1}{g'}) \\ &- \overline{N_0}(r, \frac{1}{f'}) + S(r, f) + S(r, g). \end{aligned} \tag{2.15}$$

Similarly, for the case of g,

$$\begin{aligned} pT(r, g) &\leq (\frac{m}{n+m-2} + \frac{m+p}{(n+m)s+2})(T(r, f) + T(r, g)) + \overline{N_0}(r, \frac{1}{f'}) \\ &- \overline{N_0}(r, \frac{1}{g'}) + S(r, f) + S(r, g). \end{aligned} \tag{2.16}$$

It follows from (2.15) and (2.16) that

$$(p - \frac{2m}{n+m-2} - \frac{2(m+p)}{(n+m)s+2})(T(r, f) + T(r, g)) \leq S(r, f) + S(r, g).$$

This is a contradiction with our assumption that $(n+m-2)p > 2m(1+\frac{1}{s})$, and hence the proof is complete. □

3. Proof of Theorem 1.7

Let $F = \dfrac{(f^nP(f))^{(k)}}{h}$, $G = \dfrac{(g^nP(g))^{(k)}}{h}$. Then F and G share 1 CM. Applying Lemma 2.3,

$$T(r,h) = o(T(r,F)) = S(r,f), \quad T(r,h) = o(T(r,G)) = S(r,g). \tag{3.1}$$

It follows from (2.2) and (3.1) that

$$\begin{aligned} N_2(r,\frac{1}{F}) &\le N_2(r,\frac{1}{(f^nP(f))^{(k)}}) + N_2(r,h) + S(r,f) \\ &\le N_2(r,\frac{1}{(f^nP(f))^{(k)}}) + S(r,f) \\ &\le T(r,(f^nP(f)) - (n+m)T(r,f) + N_{k+2}(r,\frac{1}{f^nP(f)}) + S(r,f) \\ &\le T(r,F) - (n+m)T(r,f) + N_{k+2}(r,\frac{1}{f^nP(f)}) + S(r,f). \end{aligned} \tag{3.2}$$

We deduce from (2.3) that

$$\begin{aligned} N_2(r,\frac{1}{F}) &\le N_2(r,\frac{1}{(f^nP(f))^{(k)}}) + S(r,f) \\ &\le k\overline{N}(r,f^nP(f))^{(k)}) + N_{k+2}(r,\frac{1}{f^nP(f)}) + S(r,f) \\ &\le k\overline{N}(r,f) + N_{k+2}(r,\frac{1}{f^nP(f)}) + S(r,f). \end{aligned} \tag{3.3}$$

It follows from (3.2) that

$$(n+m)T(r,f) \le T(r,F) + N_{k+2}(r,\frac{1}{f^nP(f)}) - N_2(r,\frac{1}{F}) + S(r,f). \tag{3.4}$$

Suppose that (i) of Lemma 2.1 holds, i.e.,

$$\begin{aligned} \max\{T(r,F),T(r,G)\} \le\ & N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G}) + N_2(r,F) + N_2(r,G) \\ & + S(r,f) + S(r,g). \end{aligned}$$

Combining this, (3.3) and (3.4),

$$\begin{aligned}(n+m)T(r,f) &\leq T(r,F)+N_{k+2}(r,\frac{1}{f^nP(f)})-N_2(r,\frac{1}{F})+S(r,f)+S(r,g)\\ &\leq N_2(r,\frac{1}{G})+N_2(r,F)+N_2(r,G)+N_{k+2}(r,\frac{1}{f^nP(f)})\\ &+S(r,f)+S(r,g)\\ &\leq N_{k+2}(r,\frac{1}{f^nP(f)})+N_{k+2}(r,\frac{1}{g^nP(g)})+k\overline{N}(r,g)\\ &+2\overline{N}(r,f)+2\overline{N}(r,g)+S(r,f)+S(r,g)\\ &\leq (\frac{k+2+2n}{ns}+m)T(r,f)+(\frac{(k+2)(n+1)}{ns}+m)T(r,g)\\ &+S(r,f)+S(r,g)\\ &\leq (\frac{k(n+2)+4(n+1)}{ns}+2m)T(r)+S(r). \qquad (3.5)\end{aligned}$$

Similarly, for the case of g,

$$(n+m)T(r,g)\leq (\frac{k(n+2)+4(n+1)}{ns}+2m)T(r)+S(r). \qquad (3.6)$$

It follows from (3.5) and (3.6) that

$$(n+m)T(r)\leq (\frac{k(n+2)+4(n+1)}{ns}+2m)T(r)+S(r).$$

This implies that

$$(n-m-\frac{k(n+2)+4(n+1)}{ns})T(r)\leq S(r). \qquad (3.7)$$

This contradicts with our assumption that $(n-m)>\max\{2+\frac{2m}{s},\frac{(n+2)(k+4)}{ns}\}$. So, we conclude that either $FG=1$ or $F=G$ by Lemma 2.1. Suppose that $FG=1$, then

$$(f^nP(f))^{(k)}(g^nP(g))^{(k)}=h^2.$$

This is a contradiction when $k=1$ by Lemma 2.6. So $F=G$, this implies that

$$(f^nP(f))^{(k)}=(g^nP(g))^{(k)}. \qquad (3.8)$$

Integrating for (3.8), we have

$$(f^nP(f))^{(k-1)}=(g^nP(g))^{(k-1)}+b_{k-1}, \qquad (3.9)$$

where b_{k-1} is constant. If $b_{k-1}\neq 0$, we obtain $n-m\leq\frac{(k+1)(n+2)}{ns}<\frac{(k+4)(n+2)}{ns}$ by Lemma 2.5. This is a contradiction with our assumption that $(n-m)>\max\{2+\frac{2m}{s},\frac{(n+2)(k+4)}{ns}\}$. Thus $b_{k-1}=0$. By repeating $k-$times,

$$f^nP(f)=g^nP(g). \qquad (3.10)$$

If $m = 1$ in (3.10), then $f = g$ by Lemma 2.4. Suppose that $m \geq 2$ and $b = \frac{f}{g}$. If b is a constant, putting $f = bg$ in (3.10), we get

(3.11)
$$a_m g^{n+m}(b^{n+m} - 1) + a_{m-1} g^{n+m-1}(b^{n+m-1} - 1) + \cdots + a_0 g^n(b^n - 1) = 0,$$

which implies $b^d = 1$, where $d = \mathrm{GCD}(n+m, n+m-1, \ldots, n+1, n)$. Hence $f = tg$ for a constant t such that $t^d = 1$, $d = \mathrm{GCD}(n+m, n+m-1, \ldots, n+m-i, \ldots, n+1, n)$, $i = 0, 1, \ldots, m$.

If b is not a constant, then we can see that f and g satisfy the algebraic function equation $R(f, g) = 0$ by (3.10), where $R(f, g) = f^n P(f) - g^n P(g)$. This completes the proof of theorem.

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Impulsive hybrid fractional quantum difference equations

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Abstract

This paper is concerned with the existence of solutions for impulsive hybrid fractional q-difference equations involving a q-shifting operator of the type ${}_a\Phi_q(m) = qm + (1-q)a$. A hybrid fixed point theorem for two operators in a Banach algebra due to Dhage [29] is applied to obtain the existence result. An example illustrating the main result is also presented.

Key words and phrases: Quantum calculus; impulsive fractional q-difference equations; hybrid differential equations; existence; fixed point theorem
AMS (MOS) Subject Classifications: 34A08; 34A12; 34A37

1 Introduction

Fractional differential equations have been extensively investigated by several researchers in the recent years. The overwhelming interest in this branch of mathematics is due to the application of fractional-order operators in the mathematical modelling of several phenomena occurring in a variety of disciplines of applied sciences and engineering such as biomathematics, signal and image processing, control theory, dynamical systems, etc.

Hybrid fractional differential equations dealing with the fractional derivative of an unknown function hybrid with the nonlinearity depending on it is another interesting field of research. For some recent works on this topic, we refer the reader to a series of papers ([1]-[6]).

The subject of q-difference calculus or quantum calculus dates back to the beginning of the 20th century, when Jackson [7] introduced the concept of q-difference operator. Afterwards, this field of research flourished with the contributions of researchers from different parts of the world, for instance, see ([8]-[15]). The intensive development of fractional calculus motivated several investigators to consider fractional q-difference calculus. Now a great deal of work on initial and boundary value problems involving nonlinear fractional q-difference equations is available, for example, see [16]-[24] and the references therein.

The quantum calculus, known as the calculus without limits, provides a descent approach to study nondifferentiable functions in terms of difference operators. Quantum difference operators appear in different areas of mathematics such as orthogonal polynomials, basic hyper-geometric functions, combinatorics, the calculus of variations, mechanics and the theory of relativity. For the fundamental concepts of quantum calculus, we refer the reader to a text by Kac and Cheung [25].

More recently, the topic of q_k-calculus has also gained consideration attention. The notions of q_k-derivative and q_k-integral for a function $f : J_k := [t_k, t_{k+1}] \to \mathbb{R}$, together with their properties can

be found in [26, 27]. In [28], new concepts of fractional quantum calculus were defined via a q-shifting operator of the form: ${}_a\Phi_q(m) = qm + (1-q)a$.

The purpose of the present work is to study the following impulsive hybrid fractional quantum difference equations:

$$\begin{cases} {}_{t_k}^{c}D_{q_k}^{\alpha_k}\left[\dfrac{x(t)}{f(t,x(t))}\right] = g(t,x(t)), \quad t \in J_k \subseteq [0,T],\ t \neq t_k, \\ \Delta x(t_k) = \varphi_k\left(x(t_k)\right), \quad k = 1,2,\ldots,m, \\ x(0) = \mu, \end{cases} \tag{1}$$

where $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, ${}_{t_k}^{c}D_{q_k}^{\alpha_k}$ denotes the Caputo fractional q_k-derivative of order α_k on intervals J_k, $J_0 = [0,t_1]$, $J_k = (t_k, t_{k+1}]$, $0 < \alpha_k \le 1$, $0 < q_k < 1$, $k = 0,1,\ldots,m$, $J = [0,T]$, $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $\varphi_k \in C(\mathbb{R},\mathbb{R})$, $k = 1,2,\ldots,m$, $\mu \in \mathbb{R}$ and $\Delta x(t_k) = x(t_k^+) - x(t_k)$, $x(t_k^+) = \lim_{\theta \to 0^+} x(t_k + \theta)$, $k = 1,2,\ldots,m$. Here, we emphasize that the above initial value problem contains the new q-shifting operator ${}_a\Phi_q(m) = qm + (1-q)a$ [28].

The papers is organized as follows. In Section 2, we recall some preliminary concepts and present an auxiliary lemma which is used to convert the impulsive problem (1) into an equivalent integral equation. An existence result for the problem (1) obtained by means of a hybrid fixed point theorem due to Dhage [29] is presented in Section 3, which is well illustrated with the aid of an example.

2 Preliminaries

For the convenience of the reader, we recall some preliminary concepts from [28].

First of all, we define a q-shifting operator as

$${}_a\Phi_q(m) = qm + (1-q)a \tag{2}$$

such that

$${}_a\Phi_q^k(m) = {}_a\Phi_q^{k-1}\left({}_a\Phi_q(m)\right) \quad \text{and} \quad {}_a\Phi_q^0(m) = m,$$

for any positive integer k. The power law for q-shifting operator is

$${}_a(n-m)_q^{(0)} = 1, \quad {}_a(n-m)_q^{(k)} = \prod_{i=0}^{k-1}\left(n - {}_a\Phi_q^i(m)\right), \quad k \in \mathbb{N} \cup \{\infty\}.$$

In case $\gamma \in \mathbb{R}$, the above power law takes the form

$${}_a(n-m)_q^{(\gamma)} = n^{(\gamma)} \prod_{i=0}^{\infty} \frac{1 - \frac{a}{n}\Phi_q^i(m/n)}{1 - \frac{a}{n}\Phi_q^{\gamma+i}(m/n)}.$$

The q-derivative of a function h on interval $[a,b]$ is defined by

$$({}_aD_q h)(t) = \frac{h(t) - h({}_a\Phi_q(t))}{(1-q)(t-a)}, \quad t \neq a, \quad \text{and} \quad ({}_aD_q h)(a) = \lim_{t \to a}({}_aD_q h)(t),$$

while the higher order q-derivative is given by the formula

$$({}_aD_q^0 f)(t) = f(t) \quad \text{and} \quad ({}_aD_q^k f)(t) = {}_aD_q^{k-1}({}_aD_q f)(t), \quad k \in \mathbb{N}.$$

The product and quotient formulas for q-derivative are

$${}_aD_q(h_1h_2)(t) = h_1(t)\,{}_aD_q h_2(t) + h_2({}_a\Phi_q(t))\,{}_aD_q h_1(t) = h_2(t)\,{}_aD_q h_1(t) + h_1({}_a\Phi_q(t))\,{}_aD_q h_2(t),$$

$${}_aD_q\left(\frac{h_1}{h_2}\right)(t) = \frac{h_2(t)\,{}_aD_q h_1(t) - h_1(t)\,{}_aD_q h_2(t)}{h_2(t)h_2({}_a\Phi_q(t))},$$

where h_1 and h_2 are well defined on $[a,b]$ with $h_2(t)h_2({}_a\Phi_q(t)) \neq 0$.

The q-integral of a function h defined on the interval $[a,b]$ is given by

$$({}_aI_qh)(t) = \int_a^t h(s)_a ds = (1-q)(t-a)\sum_{i=0}^{\infty} q^i h({}_a\Phi_{q^i}(t)), \quad t \in [a,b], \tag{3}$$

with

$$({}_aI_q^0h)(t) = h(t) \;\text{ and }\; ({}_aI_q^kh)(t) = {}_aI_q^{k-1}({}_aI_qh)(t), \quad k \in \mathbb{N}.$$

The fundamental theorem of calculus applies to the operator ${}_aD_q$ and ${}_aI_q$, that is,

$$({}_aD_{qa}I_qh)(t) = h(t),$$

and if h is continuous at $t = a$, then

$$({}_aI_{qa}D_qh)(t) = h(t) - h(a).$$

The q-integration by parts formula on the interval $[a,b]$ is

$$\int_a^b f(s)_aD_qg(s)_ad_qs = (fg)(t)\Big|_a^b - \int_a^b g({}_a\Phi_q(s))_aD_qf(s)_ad_qs.$$

Let us now define Riemann-Liouville fractional q-derivative and q-integral on interval $[a,b]$ and outline some of their properties [28].

Definition 2.1 *The fractional q-derivative of Riemann-Liouville type of order $\nu \geq 0$ on interval $[a,b]$ is defined by $({}_aD_q^0h)(t) = h(t)$ and*

$$({}_aD_q^\nu h)(t) = ({}_aD_{qa}^lI_q^{l-\nu}h)(t), \quad \nu > 0,$$

where l is the smallest integer greater than or equal to ν.

Definition 2.2 *Let $\alpha \geq 0$ and h be a function defined on $[a,b]$. The fractional q-integral of Riemann-Liouville type is given by $({}_aI_q^0h)(t) = h(t)$ and*

$$({}_aI_q^\alpha h)(t) = \frac{1}{\Gamma_q(\alpha)}\int_a^t {}_a(t - {}_a\Phi_q(s))_q^{(\alpha-1)}h(s)_ad_qs, \quad \alpha > 0, \quad t \in [a,b].$$

From [28], we have the following formulas

$${}_aD_q^\alpha(t-a)^\beta = \frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta-\alpha+1)}(t-a)^{\beta-\alpha}, \tag{4}$$

$${}_aI_q^\alpha(t-a)^\beta = \frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta+\alpha+1)}(t-a)^{\beta+\alpha}. \tag{5}$$

Lemma 2.3 *Let α, $\beta \in \mathbb{R}^+$ and f be a continuous function on $[a,b]$, $a \geq 0$. The Riemann-Liouville fractional q-integral has the following semi-group property*

$${}_aI_q^\beta{}_aI_q^\alpha h(t) = {}_aI_q^\alpha{}_aI_q^\beta h(t) = {}_aI_q^{\alpha+\beta}h(t).$$

Lemma 2.4 *Let h be a q-integrable function on $[a,b]$. Then the following equality holds*

$${}_aD_q^\alpha{}_aI_q^\alpha h(t) = h(t), \qquad \text{for } \alpha > 0, \; t \in [a,b].$$

Lemma 2.5 *Let $\alpha > 0$ and p be a positive integer. Then for $t \in [a,b]$ the following equality holds*

$${}_aI_q^\alpha{}_aD_q^p h(t) = {}_aD_{q}^p{}_aI_q^\alpha h(t) - \sum_{k=0}^{p-1}\frac{(t-a)^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)}{}_aD_q^kh(a).$$

We define Caputo fractional q-derivative as follows.

Definition 2.6 *The fractional q-derivative of Caputo type of order $\alpha \geq 0$ on interval $[a,b]$ is defined by $({}^c_aD^0_q f)(t) = h(t)$ and*

$$({}^c_aD^\alpha_q h)(t) = ({}_aI^{n-\alpha}_q {}_aD^n_q h)(t), \quad \alpha > 0,$$

where n is the smallest integer greater than or equal to α.

Lemma 2.7 *Let $\alpha > 0$ and n be the smallest integer greater than or equal to α. Then for $t \in [a,b]$ the following equality holds*

$${}_aI^\alpha_q {}^c_aD^\alpha_q h(t) = h(t) - \sum_{k=0}^{n-1} \frac{(t-a)^k}{\Gamma_q(k+1)} {}_aD^k_q h(a).$$

Proof. From Lemma 2.5, for $\alpha = p = m$, where m is a positive integer, we have

$${}_aI^m_q {}_aD^m_q h(t) = {}_aD^m_q {}_aI^m_q h(t) - \sum_{k=0}^{m-1} \frac{(t-a)^k}{\Gamma_q(k+1)} {}_aD^k_q h(a) = h(t) - \sum_{k=0}^{m-1} \frac{(t-a)^k}{\Gamma_q(k+1)} {}_aD^k_q h(a).$$

Then, by Definition 2.6, we have

$${}_aI^\alpha_q {}^c_aD^\alpha_q h(t) = {}_aI^\alpha_q {}_aI^{n-\alpha}_q {}_aD^n_q h(t) = {}_aI^n_q {}_aD^n_q h(t) = h(t) - \sum_{k=0}^{n-1} \frac{(t-a)^k}{\Gamma_q(k+1)} {}_aD^k_q h(a).$$

□

Now we present a lemma which plays a pivotal role in the forthcoming analysis.

Lemma 2.8 *Assume that the map $x \mapsto \dfrac{x}{f(t,x)}$ is injection for each $t \in J$. $x \in PC(J,\mathbb{R})$ is the solution of* (1) *if and only if x is a solution of the impulsive integral equation*

$$\begin{aligned} x(t) &= f(t,x(t)) \Bigg(\frac{\mu}{f(0,\mu)} \prod_{i=1}^{k} \frac{f(t_i,x(t_i))}{f(t_i,x(t_i^+))} + \sum_{i=1}^{k} \prod_{i \leq j \leq k} {}_{t_{i-1}}I^{\alpha_{i-1}}_{q_{i-1}} g(t_i,x(t_i)) \frac{f(t_j,x(t_j))}{f(t_j,x(t_j^+))} \\ &\quad + \sum_{i=1}^{k} \prod_{i<j\leq k} \frac{\varphi_i(x(t_i))}{f(t_i,x(t_i^+))} \cdot \frac{f(t_j,x(t_j))}{f(t_j,x(t_j^+))} + {}_{t_k}I^{\alpha_k}_{q_k} g(t,x(t)) \Bigg), \end{aligned} \tag{6}$$

where $\sum_{b<a}(\cdot) = 0$, $\prod_{b<a}(\cdot) = 1$ for $b > a$ and for $t \in J_k$,

$${}_{t_k}I^{\alpha_k}_{q_k} g(t,x(t)) = \frac{1}{\Gamma_{q_k}(\alpha_k)} \int_{t_k}^{t} {}_{t_k}(t - {}_{t_k}\Phi_{q_k}(s))^{(\alpha_k-1)}_{q_k} g(s,x(s)) {}_{t_k}d_{q_k}s. \tag{7}$$

Proof. Applying Riemann-Liouville fractional q_0-integral operator of order α_0 to both sides of the first equation of (1) for $t \in J_0$ and using Lemma 2.7, we get

$${}_{t_0}I^{\alpha_0}_{q_0} {}^c_{t_0}D^{\alpha_0}_{q_0} \left[\frac{x(t)}{f(t,x(t))} \right] = \frac{x(t)}{f(t,x(t))} - \frac{x(0)}{f(0,x(0))} = {}_{t_0}I^{\alpha_0}_{q_0} g(t,x(t)),$$

which, in view of the initial condition, takes the form

$$x(t) = f(t,x(t)) \left[\frac{\mu}{f(0,\mu)} + {}_{t_0}I^{\alpha_0}_{q_0} g(t,x(t)) \right].$$

At $t = t_1$, we have

$$x(t_1) = f(t_1,x(t_1)) \left[\frac{\mu}{f(0,\mu)} + {}_{t_0}I^{\alpha_0}_{q_0} g(t_1,x(t_1)) \right]. \tag{8}$$

For $t \in J_1$, operating the Riemann-Liouville fractional q_1-integral of order α_1 on (1) and using the above process together with impulsive condition, we obtain

$$\frac{x(t)}{f(t,x(t))} = \frac{x(t_1^+)}{f(t_1^+,x(t_1^+))} + {}_{t_1}I_{q_1}^{\alpha_1} g(t,x(t)) = \frac{x(t_1)+\varphi_1(x(t_1))}{f(t_1^+,x(t_1^+))} + {}_{t_1}I_{q_1}^{\alpha_1} g(t,x(t)). \tag{9}$$

By the continuity of f with respect to the variable t, the expression $f(t_1^+,x(t_1^+))$ can be written as $f(t_1,x(t_1^+))$. Substituting (8) into (9) yields

$$\begin{aligned} x(t) &= f(t,x(t))\Bigg(\frac{\mu}{f(0,\mu)}\cdot\frac{f(t_1,x(t_1))}{f(t_1,x(t_1^+))} + \frac{f(t_1,x(t_1))}{f(t_1,x(t_1^+))}\,{}_{t_0}I_{q_0}^{\alpha_0} g(t_1,x(t_1)) \\ &\quad + \frac{\varphi_1(x(t_1))}{f(t_1,x(t_1^+))} + {}_{t_1}I_{q_1}^{\alpha_1} g(t,x(t))\Bigg). \end{aligned}$$

Also, for $t \in J_2$, we have

$$\begin{aligned} x(t) &= f(t,x(t))\Bigg(\frac{\mu}{f(0,\mu)}\cdot\frac{f(t_1,x(t_1))}{f(t_1,x(t_1^+))}\cdot\frac{f(t_2,x(t_2))}{f(t_2,x(t_2^+))} \\ &\quad + \frac{f(t_1,x(t_1))}{f(t_1,x(t_1^+))}\cdot\frac{f(t_2,x(t_2))}{f(t_2,x(t_2^+))}\,{}_{t_0}I_{q_0}^{\alpha_0} g(t_1,x(t_1)) + \frac{f(t_2,x(t_2))}{f(t_2,x(t_2^+))}\,{}_{t_1}I_{q_1}^{\alpha_1} g(t_2,x(t_2)) \\ &\quad + \frac{\varphi_1(x(t_1))}{f(t_1,x(t_1^+))}\cdot\frac{f(t_2,x(t_2))}{f(t_2,x(t_2^+))} + \frac{\varphi_2(x(t_2))}{f(t_2,x(t_2^+))} + {}_{t_2}I_{q_2}^{\alpha_2} g(t,x(t))\Bigg). \end{aligned}$$

Repeating the above process, for $t \in J$, we obtain (6).

Conversely, we assume that $x(t)$ is a solution of (6). Dividing by $f(t,x(t))$ and applying ${}_{t_k}^{c}D_{q_k}^{\alpha_k}$ on both sides of (6) for $t \in J_k$, $t \neq t_k$ $k = 0,1,\ldots,m$, we get

$${}_{t_k}^{c}D_{q_k}^{\alpha_k}\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)).$$

It is easy to see that $\Delta x(t_k) = x(t_k^+) - x(t_k) = \varphi_k(x(t_k))$. Since $f(0,x(0)) \neq 0$, and using the fact that the map $x \mapsto \dfrac{x}{f(t,x)}$ is injection for each $t \in J$, we have $x(0) = \mu$. This completes the proof. □

Now we state a hybrid fixed point theorem due to Dhage [29], which we need to prove our main existence result.

Lemma 2.9 *Let S be a nonempty, closed convex and bounded subset of the Banach algebra E and let $A : E \to E$ and $B : S \to E$ be two operators such that (a) A is Lipschitzian with Lipschitz constant δ; (b) B is completely continuous; (c) $x = AxBy \Rightarrow x \in S$ for all $y \in S$; (d) $\delta M < 1$, where $M = \|B(S)\| = \sup\{\|B(x)\| : x \in S\}$. Then the operator equation $x = AxBx$ has a solution in S.*

3 Main Result

Let $PC(J,\mathbb{R}) = \{x : J \to \mathbb{R} : x(t)$ is continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k)$, $k = 1,2,\ldots,m\}$. Define a norm $\|\cdot\|$ and a multiplication in $PC(J,\mathbb{R})$ by $\|x\| = \sup_{t\in J}|x(t)|$ and $(xy)(t) = x(t)y(t)$, $\forall t \in J$.
Clearly $PC(J,\mathbb{R})$ is a Banach algebra with respect to above supremum norm and the multiplication in it.

Now, we are in the position to present the main existence result.

Theorem 3.1 *Assume that the map $x \mapsto \dfrac{x}{f(t,x)}$ is injection for each $t \in J$. In addition we suppose that:*

(H_1) *The function $f : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ is bounded, continuous and there exists a positive function ϕ with bound $\|\phi\|$ such that*

$$|f(t, x(t)) - f(t, y(t))| \leq \phi(t)|x(t) - y(t)|, \ \ \text{for } t \in J \ \text{and} x, y \in \mathbb{R}. \tag{10}$$

(H_2) *There exist a function $p \in C(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ such that*

$$|g(t, x(t))| \leq p(t)\psi(|x|), \qquad (t, x) \in J \times \mathbb{R}. \tag{11}$$

(H_3) *The functions $\varphi_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \ldots, m$, are bounded and continuous.*

(H_4) *There exists a number $r > 0$ such that*

$$r \geq \Omega_1\left(\frac{|\mu|}{|f(0,\mu)|}\left(\frac{\Omega_1}{\Omega_2}\right)^m + \|p\|\psi(r)\sum_{i=1}^{m+1}\frac{(t_i - t_{i-1})^{\alpha_{i-1}}}{\Gamma_{q_{i-1}}(\alpha_{i-1}+1)}\left(\frac{\Omega_1}{\Omega_2}\right)^{m+1-i} + \frac{\Omega_3}{\Omega_2}\sum_{i=1}^{m}\left(\frac{\Omega_1}{\Omega_2}\right)^{m-i}\right), \tag{12}$$

and

$$\|\phi\|\left(\frac{|\mu|}{|f(0,\mu)|}\left(\frac{\Omega_1}{\Omega_2}\right)^m + \|p\|\psi(r)\sum_{i=1}^{m+1}\frac{(t_i - t_{i-1})^{\alpha_{i-1}}}{\Gamma_{q_{i-1}}(\alpha_{i-1}+1)}\left(\frac{\Omega_1}{\Omega_2}\right)^{m+1-i} + \frac{\Omega_3}{\Omega_2}\sum_{i=1}^{m}\left(\frac{\Omega_1}{\Omega_2}\right)^{m-i}\right) < 1,$$

where $\Omega_1 = \sup\{|f(t, x)| : (t, x) \in J \times \mathbb{R}\}$, $\Omega_2 = \inf\{|f(t, x)| : (t, x) \in J \times \mathbb{R}\}$ and $\Omega_3 = \max\{\sup|\varphi_i(x)| : x \in \mathbb{R}, i = 1, 2, \ldots, m\}$.

Then the impulsive initial value problem (1) *has at least one solution on J.*

Proof. Let us introduce a subset S of $PC(J, \mathbb{R})$ by

$$S = \{x \in PC(J, \mathbb{R}) : \|x\| \leq r\},$$

where r satisfies inequality (12). Clearly S is closed, convex and bounded subset of the Banach space $PC(J, \mathbb{R})$. In view of Lemma 2.8, the problem (1) is equivalent to the integral equation (6). Let us define two operators $\mathcal{A} : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ by

$$\mathcal{A}x(t) = f(t, x(t)), \ \ t \in J, \tag{13}$$

and $\mathcal{B} : S \to PC(J, \mathbb{R})$ by

$$\begin{aligned}\mathcal{B}x(t) \ &= \ \frac{\mu}{f(0,\mu)}\prod_{i=1}^{k}\frac{f(t_i, x(t_i))}{f(t_i, x(t_i^+))} + \sum_{i=1}^{k}\prod_{i\leq j\leq k}{}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}}g(t_i, x(t_i))\frac{f(t_j, x(t_j))}{f(t_j, x(t_j^+))} \\ &\quad + \sum_{i=1}^{k}\prod_{i<j\leq k}\frac{\varphi_i(x(t_i))}{f(t_i, x(t_i^+))}\cdot\frac{f(t_j, x(t_j))}{f(t_j, x(t_j^+))} + {}_{t_k}I_{q_k}^{\alpha_k}g(t, x(t)), \ t \in J.\end{aligned} \tag{14}$$

Then, the problem (1) is transformed into an operator equation as

$$x = \mathcal{A}x\mathcal{B}x. \tag{15}$$

Under our assumptions, we will show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Lemma 2.9. This will be achieved in a series of steps.

Step 1. *The operator $\mathcal{A}$ is Lipschitzian on $PC(J, \mathbb{R})$.*

Let $x, y \in PC(J, \mathbb{R})$. Then by (H_1), for $t \in J$, we have

$$|\mathcal{A}x(t) - \mathcal{A}y(t)| = |f(t, x(t)) - f(t, y(t))| \leq \phi(t)|x(t) - y(t)|.$$

Taking supremum over t, we obtain $\|\mathcal{A}x-\mathcal{A}y\|\le\|\phi\|\|x-y\|$ for all $x,y\in PC(J,\mathbb{R})$. This show that $\mathcal{A}$ is a Lipschitzian on $PC(J,\mathbb{R})$ with Lipschitz constant $\|\phi\|$.

Step 2. *The operator $\mathcal{B}$ is completely continuous on S.*

In this step, we first show that the operator $\mathcal{B}$ is continuous on S. Let $\{x_n\}$ be a sequence in S converging to a point $x\in S$. Then, for all $t\in J$, we have

$$\begin{aligned}
&\lim_{n\to\infty}\mathcal{B}x_n(t)\\
=\;&\lim_{n\to\infty}\frac{\mu}{f(0,\mu)}\prod_{i=1}^{k}\frac{f(t_i,x_n(t_i))}{f(t_i,x_n(t_i^+))}+\lim_{n\to\infty}\sum_{i=1}^{k}\prod_{i\le j\le k}{}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}}g(t_i,x_n(t_i))\frac{f(t_j,x_n(t_j))}{f(t_j,x_n(t_j^+))}\\
&+\lim_{n\to\infty}\sum_{i=1}^{k}\prod_{i<j\le k}\frac{\varphi_i(x_n(t_i))}{f(t_i,x_n(t_i^+))}\cdot\frac{f(t_j,x_n(t_j))}{f(t_j,x_n(t_j^+))}+\lim_{n\to\infty}{}_{t_k}I_{q_k}^{\alpha_k}g(t,x_n(t))=\mathcal{B}x(t),
\end{aligned}$$

which implies that $\mathcal{B}x_n\to\mathcal{B}x$ point-wise on J. Further it can be shown that $\{\mathcal{B}x_n\}$ is an equicontinuous sequence of functions. So $\mathcal{B}x_n\to\mathcal{B}x$ uniformly and the operator $\mathcal{B}$ is continuous on S.

Next we will prove that $\mathcal{B}$ is a compact operator on S. It is enough to show that the set $\mathcal{B}(S)$ is uniformly bounded and equicontinuous in $PC(J,\mathbb{R})$. For any $x\in S$, on account of (5), we get

$$\begin{aligned}
|\mathcal{B}x(t)| \;\le\;& \frac{|\mu|}{|f(0,\mu)|}\prod_{i=1}^{k}\frac{|f(t_i,x(t_i))|}{|f(t_i,x(t_i^+))|}+\sum_{i=1}^{k}\prod_{i\le j\le k}{}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}}|g(t_i,x(t_i))|\frac{|f(t_j,x(t_j))|}{|f(t_j,x(t_j^+))|}\\
&+\sum_{i=1}^{k}\prod_{i<j\le k}\frac{|\varphi_i(x(t_i))|}{|f(t_i,x(t_i^+))|}\cdot\frac{|f(t_j,x(t_j))|}{|f(t_j,x(t_j^+))|}+{}_{t_k}I_{q_k}^{\alpha_k}|g(t,x(t))|\\
\le\;& \frac{|\mu|}{|f(0,\mu)|}\prod_{i=1}^{m}\frac{|f(t_i,x(t_i))|}{|f(t_i,x(t_i^+))|}+\sum_{i=1}^{m}\prod_{i\le j\le m}{}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}}|g(t_i,x(t_i))|\frac{|f(t_j,x(t_j))|}{|f(t_j,x(t_j^+))|}\\
&+\sum_{i=1}^{m}\prod_{i<j\le m}\frac{|\varphi_i(x(t_i))|}{|f(t_i,x(t_i^+))|}\cdot\frac{|f(t_j,x(t_j))|}{|f(t_j,x(t_j^+))|}+{}_{t_m}I_{q_m}^{\alpha_m}|g(T,x(T))|\\
\le\;& \frac{|\mu|}{|f(0,\mu)|}\left(\frac{\Omega_1}{\Omega_2}\right)^m+\|p\|\psi(r)\sum_{i=1}^{m+1}\frac{(t_i-t_{i-1})^{\alpha_{i-1}}}{\Gamma_{q_{i-1}}(\alpha_{i-1}+1)}\left(\frac{\Omega_1}{\Omega_2}\right)^{m+1-i}\\
&+\frac{\Omega_3}{\Omega_2}\sum_{i=1}^{m}\left(\frac{\Omega_1}{\Omega_2}\right)^{m-i}:=K,
\end{aligned}$$

for all $t\in J$. Taking supremum over t, we have $\|\mathcal{B}x\|\le K$ for all $x\in S$. This shows that $\mathcal{B}$ is uniformly bounded on S.

Further, we will show that $\mathcal{B}(S)$ is an equicontinuous set in $PC(J,\mathbb{R})$. Let $\tau_1,\tau_2\in J$ with $\tau_1<\tau_2$ and $x\in S$. Then we have

$$\begin{aligned}
&|\mathcal{B}x(\tau_2)-\mathcal{B}x(\tau_1)|\\
=\;&\Bigg|\frac{\mu}{f(0,\mu)}\prod_{i=1}^{k}\frac{f(t_i,x(t_i))}{f(t_i,x(t_i^+))}+\sum_{i=1}^{k}\prod_{i\le j\le k}{}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}}g(t_i,x(t_i))\frac{f(t_j,x(t_j))}{f(t_j,x(t_j^+))}\\
&+\sum_{i=1}^{k}\prod_{i<j\le k}\frac{\varphi_i(x(t_i))}{f(t_i,x(t_i^+))}\cdot\frac{f(t_j,x(t_j))}{f(t_j,x(t_j^+))}+{}_{t_k}I_{q_k}^{\alpha_k}g(\tau_2,x(\tau_2))\\
&-\frac{\mu}{f(0,\mu)}\prod_{i=1}^{n}\frac{f(t_i,x(t_i))}{f(t_i,x(t_i^+))}-\sum_{i=1}^{n}\prod_{i\le j\le n}{}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}}g(t_i,x(t_i))\frac{f(t_j,x(t_j))}{f(t_j,x(t_j^+))}\\
&-\sum_{i=1}^{n}\prod_{i<j\le n}\frac{\varphi_i(x(t_i))}{f(t_i,x(t_i^+))}\cdot\frac{f(t_j,x(t_j))}{f(t_j,x(t_j^+))}-{}_{t_n}I_{q_n}^{\alpha_n}g(\tau_1,x(\tau_1))\Bigg|,
\end{aligned}$$

for some $n \leq k$, $n, k \in \{0, 1, 2, \dots, m\}$. Further

$$\begin{aligned} |\mathcal{B}x(\tau_2) - \mathcal{B}x(\tau_1)| &= \left| {}_{t_k}I_{q_k}^{\alpha_k} g(\tau_2, x(\tau_2)) - {}_{t_k}I_{q_k}^{\alpha_k} g(\tau_1, x(\tau_1)) \right| \\ &\leq \|p\|\psi(r) \left| \frac{(\tau_2 - t_k)^{\alpha_k}}{\Gamma_{q_k}(\alpha_k + 1)} - \frac{(\tau_1 - t_k)^{\alpha_k}}{\Gamma_{q_k}(\alpha_k + 1)} \right| \to 0, \end{aligned}$$

independent of $x \in S$ as $\tau_1 \to \tau_2$. This shows that $\mathcal{B}(S)$ is an equicontinuous set in $PC(J, \mathbb{R})$. Therefore, it follows by the Arzelá-Ascoli theorem that $\mathcal{B}$ is a completely continuous operator on S.

Step 3. *The hypothesis* (c) *of Lemma 2.9 is satisfied.*

Let $x \in PC(J, \mathbb{R})$ and $y \in S$ be arbitrary elements such that $x = \mathcal{A}x\mathcal{B}y$. Then we have

$$\begin{aligned} & |x(t)| \\ \leq\ & |\mathcal{A}x(t)||\mathcal{B}y(t)| \\ \leq\ & |f(t, x(t))| \Bigg(\frac{|\mu|}{|f(0, \mu)|} \prod_{i=1}^{k} \frac{|f(t_i, y(t_i))|}{|f(t_i, y(t_i^+))|} + \sum_{i=1}^{k} \prod_{i \leq j \leq k} {}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}} |g(t_i, y(t_i))| \frac{|f(t_j, y(t_j))|}{|f(t_j, y(t_j^+))|} \\ & + \sum_{i=1}^{k} \prod_{i < j \leq k} \frac{|\varphi_i(y(t_i))|}{|f(t_i, y(t_i^+))|} \cdot \frac{|f(t_j, y(t_j))|}{|f(t_j, y(t_j^+))|} + {}_{t_k}I_{q_k}^{\alpha_k} |g(t, y(t))| \Bigg) \\ \leq\ & \Omega_1 \Bigg(\frac{|\mu|}{|f(0, \mu|)} \prod_{i=1}^{m} \frac{|f(t_i, y(t_i))|}{|f(t_i, y(t_i^+))|} + \sum_{i=1}^{m} \prod_{i \leq j \leq m} {}_{t_{i-1}}I_{q_{i-1}}^{\alpha_{i-1}} |g(t_i, y(t_i))| \frac{|f(t_j, y(t_j))|}{|f(t_j, y(t_j^+))|} \\ & + \sum_{i=1}^{m} \prod_{i < j \leq m} \frac{|\varphi_i(y(t_i))|}{|f(t_i, y(t_i^+))|} \cdot \frac{|f(t_j, y(t_j))|}{|f(t_j, y(t_j^+))|} + {}_{t_m}I_{q_m}^{\alpha_m} |g(T, y(T))| \Bigg) \\ \leq\ & \Omega_1 \Bigg(\frac{|\mu|}{|f(0, \mu)|} \left(\frac{\Omega_1}{\Omega_2} \right)^m + \|p\|\psi(r) \sum_{i=1}^{m+1} \frac{(t_i - t_{i-1})^{\alpha_{i-1}}}{\Gamma_{q_{i-1}}(\alpha_{i-1} + 1)} \left(\frac{\Omega_1}{\Omega_2} \right)^{m+1-i} \\ & + \frac{\Omega_3}{\Omega_2} \sum_{i=1}^{m} \left(\frac{\Omega_1}{\Omega_2} \right)^{m-i} \Bigg). \end{aligned}$$

Taking supremum over t, we have

$$\begin{aligned} \|x\| \leq\ & \Omega_1 \Bigg(\frac{|\mu|}{|f(0, \mu)|} \left(\frac{\Omega_1}{\Omega_2} \right)^m + \|p\|\psi(r) \sum_{i=1}^{m+1} \frac{(t_i - t_{i-1})^{\alpha_{i-1}}}{\Gamma_{q_{i-1}}(\alpha_{i-1} + 1)} \left(\frac{\Omega_1}{\Omega_2} \right)^{m+1-i} \\ & + \frac{\Omega_3}{\Omega_2} \sum_{i=1}^{m} \left(\frac{\Omega_1}{\Omega_2} \right)^{m-i} \Bigg) \leq r. \end{aligned}$$

Thus we deduce that $x \in S$.

Step 4. *We show that the condition* (d) *of Lemma 2.9 holds.*

As

$$\begin{aligned} M &= \|\mathcal{B}(S)\| \\ &\leq \left(\frac{|\mu|}{|f(0, \mu)|} \left(\frac{\Omega_1}{\Omega_2} \right)^m + \|p\|\psi(r) \sum_{i=1}^{m+1} \frac{(t_i - t_{i-1})^{\alpha_{i-1}}}{\Gamma_{q_{i-1}}(\alpha_{i-1} + 1)} \left(\frac{\Omega_1}{\Omega_2} \right)^{m+1-i} + \frac{\Omega_3}{\Omega_2} \sum_{i=1}^{m} \left(\frac{\Omega_1}{\Omega_2} \right)^{m-i} \right), \end{aligned}$$

therefore, by (H_4), we have $\delta M < 1$ with $\delta = \|\phi\|$.

Thus all the conditions of Lemma 2.9 are satisfied and hence the operator equation $x = \mathcal{A}x\mathcal{B}x$ has a solution in S. In consequence, we infer that the problem (1) has a solution on J. This completes the proof. □

Example 3.2 *Consider the following impulsive hybrid fractional quantum difference equation with initial condition*

$$\begin{cases} {}^{c}_{t_k}D^{\frac{k^2+2k+1}{k^2+2k+3}}_{\frac{k^2+3k+1}{2k^2+3k+2}}\left[\dfrac{x(t)}{\frac{|x(t)|+30}{|x(t)|+35}+\frac{1}{25}\left(t-\frac{1}{2}\right)^2}\right] = \dfrac{1+\sin^2 t}{2(t+5)}\left(\dfrac{x^2(t)}{4(1+|x(t)|)}+\dfrac{e^{-|x(t)|}}{2}\right), \\ \qquad\qquad\qquad\qquad\qquad\qquad t \in [0,3/2]\setminus\{t_1,\ldots,t_5\}, \\ \Delta x(t_k) = \dfrac{|x(t_k)|+1}{(k+1)(|x(t_k)|+2)}, \qquad t_k = \dfrac{k}{4},\ k=1,\ldots,5, \\ x(0) = \dfrac{1}{3}. \end{cases} \tag{16}$$

Here $\alpha_k = (k^2+2k+1)/(k^2+2k+3)$, $q_k = (k^2+3k+1)/(2k^2+3k+2)$, $k = 0,1,\ldots,5$, $t_k = k/4$, $k = 1,2,\ldots,5$, $m = 5$, $T = 3/2$, $\mu = 1/3$, $f(t,x) = ((|x|+30)/(|x|+35)) + (1/25)(t-(1/2))^2$ and $g(t,x) = ((1+\sin^2 t)/(2(t+5)))((x^2/(4(1+|x|))) + (e^{-|x|}/2))$. With the given values, we find that $\Omega_1 = 26/25$, $\Omega_2 = 6/7$. Also, we have

$$|f(t,x)-f(t,y)| \le \frac{1}{245}|x-y|,\ |g(t,x)| \le \frac{1}{t+5}\left(\frac{|x|}{4}+\frac{1}{2}\right),\ |\varphi_k(x)| \le \frac{1}{(k+1)},\ k=1,2,\ldots,5.$$

Clearly $\|\phi\| = 1/245$, $\Omega_3 = 1/2$, $\|p\| = 1/5$ and $\psi(|x|) = (|x|/4)+(1/2)$. Hence, there exists a constant r such that $6.611569689 < r < 1092.541483$ satisfying (H_4). Thus all the conditions of Theorem 3.1 are satisfied. Therefore, the conclusion of Theorem 3.1 implies that the problem (16) has at least one solution on $[0,3/2]$.

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A FIXED POINT ALTERNATIVE TO THE STABILITY OF A QUADRATIC α-FUNCTIONAL EQUATION IN FUZZY BANACH SPACES

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ABSTRACT. In this paper, we solve the following quadratic α-functional equation

$$N\left(2f(x)+2f(y)-f(x+y)-\alpha^{-2}f(\alpha(x-y)),t\right) \geq \frac{t}{t+\varphi(x,y)} \tag{0.1}$$

in fuzzy normed spaces, where ρ is a fixed real number with $\alpha^{-1} \neq \pm\sqrt{3}$.

Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic α-functional equation (0.1) in fuzzy Banach spaces.

1. Introduction and preliminaries

Katsaras [22] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [13, 26, 51]. In particular, Bag and Samanta [2], following Cheng and Mordeson [9], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [25]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [3].

We use the definition of fuzzy normed spaces given in [2, 30, 31] to investigate the Hyers-Ulam stability of additive ρ-functional inequalities in fuzzy Banach spaces.

Definition 1.1. [2, 30, 31, 32] Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N_1) $N(x,t) = 0$ for $t \leq 0$;
(N_2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
(N_3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N_4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;
(N_5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.
(N_6) for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a *fuzzy normed vector space*.

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [29, 30].

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Definition 1.2. [2, 30, 31, 32] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by N-$\lim_{n\to\infty} x_n = x$.

Definition 1.3. [2, 30, 31, 32] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [3]).

The stability problem of functional equations originated from a question of Ulam [50] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [18] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [42] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [14] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [8, 12, 16, 17, 19, 21, 23, 24, 27, 35, 36, 37, 38, 39, 40, 43, 44, 45, 46, 47, 48, 49]).

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.4. [5, 10] Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geq n_0$;
(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

In 1996, G. Isac and Th.M. Rassias [20] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [4, 6, 7, 11, 15, 29, 33, 34, 40, 41]).

In this paper, we solve the quadratic α-functional equation (0.1) and prove the Hyers-Ulam stability of the quadratic α-functional equation(0.1) in fuzzy Banach spaces by using the fixed point method.

Throughout this paper, assume that X is a real vector space and (Y, N) is a fuzzy Banach space. Assume that α is a real number with $\alpha^{-1} \neq \pm\sqrt{3}$.

2. Quadratic α-functional equation (0.1)

In this section, we prove the Hyers-Ulam stability of the quadratic α-functional equation (0.1) in fuzzy Banach spaces.

We need the following lemma to prove the main results.

We solve the quadratic α-functional equation (0.1) in vector spaces.

Lemma 2.1. *Let X and Y be vector spaces. If a mapping $f : X \to Y$ satisfies*

$$2f(x) + 2f(y) = f(x+y) + \alpha^{-2} f(\alpha(x-y)) \tag{2.1}$$

for all $x, y \in X$, then $f : X \to Y$ is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $3f(0) = \alpha^{-2} f(0)$. So $f(0) = 0$.

Letting $y = 0$ in (2.1), we get $f(x) = \alpha^{-2} f(\alpha x)$ and so $f(\alpha x) = \alpha^2 f(x)$ for all $x \in X$. Thus

$$2f(x) + 2f(y) = f(x+y) + \alpha^{-2} f(\alpha(x-y)) = f(x+y) + f(x-y)$$

for all $x, y \in X$, as desired. □

Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic α-functional equation (2.1) in fuzzy Banach spaces.

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{4} \varphi(2x, 2y)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$N\left(2f(x) + 2f(y) - f(x+y) - \alpha^{-2} f(\alpha(x-y)), t\right) \geq \frac{t}{t + \varphi(x, y)} \tag{2.2}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x) - Q(x), t\right) \geq \frac{(4-4L)t}{(4-4L)t + L\varphi(x,x)} \tag{2.3}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $y = x$ in (2.2), we get

$$N\left(f\left(2x\right) - 4f(x), t\right) \geq \frac{t}{t + \varphi(x,x)} \tag{2.4}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t+\varphi(x,x)},\ \forall x \in X, \forall t > 0\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [28, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t+\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned}
N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right), L\varepsilon t\right) \\
&= N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{4}\varepsilon t\right) \\
&\geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)} \geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \frac{L}{2}\varphi(x,x)} \\
&= \frac{t}{t+\varphi(x,x)}
\end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.4) that

$$N\left(f(x) - 4f\left(\frac{x}{2}\right), \frac{L}{4}t\right) \geq \frac{t}{t+\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{4}$.

By Theorem 1.4, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(\frac{x}{2}\right) = \frac{1}{4}Q(x) \tag{2.5}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (2.5) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - Q(x), \mu t) \geq \frac{t}{t + \varphi(x, x)}$$

for all $x \in X$;

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f, Q) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, Q) \leq \frac{L}{4 - 4L}.$$

This implies that the inequality (2.3) holds.

By (2.2),

$$N\left(4^n\left(2f\left(\frac{x}{2^n}\right) + 2f\left(\frac{y}{2^n}\right) - f\left(\frac{x+y}{2^n}\right) - \alpha^{-2} f\left(\alpha\frac{x-y}{2^n}\right)\right), 4^n t\right) \geq \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$N\left(4^n\left(2f\left(\frac{x}{2^n}\right) + 2f\left(\frac{y}{2^n}\right) - f\left(\frac{x+y}{2^n}\right) - \alpha^{-2} f\left(\alpha\frac{x-y}{2^n}\right)\right), t\right) \geq \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n}\varphi(x, y)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$2Q(x) + 2Q(y) - Q(x + y) - \alpha^{-2} Q(\alpha(x - y)) = 0$$

for all $x, y \in X$. By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic, as desired. $\square$

Corollary 2.3. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$N\left(2f(x) + 2f(y) - f(x + y) - \alpha^{-2} f(\alpha(x - y)), t\right) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{2.6}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{(2^p - 4)t}{(2^p - 4)t + 2\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{2-p}$, and we get the desired result. $\square$

Theorem 2.4. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.2). Then $Q(x) := N$-$\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \ge \frac{(4-4L)t}{(4-4L)t + \varphi(x,x)} \tag{2.7}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S,d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4} g(2x)$$

for all $x \in X$.

It follows from (2.4) that

$$N\left(f(x) - \frac{1}{4}f(2x), \frac{1}{4}t\right) \ge \frac{t}{t + \varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \le \frac{1}{4}$. Hence

$$d(f, A) \le \frac{1}{4-4L},$$

which implies that the inequality (2.7) holds.

The rest of the proof is similar to the proof of Theorem 2.2. $\square$

Corollary 2.5. *Let $\theta \ge 0$ and let p be a real number with $0 < p < 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.6). Then $Q(x) := N$-$\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N(f(x) - Q(x), t) \ge \frac{(4-2^p)t}{(4-2^p)t + 2\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{p-2}$, and we get the desired result. $\square$

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Four-point impulsive multi-orders fractional boundary value problems

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Abstract

Four-point boundary value problem for impulsive multi-orders fractional differential equation is studied. The existence and uniqueness results are obtained for impulsive multi-orders fractional differential equation with four-point fractional boundary conditions by applying standard fixed point theorems. An example for the illustration of the main result is presented.

Keywords: fractional differential equations, fixed point theorems, multi-orders, impulse.

1 Introduction

Impulsive differential equations have extensively been studied in the past two decades. Impulsive differential equations are used to describe the dynamics of processes in which sudden, discontinuous jumps occur. Such processes are naturally seen in harvesting, earthquakes, diseases, and so forth. Recently, fractional impulsive differential equations have attracted the attention of many researchers. For the general theory and applications of such equationswe refer the interested reader to see [1]-[18] and the references therein.

In [8], Kosmatov considered the following two impulsive problems:

$$\begin{cases} {}^{C}D^{\alpha}u(t) = f(t, u(t)), \ 1 < \alpha < 2, \ t \in [0,1] \setminus \{t_1, t_2, ..., t_p\}, \\ {}^{C}D^{\gamma}u\left(t_k^{+}\right) -{}^{C} D^{\gamma}u\left(t_k^{-}\right) = I_k\left(u\left(t_k^{-}\right)\right), \quad t_k \in (0,1), \ k = 1, ..., p, \\ u(0) = u_0, \ u'(0) = u_0, \quad 0 < \gamma < 1, \end{cases}$$

and

$$\begin{cases} {}^{L}D^{\alpha}u(t) = f(t, u(t)), \ 0 < \alpha < 1, \ t \in [0,1] \setminus \{t_1, t_2, ..., t_p\}, \\ {}^{L}D^{\gamma}u\left(t_k^{+}\right) - {}^{L}D^{\gamma}u\left(t_k^{-}\right) = I_k\left(u\left(t_k^{-}\right)\right), \quad t_k \in (0,1), \ k = 1, ..., p, \\ I^{1-\alpha}u(0) = u_0, \quad 0 < \gamma < \alpha < 1. \end{cases}$$

In [4], Feckan et al. studied the impulsive problem of the following form:

$$\begin{cases} {}^{C}D^{\alpha}u(t) = f(t, u(t)), \ 0 < \alpha < 1, \ t \in [0,1] \setminus \{t_1, t_2, ..., t_p\}, \\ u\left(t_k^{+}\right) - u\left(t_k^{-}\right) = I_k\left(u\left(t_k^{-}\right)\right), \quad t_k \in (0,1), \ k = 1, ..., p, \\ u(0) = u_0, \ 0 < \gamma < \alpha < 1. \end{cases}$$

Wang et al. [17] obtained some existence and uniqueness results for the following impulsive multipoint fractional integral boundary value problem involving multi-orders fractional derivatives and deviating argument

$$\begin{cases} {}^{C}D^{\alpha_k}_{t_k}u(t) = f(t, u(t), u(\theta(t))), \ 1 < \alpha_k \leq 2, \ t \in [0,T] \setminus \{t_1, t_2, ..., t_p\}, \\ \Delta u(t_k) = I_k\left(u\left(t_k^{-}\right)\right), \quad \Delta u'(t_k) = J_k\left(u\left(t_k^{-}\right)\right), \quad t_k \in (0,T), \ k = 1, ..., p, \\ u(0) = \sum_{k=0}^{p} \lambda_k I^{\beta_k}_{t_k} u(\eta_k), \quad t_k < \eta_k < t_{k+1}, \\ u'(0) = 0. \end{cases}$$

Yukunthorn et.al. [18] studied the similar problem for multi-order Caputo–Hadamard fractional differential equations equipped with nonlinear integral boundary conditions.

Motivated by the above works, in this paper, we study the existence of solutions for the four-point nonlocal boundary value problems of nonlinear impulsive equations of fractional order

$$\begin{cases} {}^{C}D_{t_k}^{\alpha_k}u(t)=f(t,u(t),u'(t)),\ 1+\beta\leq\alpha_k\leq 2,\ t\in[t_k,t_{k+1}), \\ \Delta u(t_k)=I_k\left(u\left(t_k^-\right)\right),\quad \Delta u'(t_k)=J_k\left(u'\left(t_k^-\right)\right),\quad t_k\in(0,T),\ k=1,...,p, \\ u(0)+\mu_1\ {}^{C}D_{0+}^{\beta}u(0)=\sigma_1 u(\eta_1),\quad 0<\eta_1<t_1<T, \\ u(T)+\mu_2\ {}^{C}D_{t_p}^{\beta}u(T)=\sigma_2 u(\eta_2),\quad 0<t_p<\eta_2<T,\ 0<\beta<1, \end{cases} \tag{1}$$

where ${}^{C}D_t^{\alpha}$ is the Caputo derivative, $f:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function, $I_k,J_k:\mathbb{R}\to\mathbb{R}$, $\Delta u(t_k)=u\left(t_k^+\right)-u\left(t_k^-\right)$, $\Delta u'(t_k)=u'\left(t_k^+\right)-u'\left(t_k^-\right)$, $u\left(t_k^+\right)$ and $u\left(t_k^-\right)$ represent the right hand limit and the left hand limit of the function $u(t)$ at $t=t_k$; and the sequence $\{t_k\}$ satisfies that $0=t_0<t_1<...<t_p<t_{p+1}=T$.

To the best of our knowledge, there is no paper that consider the four-point impulsive boundary value problem involving nonlinear differential equations of fractional order (1). The main difficulty of this problem is that the corresponding integral equation is very complex because of the impulse effects. In this paper, we study the existence and uniqueness of solutions for four-point impulsive boundary value problem (1). By use of Banach's fixed point theorem and Schauder's fixed point theorem, some existence and uniqueness results are obtained.

2 Preliminaries

Let $[0,T]^-=[0,T]\setminus\{t_1,t_2,...,t_p\}$ and

$$\begin{aligned} PC([0,T],\mathbb{R})=\{x:[0,T]\to\mathbb{R}:x(t)\ &\text{is continuous everywhere except for some} \\ &t_k \text{ at which } x\left(t_k^+\right),x\left(t_k^-\right) \text{ exist and } x\left(t_k^-\right)=x(t_k),\ k=1,...,p\}, \end{aligned}$$

and

$$\begin{aligned} PC^1([0,T],\mathbb{R})=\{x\in PC([0,T],\mathbb{R}):x'(t)\ &\text{is continuous everywhere except for some} \\ &t_k \text{ at which } x'\left(t_k^+\right),x'\left(t_k^-\right) \text{ exist and } x'\left(t_k^-\right)=x'(t_k),\ k=1,...,p\}. \end{aligned}$$

$PC([0,T],\mathbb{R})$ and $PC^1([0,T],\mathbb{R})$ are Banach spaces with the norms $\|x\|_{PC}=\sup\{|x(t)|:t\in[0,T]\}$ and $\|x\|_{PC^1}=\max\{\|x\|_{PC},\|x'\|_{PC}\}$, respectively. Let $X=PC^1([0,T],\mathbb{R})\cap C^2\left([0,T]^-,\mathbb{R}\right)$. A function $x\in X$ is called a solution of problem (1) if it satisfies (1).

Throughout the paper we will use the following notations.

$$\begin{aligned} \rho&=\sigma_1\eta_1+\left(T+\mu_2\frac{T^{1-\beta}}{\Gamma(2-\beta)}\right)(1-\sigma_1), \\ A_0&=\frac{\sigma_1}{1-\sigma_1}-\frac{\sigma_1}{\rho}\frac{\sigma_1\eta_1}{1-\sigma_1},\quad B_0=\frac{\sigma_1}{\rho}, \\ A_p&=\frac{(1-\sigma_1)}{\rho}\frac{\sigma_1\eta_1}{1-\sigma_1},\quad B_p=\frac{1-\sigma_1}{\rho}. \end{aligned}$$

$$
\begin{aligned}
F_k\left(y,u,u'\right)(t) &= \frac{1}{\Gamma(\alpha_k)}\int_{t_k}^{t}(t-s)^{\alpha_k-1}y(s)\,ds \\
&+\sum_{j=1}^{k}\frac{1}{\Gamma(\alpha_{j-1})}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha_{j-1}-1}y(s)\,ds+\sum_{j=1}^{k}I_j\left(u\left(t_j^-\right)\right) \\
&+\sum_{j=1}^{k}(t-t_j)\frac{1}{\Gamma(\alpha_{j-1}-1)}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha_{j-1}-2}y(s)\,ds+\sum_{j=1}^{k}(t-t_j)J_j\left(u'\left(t_j^-\right)\right), \\
G_k\left(y,u,u'\right)(t) &= \frac{1}{\Gamma(\alpha_k-\beta)}\int_{t_k}^{t}(t-s)^{\alpha_k-\beta-1}y(s)\,ds \\
&+\frac{t^{1-\beta}}{\Gamma(2-\beta)}\sum_{j=1}^{k}\frac{1}{\Gamma(\alpha_{j-1}-1)}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha_{j-1}-2}y(s)\,ds \\
&+\frac{t^{1-\beta}}{\Gamma(2-\beta)}\sum_{j=1}^{k}J_j\left(u'\left(t_j^-\right)\right).
\end{aligned}
$$

$$
\begin{aligned}
F_k'\left(y,u,u'\right)(t) &= \frac{1}{\Gamma(\alpha_k-1)}\int_{t_k}^{t}(t-s)^{\alpha_k-2}y(s)\,ds \\
&+\sum_{j=1}^{k}\frac{1}{\Gamma(\alpha_{j-1}-1)}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha_{j-1}-2}y(s)\,ds+\sum_{j=1}^{k}J_j\left(u'\left(t_j^-\right)\right).
\end{aligned}
$$

Lemma 1 *Let $y\in C[0,T]$. A function $u\in PC^1[0,T]$ is a solution of the boundary value problem*

$$
\begin{cases}
{}^{C}D_{t_k}^{\alpha_k}u(t)=y(t),\ 1+\beta<\alpha_k\leq 2,\ t\in[0,T]\setminus\{t_1,t_2,...,t_p\}, \\
\Delta u(t_k)=I_k\left(u\left(t_k^-\right)\right),\quad \Delta u'(t_k)=J_k\left(u'\left(t_k^-\right)\right),\quad t_k\in(0,T),\ k=1,...,p, \\
u(0)+\mu_1\ {}^{C}D_{0+}^{\beta}u(0)=\sigma_1 u(\eta_1),\quad 0<\eta_1<t_1<T, \\
u(T)+\mu_2\ {}^{C}D_{t_p}^{\beta}u(T)=\sigma_2 u(\eta_2)=\sigma_2 u(\eta_2),\quad 0<t_p<\eta_2<T,\ 0<\beta<1,
\end{cases}
\tag{2}
$$

if and only if

$$
\begin{aligned}
u(t) &= F_k(y,u)(t)+\frac{\sigma_1}{1-\sigma_1}F_0(y,u)(\eta_1) \\
&-\frac{\sigma_1}{\rho}\left(\frac{\sigma_1\eta_1}{1-\sigma_1}+t\right)F_0(y,u)(\eta_1) \\
&+\frac{\sigma_2(1-\sigma_1)}{\rho}\left(\frac{\sigma_1\eta_1}{1-\sigma_1}+t\right)F_p(y,u)(\eta_2) \\
&-\frac{(1-\sigma_1)}{\rho}\left(\frac{\sigma_1\eta_1}{1-\sigma_1}+t\right)F_p(y,u)(T) \\
&-\frac{\mu_2(1-\sigma_1)}{\rho}\left(\frac{\sigma_1\eta_1}{1-\sigma_1}+t\right)G_p(y,u)(T).
\end{aligned}
\tag{3}
$$

Proof. Suppose that u is a solution of (2). For $0\leq t\leq t_1$, we have

$$
u(t)=I_{0+}^{\alpha_0}y(t)-c_1-c_2t=\frac{1}{\Gamma(\alpha_0)}\int_0^t(t-s)^{\alpha_0-1}y(s)\,ds-c_1-c_2t,\quad c_1,c_2\in\mathbb{R}. \tag{4}
$$

Then differentiating (4), we get

$$
\begin{aligned}
D_{0+}^{\beta}u(t) &= \frac{1}{\Gamma(\alpha_0-\beta)}\int_0^t(t-s)^{\alpha_0-\beta-1}y(s)\,ds-c_2\frac{t^{1-\beta}}{\Gamma(2-\beta)}, \\
u'(t) &= \frac{1}{\Gamma(\alpha_0-1)}\int_0^t(t-s)^{\alpha_0-2}y(s)\,ds-c_2.
\end{aligned}
$$

If $t_1 < t \leq t_2$, then for some $d_1, d_2 \in \mathbb{R}$, we have

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha_1)} \int_{t_1}^{t} (t-s)^{\alpha_1 - 1} y(s)\, ds - d_1 - d_2 (t - t_1), \\ u'(t) &= \frac{1}{\Gamma(\alpha_1 - 1)} \int_{t_1}^{t} (t-s)^{\alpha_1 - 2} y(s)\, ds - d_2, \\ D_{t_1^+}^{\beta} u(t) &= \frac{1}{\Gamma(\alpha_1 - \beta)} \int_{t_1}^{t} (t-s)^{\alpha_1 - \beta - 1} y(s)\, ds - d_2 \frac{t^{1-\beta}}{\Gamma(2-\beta)}. \end{aligned}$$

Thus

$$\begin{aligned} u\left(t_1^-\right) &= \frac{1}{\Gamma(\alpha_0)} \int_0^{t_1} (t_1 - s)^{\alpha_0 - 1} y(s)\, ds - c_1 - c_2 t_1, \quad u\left(t_1^+\right) = -d_1 \\ u'\left(t_1^-\right) &= \frac{1}{\Gamma(\alpha_0 - 1)} \int_0^{t_1} (t_1 - s)^{\alpha_0 - 2} y(s)\, ds - c_2, \quad u'\left(t_1^+\right) = -d_2. \end{aligned}$$

In view of

$$u\left(t_1^+\right) - u\left(t_1^-\right) = I_1\left(u\left(t_1^-\right)\right), \quad u'\left(t_1^+\right) - u'\left(t_1^-\right) = J_1\left(u'\left(t_1^-\right)\right),$$

we find that

$$\begin{aligned} -d_1 &= \frac{1}{\Gamma(\alpha_0)} \int_0^{t_1} (t_1 - s)^{\alpha_0 - 1} y(s)\, ds + I_1\left(u\left(t_1^-\right)\right) - c_1 - c_2 t_1, \\ -d_2 &= \frac{1}{\Gamma(\alpha_0 - 1)} \int_0^{t_1} (t_1 - s)^{\alpha_0 - 2} y(s)\, ds + J_1\left(u'\left(t_1^-\right)\right) - c_2. \end{aligned}$$

Hence we obtain for $t_1 < t \leq t_2$

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha_1)} \int_{t_1}^{t} (t-s)^{\alpha_1 - 1} y(s)\, ds \\ &+ \frac{1}{\Gamma(\alpha_0)} \int_0^{t_1} (t_1 - s)^{\alpha_0 - 1} y(s)\, ds + I_1\left(u\left(t_1^-\right)\right) \\ &+ (t - t_1) \frac{1}{\Gamma(\alpha_0 - 1)} \int_0^{t_1} (t_1 - s)^{\alpha_0 - 2} y(s)\, ds + (t - t_1) J_1\left(u'\left(t_1^-\right)\right) \\ &- c_1 - c_2 t, \qquad t_1 < t \leq t_2. \end{aligned}$$

In a similar way, for $k = 1, 2, ..., p$ we can obtain

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha_k)} \int_{t_k}^{t} (t-s)^{\alpha_k - 1} y(s)\, ds + \sum_{j=1}^{k} \frac{1}{\Gamma(\alpha_{j-1})} \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha_{j-1} - 1} y(s)\, ds + \sum_{j=1}^{k} I_j\left(u\left(t_j^-\right)\right) \\ &+ \sum_{j=1}^{k} (t - t_j) \frac{1}{\Gamma(\alpha_{j-1} - 1)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha_{j-1} - 2} y(s)\, ds + \sum_{j=1}^{k} (t - t_j) J_j\left(u'\left(t_j^-\right)\right) \\ &- c_1 - c_2 t, \qquad t_k < t \leq t_{k+1}. \end{aligned} \tag{5}$$

Moreover,

$$\begin{aligned} {}^C D_{t_k}^{\beta} u(t) &= \frac{1}{\Gamma(\alpha_k - \beta)} \int_{t_k}^{t} (t-s)^{\alpha_k - \beta - 1} y(s)\, ds \\ &\quad + \frac{t^{1-\beta}}{\Gamma(2-\beta)} \sum_{j=1}^{k} \frac{1}{\Gamma(\alpha_{j-1} - 1)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha_{j-1} - 2} y(s)\, ds \\ &\quad + \frac{t^{1-\beta}}{\Gamma(2-\beta)} \sum_{j=1}^{k} J_j\left(u'\left(t_j^-\right)\right) - c_2 \frac{t^{1-\beta}}{\Gamma(2-\beta)}. \end{aligned}$$

Now applying the boundary conditions

$$u(0)+\mu_1\ {}^{C}D_{0+}^{\beta}u(0)=\sigma_1 u(\eta_1),\quad 0<\eta_1<t_1<T,$$
$$u(T)+\mu_2\ {}^{C}D_{t_p}^{\beta}u(T)=\sigma_2 u(\eta_2),\quad 0<t_p<\eta_2<T,\ 0<\beta<1,$$

we get

$$-c_1=\sigma_1 F_0(y,u,u')(\eta_1)-\sigma_1 c_1-c_2\sigma_1\eta_1,$$
$$F_p(y,u,u')(T)-c_1-c_2T+\mu_2 G_p(y,u)(T)-\mu_2 c_2\frac{T^{1-\beta}}{\Gamma(2-\beta)}=\sigma_2 F_p(y,u,u')(\eta_2).$$

Solving this system for c_1, c_2 and inserting these values into (5) we get

$$\begin{aligned}
u(t) &= F_k(y,u,u')(t)+\frac{\sigma_1}{1-\sigma_1}F_0(y,u,u')(\eta_1)-c_2\left(\frac{\sigma_1\eta_1}{1-\sigma_1}+t\right)\\
&= F_k(y,u,u')(t)+\frac{\sigma_1}{1-\sigma_1}F_0(y,u,u')(\eta_1)-\frac{\sigma_1}{\rho}\left(\frac{\sigma_1\eta_1}{1-\sigma_1}+t\right)F_0(y,u,u')(\eta_1)\\
&+\frac{\sigma_2(1-\sigma_1)}{\rho}\left(\frac{\sigma_1\eta_1}{1-\sigma_1}+t\right)F_p(y,u,u')(\eta_2)\\
&-\frac{(1-\sigma_1)}{\rho}\left(\frac{\sigma_1\eta_1}{1-\sigma_1}+t\right)F_p(y,u,u')(T)\\
&-\frac{\mu_2(1-\sigma_1)}{\rho}\left(\frac{\sigma_1\eta_1}{1-\sigma_1}+t\right)G_p(y,u,u')(T).
\end{aligned}$$

Conversely, assume that u is a solution of the impulsive fractional integral equation (3). Then by a direct computation, it follows that the solution given by (3) satisfies (2). This completes the proof. ■

3 Existence and Uniqueness

In the sequel,we assume that

(A$_1$) $f:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous function and such that

$$|f(t,x,x_1)-f(t,y,y_1)|\le l_f(|x-y|+|x_1-y_1|),\quad l_f>0,\ 0\le t\le T,\quad x,y,x_1,y_1\in\mathbb{R}.$$

(A$_2$) $I_k,J_k:\mathbb{R}\to\mathbb{R}$ are continuous functions and satisfy

$$|I_k(x)-I_k(y)|\le l_1|x-y|,$$
$$|J_k(x)-J_k(y)|\le l_2|x-y|,\quad l_1>0,\ l_2>0,\ 0\le t\le T,\quad x,y\in\mathbb{R}.$$

For convenience, we will give some notations:

$$T^*=\max\{T^{\alpha_k}:0\le k\le p\},\qquad \Gamma^*=\min\{\Gamma(\alpha_k):0\le k\le p\},$$
$$\Delta_1=\sum_{j=1}^{p}\frac{(t_j-t_{j-1})^{\alpha_{j-1}}}{\Gamma(\alpha_{j-1}+1)},\quad \Delta_2=\sum_{j=1}^{p}\frac{(T-t_j)(t_j-t_{j-1})^{\alpha_{j-1}-1}}{\Gamma(\alpha_{j-1})},$$
$$\Delta_3=\frac{T^{1-\beta}}{\Gamma(2-\beta)}\sum_{j=1}^{p}\frac{(t_j-t_{j-1})^{\alpha_{j-1}-1}}{\Gamma(\alpha_{j-1})},\quad \Delta_4=\sum_{j=1}^{p}\frac{(t_j-t_{j-1})^{\alpha_{j-1}-1}}{\Gamma(\alpha_{j-1})}.$$

$$\begin{aligned}
\Lambda_F &:= l_f \frac{T^*}{\Gamma^*} + l_f \Delta_1 + l_f \Delta_2 + p l_1 + l_2 p T, \\
\Lambda_G &:= l_f \frac{T^*}{\Gamma^*} + l_f \Delta_1 \frac{T^{1-\beta}}{\Gamma(2-\beta)} + l_2 p \frac{T^{1-\beta}}{\Gamma(2-\beta)}, \\
\Lambda_{F'} &:= l_f \frac{T^*}{\Gamma^*} + l_f \Delta_4 + l_2 p.
\end{aligned}$$

Lemma 2 *$F_k(f,u,u')$ and $G_k(f,u,u')$ are Lipschitzian operators.*

$$\begin{aligned}
&|F_k(f,u,u') - F_k(f,v,v')| \le \Lambda_F \|u-v\|_{PC^1}, \quad L_{F_k} > 0, \\
&|G_k(f,u,u') - G_k(f,v,v')| \le \Lambda_G \|u-v\|_{PC^1}, \quad L_{G_k} > 0, \quad u, v \in PC^1([0,T], \mathbb{R}).
\end{aligned}$$

Proof. For $u, v \in PC^1([0,T], \mathbb{R})$, we have

$$\begin{aligned}
&|F_k(f,u,u')(t) - F_k(f,v,v')(t)| \\
&\le \frac{1}{\Gamma(\alpha_k)} \int_{t_k}^{t} (t-s)^{\alpha_k - 1} |f(s,u(s),u'(s)) - f(s,v(s),v'(s))| \, ds \\
&+ \sum_{j=1}^{k} \frac{1}{\Gamma(\alpha_{j-1})} \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha_{j-1}-1} |f(s,u(s),u'(s)) - f(s,v(s),v'(s))| \, ds \\
&+ \sum_{j=1}^{k} \left| I_j\left(u\left(t_j^-\right)\right) - I_j\left(v\left(t_j^-\right)\right) \right| \\
&+ \sum_{j=1}^{k} \frac{(t-t_j)}{\Gamma(\alpha_{j-1}-1)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha_{j-1}-2} |f(s,u(s),u'(s)) - f(s,v(s),v'(s))| \, ds \\
&+ \sum_{j=1}^{k} (t-t_j) \left| J_j\left(u'\left(t_j^-\right)\right) - J_j\left(v'\left(t_j^-\right)\right) \right|
\end{aligned}$$

$$\begin{aligned}
&\le l_f \frac{1}{\Gamma(\alpha_k)} \int_{t_k}^{t} (t-s)^{\alpha_k - 1} (|u(s) - v(s)| + |u'(s) - v'(s)|) \, ds \\
&+ l_f \sum_{j=1}^{k} \frac{1}{\Gamma(\alpha_{j-1})} \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha_{j-1}-1} (|u(s) - v(s)| + |u'(s) - v'(s)|) \, ds \\
&+ l_1 \sum_{j=1}^{k} \left| u\left(t_j^-\right) - v\left(t_j^-\right) \right| + l_f \sum_{j=1}^{k} \frac{1}{\Gamma(\alpha_{j-1}-1)} (t-t_j)
\end{aligned}$$

$$\begin{aligned}
&\times \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha_{j-1}-2} (|u(s) - v(s)| + |u'(s) - v'(s)|) \, ds \\
&+ l_2 \sum_{j=1}^{k} (t-t_j) \left| u'\left(t_j^-\right) - v'\left(t_j^-\right) \right| \\
&\le \Lambda_F \|u-v\|_{PC^1}.
\end{aligned}$$

Similarly,

$$
\begin{aligned}
&\left|G_k\left(f,u,u'\right)(t)-G_k\left(f,v,v'\right)(t)\right| \\
&\leq \frac{1}{\Gamma\left(\alpha_k-\beta\right)}\int_{t_k}^{t}(t-s)^{\alpha_k-\beta-1}\left|f\left(s,u\left(s\right),u'\left(s\right)\right)-f\left(s,v\left(s\right),v'\left(s\right)\right)\right|ds \\
&+\frac{T^{1-\beta}}{\Gamma\left(2-\beta\right)}\sum_{j=1}^{k}\frac{1}{\Gamma\left(\alpha_{j-1}-1\right)}\int_{t_{j-1}}^{t_j}\left(t_j-s\right)^{\alpha_{j-1}-2}\left|f\left(s,u\left(s\right),u'\left(s\right)\right)-f\left(s,v\left(s\right),v'\left(s\right)\right)\right|ds \\
&+\frac{t^{1-\beta}}{\Gamma\left(2-\beta\right)}\sum_{j=1}^{k}\left|J_j\left(u'\left(t_j^-\right)\right)-J_j\left(v'\left(t_j^-\right)\right)\right| \\
&\leq\left(l_f\frac{\left(T-t_k\right)^{\alpha_k-\beta}}{\Gamma\left(\alpha_k-\beta+1\right)}+l_f\frac{T^{1-\beta}}{\Gamma\left(2-\beta\right)}\sum_{j=1}^{p}\frac{\left(t_j-t_{j-1}\right)^{\alpha_{j-1}-1}}{\Gamma\left(\alpha_{j-1}\right)}+\frac{T^{1-\beta}}{\Gamma\left(2-\beta\right)}l_2\right)\left\|u-v\right\|_{PC^1} \\
&\leq \Lambda_G\left\|u-v\right\|_{PC^1}.
\end{aligned}
$$

Also, we have

$$\left|F_k'\left(f,u,u'\right)(t)-F_k'\left(f,v,v'\right)(t)\right|\leq\Lambda_{F'}\left\|u-v\right\|_{PC^1}.$$

■

In view of Lemma 1 we define an operator $\Theta : X \to X$ by

$$
\begin{aligned}
(\Theta u)(t) &= F_k(f,u)(t)-\left(A_0-B_0t\right)F_0(f,u)\left(\eta_1\right) \\
&+\sigma_2\left(A_p+B_pt\right)F_p(f,u)\left(\eta_2\right)-\left(A_p+B_pt\right)F_p(f,u)(T) \\
&-\mu_2\left(A_p+B_pt\right)G_p(f,u)(T),
\end{aligned}
$$

where

$$
\begin{aligned}
A_0&=\frac{\sigma_1}{1-\sigma_1}-\frac{\sigma_1}{\rho}\frac{\sigma_1\eta_1}{1-\sigma_1}, \quad B_0=\frac{\sigma_1}{\rho}, \\
A_p&=\frac{\left(1-\sigma_1\right)}{\rho}\frac{\sigma_1\eta_1}{1-\sigma_1}, \quad B_p=\frac{1-\sigma_1}{\rho}.
\end{aligned}
$$

Let

$$\Lambda_\Theta := \max\left\{\Lambda_F,\Lambda_G,\Lambda_{F'}\right\}.$$

Theorem 3 *Suppose that the assumption (A_1), (A_2) are satisfied. If*

$$
\begin{aligned}
\Lambda &:= \Lambda_\Theta\max\left\{\left(1+\left|A_0\right|+\left|B_0\right|T+\left(\left|\sigma_2\right|+\left|\mu_2\right|+1\right)\left(\left|A_p\right|+\left|B_p\right|T\right)\right)\right. \\
&\left.,\left(1+\left|B_0\right|+\left(\left|\sigma_2\right|+\left|\mu_2\right|+1\right)\left|B_p\right|\right)\right\}<1,
\end{aligned}
$$

then the boundary value problem (1) has a unique solution.

Proof. Let $u,v\in PC^1\left([0,T],\mathbb{R}\right)$. For $u,v\in\left(t_k,t_{k+1}\right]$, $k=0,...,p$, we have

$$
\begin{aligned}
\left|(\Theta u)(t)-(\Theta v)(t)\right| &\leq \left|F_k\left(f,u,u'\right)(t)-F_k\left(f,v,v'\right)(t)\right| \\
&+\left|A_0-B_0t\right|\left|F_0\left(f,u,u'\right)\left(\eta_1\right)-F_0\left(f,v,v'\right)\left(\eta_1\right)\right| \\
&+\left|\sigma_2\right|\left|A_p+B_pt\right|\left|F_p\left(f,u,u'\right)\left(\eta_2\right)-F_p\left(f,v,v'\right)\left(\eta_2\right)\right| \\
&+\left|A_p+B_pt\right|\left|F_p\left(f,u,u'\right)(T)-F_p\left(f,v,v'\right)(T)\right| \\
&+\left|\mu_2\right|\left|A_p+B_pt\right|\left|G_p\left(f,u,u'\right)(T)-G_p\left(f,v,v'\right)(T)\right| \\
&\leq \Lambda_\Theta\left(1+\left|A_0\right|+\left|B_0\right|T+\left(\left|\sigma_2\right|+\left|\mu_2\right|+1\right)\left(\left|A_p\right|+\left|B_p\right|T\right)\right)\left\|u-v\right\|_{PC^1}.
\end{aligned}
$$

Similarly, for $u, v \in (t_k, t_{k+1}]$ we have

$$\begin{aligned}\left|(\Theta u)'(t) - (\Theta v)'(t)\right| &\leq \left|F_k'(f,u,u')(t) - F_k'(f,v,v')(t)\right| \\ &+ |B_0| \left|F_0(f,u,u')(\eta_1) - F_0(f,v,v')(\eta_1)\right| \\ &+ |\sigma_2| |B_p| \left|F_p(f,u,u')(\eta_2) - F_p(f,v,v')(\eta_2)\right| \\ &+ |B_p| \left|F_p(f,u,u')(T) - F_p(f,v,v')(T)\right| \\ &+ |\mu_2| |B_p| \left|G_p(f,u,u')(T) - G_p(f,v,v')(T)\right| \\ &\leq \Lambda_\Theta \left(1 + |B_0| + (|\sigma_2| + |\mu_2| + 1)|B_p|\right) \|u - v\|_{PC^1}.\end{aligned}$$

It follows that

$$\|\Theta u - \Theta v\|_{PC^1} \leq \Lambda \|u - v\|_{PC^1}.$$

Since $\Lambda < 1$, Θ is a contraction. According to the Banach fixed point theorem Θ has a unique fixed point, that is the problem (1) has a unique solution. ■

4 Existence

In this section, we assume that

(A$_3$) $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous function and there exists $h \in C([0,T], \mathbb{R}^+)$ such that

$$|f(t,u,v)| \leq h(t) + b_1 |u|^\rho + b_2 |v|^\varrho, \quad (t,u,v) \in [0,T] \times \mathbb{R} \times \mathbb{R}, \quad 0 < \rho, \varrho < 1.$$

(A$_4$) $I_k, J_k : \mathbb{R} \to \mathbb{R}$ are continuous functions and there $L_2 > 0$, $L_3 > 0$ such that

$$|I_k(x)| \leq L_2, \quad |J_k(x)| \leq L_3, \quad x \in \mathbb{R}.$$

For convenience, we will give some notations:

$$\begin{aligned} C_1 &:= (1 + |A_0| + |B_0| T + (|\sigma_2| + 1)(|A_p| + |B_p| T))(pL_2 + pTL_3)\|h\| \\ &\quad + |\mu_2| (|A_p| + |B_p| T) \frac{T^{1-\beta}}{\Gamma(2-\beta)} L_3 \|h\|, \\ C_2 &:= (1 + |A_0| + |B_0| T + (|\sigma_2| + 1)(|A_p| + |B_p| T)) \left(\frac{T^*}{\Gamma^*} + \Delta_1 + \Delta_2\right) \\ &\quad + |\mu_2| (|A_p| + |B_p| T) \left(\frac{T^*}{\Gamma^*} + \Delta_3\right). \end{aligned}$$

Lemma 4 *If*

$$R \geq \max\left\{3C_1, (3b_1 C_2)^{\frac{1}{1-\rho}}, (3b_1 C_2)^{\frac{1}{1-\varrho}}\right\},$$

then Θ maps $B(0,R) := \left\{u \in PC^1([0,T], \mathbb{R}) : \|u\|_{PC^1} \leq R\right\}$ into itself.

Proof. Assume that

$$R \geq \max\left\{3C_1, (3b_1 C_2)^{\frac{1}{1-\rho}}, (3b_1 C_2)^{\frac{1}{1-\varrho}}\right\}.$$

Then for $t \in (t_k, t_{k+1}]$, $k = 0, ..., p$, we have

$$\begin{aligned} &|F_k(f,u,u')(t)| \\ &\leq \frac{1}{\Gamma(\alpha_k)} \int_{t_k}^{t} (t-s)^{\alpha_k - 1} |f(s, u(s), u'(s))| \, ds \\ &+ \sum_{j=1}^{k} \frac{1}{\Gamma(\alpha_{j-1})} \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha_{j-1}-1} |f(s,u(s),u'(s))| \, ds + \sum_{j=1}^{k} \left|I_j\left(u\left(t_j^-\right)\right)\right| \\ &+ \sum_{j=1}^{k} \frac{(t-t_j)}{\Gamma(\alpha_{j-1}-1)} \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha_{j-1}-2} |f(s,u(s),u'(s))| \, ds + \sum_{j=1}^{k} (t-t_j) \left|J_j\left(v'\left(t_j^-\right)\right)\right|, \end{aligned}$$

$$
\begin{aligned}
&\left|F_k\left(f,u,u'\right)(t)\right| \\
&\leq \frac{1}{\Gamma\left(\alpha_k\right)}\int_{t_k}^{t}(t-s)^{\alpha_k-1}\left(h(s)+b_1\left|u(s)\right|^{\rho}+b_2\left|u'(s)\right|^{\varrho}\right)ds \\
&+\sum_{j=1}^{k}\frac{1}{\Gamma\left(\alpha_{j-1}\right)}\int_{t_{j-1}}^{t_j}\left(t_j-s\right)^{\alpha_{j-1}-1}\left(h(s)+b_1\left|u(s)\right|^{\rho}+b_2\left|u'(s)\right|^{\varrho}\right)ds \\
&+\sum_{j=1}^{k}\left|I_j\left(u\left(t_j^-\right)\right)\right| \\
&+\sum_{j=1}^{k}\frac{\left(t-t_j\right)}{\Gamma\left(\alpha_{j-1}-1\right)}\int_{t_{j-1}}^{t_j}\left(t_j-s\right)^{\alpha_{j-1}-2}\left(h(s)+b_1\left|u(s)\right|^{\rho}+b_2\left|v(s)\right|^{\varrho}\right)ds \\
&+\sum_{j=1}^{k}\left(t-t_j\right)\left|J_j\left(u\left(t_j^-\right)\right)\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{T^{\alpha_k}}{\Gamma\left(\alpha_k+1\right)}\left(\|h\|+b_1\|u\|^{\rho}+b_2\|u'\|^{\varrho}\right) \\
&+\sum_{j=1}^{p}\frac{\left(t_j-t_{j-1}\right)^{\alpha_{j-1}}}{\Gamma\left(\alpha_{j-1}+1\right)}\left(\|h\|+b_1\|u\|^{\rho}+b_2\|u'\|^{\varrho}\right)+pL_2 \\
&+\sum_{j=1}^{p}\frac{\left(t-t_j\right)\left(t_j-t_{j-1}\right)^{\alpha_{j-1}-1}}{\Gamma\left(\alpha_{j-1}\right)}\left(\|h\|+b_1\|u\|^{\rho}+b_2\|u'\|^{\varrho}\right)+pTL_3 \\
&\leq \left(\frac{T^*}{\Gamma^*}+\Delta_1+\Delta_2\right)\left(\|h\|+b_1\|u\|^{\rho}+b_2\|u'\|^{\varrho}\right)+pL_2+pTL_3,
\end{aligned}
$$

$$
\begin{aligned}
\left|G_k\left(y,u,u'\right)(t)\right| &\leq \frac{T^{\alpha_k-\beta}}{\Gamma\left(\alpha_k-\beta+1\right)}\left(\|h\|+b_1\|u\|^{\rho}+b_2\|u'\|^{\varrho}\right) \\
&+\frac{t^{1-\beta}}{\Gamma(2-\beta)}\sum_{j=1}^{p}\frac{\left(t_j-t_{j-1}\right)^{\alpha_{j-1}-1}}{\Gamma\left(\alpha_{j-1}\right)}\left(\|h\|+b_1\|u\|^{\rho}+b_2\|u'\|^{\varrho}\right)+\frac{t^{1-\beta}}{\Gamma(2-\beta)}L_3 \\
&\leq \left(\frac{T^*}{\Gamma^*}+\Delta_3\right)\left(\|h\|+b_1\|u\|^{\rho}+b_2\|u'\|^{\varrho}\right)+\frac{t^{1-\beta}}{\Gamma(2-\beta)}L_3,
\end{aligned}
$$

$$
\begin{aligned}
\left|F_k'\left(y,u,u'\right)(t)\right| \leq &\left(\frac{T^{\alpha_k-1}}{\Gamma\left(\alpha_k\right)}+\sum_{j=1}^{k}\frac{\left(t_j-t_{j-1}\right)^{\alpha_{j-1}-1}}{\Gamma\left(\alpha_{j-1}\right)}\right)\left(\|h\|+b_1\|u\|^{\rho}+b_2\|u'\|^{\varrho}\right) \\
&+L_3.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\left|(\Theta u)(t)\right| &\leq \left(1+\left|A_0\right|+\left|B_0\right|T+\left(\left|\sigma_2\right|+1\right)\left(\left|A_p\right|+\left|B_p\right|T\right)\right) \\
&\times\left(\left(\frac{T^*}{\Gamma^*}+\Delta_1+\Delta_2\right)\left(\|h\|+b_1\|u\|^{\rho}+b_2\|u'\|^{\varrho}\right)+pL_2+pTL_3\right) \\
&+\left|\mu_2\right|\left(\left|A_p\right|+\left|B_p\right|T\right)\left(\left(\frac{T^*}{\Gamma^*}+\Delta_3\right)\left(\|h\|+b_1\|u\|^{\rho}+b_2\|u'\|^{\varrho}\right)+\frac{T^{1-\beta}}{\Gamma(2-\beta)}L_3\right) \\
&\leq C_1+C_2b_1\|u\|^{\rho}+C_2b_2\|u'\|^{\varrho},
\end{aligned}
$$

and

$$\begin{aligned}\left|(\Theta u)'(t)\right| &\leq \left(\frac{T^{\alpha_k-1}}{\Gamma(\alpha_k)}+\sum_{j=1}^{k}\frac{(t_j-t_{j-1})^{\alpha_{j-1}-1}}{\Gamma(\alpha_{j-1})}\right)\left(\|h\|+b_1\|u\|^{\rho}+b_2\|u'\|^{\varrho}\right)+L_3\\
&+\left(|B_0|+|\sigma_2||B_p|+|B_p|\right)\left(\frac{T^*}{\Gamma^*}+\Delta_1+\Delta_2\right)\left(\|h\|+b_1\|u\|^{\rho}+b_2\|u'\|^{\varrho}\right)+pL_2+pTL_3\\
&+|\mu||B_p|\left(\frac{T^*}{\Gamma^*}+\Delta_3\right)\left(\|h\|+b_1\|u\|^{\rho}+b_2\|u'\|^{\varrho}\right)+\frac{T^{1-\beta}}{\Gamma(2-\beta)}L_3\\
&\leq C_1+C_2b_1\|u\|^{\rho}+C_2b_2\|u'\|^{\varrho}.\end{aligned}$$

Thus

$$\|(\Theta u)\|_{PC^1}\leq C_1+C_2b_1R^{\rho}+C_2b_2R^{\varrho}\leq\frac{R}{3}+\frac{R}{3}+\frac{R}{3}=R.$$

■

Theorem 5 *Assume that the conditions (A_3) and (A_4) are satisfied. Then the problem (1) has at least one solution.*

Proof. Firstly, we prove that Θ: $PC^1([0,T],R)\to PC^1([0,T],R)$ is completely continuous operator. It is clear that, the continuity of functions $f, I_k$and J_k implies the continuity of the operator Θ.

Let $\Omega\subset PC^1([0,T],R)$ be bounded. Then there exist positive constants such that

$$|f(t,u,u')|\leq L_1,\quad |I_k(u)|\leq L_2,\quad |J_k(u)|\leq L_3,$$

for all $u\in\Omega$. Thus, for any $u\in\Omega$,we have

$$\left|F_k(f,u,u^{'})\right|\leq L_1\left(\frac{T^*}{\Gamma^*}+\Delta_1+\Delta_2\right)+pL_2+L_3pT,$$

Similarly,

$$\left|G_k\left(f,u,u^{'}\right)(t)\right|\leq L_1\frac{T^*}{\Gamma^*}+L_1\Delta_1\frac{T^{1-\beta}}{\Gamma(2-\beta)}+\frac{T^{1-\beta}}{\Gamma(2-\beta)}pL_3.$$

It follows that

$$|(\Theta u)(t)|\leq\Lambda_{\Theta}^{1}(\text{constant}).$$

In a like manner,

$$\left|F_k'\left(f,u,u^{'}\right)(t)\right|\leq L_1\left(\frac{T^*}{\Gamma^*}+\Delta_4\right)+L_3p.$$

It follows that

$$\begin{aligned}\left|(\Theta u)^{'}(t)\right| &\leq L_1\left(\frac{T^*}{\Gamma^*}+\Delta_4\right)+L_3p\\
&+\left(|\sigma_0|+|\sigma_2||B_p|+|B_p|\right)\Lambda_F+|\mu_2||B_p|\Lambda_G=:\Lambda_{\Theta}^{2}\end{aligned}$$

Thus

$$\|\Theta u\|_{PC^1}\leq\Lambda_{\Theta}^{1}+\Lambda_{\Theta}^{2}\ = constant.$$

On the other hand, for τ_1 , $\tau_2\in[t_k,t_{k+1}]$ with $\tau_1\leq\tau_2$ and we have

$$|(\Theta u)(\tau_1)-(\Theta u)\tau_2|\leq\int_{\tau_1}^{\tau_2}\left|(\Theta u)^{'}(s)\right|ds\leq\Lambda_{\Theta}(\tau_2-\tau_1).$$

Similarly

$$(\Theta u)^{'}(\tau_2) - (\Theta u)^{'}(\tau_1) \leq \Pi_{\Theta}(\tau_2 - \tau_1),$$

where Π_{Θ} is a constant. This implies that Θu is equicontinuous on all $(t_k, t_{k+1}]$, $k = 0, 1, ..., p$. Consequently, Arzela-Ascoli theorem ensures us that the operator Θ is a completely continuous operator and by Lemma 4 Θ: $B(0, R) \to B(0, R)$. Hence, we conclude that Θ: $B(0, R) \to B(0, R)$ is completely continuous. It follows from the Schauder fixed point theorem that Θhas at least one fixed point. That is problem (1) has at least one solution. ■

Example 1. For $p = 1$, $t_1 = \frac{1}{4}$, $T = 1$, $\beta = \frac{1}{2}$, $\mu_1 = 2$, $\sigma_1 = \frac{1}{2}$, $\mu_2 = 3$, $\sigma_1 = \frac{1}{10}$, $\eta_1 = \frac{1}{5}$, $\eta_2 = \frac{2}{3}$, $\alpha_0 = \frac{3}{2}$, $\alpha_k = \frac{3}{2}$, we consider the following impulsive multi-orders fractional differential equation:

$$\begin{cases} {}^{C}D_{t_k}^{\alpha_k} u(t) = \frac{1}{100} \cos u(t) + \frac{|u'(t)|}{|u'(t)|+100} + t, & 0 < t < 1, \ t \neq \frac{1}{4}, \\ \Delta u\left(\frac{1}{4}\right) = \frac{|u(\frac{1}{4})|}{|u(\frac{1}{4})|+50}, \quad \Delta u'\left(\frac{1}{4}\right) = \frac{|u'(\frac{1}{4})|}{|u'(\frac{1}{4})|+70}, \\ u(0) + 2\,{}^{C}D_{0^+} u(0) = \frac{1}{2} u\left(\frac{1}{5}\right), \\ u(1) + 2\,{}^{C}D_{0^+} u(1) = \frac{1}{2} u\left(\frac{2}{3}\right). \end{cases} \tag{6}$$

It is clear that

$$|f(t, x, x_1) - f(t, y, y_1)| \leq 0.02\,(|x - y| + |x_1 - y_1|), \ 0 \leq t \leq 1, \quad x, y, x_1, y_1 \in \mathbb{R}.$$

One can easily calculate that

$$\Lambda = 0.2178 < 1.$$

Therefore, all the assumptions of Theorem 3 hold. Thus, by Theorem 3, the impulsive multi-orders fractional boundary value problem (6) has a unique solution on $[0, 1]$.

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Convergence of modification of the Kantorovich-type q-Bernstein-Schurer operators

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Abstract. In this paper, we introduce a new modification of Kantorovich-type Bernstein-Schurer operators $K^*_{n,p,q}(f;x)$ based on the concept of q-integers. We investigate statistical approximation properties, establish a local approximation theorem, give a convergence theorem for the Lipschitz continuous functions and obtain a Voronovskaja-type theorem. Furthermore, we also give some illustrative graphics and some numerical examples for comparisons for the convergence of operators to some function.

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1 Introduction

In 2015, Agrawal, Finta and Kumar [1] introduced a new Kantorovich-type generalization of the q-Bernstein-Schurer operators, they gave the basic convergence theorem, obtained the local direct results, estimated the rate of convergence and so on. The operators are defined as

$$K_{n,p,q}(f;x) = [n+1]_q \sum_{k=0}^{n+p} b_{n+p,k}(q;x) q^{-k} \int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} f(t) d_q^R t, \tag{1}$$

where $b_{n+p,k}(q;x)$ is defined by

$$b_{n+p,k}(q;x) = \left[\begin{array}{c} n+p \\ k \end{array} \right]_q x^k (1-x)_q^{n+p-k}. \tag{2}$$

They obtained the following lemma of the moments.

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Lemma 1.1. *(See [1], Lemma 2.1) The following equalities hold*

$$K_{n,p,q}(1;x) = 1; \tag{3}$$

$$K_{n,p,q}(t;x) = \frac{2q[n+p]_q}{[2]_q[n+1]_q}x + \frac{1}{[2]_q[n+1]_q}; \tag{4}$$

$$K_{n,p,q}\left(t^2;x\right) = \frac{q^2(1+q+4q^2)[n+p]_q[n+p-1]_q}{[2]_q[3]_q[n+1]_q^2}x^2 + \frac{q(3+5q+4q^2)[n+p]_q}{[2]_q[3]_q[n+1]_q^2}x + \frac{1}{[3]_q[n+1]_q^2}. \tag{5}$$

Apparently, these operators reproduce only constant functions. In present paper, we will introduce a new modification of Kantorovich-type q-Bernstein-Schurer operators $K^*_{n,p,q}(f;x)$ which will be defined in (7). The advantage of these new operators is that they reproduce not only constant functions but also linear functions. We will investigate statistical approximation properties, establish a local approximation theorem, give a convergence theorem for the Lipschitz continuous functions and obtain a Voronovskaja-type theorem. Furthermore, we will give some illustrative graphics and some numerical examples for comparisons for the convergence of operators to some function. We may observe that the new operators $K^*_{n,p,q}(f;x)$ give a better approximation to $f(x)$ than $K_{n,p,q}(f;x)$.

Before introducing the operators, we mention certain definitions based on q-integers, detail can be found in [4, 5]. For any fixed real number $0 < q \leq 1$ and each nonnegative integer k, we denote q-integers by $[k]_q$, where

$$[k]_q = \begin{cases} \frac{1-q^k}{1-q}, & q \neq 1; \\ k, & q = 1. \end{cases}$$

Also q-factorial and q-binomial coefficients are defined as follows:

$$[k]_q! = \begin{cases} [k]_q[k-1]_q...[1]_q, & k = 1,2,...; \\ 1, & k = 0, \end{cases} , \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad (n \geq k \geq 0).$$

For $x \in [0,1]$ and $n \in \mathbb{N}_0$, we recall that

$$(1-x)_q^n = \begin{cases} 1, & n = 0; \\ \prod_{j=0}^{n-1}\left(1-q^jx\right) = (1-x)(1-qx)...\left(1-q^{n-1}x\right), & n = 1,2,... \end{cases} .$$

The Riemann-type q-integral is defined by

$$\int_a^b f(t)d_q^Rt = (1-q)(b-a)\sum_{j=0}^{\infty} f\left(a+(b-a)q^j\right)q^j, \tag{6}$$

where the real numbers a, b and q satisfy that $0 \leq a < b$ and $0 < q < 1$.

For $f \in C(I)$, $I = [0, 1+p]$, $p \in \mathbb{N}_0$, $q \in (0,1)$ and $n \in \mathbb{N}$, we introduce the modification of Kantorovich-type q-Bernstein-Schurer operators as follows:

$$K^*_{n,p,q}(f;x) = [n+1]_q \sum_{k=0}^{n+p} b_{n+p,k}\left(q;u(x)\right)q^{-k}\int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} f(t)d_q^Rt, \tag{7}$$

where $b_{n+p,k}(q;x)$ is defined by (2), and

$$u(x)=\frac{[2]_q[n+1]_qx-1}{2q[n+p]_q},\quad \left(\frac{1}{[2]_q[n+1]_q}\le x\le \frac{1+2q[n+p]_q}{[2]_q[n+1]_q}\right). \tag{8}$$

2 Auxiliary Results

In order to obtain the approximation properties, We need the following lemmas:

Lemma 2.1. *For the modification of Kantorovich-type q-Bernstein-Schurer operators* (7), *using lemma 1.1, by some easily computations we have*

$$K^*_{n,p,q}(1;x) = 1, \tag{9}$$

$$K^*_{n,p,q}(t;x) = x, \tag{10}$$

$$\begin{aligned} K^*_{n,p,q}\left(t^2;x\right) =\ & \frac{[2]_q\left(1+q+4q^2\right)[n+p-1]_qx^2}{4[3]_q[n+p]_q}+\frac{\left(3+5q+4q^2\right)x}{2[3]_q[n+1]_q} \\ & -\frac{\left(1+q+4q^2\right)[n+p-1]_qx}{2[3]_q[n+1]_q[n+p]_q}+\frac{\left(1+q+4q^2\right)[n+p-1]_q}{4[2]_q[3]_q[n+1]_q^2[n+p]_q} \\ & -\frac{3+5q+4q^2}{2[2]_q[3]_q[n+1]_q^2}+\frac{1}{[3]_q[n+1]_q^2}. \end{aligned} \tag{11}$$

Remark 2.2. *Let* $\{q_n\}$ *denotes a sequence such that* $0<q_n<1$. *Then, by Bohman and Korovkin Theorem, for any* $f\in C(I)$, *operators* $K^*_{n,p,q}(f;x)$ *converge uniformly to* $f(x)$, *if and only if* $\lim_{n\to\infty}q_n=1$.

Lemma 2.3. *For the modification of Kantorovich-type q-Bernstein-Schurer operators* (7), *we have*

$$K^*_{n,p,q}(t-x;x) = 0, \tag{12}$$

$$K^*_{n,p,q}\left((t-x)^2;x\right) \le \frac{\left(q^2+4q^3-2q-3\right)x^2}{4[3]_q}+\frac{(1+2q)x}{[3]_q[n+1]_q}+\frac{(3+5q+4q^2)x}{2[3]_q[n+1]_q[n+p]_q} \tag{13}$$

$$\le \frac{(1+2q)x}{[3]_q[n+1]_q}+\frac{(3+5q+4q^2)x}{2[3]_q[n+1]_q[n+p]_q}, \tag{14}$$

$$K^*_{n,p,q}\left((t-x)^4;x\right) \le O\left(\frac{1}{[n]_q^2}\right). \tag{15}$$

Proof. By (9) and (10), we get (12). Using (10), (11) and some computations, we have

$$\begin{aligned} & K^*_{n,p,q}\left((t-x)^2;x\right) \\ =\ & K^*_{n,p,q}(t^2;x)-2xK^*_{n,p,q}(t;x)+x^2 \\ \le\ & \frac{\left(q^2+4q^3-2q-3\right)x^2}{4[3]_q}+\frac{(1+2q)[n+p-1]_qx}{[3]_q[n+1]_q[n+p]_q}+\frac{q^{n+p-1}(3+5q+4q^2)x}{2[3]_q[n+1]_q[n+p]_q} \\ \le\ & \frac{\left(q^2+4q^3-2q-3\right)x^2}{4[3]_q}+\frac{(1+2q)x}{[3]_q[n+1]_q}+\frac{(3+5q+4q^2)x}{2[3]_q[n+1]_q[n+p]_q}. \end{aligned}$$

Indeed, using the similar method for estimate $K_{n,p,q}\left((t-x)^4;x\right)$ in [1, P. 229], we have

$$\begin{aligned}
&K^*_{n,p,q}\left((t-x)^4;x\right)\\
&\le 64\sum_{k=0}^{n+p} b_{n+p,k}(q;u(x))\left(\frac{[k]_q}{[n+1]_q}-\frac{[k]_q}{[n+p]_q}\right)^4+64\sum_{k=0}^{n+p} b_{n+p,k}(q;u(x))\left(\frac{[k]_q}{[n+p]_q}-x\right)^4\\
&\quad+\frac{8}{[5]_q}\sum_{k=0}^{n+p} b_{n+p,k}(q;u(x))\left(\frac{q^k}{[n+1]_q}\right)^4\\
&\le 64\sum_{k=0}^{n+p} b_{n+p,k}(q;u(x))\left(\frac{[k]_q}{[n+p]_q}\right)^4\left[\frac{q^n([p]_q-1)}{[n+1]_q}\right]^4\\
&\quad+64\sum_{k=0}^{n+p} b_{n+p,k}(q;u(x))\left(\frac{[k]_q}{[n+p]_q}-\frac{[2]_q[n+1]_qx-1}{2q[n+p]_q}+\frac{[2]_q[n+1]_qx-1}{2q[n+p]_q}-x\right)^4\\
&\quad+\frac{8}{[5]_q}\sum_{k=0}^{n+p} b_{n+p,k}(q;u(x))\left(\frac{q^k}{[n+1]_q}\right)^4\\
&\le C_1\frac{([p]_q-1)^4}{[n]_q^2}+512\sum_{k=0}^{n+p} b_{n+p,k}(q;u(x))\left((t-u(x))^4;x\right)\\
&\quad+512\sum_{k=0}^{n+p} b_{n+p,k}(q;u(x))\left(\frac{[2]_q[n+1]_qx-1}{2q[n+p]_q}-x\right)^4+\frac{C_2}{[n]_q^2},
\end{aligned}$$

where $u(x)$ is defined in (8), C_1 and C_2 are some positive constants. Thus,

$$\begin{aligned}
&K^*_{n,p,q}\left((t-x)^4;x\right)\\
&\le C_1\frac{([p]_q-1)^4}{[n]_q^2}+512\frac{C_3}{[n]_q^2}+512\left[\frac{[2]_q\left([n]_q+q^n\right)x-1-2q\left([n]_q+q^n[p]_q\right)x}{2q[n+p]_q}\right]^4+\frac{C_2}{[n]_q^2}\\
&= C_1\frac{([p]_q-1)^4}{[n]_q^2}+512\frac{C_3}{[n]_q^2}+512\left[\frac{\left(1+q^{n+1}-2q^{n+1}[p]_q\right)x-1}{2q[n+p]_q}\right]^4+\frac{C_2}{[n]_q^2}=O\left(\frac{1}{[n]_q^2}\right),
\end{aligned}$$

where C_3 is a positive constant, lemma 2.3 is proved. □

3 Statistical approximation properties

In this section, we present the statistical approximation properties of the operator $K^*_{n,p,q}(f;x)$.

Let K be a subset of $\mathbb{N}$, the set of all natural numbers. The density of K is defined by $\delta(K):=\lim_n\frac{1}{n}\sum_{k=1}^n\chi_K(k)$ provided the limit exists, where χ_K is the characteristic function of K. A sequence $x:=\{x_n\}$ is called statistically convergent to a number L if, for every $\varepsilon>0$, $\delta\{n\in\mathbb{N}:|x_n-L|\ge\varepsilon\}=0$. Let $A:=(a_{jn}), j,n=1,2,...$ be an infinite summability matrix. For a given sequence $x:=\{x_n\}$, the $A-$transform of x, denoted by $Ax:=((Ax)_j)$, is given by $(Ax)_j=\sum_{k=1}^\infty a_{jn}x_n$ provided the series converges for each

j. We say that A is regular if $\lim_n (Ax)_j = L$ whenever $\lim x = L$. Assume that A is a non-negative regular summability matrix. A sequence $x = \{x_n\}$ is called A-statistically convergent to L provided that for every $\varepsilon > 0$, $\lim_j \sum_{n:|x_n - L| \geq \varepsilon} a_{jn} = 0$. We denote this limit by $st_A - \lim_n x_n = L$. For $A = C_1$, the Cesàro matrix of order one, A-statistical convergence reduces to statistical convergence. It is easy to see that every convergent sequence is statistically convergent but not conversely.

We consider a sequence $q := \{q_n\}$ for $0 < q_n < 1$ satisfying

$$st_A - \lim_n q_n = 1. \tag{16}$$

If $e_i = t^i,\ t \in \mathbb{R}^+,\ i = 0, 1, 2, ...$ stands for the ith monomial, then we have

Theorem 3.1. *Let $A = (a_{nk})$ be a non-negative regular summability matrix and $q := \{q_n\}$ be a sequence satisfying* (16)*, then for all $f \in C(I)$, $x \in [0, 1]$, we have*

$$st_A - \lim_n ||K^*_{n,p,q} f - f||_{C(I)} = 0. \tag{17}$$

Proof. Obviously

$$st_A - \lim_n ||K^*_{n,p,q_n}(e_i) - e_i||_{C(I)} = 0. \quad (i = 0, 1) \tag{18}$$

By (11) and (13), we have

$$\left|K^*_{n,p,q_n}(e_2; x) - e_2(x)\right| \leq \frac{1 + 2q_n}{[3]_{q_n}[n+1]_{q_n}} + \frac{3 + 5q_n + 4q_n^2}{2[3]_{q_n}[n+1]_{q_n}[n+p]_{q_n}}.$$

Now for a given $\varepsilon > 0$, let us define the following sets:

$$U := \left\{k : \left|\left|K^*_{n,p,q_k}(e_2) - e_2\right|\right|_{C(I)} \geq \varepsilon\right\},\ U_1 := \left\{k : \frac{1 + 2q_k}{[3]_{q_k}[n+1]_{q_k}} \geq \frac{\varepsilon}{2}\right\},$$
$$U_2 := \left\{k : \frac{3 + 5q_k + 4q_k^2}{2[3]_{q_k}[n+1]_{q_k}[n+p]_{q_k}} \geq \frac{\varepsilon}{2}\right\}.$$

Then one can see that $U \subseteq U_1 \cup U_2$, so we have

$$\begin{aligned} \delta\left\{k \leq n : ||K^*_{n,p,q_k}(e_2) - e_2||_{C(I)}\right\} \quad \leq \quad & \delta\left\{k \leq n : \frac{1 + 2q_k}{[3]_{q_k}[n+1]_{q_k}} \geq \frac{\varepsilon}{2}\right\} \\ & + \delta\left\{k \leq n : \frac{3 + 5q_k + 4q_k^2}{2[3]_{q_k}[n+1]_{q_k}[n+p]_{q_k}} \geq \frac{\varepsilon}{2}\right\}, \end{aligned}$$

since $st_A - \lim_n q_n = 1$, we have

$$st_A - \lim_n \frac{1 + 2q_n}{[3]_{q_n}[n+1]_{q_n}} = 0, \quad st_A - \lim_n \frac{3 + 5q_n + 4q_n^2}{2[3]_{q_n}[n+1]_{q_n}[n+p]_{q_n}} = 0,$$

which implies that the right-hand side of the above inequality is zero, thus we have

$$st_A - \lim_n ||K^*_{n,p,q_n}(e_2) - e_2||_{C(I)} = 0. \tag{19}$$

Combining (18) and (19), theorem 3.1 follows from the Korovkin-type statistical approximation theorem established in [3], the proof is completed. □

4 Local approximation properties

Let $f \in C(I)$, endowed with the norm $||f|| = \sup_{x\in I} |f(x)|$. The Peetre's $K-$functional is defined by

$$K_2(f;\delta) = \inf_{g\in C^2} \left\{ ||f-g|| + \delta ||g''|| \right\},$$

where $\delta > 0$ and $C^2 = \{g \in C(I) : g', g'' \in C(I)\}$. By [2, p.177, Theorem 2.4], there exits an absolute constant $C > 0$ such that

$$K_2(f;\delta) \leq C\omega_2(f;\sqrt{\delta}), \tag{20}$$

where

$$\omega_2(f;\delta) = \sup_{0<h\leq\delta} \sup_{x,x+h,x+2h\in I} |f(x+2h) - 2f(x+h) + f(x)|$$

is the second order modulus of smoothness of $f \in C(I)$.

Now we give a direct local approximation theorem for the operators $K^*_{n,p,q}(f,x)$.

Theorem 4.1. *For $q \in (0,1)$, $x \in [0,1]$ and $f \in C(I)$, we have*

$$\left|K^*_{n,p,q}(f,x) - f(x)\right| \leq C\omega_2\left(f; \sqrt{\frac{\left(q^2+4q^3-2q-3\right)x^2}{8[3]_q} + \frac{(1+2q)x}{2[3]_q[n+1]_q} + \frac{(3+5q+4q^2)x}{4[3]_q[n+1]_q[n+p]_q}}\right), \tag{21}$$

where C is a positive constant.

Proof. Let $g \in C^2$. By Taylor's expansion

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du,$$

and (12), we get

$$K^*_{n,p,q}(g;x) = g(x) + K^*_{n,p,q}\left(\int_x^t (t-u)g''(u)du; x\right).$$

Hence, by (13), we have

$$\begin{aligned}
&\left|K^*_{n,p,q}(g;x) - g(x)\right| \\
\leq\ & \left|K^*_{n,p,q}\left(\int_x^t (t-u)g''(u)du; x\right)\right| \\
\leq\ & K^*_{n,p,q}\left(\left|\int_x^t (t-u)|g''(u)|du\right|; x\right) \\
\leq\ & K^*_{n,p,q}\left((t-x)^2; x\right)||g''|| \\
\leq\ & \left[\frac{\left(q^2+4q^3-2q-3\right)x^2}{4[3]_q} + \frac{(1+2q)x}{[3]_q[n+1]_q} + \frac{(3+5q+4q^2)x}{2[3]_q[n+1]_q[n+p]_q}\right]||g''||. \qquad (22)
\end{aligned}$$

On the other hand, using lemma 2.1, we have

$$\left|K^*_{n,p,q}(f;x)\right| \leq [n+1]_q \sum_{k=0}^{n+p} b_{n+p,k}\left(q;u(x)\right) q^{-k} \int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} |f(t)| d_q^R t \leq ||f||. \tag{23}$$

Now (22) and (23) imply

$$\begin{aligned} & \left|K^*_{n,p,q}(f;x) - f(x)\right| \\ \leq\ & \left|K^*_{n,p,q}(f-g;x) - (f-g)(x)\right| + \left|K^*_{n,p,q}(g;x) - g(x)\right| \\ \leq\ & 2||f-g|| + \left[\frac{\left(q^2+4q^3-2q-3\right)x^2}{4[3]_q} + \frac{(1+2q)x}{[3]_q[n+1]_q} + \frac{(3+5q+4q^2)x}{2[3]_q[n+1]_q[n+p]_q}\right] ||g''||. \end{aligned}$$

Hence taking infimum on the right hand side over all $g \in C^2$, we get

$$\left|K^*_{n,p,q}(f;x) - f(x)\right| \leq 2K_2\left(f; \frac{\left(q^2+4q^3-2q-3\right)x^2}{8[3]_q} + \frac{(1+2q)x}{2[3]_q[n+1]_q} + \frac{(3+5q+4q^2)x}{4[3]_q[n+1]_q[n+p]_q}\right).$$

By (20), for every $q \in (0,1)$, we have

$$|K^*_{n,p,q}(f;x) - f(x)| \leq C\omega_2\left(f; \sqrt{\frac{\left(q^2+4q^3-2q-3\right)x^2}{8[3]_q} + \frac{(1+2q)x}{2[3]_q[n+1]_q} + \frac{(3+5q+4q^2)x}{4[3]_q[n+1]_q[n+p]_q}}\right).$$

This completes the proof of theorem 4.1. $\square$

Remark 4.2. *For any fixed $x \in [0,1]$, $p \in \mathbb{N}_0$ and $n \in \mathbb{N}$, let $q := \{q_n\}$ be a sequence satisfying $0 < q_n < 1$ and $\lim_n q_n = 1$, we have*

$$\lim_{n\to\infty}\left[\frac{\left(q_n^2+4q_n^3-2q_n-3\right)x^2}{8[3]_{q_n}} + \frac{(1+2q_n)x}{2[3]_{q_n}[n+1]_{q_n}} + \frac{(3+5q_n+4q_n^2)x}{4[3]_{q_n}[n+1]_{q_n}[n+p]_{q_n}}\right] = 0.$$

These gives us a rate of pointwise convergence of the operators $K^*_{n,p,q_n}(f;x)$ to $f(x)$.

Next we study the rate of convergence of the operators $K^*_{n,p,q}(f;x)$ with the help of functions of Lipschitz class $Lip_M(\alpha)$, where $M > 0$ and $0 < \alpha \leq 1$. A function f belongs to $Lip_M(\alpha)$ if

$$|f(y) - f(x)| \leq M|y-x|^\alpha \quad (y, x \in \mathbb{R}). \tag{24}$$

We have the following theorem.

Theorem 4.3. *Let $q := \{q_n\}$ be a sequence satisfying $0 < q_n < 1$, $\lim\limits_n q_n = 1$ and $f \in Lip_M(\alpha)$, $0 < \alpha \leq 1$. Then we have*

$$\left|K^*_{n,p,q}(f;x) - f(x)\right| \leq M\left[\frac{\left(q^2+4q^3-2q-3\right)x^2}{4[3]_q} + \frac{(1+2q)x}{[3]_q[n+1]_q} + \frac{(3+5q+4q^2)x}{2[3]_q[n+1]_q[n+p]_q}\right]^{\frac{\alpha}{2}}. \tag{25}$$

Proof. Since $K^*_{n,p,q}$ is a linear positive operator and $f \in Lip_M(\alpha)$ $(0 < \alpha \leq 1)$, we have

$$\begin{aligned}
& |K^*_{n,p,q}(f;x) - f(x)| \\
\leq\ & K^*_{n,p,q}\left(|f(t) - f(x)|;x\right) \\
=\ & [n+1]_q \sum_{k=0}^{n+p} b_{n+p,k}(q;u(x)) q^{-k} \int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} |f(t) - f(x)| d_q^R t \\
\leq\ & M[n+1]_q \sum_{k=0}^{n+p} b_{n+p,k}(q;u(x)) q^{-k} \int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} |t - x|^\alpha d_q^R t \\
\leq\ & M[n+1]_q \sum_{k=0}^{n+p} b_{n+p,k}(q;u(x)) q^{-k} \left(\int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} \left[|t-x|^\alpha\right]^{\frac{2}{\alpha}} d_q^R t \right)^{\frac{\alpha}{2}} \left(\int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} d_q^R t \right)^{\frac{2-\alpha}{2}} \\
=\ & M[n+1]_q \sum_{k=0}^{n+p} b_{n+p,k}(q;u(x)) q^{-k} \left(\int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} (t-x)^2 d_q^R t \right)^{\frac{\alpha}{2}} \left(\frac{q^k}{[n+1]_q} \right)^{\frac{2-\alpha}{2}} \\
=\ & M \sum_{k=0}^{n+p} b_{n+p,k}(q;u(x)) \left(\int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} (t-x)^2 d_q^R t \right)^{\frac{\alpha}{2}} \left(\frac{[n+1]_q}{q^k} \right)^{\frac{\alpha}{2}} \\
=\ & M \sum_{k=0}^{n+p} \left[b_{n+p,k}(q;u(x))\right]^{\frac{2-\alpha}{2}} \left([n+1]_q b_{n+p,k}(q;u(x)) q^{-k} \int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} (t-x)^2 d_q^R t \right)^{\frac{\alpha}{2}}.
\end{aligned}$$

Applying Hölder's inequality for sums, we obtain

$$\begin{aligned}
& |K^*_{n,p,q}(f;x) - f(x)| \\
\leq\ & M \left(\sum_{k=0}^{n+p} b_{n+p,k}(q;u(x)) \right)^{\frac{2-\alpha}{2}} \left(\sum_{k=0}^{n+p} [n+1]_q b_{n+p,k}(q;u(x)) q^{-k} \int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} (t-x)^2 d_q^R t \right)^{\frac{\alpha}{2}} \\
=\ & M \left[K^*_{n,p,q}\left((t-x)^2;x\right) \right]^{\frac{\alpha}{2}}.
\end{aligned}$$

Thus, theorem 4.3 is proved. □

Now, we give a Voronovskaja-type asymptotic formula for $K^*_{n,p,q}(f;x)$ by means of the second and fourth central moments.

Theorem 4.4. *Let $q := \{q_n\}$ be a sequence satisfying $0 < q_n < 1$, $\lim_n q_n = 1$. For $f \in C^2(I)$, ($f(x)$ is a twice differentiable function in I), the following equality holds*

$$\lim_{n\to\infty} [n]_q \left(K^*_{n,p,q}(f;x) - f(x) \right) = \frac{f''(x)}{2} x. \tag{26}$$

Proof. Let $x \in [0,1]$ be fixed. By the Taylor formula, we may write

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2} f''(x)(t-x)^2 + r(t;x)(t-x)^2, \tag{27}$$

where $r(t;x)$ is the Peano form of the remainder, $r(t;x) \in C(I)$, using L'Hopital's rule, we have

$$\begin{aligned}\lim_{t\to x} r(t;x) &= \lim_{t\to x} \frac{f(t)-f(x)-f'(x)(t-x)-\frac{1}{2}f''(x)(t-x)^2}{(t-x)^2} \\ &= \lim_{t\to x} \frac{f'(t)-f'(x)-f''(x)(t-x)}{2(t-x)} = \lim_{t\to x} \frac{f''(t)-f''(x)}{2} = 0.\end{aligned}$$

Since (12), applying $K^*_{n,p,q}(f;x)$ to (27), we obtain

$$[n]_q \left(K^*_{n,p,q}(f;x)-f(x)\right) = \frac{1}{2}[n]_q f''(x) K^*_{n,p,q}\left((t-x)^2;x\right) + [n]_q K^*_{n,p,q}\left(r(t;x)(t-x)^2;x\right).$$

By the Cauchy-Schwarz inequality, we have

$$K^*_{n,p,q}\left(r(t;x)(t-x)^2;x\right) \le \sqrt{K^*_{n,p,q}\left(r^2(t;x);x\right)}\sqrt{K^*_{n,p,q}\left((t-x)^4;x\right)}. \tag{28}$$

Since $r^2(x;x)=0$, then it is obtained easily that $\lim_n K_{n,p,q}\left(r^2(t;x);x\right) = r^2(x;x) = 0$ by remark 2.2. Now, from (15), (28) and (14), we get immediately

$$\lim_{n\to\infty}[n]_q K^*_{n,p,q}\left(r(t;x)(t-x)^2;x\right) = 0, \quad \lim_{n\to\infty}\frac{1}{2}[n]_q f''(x) K^*_{n,p,q}\left((t-x)^2;x\right) = \frac{f''(x)}{2}x.$$

Thus, theorem 4.4 is proved. □

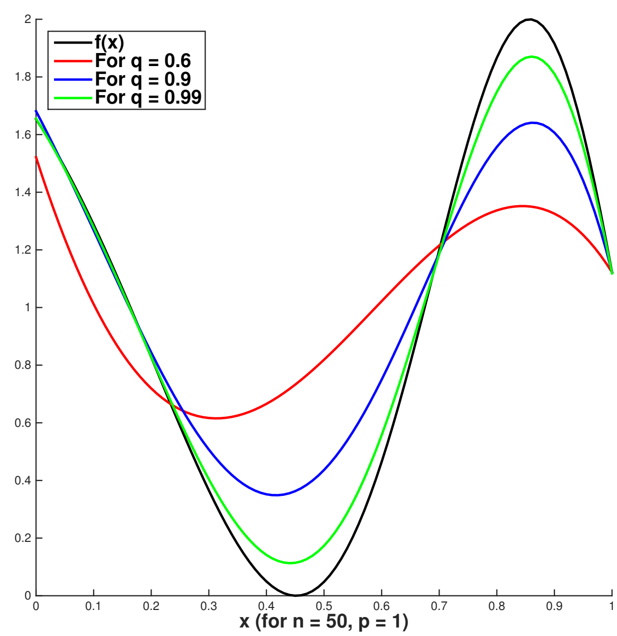

Figure 1: Convergence of $K^*_{n,p,q}(f;x)$ for $n=50$, $p=1$ and different values of q.

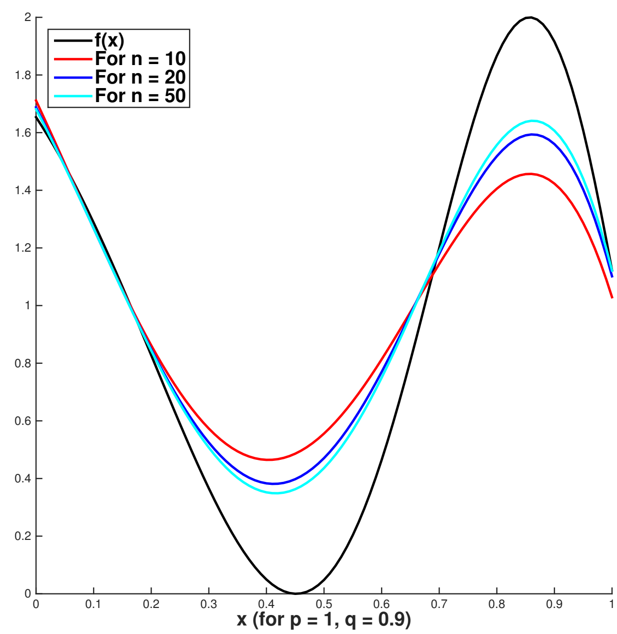

Figure 2: Convergence of $K^*_{n,p,q}(f;x)$ for $p = 1$, $q = 0.9$ and different values of n.

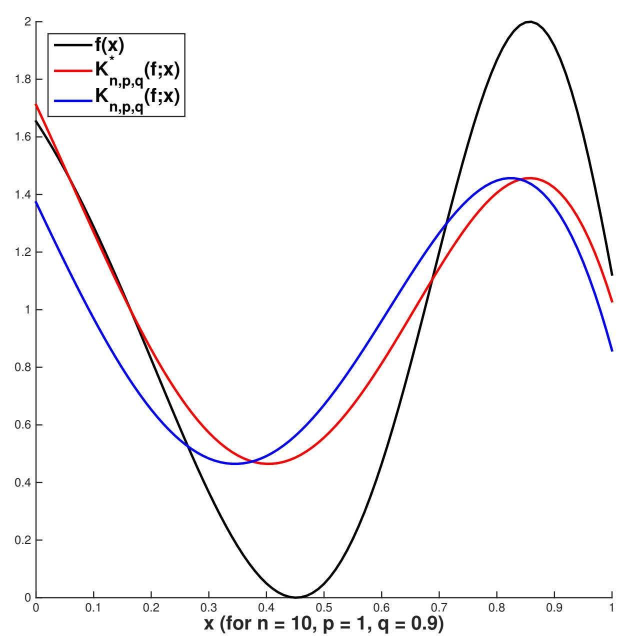

Figure 3: The graphs of $K^*_{n,p,q}(f;x)$ (red) and $K_{n,p,q}(f;x)$ (blue) for $n = 10$, $p = 1$ and $q = 0.9$.

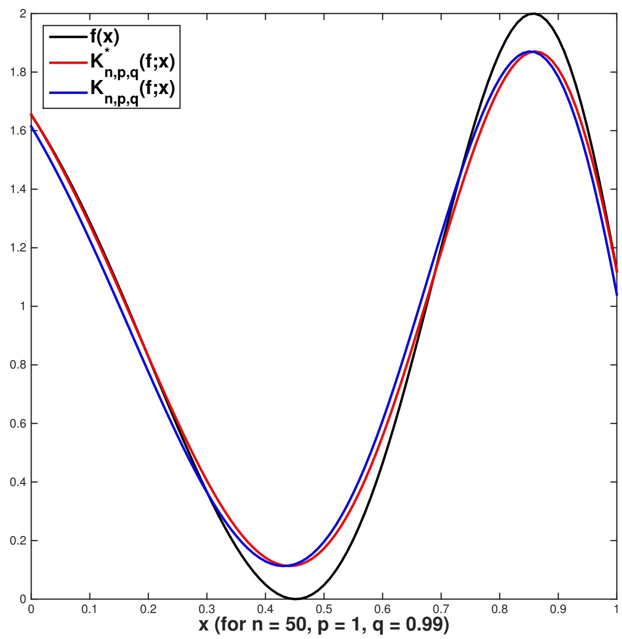

Figure 4: The graphs of $K^*_{n,p,q}(f;x)$ (red) and $K_{n,p,q}(f;x)$ (blue) for $n = 50$, $p = 1$ and $q = 0.99$.

5 Graphical and numerical examples analysis

In this section, we give several graphs and numerical examples to show the convergence of $K^*_{n,p,q}(f;x)$ to $f(x)$ with different values of n and q, and also compare the operators $K^*_{n,p,q}(f;x)$ with $K_{n,p,q}(f;x)$.

Let $f(x) = 1 - cos(4e^x)$, for $n = 50$ and $p = 1$, the graphs of $K^*_{n,p,q}(f;x)$ with different values of q are shown in figure 1. Moreover, for $p = 1$ and $q = 0.9$, the graphs of $K^*_{n,p,q}(f;x)$ with different values of n are shown in figure 2.

Figure 3 shows the graphs of $K^*_{n,p,q}(f;x)$ (red) and $K_{n,p,q}(f;x)$ (blue) for $n = 10$, $p = 1$ and $q = 0.9$. In figure 4, the values of n and q are replaced by 50 and 0.99, respectively.

In Table 1, we give the errors of the approximation to $f(x)$ of $K^*_{n,p,q}(f;x)$ and $K_{n,p,q}(f;x)$ with different values of q and n. We may observe that operators $K^*_{n,p,q}(f;x)$ give a better estimate than $K_{n,p,q}(f;x)$.

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Table 1: The errors of the approximation to $f(x)$ of $K^*_{n,p,q}(f;x)$ and $K_{n,p,q}(f;x)$.

$q = 1 - 1/m$	$\|f - K^*_{n,p,q}(f)\|_\infty$		$\|f - K_{n,p,q}(f)\|_\infty$	
	$n = 10$	$n = 50$	$n = 10$	$n = 50$
$m = 100$	0.4890	0.1318	0.5628	0.1587
$m = 200$	0.4856	0.1201	0.5638	0.1471
$m = 300$	0.4844	0.1163	0.5642	0.1436
$m = 400$	0.4838	0.1145	0.5645	0.1419
$m = 500$	0.4835	0.1134	0.5646	0.1409
$m = 600$	0.4832	0.1126	0.5647	0.1402
$m = 700$	0.4831	0.1121	0.5648	0.1397
$m = 800$	0.4829	0.1117	0.5648	0.1394
$m = 900$	0.4829	0.1114	0.5649	0.1391
$m = 1000$	0.4828	0.1112	0.5649	0.1389

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BARNES-TYPE DEGENERATE BERNOULLI AND EULER MIXED-TYPE POLYNOMIALS

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Abstract. In this paper, we consider the Barnes-type degenerate Bernoulli and Euler mixed-type polynomials. We present several explicit formulas and recurrence relations for these polynomials. Also, we establish a connection between our polynomials and several known families of polynomials.

1. Introduction

The goals of this paper are to use umbral calculus to obtain several new and interesting identities of Barnes-type degenerate Bernoulli and Euler mixed-type polynomials. The use of umbral calculus technique has been very attractive in numerous problems of mathematics (for example, see [1, 6, 8, 14, 18–21, 24]) and used in different areas of physics (for example, see [4, 5, 19]).

Let $r, s \in \mathbb{Z}_{>0}$. Throughout the paper we assume that $\mathbf{a} = a_1, \ldots, a_r$ and $\mathbf{b} = b_1, b_2, \ldots, b_s$. The *Barnes-type degenerate Bernoulli and Euler mixed-type polynomials* $\beta\mathcal{E}_n(\lambda, x|\mathbf{a}; \mathbf{b})$ with $a_1, \ldots, a_r; b_1, \ldots, b_s \neq 0$ are defined by the generating function

$$\prod_{i=1}^{r}\left(\frac{t}{(1+\lambda t)^{a_i/\lambda}-1}\right)\prod_{i=1}^{s}\left(\frac{2}{(1+\lambda t)^{b_i/\lambda}+1}\right)(1+\lambda t)^{x/\lambda} = \sum_{n\geq 0}\beta\mathcal{E}_n(\lambda, x|\mathbf{a}; \mathbf{b})\frac{t^n}{n!}. \tag{1}$$

If $x = 0$, $\beta\mathcal{E}_n(\lambda|\mathbf{a}; \mathbf{b}) = \beta\mathcal{E}_n(\lambda, 0|\mathbf{a}; \mathbf{b})$ are called the *Barnes-type degenerate Bernoulli and Euler mixed-type numbers*. Here, we recall that the polynomial $\beta_n(\lambda, x|\mathbf{a})$ with $a_1, \ldots, a_r \neq 0$ are given by

$$\prod_{i=1}^{r}\left(\frac{t}{(1+\lambda t)^{a_i/\lambda}-1}\right)(1+\lambda t)^{x/\lambda} = \sum_{n\geq 0}\beta_n(\lambda, x|\mathbf{a})\frac{t^n}{n!} \tag{2}$$

are called the *Barnes-type degenerate Bernoulli polynomials* and studied in [7]. We note here that

$$\lim_{\lambda\to 0}\beta_n(\lambda, x|\mathbf{a}) = B_n(x|\mathbf{a}),$$
$$\lim_{\lambda\to\infty}\lambda^{-n}\beta_n(\lambda, \lambda x|\mathbf{a}) = (a_1a_2\cdots a_r)^{-1}b_n^{(r)}(x),$$

where $B_n(x|\mathbf{a})$ are the *Barnes-type Bernoulli polynomials* given by $\prod_{i=1}^{r}\left(\frac{t}{e^{a_it}-1}\right)e^{tx} = \sum_{n\geq 0}B_n(x|\mathbf{a})\frac{t^n}{n!}$ and $b_n^{(r)}(x)$ are the *Bernoulli polynomials of the second kind of order*

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r given by $\left(\frac{t}{\log(1+t)}\right)^r (1+t)^x = \sum_{n\geq 0} b_n^{(r)}(x)\frac{t^n}{n!}$ (see [12, 22]). Also, we recall that the polynomial $\mathcal{E}_n(\lambda, x|\mathbf{b})$ with $b_1, \dots, b_s \neq 0$ are given by

$$\prod_{i=1}^{s}\left(\frac{2}{(1+\lambda t)^{b_i/\lambda}+1}\right)(1+\lambda t)^{x/\lambda} = \sum_{n\geq 0}\mathcal{E}_n(\lambda, x|\mathbf{b})\frac{t^n}{n!} \tag{3}$$

are called the *Barnes-type degenerate Euler polynomials* and studied in [11, 17, 25]. We denote $\mathcal{E}_n(\lambda, 0|\mathbf{b})$ by $\mathcal{E}_n(\lambda|\mathbf{b})$. We note here that

$$\lim_{\lambda\to 0}\mathcal{E}_n(\lambda, x|\mathbf{b}) = E_n(x|\mathbf{b}),$$
$$\lim_{\lambda\to\infty}\lambda^{-n}\mathcal{E}_n(\lambda, \lambda x|\mathbf{a}) = (x)_n = x(x-1)\cdots(x-n+1),$$

where $E_n(x|\mathbf{a})$ are the *Barnes-type Euler polynomials* given by (see [3])

$$\prod_{i=1}^{s}\left(\frac{2}{e^{b_i t}+1}\right)e^{tx} = \sum_{n\geq 0}E_n(x|\mathbf{b})\frac{t^n}{n!}.$$

In order to study the Barnes-type degenerate Bernoulli and Euler mixed-type polynomials, we use the umbral calculus technique. We denote the algebra of polynomials in a single variable x over $\mathbb{C}$ by Π. Let Π^* be the vector space of all linear functionals on Π. Let $\langle L|p(x)\rangle$ be the action of a linear functional $L \in \Pi^*$ on a polynomial $p(x)$, where we extend it as $\langle cL + c'L'|p(x)\rangle = c\langle L|p(x)\rangle + c'\langle L'|p(x)\rangle$, where $c, c' \in \mathbb{C}$ (see [22, 23]). Define

$$\mathcal{H} = \left\{ f(t) = \sum_{k\geq 0} a_k\frac{t^k}{k!} \mid a_k \in \mathbb{C}\right\} \tag{4}$$

to be the algebra of formal power series in a single variable t. The formal power series in the variable t defines a linear functional on Π by setting $\langle f(t)|x^n\rangle = a_n$, for all $n \geq 0$ (see [22, 23]). By (4), we have

$$\langle t^k|x^n\rangle = n!\delta_{n,k}, \text{ for all } n, k \geq 0, \text{ (see [22, 23])}, \tag{5}$$

where $\delta_{n,k}$ is the Kronecker's symbol. For $f_L(t) = \sum_{n\geq 0}\langle L|x^n\rangle\frac{t^n}{n!}$, by (5), we have that $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. Thus, the map $L \mapsto f_L(t)$ is a vector space isomorphism from Π^* onto $\mathcal{H}$, namely $\mathcal{H}$ is thought of as set of both formal power series and linear functionals. We call $\mathcal{H}$ the *umbral algebra*. The *umbral calculus* is the study of umbral algebra.

The *order* $O(f(t))$ of the non-zero power series $f(t)$ is the smallest integer ℓ for which the coefficient of t^ℓ does not vanish (see [22, 23]). If $O(f(t)) = 1$ ($O(f(t)) = 0$) then $f(t)$ is called a *delta* (an *invertable*) series. If $O(f(t)) = 1$ and $O(g(t)) = 0$, then there exists a unique sequence $s_n(x)$ of polynomials such that $\langle g(t)(f(t))^k|s_n(x)\rangle = n!\delta_{n,k}$, where $n, k \geq 0$. The sequence $s_n(x)$ is called the *Sheffer sequence* for $(g(t), f(t))$, and we write $s_n(x) \sim (g(t), f(t))$ (see [22, 23]). For $f(t) \in \mathcal{H}$ and $p(x) \in \Pi$, we have that $\langle e^{yt}|p(x)\rangle = p(y)$, $\langle f(t)g(t)|p(x)\rangle = \langle g(t)|f(t)p(x)\rangle$, $f(t) = \sum_{n\geq 0}\langle f(t)|x^n\rangle\frac{t^n}{n!}$ and $p(x) = \sum_{n\geq 0}\langle t^n|p(x)\rangle\frac{x^n}{n!}$. Thus,

$$\langle t^k|p(x)\rangle = p^{(k)}(0), \quad \langle 1|p^{(k)}(x)\rangle = p^{(k)}(0), \tag{6}$$

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where $p^{(k)}(0)$ denotes the k-th derivative of $p(x)$ with respect to x at $x = 0$. So, by (6), we get that $t^k p(x) = p^{(k)}(x) = \frac{d^k}{dx^k} p(x)$, for all $k \geq 0$, (see [22, 23]). Let $s_n(x) \sim (g(t), f(t))$. Then we have

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{n\geq 0} s_n(y) \frac{t^n}{n!}, \tag{7}$$

for all $y \in \mathbb{C}$, where $\bar{f}(t)$ is the compositional inverse of $f(t)$ (see [22, 23]). For $s_n(x) \sim (g(t), f(t))$ and $r_n(x) \sim (h(t), \ell(t))$, let $s_n(x) = \sum_{k=0}^{n} c_{n,k} r_k(x)$, then we have

$$c_{n,k} = \frac{1}{k!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} (\ell(\bar{f}(t)))^k | x^n \right\rangle, \tag{8}$$

(see [22, 23]).

By the theory of Sheffer sequences, it is immediate that the Barnes-type degenerate Bernoulli and Euler mixed-type polynomial is the Sheffer sequence for the pair $g(t) = \left(\frac{\lambda}{e^{\lambda t}-1}\right)^r \prod_{i=1}^{r} \left(e^{a_i t} - 1\right) \prod_{i=1}^{s} \left(\frac{e^{b_i t}+1}{2}\right)$ and $f(t) = \frac{1}{\lambda}(e^{\lambda t} - 1)$. Thus

$$\beta\mathcal{E}_n(\lambda, x|\mathbf{a}; \mathbf{b}) \sim \left(\left(\frac{\lambda}{e^{\lambda t} - 1}\right)^r \prod_{i=1}^{r} (e^{a_i t} - 1) \prod_{i=1}^{s} \left(\frac{e^{b_i t} + 1}{2}\right), \frac{1}{\lambda}(e^{\lambda t} - 1) \right). \tag{9}$$

The aim of the present paper is to present several new identities for Barnes-type degenerate Bernoulli and Euler mixed-type polynomials by the use of umbral calculus.

2. Explicit Expressions

In this section we suggest several explicit formulas for the Barnes-type degenerate Bernoulli and Euler mixed-type polynomials. To do so, we recall that the Stirling numbers $S_1(n, m)$ of the first kind are defined as $(x)_n = \sum_{m=0}^{n} S_1(n, m) x^m \sim (1, e^t - 1)$ or $\frac{1}{j!}(\log(1 + t))^j = \sum_{\ell\geq j} S_1(\ell, j) \frac{t^\ell}{\ell!}$. Also, we recall that the Stirling numbers $S_2(n, m)$ of the second kind are defined by $\frac{(e^t-1)^k}{k!} = \sum_{\ell\geq k} S_2(\ell, k) \frac{t^\ell}{\ell!}$. Define $(x|\lambda)_n = \lambda^n (x/\lambda)_n$ to be $(x|\lambda)_n = x(x - \lambda)(x - 2\lambda) \cdots (x - (n - 1)\lambda)$ with $(x|\lambda)_0 = 1$. Also, we define

$$P_{r,s}(t) = \prod_{i=1}^{r} \left(\frac{t}{(1 + \lambda t)^{a_i/\lambda} - 1}\right) \prod_{i=1}^{s} \left(\frac{2}{(1 + \lambda t)^{b_i/\lambda} + 1}\right)$$

and

$$Q_{r,s}(t) = \prod_{i=1}^{r} \left(\frac{t}{e^{a_i t} - 1}\right) \prod_{i=1}^{s} \left(\frac{2}{e^{b_i t} + 1}\right).$$

Theorem 2.1. *For all $n \geq 0$,*

$$\beta\mathcal{E}_n(\lambda, x|\mathbf{a}; \mathbf{b}) = \sum_{j=0}^{n} \left(\sum_{\ell=j}^{n} \binom{n}{\ell} S_1(\ell, j) \lambda^{\ell-j} \beta\mathcal{E}_{n-\ell}(\lambda|\mathbf{a}; \mathbf{b}) \right) x^j.$$

Proof. By applying the conjugation representation for $s_n(x) \sim (g(t), f(t))$, that is,

$$s_n(x) = \sum_{j=0}^{n} \frac{1}{j!} \langle g(\bar{f}(t))^{-1} \bar{f}(t)^j | x^n \rangle x^j,$$

(see [22, 23]) we obtain

$$\begin{aligned}\langle g(\bar{f}(t))^{-1} \bar{f}(t)^j | x^n \rangle &= \left\langle P_{r,s}(t) \left(\frac{\log(1+\lambda t)}{\lambda} \right)^j |x^n \right\rangle = \lambda^{-j} \left\langle P_{r,s}(t) | \left(\log(1+\lambda t)\right)^j x^n \right\rangle \\ &= \lambda^{-j} \left\langle P_{r,s}(t) | j! \sum_{\ell \geq j} S_1(\ell, j) \lambda^\ell \frac{t^\ell}{\ell!} x^n \right\rangle.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1}{j!} \langle g(\bar{f}(t))^{-1} \bar{f}(t)^j | x^n \rangle &= \sum_{\ell=j}^{n} \binom{n}{\ell} S_1(\ell, j) \lambda^{\ell-j} \left\langle P_{r,s}(t) | x^{n-\ell} \right\rangle \\ &= \sum_{\ell=j}^{n} \binom{n}{\ell} S_1(\ell, j) \lambda^{\ell-j} \left\langle \sum_{m\geq 0} \beta\mathcal{E}_m(\lambda|\mathbf{a};\mathbf{b}) \frac{t^m}{m!} | x^{n-\ell} \right\rangle \\ &= \sum_{\ell=j}^{n} \binom{n}{\ell} S_1(\ell, j) \lambda^{\ell-j} \beta\mathcal{E}_{n-\ell}(\lambda|\mathbf{a};\mathbf{b}),\end{aligned}$$

which completes the proof. □

Theorem 2.2. *For all $n \geq 0$,*

$$\beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = \lambda^n \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{k+r}{r}} S_1(n, m) S_2(k+r, r) \lambda^{k-m} BE_{m-k}(x|\mathbf{a};\mathbf{b}),$$

where $BE_n(x|\mathbf{a};\mathbf{b})$ are the Barnes-type Bernoulli and Euler mixed-type polynomials with

$$\prod_{i=1}^{r} \left(\frac{t}{e^{a_i t} - 1} \right) \prod_{i=1}^{s} \left(\frac{2}{e^{b_i t} + 1} \right) e^{xt} = \sum_{n=0}^{\infty} BE_n(x|\mathbf{a};\mathbf{b}) \frac{t^n}{n!}$$

(see [26]).

Proof. By (9), we have

$$\left(\frac{\lambda}{e^{\lambda t} - 1} \right)^r \prod_{i=1}^{r} \left(e^{a_i t} - 1 \right) \prod_{i=1}^{s} \left(\frac{e^{b_i t} + 1}{2} \right) \beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) \sim \left(1, \frac{1}{\lambda}(e^{\lambda t} - 1) \right), \tag{10}$$

which implies

$$\begin{aligned}
\beta\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b}) &= \sum_{m=0}^{n} S_1(n,m)\lambda^{n-m}Q_{r,s}(t)\left(\frac{e^{\lambda t}-1}{\lambda t}\right)^r x^m \\
&= \sum_{m=0}^{n} S_1(n,m)\lambda^{n-m}Q_{r,s}(t)\left(r!\sum_{k\geq 0} S_2(k+r,r)\frac{\lambda^k}{(k+r)!}t^k\right)x^m \\
&= \lambda^n\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{k+r}{r}}S_1(n,m)S_2(k+r,r)\lambda^{k-m}Q_{r,s}(t)x^{m-k} \\
&= \lambda^n\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{k+r}{r}}S_1(n,m)S_2(k+r,r)\lambda^{k-m}BE_{m-k}(x|\mathbf{a};\mathbf{b}),
\end{aligned}$$

which completes the proof. □

Theorem 2.3. *For all $n \geq 1$,*

$$\beta\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b}) = \sum_{\ell=0}^{n-1}\sum_{k=0}^{n-\ell}\frac{\binom{n-1}{\ell}\binom{n-\ell}{k}}{\binom{k+r}{r}}\lambda^{k+\ell}S_2(k+r,r)B_\ell^{(n)}BE_{n-\ell-k}(x|\mathbf{a};\mathbf{b}).$$

Proof. We proceed the proof by invoking the following transfer formula (see (7) and (8)): for $p_n(x) \sim (1,f(t))$ and $q_n(x) \sim (1,g(t))$, then $q_n(x) = x\left(\frac{f(t)}{g(t)}\right)^n x^{-1}p_n(x)$, for all $n \geq 1$. In our case, by $x^n \sim (1,t)$ and (10), we have

$$\begin{aligned}
&\left(\frac{\lambda}{e^{\lambda t}-1}\right)^r\prod_{i=1}^{r}(e^{a_i t}-1)\prod_{i=1}^{s}\left(\frac{e^{b_i t}+1}{2}\right)\beta\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b}) \\
&= x\left(\frac{\lambda t}{e^{\lambda t}-1}\right)^n x^{n-1} = x\sum_{\ell\geq 0}B_\ell^{(n)}\frac{\lambda^\ell}{\ell!}t^\ell x^{n-1} = \sum_{\ell=0}^{n-1}\binom{n-1}{\ell}\lambda^\ell B_\ell^{(n)}x^{n-\ell}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\beta\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b}) &= \sum_{\ell=0}^{n-1}\binom{n-1}{\ell}\lambda^\ell B_\ell^{(n)}\left(Q_{r,s}(t)\left(\frac{e^{\lambda t}-1}{\lambda t}\right)^r x^{n-\ell}\right) \\
&= \sum_{\ell=0}^{n-1}\binom{n-1}{\ell}\lambda^\ell B_\ell^{(n)}\left(Q_{r,s}(t)\sum_{k\geq 0}S_2(k+r,r)\frac{r!\lambda^k t^k}{(k+r)!}x^{n-\ell}\right) \\
&= \sum_{\ell=0}^{n-1}\sum_{k=0}^{n-\ell}\frac{\binom{n-1}{\ell}\binom{n-\ell}{k}}{\binom{k+r}{r}}\lambda^{k+\ell}S_2(k+r,r)B_\ell^{(n)}Q_{r,s}(t)x^{n-\ell-k} \\
&= \sum_{\ell=0}^{n-1}\sum_{k=0}^{n-\ell}\frac{\binom{n-1}{\ell}\binom{n-\ell}{k}}{\binom{k+r}{r}}\lambda^{k+\ell}S_2(k+r,r)B_\ell^{(n)}BE_{n-\ell-k}(x|\mathbf{a};\mathbf{b}),
\end{aligned}$$

as claimed. □

Note that the *Barnes-type Daehee polynomials with λ-parameter* $D_{n,\lambda}(x|\mathbf{a})$ with $a_1,\ldots,a_r\neq 0$ was defined as

$$\prod_{i=1}^{r}\frac{\log(1+\lambda t)}{\lambda(1+\lambda t)^{a_i/\lambda}-1}(1+\lambda t)^{x/\lambda}=\sum_{n\geq 0}D_{n,\lambda}(x|\mathbf{a})\frac{t^n}{n!},\tag{11}$$

see [15, 16]. When $x=0$ we write $D_{n,\lambda}(\mathbf{a})=D_{n,\lambda}(0|\mathbf{a})$; the *Barnes-type Daehee numbers.*

Theorem 2.4. *For all $n\geq 0$,*

$$\begin{aligned}\beta\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b})&=\sum_{\ell=0}^{n}\sum_{k=0}^{n-\ell}\binom{n}{\ell}\binom{n-\ell}{k}\lambda^\ell b_\ell^{(r)}(x/\lambda)D_{k,\lambda}(\mathbf{a})\mathcal{E}_{n-\ell-k}(\lambda|\mathbf{b})\\&=\sum_{\ell=0}^{n}\sum_{k=0}^{n-\ell}\binom{n}{\ell}\binom{n-\ell}{k}\lambda^\ell b_\ell^{(r)}D_{k,\lambda}(x|\mathbf{a})\mathcal{E}_{n-\ell-k}(\lambda|\mathbf{b})\\&=\sum_{\ell=0}^{n}\sum_{k=0}^{n-\ell}\binom{n}{\ell}\binom{n-\ell}{k}\lambda^\ell b_\ell^{(r)}D_{k,\lambda}(\mathbf{a})\mathcal{E}_{n-\ell-k}(x,\lambda|\mathbf{b}).\end{aligned}$$

Proof. By (9) we have

$$\begin{aligned}&\beta\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b})=\left\langle P_{r,s}(t)(1+\lambda t)^{x/\lambda}|x^n\right\rangle\\&=\left\langle\prod_{i=1}^{r}\left(\frac{\log(1+\lambda t)}{\lambda((1+\lambda t)^{a_i/\lambda}-1)}\right)\prod_{i=1}^{s}\left(\frac{2}{(1+\lambda t)^{b_i/\lambda}+1}\right)\left(\frac{\lambda t}{\log(1+\lambda t)}\right)^r(1+\lambda t)^{x/\lambda}|x^n\right\rangle\\&=\left\langle\prod_{i=1}^{r}\left(\frac{\log(1+\lambda t)}{\lambda((1+\lambda t)^{a_i/\lambda}-1)}\right)\prod_{i=1}^{s}\left(\frac{2}{(1+\lambda t)^{b_i/\lambda}+1}\right)|\sum_{\ell\geq 0}b_\ell^{(r)}(x/\lambda)\frac{\lambda^\ell}{\ell!}t^\ell x^n\right\rangle\\&=\sum_{\ell=0}^{n}\binom{n}{\ell}b_\ell^{(r)}(x/\lambda)\lambda^\ell\left\langle\prod_{i=1}^{r}\left(\frac{\log(1+\lambda t)}{\lambda((1+\lambda t)^{a_i/\lambda}-1)}\right)\prod_{i=1}^{s}\left(\frac{2}{(1+\lambda t)^{b_i/\lambda}+1}\right)|x^{n-\ell}\right\rangle.\end{aligned}$$

Thus, by (11), we obtain

$$\begin{aligned}\beta\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b})&=\sum_{\ell=0}^{n}\binom{n}{\ell}b_\ell^{(r)}(x/\lambda)\lambda^\ell\left\langle\prod_{i=1}^{s}\left(\frac{2}{(1+\lambda t)^{b_i/\lambda}+1}\right)|\sum_{k\geq 0}D_{k,\lambda}(\mathbf{a})\frac{t^k}{k!}x^{n-\ell}\right\rangle\\&=\sum_{\ell=0}^{n}\sum_{k=0}^{n-\ell}\binom{n}{\ell}\binom{n-\ell}{k}\lambda^\ell b_\ell^{(r)}(x/\lambda)D_{k,\lambda}(\mathbf{a})\left\langle\prod_{i=1}^{s}\left(\frac{2}{(1+\lambda t)^{b_i/\lambda}+1}\right)|x^{n-\ell-k}\right\rangle\\&=\sum_{\ell=0}^{n}\sum_{k=0}^{n-\ell}\binom{n}{\ell}\binom{n-\ell}{k}\lambda^\ell b_\ell^{(r)}(x/\lambda)D_{k,\lambda}(\mathbf{a})\mathcal{E}_{n-\ell-k}(\lambda|\mathbf{b}),\end{aligned}$$

which proves the first formula. Similar techniques show the second and the third formulas. □

3. Recurrence relations

In this section, we present several recurrence relations for the Barnes-type degenerate Bernoulli and Euler mixed-type polynomials.

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Theorem 3.1. *For all $n \geq 0$,*

$$\beta\mathcal{E}_n(\lambda, x+y|\mathbf{a};\mathbf{b}) = \sum_{j=0}^{n} \binom{n}{j} \beta\mathcal{E}_j(\lambda, x|\mathbf{a};\mathbf{b})(y|\lambda)_{n-j}.$$

Proof. By (9) we have $\left(\frac{\lambda t}{e^{\lambda t}-1}\right)^r \frac{1}{Q_{r,s}(t)} \beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = (x|\lambda)_n \sim \left(1, \frac{e^{\lambda t}-1}{\lambda}\right)$, which implies the result. □

Theorem 3.1 with $x = 0$, gives the following result.

Corollary 3.2. *For all $n \geq 0$,*

$$\beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = \sum_{j=0}^{n} \binom{n}{j} \beta\mathcal{E}_{n-j}(\lambda|\mathbf{a};\mathbf{b})(x|\lambda)_j.$$

Theorem 3.3. *For all $n \geq 1$,*

$$\beta\mathcal{E}_n(\lambda, x+\lambda|\mathbf{a};\mathbf{b}) = \beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) + n\lambda\beta\mathcal{E}_{n-1}(\lambda, x|\mathbf{a};\mathbf{b}).$$

Proof. By (7) we have that $f(t)s_n(x) = ns_{n-1}(x)$ when $s_n(x) \sim (g(t), f(t))$. In our case, from (9), we have

$$\frac{e^{\lambda t}-1}{\lambda} \beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = n\beta\mathcal{E}_{n-1}(\lambda, x|\mathbf{a};\mathbf{b}),$$

which implies that $\beta\mathcal{E}_n(\lambda, x+\lambda|\mathbf{a};\mathbf{b}) - \beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = n\lambda\beta\mathcal{E}_{n-1}(\lambda, x|\mathbf{a};\mathbf{b})$, as required. □

Theorem 3.4. *For all $n \geq 1$,*

$$\begin{aligned} \frac{d}{dx}\beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) &= n! \sum_{\ell=0}^{n-1} \frac{(-\lambda)^{n-1-\ell}}{\ell!(n-\ell)} \beta\mathcal{E}_\ell(\lambda, x|\mathbf{a};\mathbf{b}) \\ &= n\lambda^{n-1} \sum_{\ell=0}^{n-1} S_1(n-1, \ell)\lambda^{-\ell} BE_\ell(x|\mathbf{a};\mathbf{b}). \end{aligned}$$

Proof. By (7) we have $\frac{d}{dx}s_n(x) = \sum_{\ell=0}^{n-1} \binom{n}{\ell} \langle \bar{f}(t)|x^{n-\ell}\rangle s_\ell(x)$ when $s_n(x) \sim (g(t), f(t))$. In our case, from (9), we have

$$\begin{aligned} \langle \bar{f}(t)|x^{n-\ell}\rangle &= \langle \frac{1}{\lambda}\log(1+\lambda t)|x^{n-\ell}\rangle = \lambda^{-1}\langle \sum_{m\geq 1} \frac{(-1)^{m-1}\lambda^m t^m}{m} |x^{n-\ell}\rangle \\ &= \lambda^{-1}(-1)^{n-\ell-1}\lambda^{n-\ell}(n-\ell-1)! = (-\lambda)^{n-\ell-1}(n-\ell-1)!. \end{aligned}$$

Thus $\frac{d}{dx}\beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = n! \sum_{\ell=0}^{n-1} \frac{(-\lambda)^{n-1-\ell}}{\ell!(n-\ell)} \beta\mathcal{E}_\ell(\lambda, x|\mathbf{a};\mathbf{b})$, as required.

To show the second formula, we note that $(x|\lambda)_n = \sum_{\ell=0}^{n} S_1(n,\ell)\lambda^{n-\ell}x^\ell \sim \left(1, \frac{e^{\lambda t}-1}{\lambda}\right)$, which shows that $\frac{e^{\lambda t}-1}{\lambda}(x|\lambda)_n = n(x|\lambda)_{n-1}$. Thus $\left(\frac{e^{\lambda t}-1}{\lambda}\right)^r (x|\lambda)_n = (n)_r(x|\lambda)_{n-r}$, for all

$n \geq r$. Thus, by (10), we have

$$\begin{aligned}\frac{d^r}{dx^r}\beta\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b}) &= t^r\beta\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b}) = Q_{r,s}(t)\left(\frac{e^{\lambda t}-1}{\lambda}\right)^r(x|\lambda)_n \\ &= (n)_rQ_{r,s}(t)(x|\lambda)_{n-r} = (n)_r\lambda^{n-r}\sum_{m=0}^{n-r}S_1(n-r,m)\lambda^{-m}BE_m(x|\mathbf{a};\mathbf{b}),\end{aligned}$$

which completes the proof. □

Theorem 3.5. *For all $n \geq 1$,*

$$\begin{aligned}(1-r/n)\beta\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b}) &= \left(x-\sum_{i=1}^r a_i - \sum_{j=1}^s b_j\right)\beta\mathcal{E}_{n-1}(\lambda,x-\lambda|\mathbf{a};\mathbf{b}) \\ &\quad - \frac{1}{n}\sum_{i=1}^r a_i\beta\mathcal{E}_n(\lambda,x-\lambda|a_i,\mathbf{a};\mathbf{b}) + \frac{1}{2}\sum_{i=1}^s b_i\beta\mathcal{E}_{n-1}(\lambda,x-\lambda|\mathbf{a};b_i,\mathbf{b}).\end{aligned}$$

Proof. Let $n \geq 1$. By (9), we have

$$\begin{aligned}\beta\mathcal{E}_n(\lambda,y|\mathbf{a};\mathbf{b}) &= \left\langle P_{r,s}(t)(1+\lambda t)^{y/\lambda}|x^n\right\rangle \\ &= \left\langle \frac{d}{dt}\left(\prod_{i=1}^r\left(\frac{t}{(1+\lambda t)^{a_i/\lambda}-1}\right)\prod_{i=1}^s\left(\frac{2}{(1+\lambda t)^{b_i/\lambda}+1}\right)(1+\lambda t)^{y/\lambda}\right)|x^{n-1}\right\rangle \\ &= \left\langle \frac{d}{dt}\prod_{i=1}^r\left(\frac{t}{(1+\lambda t)^{a_i/\lambda}-1}\right)\prod_{i=1}^s\left(\frac{2}{(1+\lambda t)^{b_i/\lambda}+1}\right)(1+\lambda t)^{y/\lambda}|x^{n-1}\right\rangle \quad (12)\\ &\quad + \left\langle \prod_{i=1}^r\left(\frac{t}{(1+\lambda t)^{a_i/\lambda}-1}\right)\frac{d}{dt}\prod_{i=1}^s\left(\frac{2}{(1+\lambda t)^{b_i/\lambda}+1}\right)(1+\lambda t)^{y/\lambda}|x^{n-1}\right\rangle \quad (13)\\ &\quad + \left\langle \prod_{i=1}^r\left(\frac{t}{(1+\lambda t)^{a_i/\lambda}-1}\right)\prod_{i=1}^s\left(\frac{2}{(1+\lambda t)^{b_i/\lambda}+1}\right)\frac{d}{dt}(1+\lambda t)^{y/\lambda}|x^{n-1}\right\rangle. \quad (14)\end{aligned}$$

The term in (14) is given by

$$y\left\langle P_{r,s}(t)(1+\lambda t)^{y/\lambda-1}|x^{n-1}\right\rangle = y\beta\mathcal{E}_{n-1}(\lambda,y-\lambda|\mathbf{a};\mathbf{b}). \quad (15)$$

In order to find the first term, namely (12), we note that

$$\begin{aligned}&\frac{d}{dt}\prod_{i=1}^r\left(\frac{t}{(1+\lambda t)^{a_i/\lambda}-1}\right) \\ &= \prod_{i=1}^r\left(\frac{t}{(1+\lambda t)^{a_i/\lambda}-1}\right)\sum_{i=1}^r\left(-\frac{a_i}{1+\lambda t}+\frac{1}{t}\left(1-\frac{a_i}{1+\lambda t}\frac{t}{(1+\lambda t)^{a_i/\lambda}-1}\right)\right),\end{aligned}$$

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where the order of $1-\frac{a_i}{1+\lambda t}\frac{t}{(1+\lambda t)^{a_i/\lambda}-1}$ is at least 1. Thus the term in (12) is given by

$$-\sum_{i=1}^{r} a_i \left\langle P_{r,s}(t)(1+\lambda t)^{y/\lambda-1}|x^{n-1}\right\rangle + \left\langle P_{r,s}(t)(1+\lambda t)^{y/\lambda}|\frac{1}{t}\sum_{i=1}^{r}\left(1-\frac{a_i}{1+\lambda t}\frac{t}{(1+\lambda t)^{a_i/\lambda}-1}\right)x^{n-1}\right\rangle$$

which equals

$$\begin{aligned} &-\sum_{i=1}^{r} a_i\beta\mathcal{E}_{n-1}(\lambda,y-\lambda|\mathbf{a};\mathbf{b})+\frac{r}{n}\left\langle P_{r,s}(t)(1+\lambda t)^{y/\lambda}|x^{n}\right\rangle \\ &\quad -\frac{1}{n}\sum_{i=1}^{r} a_i\left\langle \frac{t}{(1+\lambda t)^{a_i/\lambda}-1}P_{r,s}(t)(1+\lambda t)^{y/\lambda-1}|x^{n}\right\rangle \\ &=-\sum_{i=1}^{r} a_i\beta\mathcal{E}_{n-1}(\lambda,y-\lambda|\mathbf{a};\mathbf{b})+\frac{r}{n}\beta\mathcal{E}_{n}(\lambda,y|\mathbf{a};\mathbf{b})-\frac{1}{n}\sum_{i=1}^{r} a_i\beta\mathcal{E}_{n}(\lambda,y-\lambda|a_i,\mathbf{a};\mathbf{b}). \end{aligned} \tag{16}$$

In order to find the second term, namely (13), we note that

$$\begin{aligned} &\frac{d}{dt}\prod_{i=1}^{s}\left(\frac{2}{(1+\lambda t)^{b_i/\lambda}+1}\right) \\ &=\prod_{i=1}^{s}\left(\frac{2}{(1+\lambda t)^{b_i/\lambda}+1}\right)\sum_{i=1}^{s}\left(-\frac{b_i}{1+\lambda t}+\frac{b_i}{2(1+\lambda t)}\frac{2}{(1+\lambda t)^{b_i/\lambda}+1}\right). \end{aligned}$$

Thus the term in (13) is given by

$$\begin{aligned} &\sum_{i=1}^{s} b_i\left\langle\left(-1+\frac{1}{(1+\lambda t)^{b_i/\lambda}+1}\right)P_{r,s}(t)(1+\lambda t)^{y/\lambda-1}|x^{n-1}\right\rangle \\ &=-\sum_{i=1}^{s} b_i\beta\mathcal{E}_{n-1}(\lambda,y-\lambda|\mathbf{a};\mathbf{b})+\frac{1}{2}\sum_{i=1}^{s} b_i\beta\mathcal{E}_{n-1}(\lambda,y-\lambda|\mathbf{a};b_i,\mathbf{b}). \end{aligned} \tag{17}$$

Altogether, namely by (15), (16) and (17), we complete the proof. □

Theorem 3.6. *For $n \geq 0$,*

$$\begin{aligned} &\beta\mathcal{E}_{n+1}(\lambda,x|\mathbf{a};\mathbf{b})=x\beta\mathcal{E}_{n}(\lambda,x-\lambda|\mathbf{a};\mathbf{b}) \\ &-\lambda^{n}\sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{\ell=0}^{k}\frac{\lambda^{-k}S_1(n,m)S_2(m-k+r,r)\binom{m}{k}\binom{k}{\ell}}{\binom{m-k+r}{r}}\cdot \\ &\cdot\left(\frac{B_{k-\ell+1}(1)}{k-\ell+1}\left(\sum_{i=1}^{r}a_i^{k-\ell+1}-r\lambda^{k-\ell+1}\right)+\frac{E_{k-\ell}(1)}{2}\left(\sum_{j=1}^{r}b_j^{k-\ell+1}\right)\right)BE_{\ell}(x-\lambda|\mathbf{a};\mathbf{b}). \end{aligned}$$

Proof. By (7), we have that $s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)}\right)\frac{1}{f'(t)}s_n(x)$ when $s_n(x) \sim (g(t), f(t))$. In our case, see (9), we have

$$\beta\mathcal{E}_{n+1}(\lambda, x|\mathbf{a};\mathbf{b}) = x\beta\mathcal{E}_n(\lambda, x-\lambda|\mathbf{a};\mathbf{b}) - e^{-\lambda t}\frac{g'(t)}{g(t)}\beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}),$$

where

$$\begin{aligned}
\frac{g'(t)}{g(t)} &= (\log g(t))' \\
&= \left(r\log\lambda - r\log(e^{\lambda t}-1) + \sum_{i=1}^{r}\log(e^{a_it}-1) + \sum_{j=1}^{s}\log(e^{b_jt}+1) - s\log 2\right)' \\
&= -\frac{r\lambda e^{\lambda t}}{e^{\lambda t}-1} + \sum_{i=1}^{r}\frac{a_ie^{a_it}}{e^{a_it}-1} + \sum_{j=1}^{s}\frac{b_je^{b_jt}}{e^{b_jt}+1} \\
&= \frac{1}{t}\left(-\frac{r\lambda te^{\lambda t}}{e^{\lambda t}-1} + \sum_{i=1}^{r}\frac{a_ite^{a_it}}{e^{a_it}-1}\right) + \frac{1}{2}\sum_{j=1}^{s}\frac{2b_je^{b_jt}}{e^{b_jt}+1} \\
&= \frac{1}{t}\left(-r\sum_{\ell\geq 0}B_\ell(1)\frac{\lambda^\ell t^\ell}{\ell!} + \sum_{i=1}^{r}\sum_{\ell\geq 0}B_\ell(1)\frac{a_i^\ell t^\ell}{\ell!}\right) + \frac{1}{2}\sum_{j=1}^{s}\sum_{\ell\geq 0}E_\ell(1)\frac{b_j^{\ell+1}t^\ell}{\ell!} \\
&= \sum_{\ell\geq 0}\frac{B_{\ell+1}(1)}{(\ell+1)!}\left(\sum_{i=1}^{r}a_i^{\ell+1} - r\lambda^{\ell+1}\right)t^\ell + \frac{1}{2}\sum_{\ell\geq 0}\left(\frac{E_\ell(1)}{\ell!}\sum_{j=1}^{s}b_j^{\ell+1}\right)t^\ell.
\end{aligned}$$

Therefore, by Theorem 2.2, we obtain

$$\begin{aligned}
&\frac{g'(t)}{g(t)}\beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) \\
&= \lambda^n\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{\lambda^{k-m}S_1(n,m)S_2(k+r,r)\binom{m}{k}}{\binom{k+r}{r}}\sum_{\ell=0}^{m-k}\frac{B_{\ell+1}(1)}{(\ell+1)!}\left(\sum_{i=1}^{r}a_i^{\ell+1} - r\lambda^{\ell+1}\right)t^\ell BE_{m-k}(x|\mathbf{a};\mathbf{b}) \\
&+ \frac{\lambda^n}{2}\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{\lambda^{k-m}S_1(n,m)S_2(k+r,r)\binom{m}{k}}{\binom{k+r}{r}}\sum_{\ell=0}^{m-k}\frac{E_\ell(1)}{\ell!}\left(\sum_{j=1}^{s}b_j^{\ell+1}\right)t^\ell BE_{m-k}(x|\mathbf{a};\mathbf{b}) \\
&= \lambda^n\sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{\ell=0}^{k}\frac{\lambda^{-k}S_1(n,m)S_2(m-k+r,r)\binom{m}{k}\binom{k}{\ell}}{\binom{m-k+r}{r}}\cdot \\
&\qquad\cdot\left(\frac{B_{k-\ell+1}(1)}{k-\ell+1}\left(\sum_{i=1}^{r}a_i^{k-\ell+1} - r\lambda^{k-\ell+1}\right) + \frac{E_{k-\ell}(1)}{2}\left(\sum_{j=1}^{r}b_j^{k-\ell+1}\right)\right)BE_\ell(x|\mathbf{a};\mathbf{b}).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\beta\mathcal{E}_{n+1}(\lambda, x|\mathbf{a};\mathbf{b})\\
&= x\beta\mathcal{E}_n(\lambda, x-\lambda|\mathbf{a};\mathbf{b})\\
&- \lambda^n \sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{\ell=0}^{k} \frac{\lambda^{-k}S_1(n,m)S_2(m-k+r,r)\binom{m}{k}\binom{k}{\ell}}{\binom{m-k+r}{r}}\cdot\\
&\cdot\left(\frac{B_{k-\ell+1}(1)}{k-\ell+1}\left(\sum_{i=1}^{r} a_i^{k-\ell+1} - r\lambda^{k-\ell+1}\right) + \frac{E_{k-\ell}(1)}{2}\left(\sum_{j=1}^{r} b_j^{k-\ell+1}\right)\right) BE_\ell(x-\lambda|\mathbf{a};\mathbf{b}),
\end{aligned}$$

as claimed. □

4. Connections with families of polynomials

The Bernoulli polynomials $B_n^{(\alpha)}(x)$ of order α are defined by the generating function

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n\geq 0} B_n^{(\alpha)}(x)\frac{t^n}{n!},$$

equivalently, $B_n^{(\alpha)}(x) \sim \left(\left(\frac{e^t-1}{t}\right)^{\alpha}, t\right)$ (see [3, 9, 10]). In the next result, we express our polynomials $\beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b})$ in terms of Bernoulli polynomials of order α.

Theorem 4.1. *For $n \geq 0$,*

$$\beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = \sum_{m=0}^{n} \lambda^{-m} d_{n,m} B_m^{(\alpha)}(x),$$

where

$$\begin{aligned}
d_{n,m} = &\sum_{\ell=m}^{n}\sum_{k=0}^{n-\ell}\Bigg[\binom{n}{\ell}\binom{n-\ell}{k}\lambda^{k+\ell}S_1(\ell,m)b_k^{(\alpha)}\cdot\\
&\cdot\sum_{q=0}^{n-\ell-k}\sum_{p=0}^{q}\frac{\binom{n-\ell-k}{q}}{\binom{q+\alpha}{\alpha}}S_1(q+\alpha, q-p+\alpha)S_2(q-p+\alpha,\alpha)\lambda^p\beta\mathcal{E}_{n-\ell-k-q}(\lambda|\mathbf{a};\mathbf{b})\Bigg].
\end{aligned}$$

Proof. Let $\beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = \sum_{m=0}^{n} c_{n,m}B_m^{(\alpha)}(x)$. By (8) and (9), we have

$$\begin{aligned}
c_{n,m} &= \frac{1}{m!\lambda^m}\left\langle P_{r,s}(t)\left(\frac{(1+\lambda t)^{1/\lambda}-1}{t}\right)^{\alpha}\left(\frac{\lambda t}{\log(1+\lambda t)}\right)^{\alpha}(\log(1+\lambda t))^m|x^n\right\rangle\\
&= \frac{1}{m!\lambda^m}\left\langle P_{r,s}(t)\left(\frac{(1+\lambda t)^{1/\lambda}-1}{t}\right)^{\alpha}\left(\frac{\lambda t}{\log(1+\lambda t)}\right)^{\alpha}|m!\sum_{\ell\geq m}S_1(\ell,m)\frac{\lambda^\ell t^\ell}{\ell!}x^n\right\rangle\\
&= \lambda^{-m}\sum_{\ell=m}^{n}\binom{n}{\ell}\lambda^\ell S_1(\ell,m)\left\langle P_{r,s}(t)\left(\frac{(1+\lambda t)^{1/\lambda}-1}{t}\right)^{\alpha}|\left(\frac{\lambda t}{\log(1+\lambda t)}\right)^{\alpha}x^{n-\ell}\right\rangle\\
&= \lambda^{-m}\sum_{\ell=m}^{n}\sum_{k=0}^{n-\ell}\binom{n}{\ell}\binom{n-\ell}{k}\lambda^{k+\ell}S_1(\ell,m)b_k^{(\alpha)}\left\langle P_{r,s}(t)|\left(\frac{(1+\lambda t)^{1/\lambda}-1}{t}\right)^{\alpha}x^{n-\ell-k}\right\rangle.
\end{aligned}$$

Before proceeding further, we note that

$$\left(\frac{(1+\lambda t)^{1/\lambda}-1}{t}\right)^{\alpha}=\left(\frac{e^{\frac{1}{\lambda}\log(1+\lambda t)}-1}{t}\right)^{\alpha}$$
$$=\sum_{q\ge0}\sum_{p=0}^{q}\binom{q+\alpha}{\alpha}^{-1}S_1(q+\alpha,q-p+\alpha)S_2(q-p+\alpha,\alpha)\frac{\lambda^p t^q}{q!}.$$

Thus,

$$c_{n,m}=\lambda^{-m}\sum_{\ell=m}^{n}\sum_{k=0}^{n-\ell}\Bigg[\binom{n}{\ell}\binom{n-\ell}{k}\lambda^{k+\ell}S_1(\ell,m)b_k^{(\alpha)}\cdot$$
$$\cdot\sum_{q=0}^{n-\ell-k}\sum_{p=0}^{q}\frac{\binom{n-\ell-k}{q}}{\binom{q+\alpha}{\alpha}}S_1(q+\alpha,q-p+\alpha)S_2(q-p+\alpha,\alpha)\lambda^p\left\langle P_{r,s}(t)|x^{n-\ell-k-q}\right\rangle\Bigg],$$

which gives

$$c_{n,m}=\lambda^{-m}\sum_{\ell=m}^{n}\sum_{k=0}^{n-\ell}\Bigg[\binom{n}{\ell}\binom{n-\ell}{k}\lambda^{k+\ell}S_1(\ell,m)b_k^{(\alpha)}\cdot$$
$$\cdot\sum_{q=0}^{n-\ell-k}\sum_{p=0}^{q}\frac{\binom{n-\ell-k}{q}}{\binom{q+\alpha}{\alpha}}S_1(q+\alpha,q-p+\alpha)S_2(q-p+\alpha,\alpha)\lambda^p\beta\mathcal{E}_{n-\ell-k-q}(\lambda|\mathbf{a};\mathbf{b})\Bigg],$$

which completes the proof. □

The degenerate Bernoulli polynomials $\beta_n^{(\alpha)}(\lambda,x)$ of order α are defined by the generating function

$$\left(\frac{t}{(1+\lambda t)^{1/\lambda}-1}\right)^{\alpha}(1+\lambda t)^{x/\lambda}=\sum_{n\ge0}\beta_n^{(\alpha)}(\lambda,x)\frac{t^n}{n!},$$

equivalently, $B_n^{(\alpha)}(\lambda,x)\sim\left(\left(\frac{\lambda(e^t-1)}{e^{\lambda t}-1}\right)^{\alpha},\frac{1}{\lambda}(e^{\lambda t}-1)\right)$. Then by using similar arguments as in the proof of Theorem 4.1, we obtain the following result.

Theorem 4.2. *For $n\ge0$,*

$$\beta\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b})=\sum_{m=0}^{n}\binom{n}{m}d_{n,m}\beta_m^{(\alpha)}(\lambda,x),$$

where

$$d_{n,m}=\sum_{q=0}^{n-m}\sum_{p=0}^{q}\frac{\binom{n-m}{q}}{\binom{q+\alpha}{\alpha}}S_1(q+\alpha,q-p+\alpha)S_2(q-p+\alpha,\alpha)\lambda^p\beta\mathcal{E}_{n-m-q}(\lambda|\mathbf{a};\mathbf{b}).$$

The *Frobenius-Euler polynomials* of order α are defined by the generating function

$$\left(\frac{1-\mu}{e^t-\mu}\right)^{\alpha}e^{xt}=\sum_{n\ge0}H_n^{(\alpha)}(x|\mu)\frac{t^n}{n!},$$

equivalently, $H_n^{(\alpha)}(x|\mu)\sim\left(\left(\frac{e^t-\mu}{1-\mu}\right)^{\alpha},t\right)$ (see [2, 13]). In the next result, we express our polynomials $\beta\mathcal{E}_n(\lambda,x|\mathbf{a};\mathbf{b})$ in terms of Frobenius-Euler polynomials.

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Theorem 4.3. *For $n \geq 0$,*

$$\beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = (1-\mu)^{-\alpha}\sum_{m=0}^{n}\lambda^{-m}d_{n,m}H_m^{(\alpha)}(x|\mu),$$

where

$$d_{n,m} = \sum_{\ell=m}^{n}\sum_{k=0}^{n-\ell}\sum_{p=0}^{\alpha}\binom{n}{\ell}\binom{n-\ell}{k}\binom{\alpha}{p}S_1(\ell,m)\lambda^{\ell}(-\mu)^{\alpha-p}\beta\mathcal{E}_k(\lambda|\mathbf{a};\mathbf{b})(p|\lambda)_{n-\ell-k}.$$

Proof. Let $\beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = \sum_{m=0}^{n} c_{n,m}H_m^{(\alpha)}(x|\mu)$. Then

$$\begin{aligned}
c_{n,m} &= \frac{1}{m!(1-\mu)^{\alpha}\lambda^{m}}\left\langle ((1+\lambda t)^{1/\lambda}-\mu)^{\alpha}P_{r,s}(t)|(\log(1+\lambda t))^m x^n\right\rangle \\
&= \frac{1}{m!(1-\mu)^{\alpha}\lambda^{m}}\left\langle ((1+\lambda t)^{1/\lambda}-\mu)^{\alpha}P_{r,s}(t)|m!\sum_{\ell\geq m}S_1(\ell,m)\frac{\lambda^{\ell}t^{\ell}}{\ell!}x^n\right\rangle \\
&= \frac{1}{(1-\mu)^{\alpha}\lambda^{m}}\sum_{\ell=m}^{n}\binom{n}{\ell}\lambda^{\ell}S_1(\ell,m)\left\langle ((1+\lambda t)^{1/\lambda}-\mu)^{\alpha}|P_{r,s}(t)x^{n-\ell}\right\rangle \\
&= \frac{1}{(1-\mu)^{\alpha}\lambda^{m}}\sum_{\ell=m}^{n}\sum_{k=0}^{n-\ell}\binom{n}{\ell}\binom{n-\ell}{k}\lambda^{\ell}S_1(\ell,m)\beta\mathcal{E}_k(\lambda|\mathbf{a};\mathbf{b})\left\langle ((1+\lambda t)^{1/\lambda}-\mu)^{\alpha}|x^{n-\ell-k}\right\rangle.
\end{aligned}$$

Note that

$$\begin{aligned}
\left\langle ((1+\lambda t)^{1/\lambda}-\mu)^{\alpha}|x^{n-\ell-k}\right\rangle &= \left\langle \sum_{p=0}^{\alpha}\binom{\alpha}{p}(-\mu)^{\alpha-p}(1+\lambda t)^{p/\lambda}|x^{n-\ell-k}\right\rangle \\
&= \sum_{p=0}^{\alpha}\binom{\alpha}{p}(-\mu)^{\alpha-p}\left\langle \sum_{q\geq 0}(p|\lambda)_q\frac{t^q}{q!}|x^{n-\ell-k}\right\rangle \\
&= \sum_{p=0}^{\alpha}\binom{\alpha}{p}(-\mu)^{\alpha-p}(p|\lambda)_{n-\ell-k}.
\end{aligned}$$

Thus,

$$c_{n,m} = \frac{1}{(1-\mu)^{\alpha}\lambda^{m}}\sum_{\ell=m}^{n}\sum_{k=0}^{n-\ell}\sum_{p=0}^{\alpha}\binom{n}{\ell}\binom{n-\ell}{k}\binom{\alpha}{p}\lambda^{\ell}(-\mu)^{\alpha-p}S_1(\ell,m)\beta\mathcal{E}_k(\lambda|\mathbf{a};\mathbf{b})(p|\lambda)_{n-\ell-k},$$

which completes the proof. □

The degenerate Euler polynomials $\mathcal{E}_n^{(\alpha)}(\lambda, x)$ of order α are defined by the generating function

$$\left(\frac{2}{(1+\lambda t)^{1/\lambda}+1}\right)^{\alpha}(1+\lambda t)^{x/\lambda} = \sum_{n\geq 0}\mathcal{E}_n^{(\alpha)}(\lambda, x)\frac{t^n}{n!},$$

equivalently, $E_n^{(\alpha)}(\lambda, x) \sim \left(\left(\frac{e^t+1}{2}\right)^{\alpha}, \frac{e^{\lambda t}-1}{\lambda}\right)$. Then by using similar arguments as in the proof of Theorems 4.1 and 4.3, we obtain the following result.

Theorem 4.4. *For $n \geq 0$,*

$$\beta\mathcal{E}_n(\lambda, x|\mathbf{a};\mathbf{b}) = 2^{-\alpha}\sum_{m=0}^{n}\binom{n}{m}d_{n,m}\mathcal{E}_m^{(\alpha)}(\lambda, x),$$

where

$$d_{n,m} = \sum_{q=0}^{n-m}\sum_{p=0}^{\alpha}\binom{n-m}{q}\binom{\alpha}{p}\beta\mathcal{E}_{n-m-q}(\lambda|\mathbf{a};\mathbf{b})(p|\lambda)_q.$$

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Ground state solutions for second order nonlinear p-Laplacian difference equations with periodic coefficients

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Abstract

We study the existence of homoclinic solutions for nonlinear p-Laplacian difference equations with periodic coefficients. The proof of the main result is based on the critical point theory in combination with the Nehari manifold approach. Under rather weaker conditions, we obtain the existence of ground state solutions and considerably improve some existing ones even for some special cases.

Key words: P-Laplacian Difference equations; Nehari manifold; Ground state solutions; Critical point theory.

1 Introduction

Difference equations represent the discrete counterpart of ordinary differential equations, have been widely used in many fields such as computer science, economics, neural network, ecology, cybernetics, etc. In the past decades, the existence of homoclinic solutions for difference equations with p-Laplacian has been extensively studied, The classical method used is fixed point theory, to mention a few, see [1–3] and references therein for details. As it is well known, the critical point theory is used to deal with the existence of solutions of difference equations [4–10]. Here we mention the works of Cabada, Iannizzotto and Tersian [4], Jiang and Zhou [5], Long and Shi [6]. In these papers, critical point theory is applied on bound discrete intervals, which leads to the study of critical points of an energy functional defined on a finite-dimensional Banach space. For unbounded discrete intervals such as the whole set of integers $\mathbb{Z}$, Ma and Guo used critical point theory in combination with periodic approximation to deal with such problems [7]. In the present paper, under convenient assumption,

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without periodic approximation and without verifying Palais-Smale condition, we not only prove the existence of homoclinic solution, but also obtain the ground state solution. we extend [11] to the case of the p-lapacian difference equation with periodic coefficients.

In this paper, our work focus on the existence of homoclinic solution for the following second order nonlinear difference equations with p-Laplacian

$$-\Delta[a(k)\phi_p(\Delta u(k-1))] + b(k)\phi_p(u(k)) = f(k, u(k)), \quad k \in \mathbb{Z}, \tag{1.1}$$

where $\phi_p(t) = |t|^{p-2}t$ for all $t \in \mathbb{R}$, $p > 1$. $a(k), b(k)$ are positive and $T-$periodic sequences, T is a fixed positive integer. $f(k, u) : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}$ is a continuous function on u and $T-$periodic on k. The forward difference operator Δ is defined by

$$\Delta u(k-1) = u(k) - u(k-1), \qquad \text{for all } k \in \mathbb{Z}.$$

where $\mathbb{Z}$ and $\mathbb{R}$ denote the set of all integers and real numbers, respectively.

In addition, we are interested in the existence of nontrivial homoclinic solution for (1.1), that is, solutions that are not equal to 0 identically. We call that a solution $u = \{u(k)\}$ of (1.1) is homoclinic (to 0) if

$$\lim_{|k|\to\infty} u(k) = 0. \tag{1.2}$$

Throughout this paper, we always suppose that the following conditions hold.

(A) $a(k) > 0$ and $a(k + T) = a(k)$ for all $k \in \mathbb{Z}$.

(B) $b(k) > 0$ and $b(k + T) = b(k)$ for all $k \in \mathbb{Z}$.

(f_1) $f \in C(\mathbb{Z} \times \mathbb{R}, \mathbb{R})$, and there exist $C > 0$, $q \in (p, \infty)$ such that

$$|f(k, u)| \le C(1 + |u|^{q-1}), \quad \text{for all } k \in \mathbb{Z}, u \in \mathbb{R}.$$

(f_2) $\lim_{|u|\to 0} f(k, u)/|u|^{p-1} = 0$ uniformly for $k \in \mathbb{Z}$.

(f_3) $\lim_{|u|\to\infty} F(k, u)/|u|^p = +\infty$ uniformly for $k \in \mathbb{Z}$, where $F(k, u)$ is the primitive function of $f(k, u)$, *i.e.*,

$$F(k, u) = \int_0^u f(k, s)ds.$$

(f_4) $u \mapsto f(k, u)/|u|^{p-1}$ is strictly increasing on $(-\infty, 0)$ and $(0, \infty)$.

The main result in this paper is the following theorem:

Theorem 1.1. *Suppose conditions* $(A), (B)$ *and* $(f_1)-(f_4)$ *are satisfied. Then equation* (1.1) *has at least a nontrivial ground state solution.*

Remark 1.1. In [7], Ma and Guo considered the special case of (1.1) with $p = 2$ and obtained the following theorem:

Theorem A *Suppose conditions* $(A), (B)$, (f_2) *and the following generalized Ambrosetti-Rabinowitz superlinear condition are satisfied:*

(GAR) *there exists a constant* $\mu > p$ *such that*

$$0 < \mu F(k,u) \leq f(k,u)u, \quad \text{for all } k \text{ and } u \neq 0, \tag{1.3}$$

Then equation (1.1) *has a nontrivial ground state solution.*

It is easy to see that (1.3) implies (f_3). There exists a p-superlinear function, such as

$$f(k,u) = |u|^{p-2}u\ln(1+|u|),$$

does not satisfy (1.3). However, it satisfies the condition $(f_1)-(f_4)$. So our conditions are weaker than conditions in [7]. And we do not need periodic approximation technique to obtain homoclinic solutions. Furthermore, we obtain the existence of a ground state solution. Therefore, our result not only extends the main result in [7] to difference equations with p-Laplacian but also improves it.

Remark 1.2. In [12], the authors considered the following second order nonlinear difference equations with p-Laplacian

$$-\Delta\phi_p(\Delta u(k-1)) + b(k)\phi_p(u(k)) = f(k,u(k)), \quad k \in \mathbb{Z}, \tag{1.4}$$

without any periodic assumption, they obtained the homoclinic solutions of the equation. However, PS condition need to be proved in [12], in this paper, we only prove the coercive condition (below Lemma 3.2) is satisfied.

Example 1.1. Let

$$f(k,u) = \begin{cases} 0, & u = 0, \\ |u|^{p-2}u\ln(1+|u|), & u \neq 0, \end{cases}$$

for all $k \in \mathbb{Z}$, If (A) and (B) are satisfied, then it is easy to check that all the conditions of our Theorem 1.1 are satisfied. Therefore, the nontrivial homoclinic solution is obtained at once.

The rest of the paper is organized as follows: In Section 2, we establish the variational framework associated with (1.1), then present the main results of this paper. Section 3 is devoted to prove the main result.

2 Preliminaries

In this section, we shall establish the corresponding variational framework associated with (1.1). We are going to define a suitable space E and an energy functional $J \in E$, such that critical points of J in E are exactly solutions of (1.1).

Consider the real sequence spaces

$$l^p \equiv l^p(\mathbb{Z}) = \left\{ u = \{u(k)\}_{k\in\mathbb{Z}} : \forall\ k \in \mathbb{Z}, u(k) \in \mathbb{R}, \|u\|_{l^p} = \left(\sum_{k\in\mathbb{Z}} |u(k)|^p\right)^{\frac{1}{p}} < \infty \right\}. \quad (2.1)$$

Then the following embedding between l^p spaces holds,

$$l^q \subset l^p, \|u\|_{l^p} \leq \|u\|_{l^q}, 1 \leq q \leq p \leq \infty. \quad (2.2)$$

Define the space

$$E := \{\ u \in l^p : \sum_{k\in\mathbb{Z}} [a(k)|\Delta u(k-1)|^p + b(k)|u(k)|^p] < \infty\}.$$

Then E is a Hilbert space equipped with the norm

$$\|u\|^p = \sum_{k\in\mathbb{Z}} \left[a(k)|\Delta u(k-1)|^p + b(k)|u(k)|^p\right].$$

$|\cdot|$ is the usual absolute value in $\mathbb{R}$.

Now we consider the variational functional J defined on E by

$$\begin{aligned} J(u) &= \frac{1}{p}\sum_{k\in\mathbb{Z}} [a(k)|\Delta u(k-1)|^p + b(k)|u(k)|^p] - \sum_{k\in\mathbb{Z}} F(k, u(k)) \\ &= \frac{1}{p}\|u\|^p - \sum_{k\in\mathbb{Z}} F(k, u(k)). \end{aligned}$$

Then $J \in C^1(E, \mathbb{R})$ with for all $v \in E$,

$$\begin{aligned} (J'(u), v) &= \lim_{t\to 0} \frac{J(u + tv) - J(u)}{t} \\ &= \sum_{k\in\mathbb{Z}} [a(k)\phi_p(\Delta u(k-1))\Delta v(k-1) + b(k)\phi_p(u(k))v(k)] \\ &\quad - \sum_{k\in\mathbb{Z}} f(k, u(k))v(k). \end{aligned}$$

and

$$\frac{\partial J(u)}{\partial u(k)} = -a(k)\Delta\phi_p(\Delta u(k-1)) + b(k)\phi_p(u(k)) - f(k, u(k)), \quad k \in \mathbb{Z}.$$

Thus, u is a critical point of J on E only if u is a homoclinic solutions of equation (1.1).

Let

$$c_{min} = \inf\{J(u) : J'(u) = 0, u \in E\backslash\{0\}\}.$$

Then $u_0 \neq 0$ with $J(u_0) = c_{min}$ is said to be a ground state solution of (1.1).

3 Proofs of main result

We define the Nehari manifold

$$\mathcal{N} = \{u \in E \setminus \{0\} : (J'(u), u) = 0\}.$$

To prove the main results, we need some lemmas.

Lemma 3.1. *Assume that* (B), $(f_1) - (f_4)$ *are satisfied. Then for each* $w \in E \setminus \{0\}$, *there exists a unique* $s_w > 0$ *such that* $s_w w \in \mathcal{N}$.

Proof. Let $I(u) = \sum_{k\in\mathbb{Z}} F(k, u(k))$. By (f_2), we have

$$I'(u) = o(\|u\|^{p-1}) \quad \text{as} \quad u \to 0. \tag{3.1}$$

From (f_4), for all $u \neq 0$ and $s > 0$, we have

$$s \mapsto I'(su)u/s^{p-1} \quad \text{is strictly increasing.} \tag{3.2}$$

Let $W \subset E \setminus \{0\}$ be a weakly compact subset and $s > 0$, we claim that

$$I(su)/s^p \to \infty \text{ uniformly for } u \text{ on } W, \text{ as } s \to \infty. \tag{3.3}$$

Indeed, let $\{u_n\} \subset W$. It suffices to show that

$$\text{if } s_n \to \infty, \quad I(s_n u_n)/(s_n)^p \to \infty.$$

as $n \to \infty$. Passing to a subsequence if necessary, $u_n \rightharpoonup u \in E \setminus \{0\}$ and $u_n(k) \to u(k)$ for every k, as $n \to \infty$.

Note that from (f_2) and (f_4), it is easy to get that

$$F(k, u) > 0, \quad \text{for all } u \neq 0. \tag{3.4}$$

Since $|s_n u_n(k)| \to \infty$ and $u_n \neq 0$, by (f_3) and (3.4), we have

$$\frac{I(s_n u_n)}{(s_n)^p} = \sum_{k\in\mathbb{Z}} \frac{F(k, s_n u_n(k))}{|s_n u_n(k)|^p} |u_n(k)|^p \to \infty \quad \text{as } n \to \infty.$$

Therefore, (3.3) holds.

Let $g(s) := J(sw)$, $s > 0$. Then

$$g'(s) = J'(sw)w = s^{p-1}(\|w\|^p - s^{1-p} I'(sw)w),$$

from (3.1)-(3.3), then there exists a unique s_w, such that $g'(s) > 0$ whenever $0 < s < s_w$, $g'(s) < 0$ whenever $s > s_w$ and $g'(s_w) = J'(s_w w)w = 0$. So $s_w w \in \mathcal{N}$. □

Lemma 3.2. *J is coercive on $\mathcal{N}$, i.e., $J(u) \to \infty$ as $\|u\| \to \infty$, $u \in \mathcal{N}$.*

Proof. Suppose by contradiction, there exists a sequence $\{u_n\} \subset \mathcal{N}$ such that $\|u_n\| \to \infty$ and $J(u_n) \leq d$. Let $v_n = \frac{u_n}{\|u_n\|}$, then there exists a subsequence, still denoted by the same notation, such that $v_n \rightharpoonup v$ and $v_n(k) \to v(k)$ for every k, as $n \to \infty$.

First we know that there exist $\delta > 0$ and $k_j \in \mathbb{Z}$ such that

$$|v_n(k_j)| \geq \delta. \tag{3.5}$$

Indeed, if not, then $v_n \to 0$ in l^∞ as $n \to \infty$. For $r > p$,

$$\|v_n\|_{l^r}^r \leq \|v_n\|_{l^\infty}^{r-p} \|v_n\|_{l^p}^p$$

we have $v_n \to 0$ in all $l^r,\ r > p$.

Note that by (f_1) and (f_2), for any $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that

$$|f(k,u)| \leq \varepsilon |u|^{p-1} + c_\varepsilon |u|^{q-1} \quad \text{and} \quad |F(k,u)| \leq \varepsilon |u|^p + c_\varepsilon |u|^q. \tag{3.6}$$

Then for each $s > 0$, we have

$$\sum_{k \in \mathbb{Z}} F(k, sv_n(k)) \leq \varepsilon s^p \|v_n\|_{l^p}^p + c_\varepsilon s^q \|v_n\|_{l^q}^q$$

which implies that $\sum\limits_{k \in \mathbb{Z}} F(k, sv_n(k)) \to 0$ as $n \to \infty$. So

$$d \geq J(u_n) \geq J(sv_n) = \frac{s^p}{p} \|v^{(k)}\|^p - \sum_{k \in \mathbb{Z}} F(k, sv_n(k)) \to \frac{s^p}{p}, \tag{3.7}$$

as $n \to \infty$. This is a contradiction with $s > \sqrt[p]{pd}$.

Due to periodicity of coefficients, we know J and $\mathcal{N}$ are both invariant under T-translation. Making such shifts, we can assume that $1 \leq k_j \leq T-1$ in (3.5). Moreover, passing to a subsequence, we can assume that $k_j = k_0$ is independent of j.

Next we may extract a subsequence, still denoted by $\{v_n\}$, such that $v_n(k) \to v(k)$ for all $k \in \mathbb{Z}$. Specially, for $k = k_0$, inequality (3.5) shows that $|v(k_0)| \geq \delta$, so $v \neq 0$. Since $|u_n(k)| \to \infty$ as $n \to \infty$, it follows again from (f_3) that

$$0 \leq \frac{J(u_n)}{\|u_n\|^p} = \frac{1}{p} - \sum_{k \in \mathbb{Z}} \frac{F(k, u_n(k))}{(u_n(k))^p} (v_n(k))^p \to -\infty \qquad \text{as } n \to \infty,$$

a contradiction again. □

Proof of Theorem 1.1.

The proof consists of five steps. The proof of step 1-3 is similar to [12], for readers' convenience, we give the proof.

step 1. we claim that $\mathcal{N}$ is homeomorphic to the unit sphere S in E.

By (3.1) and (3.3), $g(s) > 0$ for $s > 0$ small and $g(s) < 0$ for $s > 0$ large. So s_w is a unique maximum of $g(s)$ and $s_w w$ is the unique point on the ray $s \mapsto sw$ $(s > 0)$ which intersects $\mathcal{N}$. That is, $u \in \mathcal{N}$ is the unique maximum of J on the ray. Therefore, by Lemma 3.1, we may define the mapping $\hat{m} : E \setminus \{0\} \to \mathcal{N}$ by setting

$$\hat{m}(w) := s_w w.$$

Next we show the mapping $\hat{m}$ is continuous. Indeed, suppose $w_n \to w \neq 0$. Since $\hat{m}(tu) = \hat{m}(u)$ for each $t > 0$, we may assume $w_n \in S$ for all n. Write $\hat{m}(w_n) = s_{w_n} w_n$. Then $\{s_{w_n}\}$ is bounded. If not, $s_{wn} \to \infty$ as $n \to \infty$.

Note that by (f_4), for all $u \neq 0$,

$$\begin{aligned} \frac{1}{p} f(k,u)u - F(k,u) &= \frac{1}{p} f(k,u)u - \int_0^u f(k,s)ds \\ &> \frac{1}{p} f(k,u)u - \frac{f(k,u)}{u^{p-1}} \int_0^u s^{p-1} ds \\ &= 0. \end{aligned}$$

Therefore, for all $u \in \mathcal{N}$, we have

$$J(u) = J(u) - \frac{1}{p} J'(u)u = \sum_{k \in \mathbb{Z}} \left(\frac{1}{p} f(k,u(k))u(k) - F(k,u(k)) \right) > 0. \tag{3.8}$$

Combining with (f_3) and Lemma 3.1, we have

$$0 < \frac{J(s_{wn}w)}{(s_{wn})^p} = \frac{1}{p}\|w\|^p - \sum_{k \in \mathbb{Z}} \frac{F(k, s_{wn}w(k))}{|s_{wn}w(k)|^p} |w(k)|^p \to -\infty, \quad \text{as } n \to \infty,$$

this is a contradiction. Therefore, $s_{w_n} \to s > 0$ after passing to a subsequence if needed. Since $\mathcal{N}$ is closed and $\hat{m}(w_n) = s_{w_n} w_n \to sw, sw \in \mathcal{N}$. Hence $sw = s_w w = \hat{m}(w)$ by the uniqueness of s_w of Lemma 3.1. Therefore, $\hat{m}$ is continuous.

Then we define a mapping $m : S \to \mathcal{N}$ by setting $m := \hat{m}|_S$, then m is a homeomorphism between S and $\mathcal{N}$, and the inverse of m is given by $m^{-1}(u) = \frac{u}{\|u\|}$.

step 2. now we define the functional $\hat{\Psi} : E \setminus \{0\} \to \mathbb{R}$ and $\Psi : S \to \mathbb{R}$ by

$$\hat{\Psi}(w) := J(\hat{m}(w)) \quad \text{and} \quad \Psi(w) := \hat{\Psi}|_S.$$

Then we have

$\hat{\Psi} \in C^1(E \setminus \{0\}, \mathbb{R})$ and $\Psi \in C^1(S, \mathbb{R})$. Moreover,

$$\hat{\Psi}'(w)z = \frac{\|\hat{m}(w)\|}{\|w\|} J'(\hat{m}(w))z \quad \text{for all } w, z \in E, w \neq 0. \tag{3.9}$$

$$\Psi'(w)z = \|m(w)\|J'(m(w))z \quad \text{for all } z \in T_w(S) = \{v \in E : (w,v) = 0\}. \tag{3.10}$$

In fact, let $w \in E \setminus \{0\}$ and $z \in E$. By Lemma 3.1 and the mean value theorem, we obtain

$$\begin{aligned}\hat{\Psi}(w+tz) - \hat{\Psi}(w) &= J(s_{w+tz}(w+tz)) - J(s_w w)\\ &\leq J(s_{w+tz}(w+tz)) - J(s_{w+tz}(w))\\ &= J'(s_{w+tz}(w+\tau_t tz))s_{w+tz}tz,\end{aligned}$$

where $|t|$ is small enough and $\tau_t \in (0,1)$. Similarly,

$$\begin{aligned}\hat{\Psi}(w+tz) - \hat{\Psi}(w) &= J(s_{w+tz}(w+tz)) - J(s_w w)\\ &\geq J(s_w(w+tz)) - J(s_w(w))\\ &= J'(s_w(w+\eta_t tz))s_w tz,\end{aligned}$$

where $\eta_t \in (0,1)$. Combining these two inequalities and the continuity of function $w \mapsto s_w$, we have

$$\lim_{t\to 0}\frac{\hat{\Psi}(w+tz) - \hat{\Psi}(w)}{t} = s_w J'(s_w w)z = \frac{\|\hat{m}(w)\|}{\|w\|}J'(\hat{m}(w))z.$$

Hence the Gâteaux derivative of $\hat{\Psi}$ is bounded linear in z and continuous in w. It follows that $\hat{\Psi}$ is a class of C^1 and (3.9) holds. Note only that since $w \in S$, $m(w) = \hat{m}(w)$, so (3.10) is clear.

step 3. $\{w_n\}$ is a Palais-Smale sequence for Ψ if and only if $\{m(w_n)\}$ is a Palais-Smale sequence for J.

Let $\{w_n\}$ be a Palais-Smale sequence for Ψ, and let $u_n = m(w_n) \in \mathcal{N}$. Since for every $w_n \in S$ we have an orthogonal splitting $E = T_{w_n}S \oplus \mathbb{R}w_n$, we have

$$\|\Psi'(w_n)\| = \sup_{\substack{z\in T_{w_n}S\\ \|z\|=1}}\Psi'(w_n)z = \|m(w_n)\|\sup_{\substack{z\in T_{w_n}S\\ \|z\|=1}}J'(m(w_n))z = \|u_n\|\sup_{\substack{z\in T_{w_n}S\\ \|z\|=1}}J'(u_n)z.$$

Then

$$\begin{aligned}\|\Psi'(w_n)\| &\leq \|u_n\|\|J'(u_n)\| = \|u_n\|\sup_{\substack{z\in T_{w_n}S,\ t\in\mathbb{R}\\ z+tw\neq 0}}\frac{J'(u_n)(z+tw)}{\|z+tw\|}\\ &\leq \|u_n\|\sup_{z\in T_{w_n}S\setminus\{0\}}\frac{J'(u_n)(z)}{\|z\|} = \|\Psi'(w_n)\|,\end{aligned}$$

Therefore

$$\|\Psi'(w_n)\| = \|u_n\|\|J'(u_n)\|. \tag{3.11}$$

By (3.8), for $u_n \in \mathcal{N}$, $J(u_n) > 0$, so there exists a constant $c_0 > 0$ such that $J(u_n) > c_0$. And since $c_0 \leq J(u_n) = \frac{1}{p}\ \|u_n\|^p - I(u_n) \leq \frac{1}{p}\ \|u_n\|^p$, $\|u_n\| \geq \sqrt[p]{pc_0}$. Together with Lemma 3.2, $\sqrt[p]{pc_0} \leq \|u_n\| \leq \sup_n \|u_n\| < \infty$. Hence $\{w_n\}$ is a Palais-Smale sequence for Ψ if and only if $\{u_n\}$ is a Palais-Smale sequence for J.

step 4. by (3.11), $\Psi'(w) = 0$ if and only if $J'(m(w)) = 0$. So w is a critical point of Ψ if and only if $m(w)$ is a nontrivial critical point of J. Moreover, the corresponding values of Ψ and J coincide and $\inf_S \Psi = \inf_{\mathcal{N}} J$.

If $u_0 \in \mathcal{N}$ satisfies $J(u_0) = c := \inf_{u\in\mathcal{N}} J(u)$, then $m^{-1}(u_0) \in S$ is a minimizer of Ψ and therefore a critical point of Ψ, so u_0 is a critical point of J. It remains to show that there exists a minimizer $u \in \mathcal{N}$ of $J|_{\mathcal{N}}$.

Let $\{w_n\} \subset S$ be a minimizing sequence for Ψ. By Ekeland's variational principle we may assume $\Psi(w_n) \to c$, $\Psi'(w_n) \to 0$ as $n \to \infty$, hence $J(u_n) \to c, J'(u_n) \to 0$ as $n \to \infty$, where $u_n := m(w_n) \in \mathcal{N}$.

We know that $\{u_n\}$ is bounded in $\mathcal{N}$ by Lemma 3.2, then there exists a subsequence, still denoted by the same notation, such that u_n weakly converges to some $u \in E$. We claim that there exist $\delta > 0$ and $k_j \in \mathbb{Z}$ such that

$$|u_n(k_j)| \geq \delta. \tag{3.12}$$

Indeed, if not, then $u_n \to 0$ in l^∞ as $n \to \infty$. From the simple fact that, for $r > p$,

$$\|u_n\|_{l^r}^r \leq \|u_n\|_{l^\infty}^{r-p}\|u_n\|_{l^p}^p$$

we have $u_n \to 0$ in all l^r, $r > p$. By (3.6), we know

$$\begin{aligned}\sum_{k\in\mathbb{Z}} f(k,u_n(k))u_n(k) &\leq \varepsilon\sum_{k\in\mathbb{Z}}|u_n(k)|^{p-1}\cdot|u_n(k)| + c_\varepsilon\sum_{k\in\mathbb{Z}}|u_n(k)|^{q-1}\cdot|u_n(k)| \\ &\leq \varepsilon\|u_n\|_{l^p}^p + c_\varepsilon\|u_n\|_{l^q}^{q-1}\end{aligned}$$

which implies that $\sum\limits_{k\in\mathbb{Z}} f(k,u_n(k))u_n(k) = o(\|u_n\|)$ as $n \to \infty$. Then

$$o(\|u_n\|) = (J'(u_n), u_n) = \|u_n\|^p - \sum_{k\in\mathbb{Z}} f(k,u_n(k))u_n(k) = \|u_n\|^p - o(\|u_n\|).$$

So $\|u_n\|^p \to 0$, as $n \to \infty$, which contradicts with $u_n \in \mathcal{N}$.

Since J and J' are both invariant under T-translation. Making such shifts, we can assume that $1 \leq k_j \leq T-1$ in (3.12). Moreover passing to a subsequence, we can assume that $k_j = k_0$ is independent of j. Extract a subsequence, still denoted by $\{u_n\}$, we have $u_n \rightharpoonup u$ and $u_n(k) \to u(k)$ for all $k \in \mathbb{Z}$. Specially, for $k = k_0$, inequality (3.12) shows that $|u(k_0)| \geq \delta$, so $u \neq 0$. Hence $u \in \mathcal{N}$.

step 5. we need to show that $J(u) = c$. By Fatou's lemma, we have

$$\begin{aligned} c &= \lim_{n\to\infty}\left(J(u_n) - \frac{1}{2}J'(u_n)u_n\right) = \lim_{n\to\infty}\sum_{k\in\mathbb{Z}}\left(\frac{1}{2}f(k,u_n(k))u_n(k) - F(k,u_n(k))\right) \\ &\geq \sum_{k\in\mathbb{Z}}\left(\frac{1}{2}f(k,u(k))u(k) - F(k,u(k))\right) = J(u) - \frac{1}{2}J'(u)u = J(u) \geq c.\end{aligned}$$

Hence $J(u) = c$. The proof of Theorem 1.1 is completed. □

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On a solutions of fourth order rational systems of difference equations

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ABSTRACT

In this paper, we get the form of the solutions of the following difference equation systems of order four

$$x_{n+1} = \frac{y_n x_{n-2}}{y_n + y_{n-3}}, \quad y_{n+1} = \frac{x_n y_{n-2}}{\pm x_n \pm x_{n-3}}, \quad n = 0, 1, 2, \cdots,$$

where the initial conditions $x_{-3},\ x_{-2},\ x_{-1},\ x_0,\ y_{-3},\ y_{-2},\ y_{-1},\ y_0$ are arbitrary non zero real numbers.

Keywords: difference equations, recursive sequences, system of difference equations, stability, periodicity, boundedness.

Mathematics Subject Classification: 39A10.

1. INTRODUCTION

Difference equations enter as approximations of continuous problems and as models describing life situations in many directions. Recently there has been a great interest in studying difference equations, see, for instance [4], [11], [30] and references cited therein, as well as in studying systems of difference equations (see, e.g. [1], [3], [6], [8]-[10]).

Some of the systems of difference equations that are of considerable interest nowadays are symmetric or those obtained from symmetric ones by modifications of their parameters or the sequence coefficients appearing in them (for the case of nonautonomous systems of difference equations). Such systems are studied, for example, in the following papers: Clark et al. [2] has investigated the global stability properties and asymptotic behavior of solutions of the system

$$x_{n+1} = \frac{x_n}{a + c y_n}, \quad y_{n+1} = \frac{y_n}{b + d x_n}.$$

Din and Elsayed [5] investigated the boundedness character, persistence, local and global behavior of positive solutions of following two directional interactive and invasive species model

$$x_{n+1} = \alpha + \beta x_n + \gamma x_{n-1} e^{-y_n}, \ y_{n+1} = \delta + \epsilon y_n + \zeta y_{n-1} e^{-x_n}.$$

Halim et al. [13] deal with the form of the solutions of the two following systems of rational difference equations

$$\begin{aligned} x_{n+1} &= \frac{y_n(x_{n-2} + y_{n-3})}{y_{n-3} + x_{n-2} - y_n}, \ y_{n+1} = \frac{x_{n-1}(x_{n-1} + y_{n-2})}{2x_{n-1} + y_{n-2}}, \\ x_{n+1} &= \frac{(y_{n-3} - x_{n-2}) y_n}{y_{n-3} - x_{n-2} + y_n}, \ y_{n+1} = \frac{(y_{n-2} - x_{n-1}) x_{n-1}}{y_{n-2}}. \end{aligned}$$

Kurbanli [21] investigated the behavior of the solution of the difference equation system

$$x_{n+1}=\frac{x_{n-1}}{x_{n-1}y_n-1},\ y_{n+1}=\frac{y_{n-1}}{y_{n-1}x_n-1},\quad z_{n+1}=\frac{1}{z_n y_n}.$$

The authors in [27] have got the form of the solutions of some systems of the following rational difference equations

$$x_{n+1}=\frac{x_{n-1}}{\alpha-x_{n-1}y_n},\ y_{n+1}=\frac{y_{n-1}}{\beta+\gamma y_{n-1}x_n}.$$

In [29] Papaschinnopoulos and Schinas studied the oscillatory behavior, the boundedness of the solutions, and the global asymptotic stability of the positive equilibrium of the system of nonlinear difference equations

$$x_{n+1}=A+\frac{y_n}{x_{n-p}},\quad y_{n+1}=A+\frac{x_n}{y_{n-q}}.$$

In [39], Yalcinkaya et al. studied the periodic character of the following two systems of difference equations

$$x^{(1)}_{n+1}=\frac{x^{(2)}_n}{x^{(2)}_n-1},x^{(2)}_{n+1}=\frac{x^{(3)}_n}{x^{(3)}_n-1},\ldots,x^{(k)}_{n+1}=\frac{x^{(1)}_n}{x^{(1)}_n-1},$$

and

$$x^{(1)}_{n+1}=\frac{x^{(k)}_n}{x^{(k)}_n-1},x^{(2)}_{n+1}=\frac{x^{(1)}_n}{x^{(1)}_n-1},\ldots,x^{(k)}_{n+1}=\frac{x^{(k-1)}_n}{x^{(k-1)}_n-1},$$

where the initial values are nonzero real numbers for $x_0^{(1)},x_0^{(2)},\ldots,x_0^{(k)}\neq 1$.

In [42]-[43] Zhang et al. studied the boundedness, the persistence and global asymptotic stability of the positive solutions of the systems of difference equations

$$x_{n+1}=A+\frac{y_{n-m}}{x_n},\qquad y_{n+1}=A+\frac{x_{n-m}}{y_n},$$

and

$$x_n=A+\frac{1}{y_{n-p}},\qquad y_n=A+\frac{y_{n-1}}{x_{n-r}y_{n-s}}.$$

Similar to difference equations and nonlinear systems of rational difference equations were investigated see [12]-[45].

In this paper, we obtain the expressions of the solutions of the following nonlinear systems of difference equations

$$x_{n+1}=\frac{y_n x_{n-2}}{y_n+y_{n-3}},\qquad y_{n+1}=\frac{x_n y_{n-2}}{\pm x_n\pm x_{n-3}},\quad n=0,1,2,\cdots,$$

where the initial values $x_{-3},\ x_{-2},\ x_{-1},\ x_0,\ y_{-3},\ y_{-2},\ y_{-1},\ y_0$ are arbitrary non zero real numbers, moreover, we take some numerical examples for the equation to illustrate the results.

2. ON THE SYSTEM $X_{N+1}=\frac{Y_N X_{N-2}}{Y_N+Y_{N-3}},\quad Y_{N+1}=\frac{X_N Y_{N-2}}{X_N+X_{N-3}}$

In this section,we study the solutions of the following system of difference equations

$$x_{n+1}=\frac{y_n x_{n-2}}{y_n+y_{n-3}},\qquad y_{n+1}=\frac{x_n y_{n-2}}{x_n+x_{n-3}},\tag{1}$$

where the initial values $x_{-3},\ x_{-2},\ x_{-1},\ x_0,\ y_{-3},\ y_{-2},\ y_{-1},\ y_0$ are arbitrary nonzero real numbers.

Theorem 1. Suppose that $\{x_n,y_n\}$ are solutions of the system (1). Then for $n=0,1,2,\cdots$, we have the following formula

$$x_{6n-3}=\frac{ad^nh^n}{\prod_{i=0}^{n-1}(e+(6i+3)h)(a+(6i)d)},\qquad x_{6n-2}=\frac{bd^nh^n}{\prod_{i=0}^{n-1}(e+(6i+1)h)(a+(6i+4)d)},$$

$$x_{6n-1}=\frac{cd^nh^n}{\prod_{i=0}^{n-1}(e+(6i+5)h)(a+(6i+2)d)},\qquad x_{6n}=\frac{d^{n+1}h^n}{\prod_{i=0}^{n-1}(e+(6i+3)h)(a+(6i+6)d)},$$

$$
\begin{aligned}
x_{6n+1} &= \frac{bd^n h^{n+1}}{(e+h)\prod_{i=0}^{n-1}(e+(6i+7)h)(a+(6i+4)d)}, \quad x_{6n+2} = \frac{cd^{n+1}h^n}{(a+2d)\prod_{i=0}^{n-1}(e+(6i+5)h)(a+(6i+8)d)},\\
y_{6n-3} &= \frac{ed^n h^n}{\prod_{i=0}^{n-1}(e+(6i)h)(a+(6i+3)d)}, \qquad y_{6n-2} = \frac{fd^n h^n}{\prod_{i=0}^{n-1}(e+(6i+4)h)(a+(6i+1)d)},\\
y_{6n-1} &= \frac{gd^n h^n}{\prod_{i=0}^{n-1}(e+(6i+2)h)(a+(6i+5)d)}, \qquad y_{6n} = \frac{d^n h^{n+1}}{\prod_{i=0}^{n-1}(e+(6i+6)h)(a+(6i+3)d)},\\
y_{6n+1} &= \frac{fd^{n+1}h^n}{(a+d)\prod_{i=0}^{n-1}(e+(6i+4)h)(a+(6i+7)d)}, \quad y_{6n+2} = \frac{gd^n h^{n+1}}{(e+2h)\prod_{i=0}^{n-1}(e+(6i+8)h)(a+(6i+5)d)},
\end{aligned}
$$

where $x_{-3}=a,\ x_{-2}=b,\ x_{-1}=c,\ x_0=d,\ y_{-3}=e,\ y_{-2}=f,\ y_{-1}=g,\ y_0=h.$

Proof. By using mathematical induction. The result holds for $n=0$. Suppose that the result holds for $n-1$

$$
\begin{aligned}
x_{6n-7} &= \frac{cd^{n-1}h^{n-1}}{\prod_{i=0}^{n-2}(e+(6i+5)h)(a+(6i+2)d)}, \qquad x_{6n-6} = \frac{d^n h^{n-1}}{\prod_{i=0}^{n-2}(e+(6i+3)h)(a+(6i+6)d)},\\
x_{6n-5} &= \frac{bd^{n-1}h^n}{(e+h)\prod_{i=0}^{n-2}(e+(6i+7)h)(a+(6i+4)d)}, \quad x_{6n-4} = \frac{cd^n h^{n-1}}{(a+2d)\prod_{i=0}^{n-2}(e+(6i+5)h)(a+(6i+8)d)},\\
y_{6n-7} &= \frac{gd^{n-1}h^{n-1}}{\prod_{i=0}^{n-2}(e+(6i+2)h)(a+(6i+5)d)}, \qquad y_{6n-6} = \frac{d^{n-1}h^n}{\prod_{i=0}^{n-2}(e+(6i+6)h)(a+(6i+3)d)},\\
y_{6n-5} &= \frac{fd^n h^{n-1}}{(a+d)\prod_{i=0}^{n-2}(e+(6i+4)h)(a+(6i+7)d)}, \quad y_{6n-4} = \frac{gd^{n-1}h^n}{(e+2h)\prod_{i=0}^{n-2}(e+(6i+8)h)(a+(6i+5)d)}.
\end{aligned}
$$

From system (1) we can prove as follow

$$
\begin{aligned}
x_{6n-3} &= \frac{y_{6n-4}x_{6n-6}}{y_{6n-4}+y_{6n-7}} = \frac{\left(\frac{gd^{n-1}h^n}{(e+2h)\prod_{i=0}^{n-2}(e+(6i+8)h)(a+(6i+5)d)}\right)\left(\frac{d^n h^{n-1}}{\prod_{i=0}^{n-2}(e+(6i+3)h)(a+(6i+6)d)}\right)}{\left(\frac{gd^{n-1}h^n}{(e+2h)\prod_{i=0}^{n-2}(e+(6i+8)h)(a+(6i+5)d)}\right)+\left(\frac{gd^{n-1}h^{n-1}}{\prod_{i=0}^{n-2}(e+(6i+2)h)(a+(6i+5)d)}\right)}\\
&= \frac{d^n h^n}{\prod_{i=0}^{n-2}(e+(6i+3)h)(a+(6i+6)d)\left(h+\frac{(e+2h)\prod_{i=0}^{n-2}(e+(6i+8)h)}{\prod_{i=0}^{n-2}(e+(6i+2)h)}\right)}\\
&= \frac{d^n h^n}{(e+(6n-3)h)\prod_{i=0}^{n-2}(e+(6i+3)h)(a+(6i+6)d)} = \frac{ad^n h^n}{\prod_{i=0}^{n-1}(e+(6i+3)h)(a+(6i)d)},
\end{aligned}
$$

$$
\begin{aligned}
y_{6n-3} &= \frac{x_{6n-4}y_{6n-6}}{x_{6n-4}+x_{6n-7}} = \frac{\frac{cd^n h^{n-1}}{(a+2d)\prod_{i=0}^{n-2}(e+(6i+5)h)(a+(6i+8)d)}\frac{d^{n-1}h^n}{\prod_{i=0}^{n-2}(e+(6i+6)h)(a+(6i+3)d)}}{\left(\frac{cd^n h^{n-1}}{(a+2d)\prod_{i=0}^{n-2}(e+(6i+5)h)(a+(6i+8)d)}+\frac{cd^{n-1}h^{n-1}}{\prod_{i=0}^{n-2}(e+(6i+5)h)(a+(6i+2)d)}\right)}\\
&= \frac{d^n h^n}{(a+2d)\prod_{i=0}^{n-2}(a+(6i+8)d)(e+(6i+6)h)(a+(6i+3)d)\left(\frac{d}{(a+2d)\prod_{i=0}^{n-2}(a+(6i+8)d)}+\frac{1}{\prod_{i=0}^{n-2}(a+(6i+2)d)}\right)}\\
&= \frac{d^n h^n}{\prod_{i=0}^{n-2}(e+(6i+6)h)(a+(6i+3)d)\left(d+\frac{(a+2d)\prod_{i=0}^{n-2}(a+(6i+8)d)}{\prod_{i=0}^{n-2}(a+(6i+2)d)}\right)}\\
&= \frac{d^n h^n}{\prod_{i=0}^{n-2}(e+(6i+6)h)(a+(6i+3)d)\left(d+\frac{(a+2d)\prod_{i=0}^{n-2}(a+(6i+8)d)}{\prod_{i=0}^{n-2}(a+(6i+2)d)}\right)}\\
&= \frac{d^n h^n}{\prod_{i=0}^{n-2}(e+(6i+6)h)(a+(6i+3)d)(a+(6n-3)d)} = \frac{ed^n h^n}{\prod_{i=0}^{n-1}(e+(6i)h)(a+(6i+3)d)}.
\end{aligned}
$$

The other relations can be proved similarly, this completes the proof.

Lemma 1. Every positive solution of system (1) is bounded and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$.

Proof: It follows from system (1), that

$$x_{n+1} = \frac{y_n x_{n-2}}{y_n + y_{n-3}} < \frac{y_n x_{n-2}}{y_n} = x_{n-2}, \qquad y_{n+1} = \frac{x_n y_{n-2}}{x_n + x_{n-3}} < \frac{x_n y_{n-2}}{x_n} = y_{n-2}.$$

Then the subsequences $\{x_{3n-2}\}_{n=0}^{\infty}$, $\{x_{3n-1}\}_{n=0}^{\infty}$, $\{x_{3n}\}_{n=0}^{\infty}$, $\{y_{3n-2}\}_{n=0}^{\infty}$, $\{y_{3n-1}\}_{n=0}^{\infty}$, $\{y_{3n}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by M, N respectively since $M = \max\{x_{-3}, x_{-2}, x_{-1}, x_0\}$, $N = \max\{y_{-3}, y_{-2}, y_{-1}, y_0\}$.

3. ON THE SYSTEM $X_{N+1} = \frac{Y_N X_{N-2}}{Y_N + Y_{N-3}}, \quad Y_{N+1} = \frac{X_N Y_{N-2}}{X_N - X_{N-3}}$

We study, in this section, the solutions formulas of the system of rational difference equations

$$x_{n+1} = \frac{y_n x_{n-2}}{y_n + y_{n-3}}, \qquad y_{n+1} = \frac{x_n y_{n-2}}{x_n - x_{n-3}}, \tag{2}$$

where the initial values $x_{-3},\ x_{-2},\ x_{-1},\ x_0,\ y_{-3},\ y_{-2},\ y_{-1},\ y_0$ are arbitrary nonzero real numbers.

Theorem 2. Assume that $\{x_n, y_n\}$ are solutions of system (2) with $x_{-3} \neq x_0,\ x_{-3} \neq 2x_0$ and $y_{-3} \neq \pm y_0$. Then for $n = 0, 1, 2, ...,$

$$\begin{aligned}
x_{12n-3} &= \frac{d^{2n}h^{2n}}{a^{n-1}(h+e)^n(h-e)^n(2d-a)^n}, & y_{12n-3} &= \frac{h^{2n}d^{2n}}{e^{2n-1}(d-a)^{2n}},\\
x_{12n-2} &= \frac{bd^{2n}h^{2n}}{a^{n}(h+e)^n(h-e)^n(2d-a)^n}, & y_{12n-2} &= \frac{fh^{2n}d^{2n}}{e^{2n}(d-a)^{2n}},\\
x_{12n-1} &= \frac{cd^{2n}h^{2n}}{a^{n}(h+e)^n(h-e)^n(2d-a)^n}, & y_{12n-1} &= \frac{gh^{2n}d^{2n}}{e^{2n}(d-a)^{2n}},\\
x_{12n} &= \frac{d^{2n+1}h^{2n}}{a^{n}(h+e)^n(h-e)^n(2d-a)^n}, & y_{12n} &= \frac{h^{2n+1}d^{2n}}{e^{2n}(d-a)^{2n}},\\
x_{12n+1} &= \frac{bd^{2n}h^{2n+1}}{a^{n}(h+e)^{n+1}(h-e)^n(2d-a)^n}, & y_{12n+1} &= \frac{fh^{2n}d^{2n+1}}{e^{2n}(d-a)^{2n+1}},\\
x_{12n+2} &= \frac{cd^{2n+1}h^{2n}}{a^{n}(h+e)^n(h-e)^n(2d-a)^{n+1}}, & y_{12n+2} &= \frac{-gh^{2n+1}d^{2n}}{e^{2n+1}(d-a)^{2n}},\\
x_{12n+3} &= \frac{d^{2n+1}h^{2n+1}}{a^{n}(h+e)^n(h-e)^{n+1}(2d-a)^n}, & y_{12n+3} &= \frac{-h^{2n+1}d^{2n+1}}{e^{2n}(d-a)^{2n+1}},\\
x_{12n+4} &= \frac{bd^{2n+1}h^{2n+1}}{a^{n+1}(h+e)^{n+1}(h-e)^n(2d-a)^n}, & y_{12n+4} &= \frac{fh^{2n+1}d^{2n+1}}{e^{2n+1}(d-a)^{2n+1}},\\
x_{12n+5} &= \frac{cd^{2n+1}h^{2n+1}}{a^{n}(h+e)^{n+1}(h-e)^n(2d-a)^{n+1}}, & y_{12n+5} &= \frac{-gh^{2n+1}d^{2n+1}}{e^{2n+1}(d-a)^{2n+1}},\\
x_{12n+6} &= \frac{d^{2n+2}h^{2n+1}}{a^{n}(h+e)^n(h-e)^{n+1}(2d-a)^{n+1}}, & y_{12n+6} &= \frac{h^{2n+2}d^{2n+1}}{e^{2n+1}(d-a)^{2n+1}},\\
x_{12n+7} &= \frac{bd^{2n+1}h^{2n+2}}{a^{n+1}(h+e)^{n+1}(h-e)^{n+1}(2d-a)^n}, & y_{12n+7} &= \frac{-fh^{2n+1}d^{2n+2}}{e^{2n+1}(d-a)^{2n+2}},\\
x_{12n+8} &= \frac{cd^{2n+2}h^{2n+1}}{a^{n+1}(h+e)^{n+1}(h-e)^n(2d-a)^{n+1}}, & y_{12n+8} &= \frac{-gh^{2n+2}d^{2n+1}}{e^{2n+2}(d-a)^{2n+1}}.
\end{aligned}$$

Proof. By using mathematical induction. The result holds for $n=0$. Suppose that the result holds for $n-1$

$$
\begin{aligned}
x_{12n-7} &= \frac{cd^{2n-1}h^{2n-1}}{a^{n-1}(h+e)^{n}(h-e)^{n-1}(2d-a)^{n}}, & y_{12n-7} &= \frac{-gh^{2n-1}d^{2n-1}}{e^{2n-1}(d-a)^{2n-1}},\\
x_{12n-6} &= \frac{d^{2n}h^{2n-1}}{a^{n-1}(h+e)^{n-1}(h-e)^{n}(2d-a)^{n}}, & y_{12n-6} &= \frac{h^{2n}d^{2n-1}}{e^{2n-1}(d-a)^{2n-1}},\\
x_{12n-5} &= \frac{bd^{2n-1}h^{2n}}{a^{n}(h+e)^{n}(h-e)^{n}(2d-a)^{n-1}}, & y_{12n-5} &= \frac{-fh^{2n-1}d^{2n}}{e^{2n-1}(d-a)^{2n}},\\
x_{12n-4} &= \frac{cd^{2n}h^{2n-1}}{a^{n}(h+e)^{n}(h-e)^{n-1}(2d-a)^{n}}, & y_{12n-4} &= \frac{-gh^{2n}d^{2n-1}}{e^{2n}(d-a)^{2n-1}},
\end{aligned}
$$

From system (2) we have

$$
\begin{aligned}
x_{12n-3} &= \frac{y_{12n-4}x_{12n-6}}{y_{12n-4}+y_{12n-7}} = \frac{\frac{-gh^{2n}d^{2n-1}}{e^{2n}(d-a)^{2n-1}}\frac{d^{2n}h^{2n-1}}{a^{n-1}(h+e)^{n-1}(h-e)^{n}(2d-a)^{n}}}{\frac{-gh^{2n}d^{2n-1}}{e^{2n}(d-a)^{2n-1}}+\frac{-gh^{2n-1}d^{2n-1}}{e^{2n-1}(d-a)^{2n-1}}}\\
&= \frac{h^{2n}d^{2n}}{a^{n-1}(h+e)^{n-1}(h-e)^{n}(2d-a)^{n}(h+e)} = \frac{d^{2n}h^{2n}}{a^{n-1}(h+e)^{n}(h-e)^{n}(2d-a)^{n}},\\
y_{12n-3} &= \frac{x_{12n-4}y_{12n-6}}{x_{12n-4}-x_{12n-7}} = \frac{\frac{cd^{2n}h^{2n-1}}{a^{n}(h+e)^{n}(h-e)^{n-1}(2d-a)^{n}}\frac{h^{2n}d^{2n-1}}{e^{2n-1}(d-a)^{2n-1}}}{\frac{cd^{2n}h^{2n-1}}{a^{n}(h+e)^{n}(h-e)^{n-1}(2d-a)^{n}}-\frac{cd^{2n-1}h^{2n-1}}{a^{n-1}(h+e)^{n}(h-e)^{n-1}(2d-a)^{n}}} = \frac{h^{2n}d^{2n}}{e^{2n-1}(d-a)^{2n}},\\
x_{12n-2} &= \frac{y_{12n-3}x_{12n-5}}{y_{12n-3}+y_{12n-6}}\\
&= \frac{d^{2n}h^{2n}bd^{2n-1}h^{2n}}{e^{2n-1}(d-a)^{2n}a^{n}(h+e)^{n}(h-e)^{n}(2d-a)^{n-1}\left[\frac{d^{2n}h^{2n}}{e^{2n}(d-a)^{2n}}+\frac{d^{2n-1}h^{2n}}{e^{2n-1}(d-a)^{2n-1}}\right]} = \frac{bd^{2n}h^{2n}}{a^{n}(h+e)^{n}(h-e)^{n}(2d-a)^{n}},\\
y_{12n-2} &= \frac{x_{12n-3}y_{12n-5}}{x_{12n-3}-x_{12n-6}}\\
&= \frac{-h^{2n}d^{2n}fd^{2n}h^{2n-1}}{a^{n-1}(h+e)^{n}(h-e)^{n}(2d-a)^{n}e^{2n-1}(d-a)^{2n}\left[\frac{h^{2n}d^{2n}}{a^{n-1}(h+e)^{n}(h-e)^{n}(2d-a)^{n}}-\frac{h^{2n-1}d^{2n}}{a^{n-1}(h+e)^{n-1}(h-e)^{n}(2d-a)^{n}}\right]} = \frac{fd^{2n}h^{2n}}{e^{2n}(d-a)^{2n}}.
\end{aligned}
$$

So, we can prove the other relations and the proof is completed.

Lemma 2. Every positive solution of the equation $x_{n+1}=\dfrac{y_n x_{n-2}}{y_n+y_{n-3}}$ is bounded and $\lim_{n\to\infty} x_n = 0$.

The following cases can be proved similarly.

4. ON THE SYSTEM $X_{N+1}=\frac{Y_N X_{N-2}}{Y_N+Y_{N-3}}$, $Y_{N+1}=\frac{X_N Y_{N-2}}{-X_N+X_{N-3}}$

In this section, we study the solutions of the system of the difference equations

$$
x_{n+1}=\frac{y_n x_{n-2}}{y_n+y_{n-3}}, \qquad y_{n+1}=\frac{x_n y_{n-2}}{-x_n+x_{n-3}}, \tag{3}
$$

where the initial values $x_{-3},\ x_{-2},\ x_{-1},\ x_0,\ y_{-3},\ y_{-2},\ y_{-1},\ y_0$ are arbitrary nonzero real numbers with $x_{-3}\neq x_0$, and $y_{-3}\neq -y_0$.

Theorem 3. Let $\{x_n, y_n\}_{n=-3}^{+\infty}$ be solutions of system (3). Then for $n=0,1,2,...,$

$$
\begin{aligned}
x_{6n-3} &= \frac{h^{n}d^{n}}{a^{n-1}(h+e)^{n}}, & y_{6n-3} &= \frac{h^{n}d^{n}}{e^{n-1}(-d+a)^{n}},\\
x_{6n-2} &= \frac{bh^{n}d^{n}}{a^{n}(h+e)^{n}}, & y_{6n-2} &= \frac{fh^{n}d^{n}}{e^{n}(-d+a)^{n}},\\
x_{6n-1} &= \frac{ch^{n}d^{n}}{a^{n}(h+e)^{n}}, & y_{6n-1} &= \frac{gh^{n}d^{n}}{e^{n}(-d+a)^{n}},
\end{aligned}
$$

$$\begin{aligned} x_{6n} &= \frac{h^n d^{n+1}}{a^n(h+e)^n}, \qquad y_{6n} = \frac{d^n h^{n+1}}{e^n(-d+a)^n}, \\ x_{6n+1} &= \frac{bd^n h^{n+1}}{a^n(h+e)^{n+1}}, \qquad y_{6n+1} = \frac{fh^n d^{n+1}}{e^n(-d+a)^{n+1}}, \\ x_{6n+2} &= \frac{ch^n d^{n+1}}{a^{n+1}(h+e)^n}, \qquad y_{6n+2} = \frac{gd^n h^{n+1}}{e^{n+1}(-d+a)^n}, \end{aligned}$$

Lemma 3. The system (3) has a periodic solutions of period 6 iff $hd = e(a-d) = a(h+e)$.

Proof. First, if $hd = e(a-d) = a(h+e)$, then from the form of the solutions of system (3), we see that

$$\begin{aligned} x_{6n-3} &= a^n(h+e)^n a^{n-1}(h+e)^n = a\ , \quad x_{6n-2} = b, \quad x_{6n-1} = c,\ x_{6n} = d, \quad x_{6n+1} = hh+e, \quad x_{6n+2} = cda, \\ y_{6n-3} &= e, \quad y_{6n-2} = f, \quad y_{6n-1} = g,\ y_{6n} = h, \quad y_{6n+1} = fda-d, \quad y_{6n+2} = hge. \end{aligned}$$

Thus system (3) has a periodic solution with period 6. Second:if we have a period 6 then

$$\begin{aligned} x_{6n-3} &= \frac{h^n d^n}{a^{n-1}(h+e)^n} = x_{-3} = a,\ x_{6n-2} = \frac{bh^n d^n}{a^n(h+e)^n} = x_{-2} = b,\ x_{6n-1} = \frac{ch^n d^n}{a^n(h+e)^n} = x_{-1} = c, \\ x_{6n} &= \frac{h^n d^{n+1}}{a^n(h+e)^n} = x_0 = d,\ x_{6n+1} = \frac{bd^n h^{n+1}}{a^n(h+e)^{n+1}} = x_1 = \frac{bh}{h+e},\ x_{6n+2} = \frac{ch^n d^{n+1}}{a^{n+1}(h+e)^n} = x_2 = \frac{cd}{a}, \\ y_{6n-3} &= \frac{h^n d^n}{e^{n-1}(-d+a)^n} = y_{-3} = e, \qquad y_{6n-2} = \frac{fh^n d^n}{e^n(-d+a)^n} = y_{-2} = f, \\ y_{6n-1} &= \frac{gh^n d^n}{e^n(-d+a)^n} = y_{-1} = g, \qquad y_{6n} = \frac{d^n h^{n+1}}{e^n(-d+a)^n} = y_0 = h, \\ y_{6n+1} &= \frac{fh^n d^{n+1}}{e^n(-d+a)^{n+1}} = y_1 = \frac{fd}{a-d},\ \ y_{6n+2} = \frac{gd^n h^{n+1}}{e^{n+1}(-d+a)^n} = y_2 = \frac{gh}{e}, \end{aligned}$$

Then we get $hd = a(h+e)$, $\quad hd = e(a-d)$, and the proof is completed.

5. ON THE SYSTEM $X_{N+1} = \frac{Y_N X_{N-2}}{Y_N + Y_{N-3}}$, $\ Y_{N+1} = \frac{X_N Y_{N-2}}{-X_N - X_{N-3}}$

In this section,we study the solutions of the system of the difference equations

$$x_{n+1} = \frac{y_n x_{n-2}}{y_n + y_{n-3}}, \qquad y_{n+1} = \frac{x_n y_{n-2}}{-x_n - x_{n-3}}, \tag{4}$$

where the initial values $x_{-3},\ x_{-2},\ x_{-1},\ x_0, y_{-3},\ y_{-2},\ y_{-1},\ y_0$ are arbitrary nonzero real numbers.

Theorem 4. If $\{x_n, y_n\}$ are solutions of difference equation system (4). Then for $n = 0, 1, 2, ...$, we have

$$\begin{aligned} x_{12n-3} &= \frac{d^{2n} h^{2n}}{a^{2n-1}(h+e)^{2n}}, \qquad y_{12n-3} = \frac{(-1)^n d^{2n} h^{2n}}{e^{n-1}(d+a)^n(d-a)^n(2h+e)^n}, \\ x_{12n-2} &= \frac{bd^{2n} h^{2n}}{a^{2n}(h+e)^{2n}}, \qquad y_{12n-2} = \frac{(-1)^n fd^{2n} h^{2n}}{e^n(d+a)^n(d-a)^n(2h+e)^n}, \\ x_{12n-1} &= \frac{cd^{2n} h^{2n}}{a^{2n}(h+e)^{2n}}, \qquad y_{12n-1} = \frac{(-1)^n gd^{2n} h^{2n}}{e^n(d+a)^n(d-a)^n(2h+e)^n}, \\ x_{12n} &= \frac{d^{2n+1} h^{2n}}{a^{2n}(h+e)^{2n}}, \qquad y_{12n} = \frac{(-1)^n d^{2n} h^{2n+1}}{e^n(d+a)^n(d-a)^n(2h+e)^n}, \end{aligned}$$

$$
\begin{aligned}
x_{12n+1} &= \frac{bd^{2n}h^{2n+1}}{a^{2n}(h+e)^{2n+1}}, & y_{12n-3} &= \frac{(-1)^{n+1}fd^{2n+1}h^{2n}}{e^{n}(d+a)^{n+1}(d-a)^{n}(2h+e)^{n}},\\
x_{12n+2} &= \frac{-cd^{2n+1}h^{2n}}{a^{2n+1}(h+e)^{2n}}, & y_{12n+2} &= \frac{(-1)^{n+1}gd^{2n}h^{2n+1}}{e^{n}(d+a)^{n}(d-a)^{n}(2h+e)^{n+1}},\\
x_{12n+3} &= \frac{-d^{2n+1}h^{2n+1}}{a^{2n}(h+e)^{2n+1}}, & y_{12n+3} &= \frac{(-1)^{n+1}d^{2n+1}h^{2n+1}}{e^{n}(d+a)^{n}(d-a)^{n+1}(2h+e)^{n}},\\
x_{12n+4} &= \frac{bd^{2n+1}h^{2n+1}}{a^{2n+1}(h+e)^{2n+1}}, & y_{12n+4} &= \frac{(-1)^{n+1}fd^{2n+1}h^{2n+1}}{e^{n+1}(d+a)^{n+1}(d-a)^{n}(2h+e)^{n}},\\
x_{12n+5} &= \frac{-cd^{2n+1}h^{2n+1}}{a^{2n+1}(h+e)^{2n+1}}, & y_{12n+5} &= \frac{(-1)^{n}gd^{2n+1}h^{2n+1}}{e^{n}(d+a)^{n+1}(d-a)^{n}(2h+e)^{n+1}},\\
x_{12n+6} &= \frac{d^{2n+2}h^{2n+1}}{a^{2n+1}(h+e)^{2n+1}}, & y_{12n+6} &= \frac{(-1)^{n}d^{2n+1}h^{2n+2}}{e^{n}(d+a)^{n}(d-a)^{n+1}(2h+e)^{n+1}},\\
x_{12n+7} &= \frac{-bd^{2n+1}h^{2n+2}}{a^{2n+1}(h+e)^{2n+2}}, & y_{12n+7} &= \frac{(-1)^{n}fd^{2n+2}h^{2n+1}}{e^{n+1}(d+a)^{n+1}(d-a)^{n+1}(2h+e)^{n}},\\
x_{12n+8} &= \frac{-cd^{2n+2}h^{2n+1}}{a^{2n+2}(h+e)^{2n+1}}, & y_{12n+8} &= \frac{(-1)^{n}gd^{2n+1}h^{2n+2}}{e^{n+1}(d+a)^{n+1}(d-a)^{n}(2h+e)^{n+1}}.
\end{aligned}
$$

6. NUMERICAL EXAMPLES

Here, we consider interesting numerical examples in order to illustrate the results of the previous sections and to support our theoretical discussions.

Example 1. We consider numerical example for the difference system (1) with the initial conditions $x_{-3} = 2,\ x_{-2} = 14,\ x_{-1} = 6,\ x_0 = 7,\ y_{-3} = 5,\ y_{-2} = 9,\ y_{-1} = 7$ *and* $y_0 = -8$. (See Fig. 1).

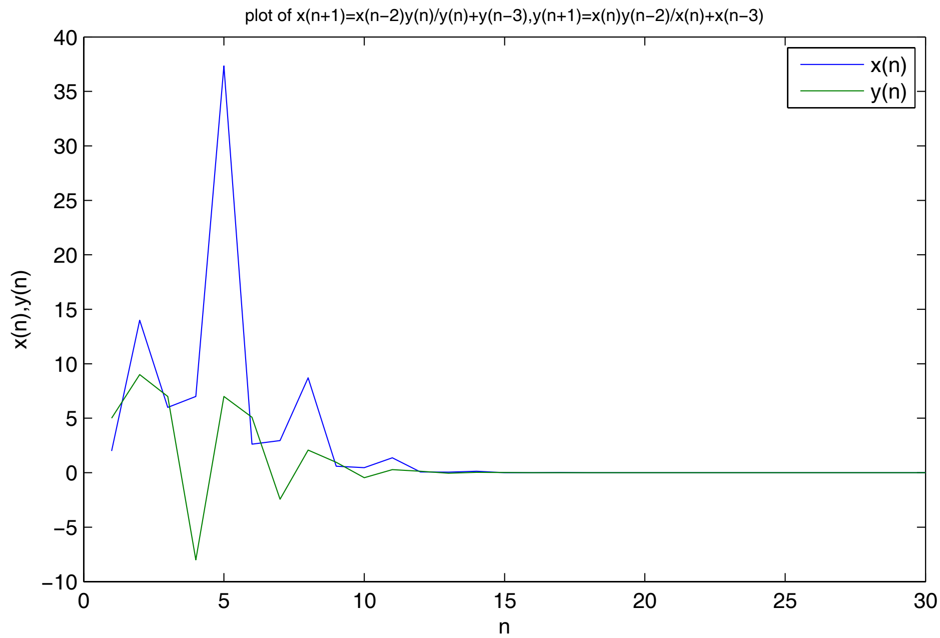

Figure 1.

Example 2. Assume for the system (2) with the initial conditions $x_{-3} = 4,\ x_{-2} = 5,\ x_{-1} = 6, x_0 = 3,\ y_{-3} = 1.8,\ y_{-2} = 9,\ y_{-1} = 2$ *and* $y_0 = 1.9$. See Figure (2).

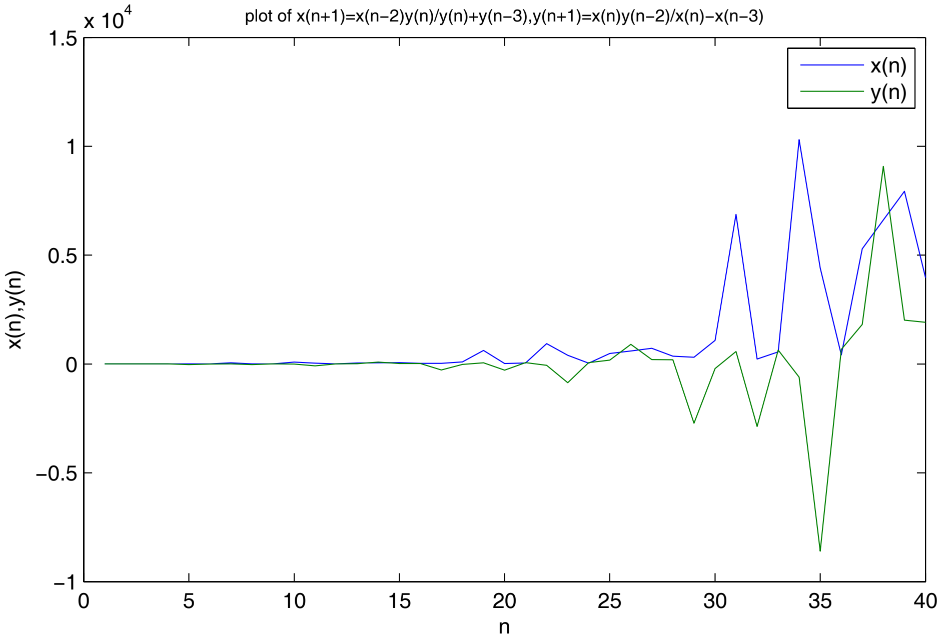

Figure 2.

Example 3. Figure (3) shows the behavior of the solution of the difference system (3) with the initial conditions $x_{-3} = 4,\ x_{-2} = 5,\ x_{-1} = 6, x_0 = 10,\ y_{-3} = 8, y_{-2} = 9,\ y_{-1} = 2\ and\ y_0 = 2.$

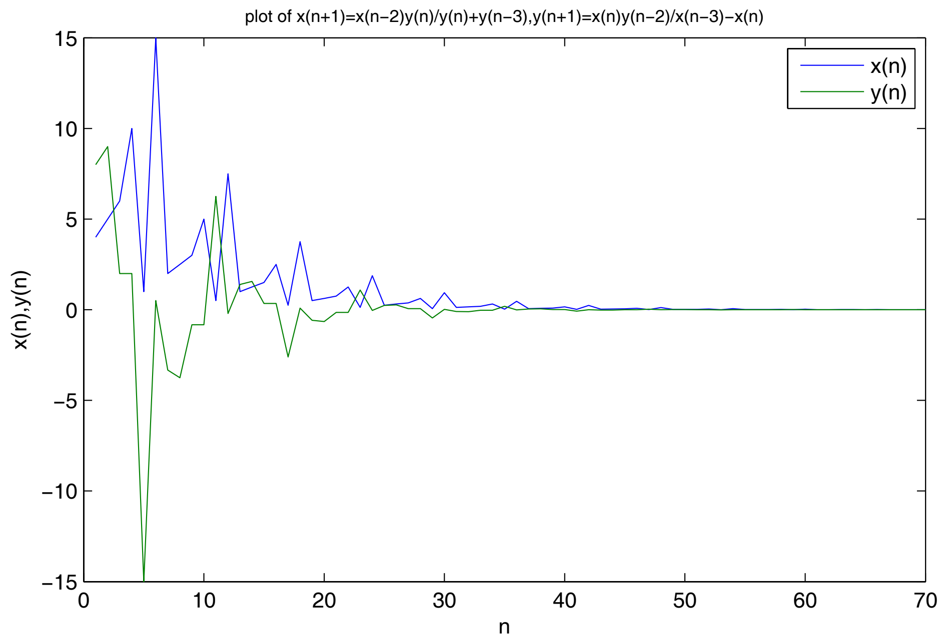

Figure 3.

Example 4. We take the initial conditions, for the system (4), as follows $x_{-3} = 3,\ x_{-2} = 5,\ x_{-1} = -9,$ $x_0 = 6,\ y_{-3} = 2, y_{-2} = 1.7,\ y_{-1} = 2.8\ and\ y_0 = 4.$ See Figure (4).

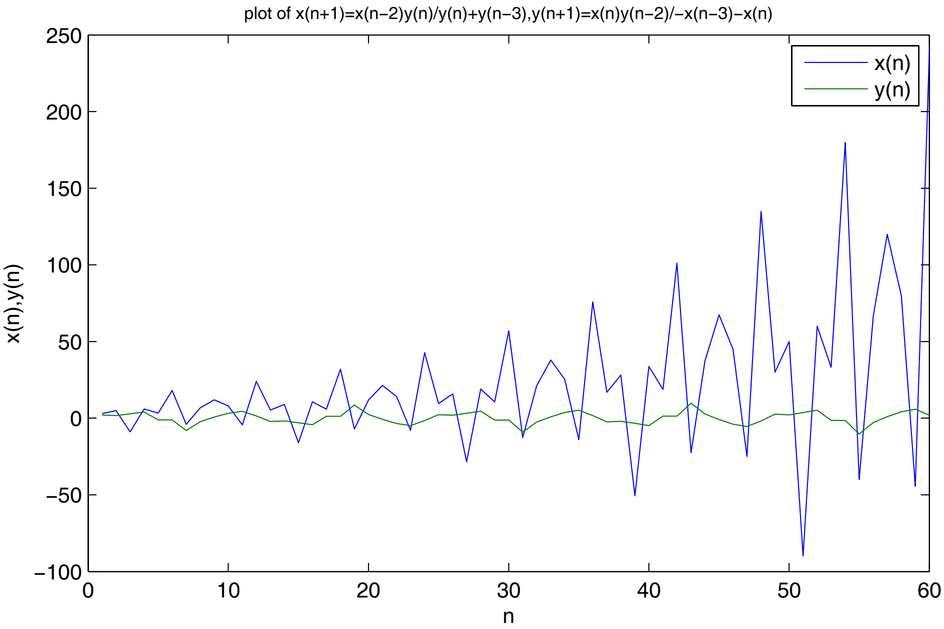

Figure 4.

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On the dynamics of higher Order difference equations $x_{n+1} = ax_n + \frac{\alpha x_n x_{n-l}}{\beta x_n + \gamma x_{n-k}}$

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Abstract

The main objective of this paper is to study the global stability of the positive solutions and the periodic character of the difference equation

$$x_{n+1} = ax_n + \frac{\alpha x_n x_{n-l}}{\beta x_n + \gamma x_{n-k}}, \qquad n = 0,\ 1,\ ...,$$

where the parameters $\alpha,\ \beta,\ \gamma$ and a are positive real numbers and the initial conditions $x_{-t},\ x_{-t+1}\ ...,\ x_{-1}$ and x_0 are positive real numbers where $t = max\{l,\ k\}$. Numerical examples to the difference equation are given to explain our results.

Keywords: difference equations, stability, global stability, boundedness, periodic solutions.
Mathematics Subject Classification: 39A10

1 Introduction and Preliminaries

Our object in this paper is to study some qualitative behavior of the positive solutions of the difference equation

$$x_{n+1} = ax_n + \frac{\alpha x_n x_{n-l}}{\beta x_n + \gamma x_{n-k}}, \quad n = 0,\ 1,\ ..., \tag{1}$$

where the parameters $\alpha,\ \beta,\ \gamma$ and δ are positive real numbers and the initial conditions $x_{-t},\ x_{-t+1}\ ...,\ x_{-1}$ and x_0 are positive real numbers where $t = max\{l,\ k\}$. In addition, we obtain the solutions of some special cases of this equation.

Many researchers have Many researchers have studied the behavior of the solution of difference equations for example: Kalabušić et al. [1] studied the global character of the solution of the nonlinear rational difference equation

$$x_{n+1} = \frac{\beta x_{n-l}+\delta x_{n-k}}{Bx_{n-l}+Dx_{n-k}}, \qquad n = 0,\ 1,\ ...,$$

with positive parameters and non-negative initial conditions.

Cinar [2] studied the solutions of the following difference equation

$$x_{n+1} = \frac{ax_{n-1}}{1+bx_nx_{n-1}}, \qquad n = 0,\ 1,\ ...,$$

where $a,\ b,\ x_{-1}$ and x_0 are non-negative real numbers.

Yang et al. [3] studied the invariant intervals, the asymptotic behavior of the solutions, and the global attractivity of equilibrium points of the recursive sequence

$$x_{n+1} = \frac{ax_{n-1}+bx_{n-2}}{c+dx_{n-1}x_{n-2}}, \qquad n = 0,\ 1,\ ...,$$

where $a \geq 0,\ b,\ c,\ d > 0$.

In [4] kenneth et al. got the global asymptotic stability for positive solutions to the difference equation

$$y_{n+1} = \frac{y_{n-k}\ +\ y_{n-m}}{1+y_{n-k}y_{n-m}}, \qquad n = 0,\ 1,\ ...,$$

with $y_{-m},\ y_{-m+1},\ ...,\ y_{-1} \in (0,\ \infty)$ and $1 \leq k \leq m$.

Raafat [5] investigated the global asymptotic stability of all solutions of the difference equation

$$x_{n+1} = \frac{Ax_{n-2}}{B+Cx_nx_{n-1}x_{n-2}}, \qquad n = 0,\ 1,\ ...,$$

where $A,\ B,\ C$ are positive real numbers and the initial conditions $x_{-2},\ x_{-1},\ x_0$ are real numbers.

Also, Raafat [6] introduced an explicit formula and discuss the global behavior of solutions of the difference equation

$$x_{n+1} = \frac{ax_{n-3}}{b+cx_{n-1}x_{n-3}}, \qquad n = 0,\ 1,\ ...,$$

where $a,\ b,\ c$ are positive real numbers and the initial conditions $x_{-3},\ x_{-2},\ x_{-1},\ x_0$ are real numbers.

In [7] Elsayed studied the behavior of the solutions of the difference equation

$$x_{n+1} = ax_{n-1} + \frac{bx_nx_{n-1}}{cx_n+dx_{n-2}}, \qquad n = 0,\ 1,\ ...,$$

where $a,\ b,\ c$ are positive constant and the initial conditions $x_{-2},\ ,\ x_{-1},\ x_0$ are arbitrary positive real numbers.

Zayed et al. [8] investigated some qualitative behavior of the solutions of the difference equation,

$$x_{n+1} = \gamma x_{n-k} + \tfrac{ax_n + bx_{n-k}}{cx_n - dx_{n-k}}, \qquad n = 0, 1, ...,$$

where the coefficients $\gamma,\ a,\ b, c$ and d are positive constants and the initial conditions $x_{-k},\ ...,\ x_{-1},\ x_0$ are arbitrary positive real numbers, while k is a positive integer number.

Other related results on rational difference equations can be found in refs. [11] - [24].

Let I be some interval of real numbers and let

$$F : I^{t+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-t}, x_{-t+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-t}), \quad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-t}^{\infty}$.

Definition 1 *The linearized equation of the difference equation (2) about the equilibrium $\overline{x}$ is the linear difference equation*

$$y_{n+1} = \sum_{i=0}^{t} \frac{\partial F(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{3}$$

Now, assume that the characteristic equation associated with (3) is

$$p(\lambda) = p_0 \lambda^t + p_1 \lambda^{t-1} + ... + p_{t-1}\lambda + p_t = 0, \tag{4}$$

where

$$p_i = \frac{\partial F(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}}.$$

Theorem 1 *[9]: Assume that $p_i \in R,\ i = 1, 2, ..., t$ and t is non-negative integer. Then*

$$\sum_{i=1}^{t} |p_i| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+t} + p_1 x_{n+t-1} + ... + p_t x_n = 0, \quad n = 0, 1, ... \ .$$

Theorem 2 *[10, 11]: Let $g : [a,b]^{t+1} \to [a,b]$, be a continuous function, where t is a positive integer, and where $[a,b]$ is an interval of real numbers. Consider the difference equation*

$$x_{n+1} = g(x_n, x_{n-1}, ..., x_{n-t}), \qquad n = 0, 1, ... \ . \tag{5}$$

Suppose that g satisfies the following conditions.

(1) For each integer i with $1 \leq i \leq t+1$; the function $g(z_1, z_2, ..., z_{t+1})$ is weakly monotonic in z_i for fixed $z_1, z_2, ..., z_{i-1}, z_{i+1}, ..., z_{t+1}$.

(2) If m, M is a solution of the system

$$m = g(m_1, m_2, ..., m_{t+1}), \qquad M = g(M_1, M_2, ..., M_{t+1}),$$

then $m = M$, where for each $i = 1, 2, ..., t+1$, we set

$$m_i = \begin{Bmatrix} m, & \text{if } g \text{ is non-decreasing in } z_i, \\ M, & \text{if } g \text{ is non-increasing in } z_i, \end{Bmatrix},$$

and

$$M_i = \begin{Bmatrix} M, & \text{if } g \text{ is non-decreasing in } z_i, \\ m, & \text{if } g \text{ is non-increasing in } z_i. \end{Bmatrix}.$$

Then there exists exactly one equilibrium point $\bar{x}$ of Equation (5), and every solution of Equation (5) converges to $\bar{x}$.

2 Stability of the Equilibrium Point of Eq. (1)

2.1 Local stability

In this subsection, we study the local stability character of the equilibrium point of Eq. (1).

Eq. (1) has equilibrium point and is given by

$$\bar{x} = a\bar{x} + \tfrac{\alpha\bar{x}^2}{\beta\bar{x}+\gamma\bar{x}}, \qquad or \qquad ((1-a)(\beta+\gamma)-\alpha)\,\bar{x}^2 = 0,$$

if $(1-a)(\beta+\gamma) \neq \alpha$, then the unique equilibrium point is $\overline{x} = 0$.

Theorem 3 *Assume that $a + \frac{2\alpha}{\beta+\gamma} < 1$,then equilibrium $\overline{x}$ of Eq. (1) is locally asymptotically stable.*

Proof: Let $f : (0,\ \infty)^3 \longrightarrow (0,\ \infty)$ be a continuous function defined by

$$f(v_0,\ v_1,\ v_2) = av_0 + \frac{\alpha v_0 v_1}{\beta v_0 + \gamma v_2}. \tag{6}$$

Therefore, it follows that

$$\begin{aligned}
\frac{\partial f(v_0,\ v_1,\ v_2)}{\partial v_0} &= a+\frac{\alpha v_1(\beta v_0+\gamma v_2)-\alpha\beta v_0 v_1}{(\beta v_0+\gamma v_2)^2}=a+\frac{\alpha\beta v_1^2}{(\beta v_0+\gamma v_2)^2},\\
\frac{\partial f(v_0,\ v_1,\ v_2)}{\partial v_1} &= \frac{\alpha v_0}{\beta v_0+\gamma v_2},\\
\frac{\partial f(v_0,\ v_1,\ v_2)}{\partial v_2} &= \frac{-\alpha v_0 v_1}{(\beta v_0+\gamma v_2)^2}=-\frac{\alpha\gamma v_0 v_1}{(\beta v_0+\gamma v_2)^2}.
\end{aligned}$$

Then, we see that

$$\frac{\partial f(\bar{x},\ \bar{x},\ \bar{x})}{\partial v_0}=a+\frac{\alpha\beta}{(\beta+\gamma)^2},\quad \frac{\partial f(\bar{x},\ \bar{x},\ \bar{x})}{\partial v_1}=\frac{\alpha}{\beta+\gamma},\quad \frac{\partial f(\bar{x},\ \bar{x},\ \bar{x})}{\partial v_2}=-\frac{\alpha\gamma}{(\beta+\gamma)^2}.$$

and the linearized equation of Eq. (1) about $\bar{x}$, is

$$y_{n+1}=\left(a+\frac{\alpha\beta}{(\beta+\gamma)^2}\right)y_n+\left(\frac{\alpha}{\beta+\gamma}\right)y_{n-l}+\left(\frac{-\alpha\gamma}{(\beta+\gamma)^2}\right)y_{n-k},$$

Under the conditions, we get

$$\left|a+\frac{\alpha\beta}{(\beta+\gamma)^2}\right|+\left|\frac{\alpha}{\beta+\gamma}\right|+\left|\frac{-\alpha\gamma}{(\beta+\gamma)^2}\right|<1,$$

and so

$$a+\frac{2\alpha}{\beta+\gamma}<1.$$

According to Theorem 1, the proof is complete.

Example 1. The solution of the difference equation (1) is local stability if $l=2, k=3,\ \alpha=0.1,\ \beta=0.2,\ \gamma=1,\ a=0.2$ and the initial conditions $x_{-3}=0.6$, $x_{-2}=0.3,\ x_{-1}=0.4$ and $x_0=0.8$ (See Fig. 1).

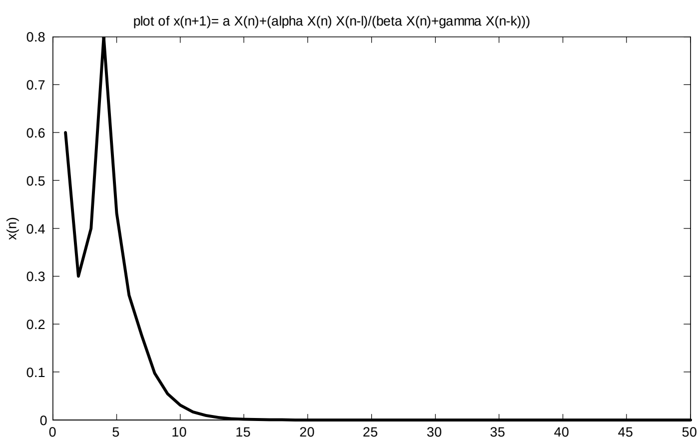

Fig. 1. Plot the behavior of the solution of equation (1).

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Example 2. See Figure (2) when we take the difference equation (1) with $l = 2, k = 3,\ \alpha = 1,\ \beta = 0.2,\ \gamma = 0.4,\ a = 0.5$ and the initial conditions $x_{-3} = 0.6$, $x_{-2} = 0.3,\ x_{-1} = 0.4$ and $x_0 = 0.8$.

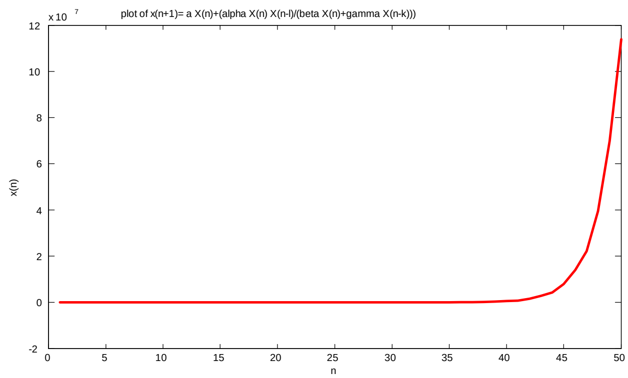

Fig. 2. Draw the behavior of the solution of equation (1).

2.2 Global Stability

In this subsection we study the global stability of the positive solutions of Eq. (1).

Theorem 4 *The equilibrium point $\bar{x}$ is a global attractor of equation (1) if*

$$(1-a)(\beta-\gamma) \neq \alpha.$$

Proof. Let r, s be nonnegative real numbers and assume that $h : [r,s]^3 \to [r,s]$ be a function defined by

$$h(v_0,\ v_1,\ v_2) = av_0 + \tfrac{\alpha v_0 v_1}{\beta v_0 + \gamma v_2}.$$

Then

$$\tfrac{\partial h(v_0,\ v_1,\ v_2)}{\partial v_0} = a + \tfrac{\alpha\beta v_1^2}{(\beta v_0+\gamma v_2)^2},\ \tfrac{\partial h(v_0,\ v_1,\ v_2)}{\partial v_1} = \tfrac{\alpha v_0}{\beta v_0+\gamma v_2} \text{ and } \tfrac{\partial h(v_0,\ v_1,\ v_2)}{\partial v_2} = -\tfrac{\alpha\gamma v_0 v_1}{(\beta v_0+\gamma v_2)^2}.$$

We can see that the function $h(v_0,\ v_1,\ v_2)$ increasing in $v_0,\ v_1$ and decreasing in v_2.

Suppose that (m, M) is a solution of the system

$$M = h(M,\ M,\ m) \quad \text{and} \quad m = h(m,\ m,\ M).$$

Then from Equation (1), we see that

$$M = aM + \tfrac{\alpha M^2}{\beta M+\gamma m},\ m = am + \tfrac{\alpha m^2}{\beta m+\gamma M},$$

then

$$\begin{aligned} \beta(1-a)M + \gamma(1-a)m &= \alpha M, \\ \beta(1-a)m + \gamma(1-a)M &= \alpha m, \end{aligned}$$

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Subtracting this two equations, we obtain

$$((1-a)(\beta-\gamma)-\alpha)(M-m)=0,$$

under the condition $(1-a)(\beta-\gamma)\neq\alpha$, we see that $M=m$. It follows from Theorem 2 that $\bar{x}$ is a global attractor of Equation (1).

Example 3. The solution of the difference equation (1) is global stability if $l=2, k=3,\ \alpha=0.01,\ \beta=0.2,\ \gamma=0.4,\ a=0.1$ and the initial conditions $x_{-3}=0.6$, $x_{-2}=0.3,\ x_{-1}=0.4$ and $x_0=0.8$ (See Fig. 3).

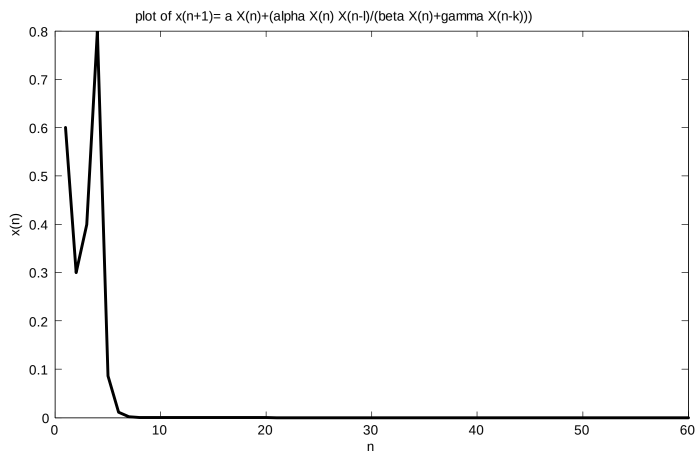

Fig. 3. Sketch the behavior of the solution of Eq. (1).

3 Boundedness of Solutions of Equation (1)

In this section we investigate the boundedness nature of the solutions of Equation (1).

Theorem 5 *Every solution of Equation (1) is bounded if* $a<1$.

Proof. Let $\{x_n\}_{n=-m}^{\infty}$ be a solution of Equation (1). It follows from Equation (1) that

$$x_{n+1}=ax_n+\frac{\alpha x_n x_{n-l}}{\beta x_n+\gamma x_{n-k}}\leq ax_n+\frac{\alpha x_n x_{n-l}}{\beta x_n}=ax_n+\left(\frac{\alpha}{\beta}\right)x_{n-l}.$$

By using a comparison, we can right hand side as follows

$$t_{n+1}=at_n+\left(\frac{\alpha}{\beta}\right)t_{n-l}.$$

and this equation is locally asymptotically stable if $a<1$, and converges to the equilibrium point $\overline{t}=0$. Therefore

$$\lim_{n\to\infty}\sup x_n\leqslant 0.$$

Example 4. Figure (4) shows that $l = 4, k = 3,\ \alpha = 0.1,\ \beta = 0.2,\ \gamma = 0.4,\ a = 1.3$, the solution of the difference equation (1) with initial conditions $x_{-3} = 0.6$, $x_{-2} = 0.3,\ x_{-1} = 0.4$ and $x_0 = 0.8$ is unbounded.

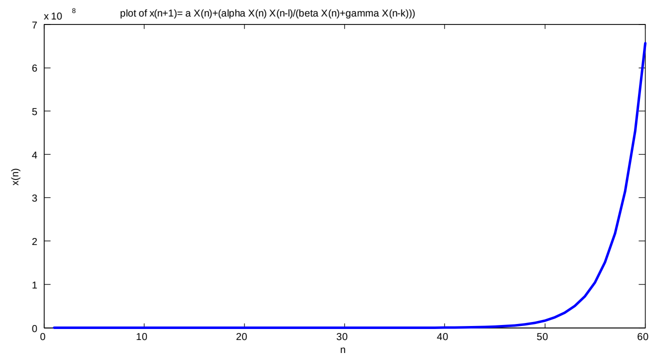

Fig. 4. Polt the behavior of the solution of equation (1) when $a > 1$.

4 Existence of Periodic Solutions

In this section we investigate the existence of periodic solutions of Eq. (5).

Theorem 6 *Equation (1) has no prime period two solutions if l and k are even when $a + \alpha \neq 0$ and $\beta + \gamma \neq 0$.*

Proof. Suppose that there exists a prime period two solution $...p, q, p, q, ...$, of Equation (1). We see from Equation (1) when l and k are even that

$$p = aq + \frac{\alpha q^2}{\beta q + \gamma q}, \qquad q = ap + \frac{\alpha p^2}{\beta p + \gamma p}.$$

$$(\beta + \gamma) pq = a (\beta + \gamma) q^2 + \alpha q^2, \tag{7}$$
$$(\beta + \gamma) pq = a (\beta + \gamma) p^2 + \alpha p^2 \tag{8}$$

Subtracting (7) from (8) gives

$$(a + \alpha) (\beta + \gamma) (p^2 - q^2) = 0,$$

Since $a + \alpha \neq 0$ and $\beta + \gamma \neq 0$, then $p = q$. This is a contradiction. Thus, the proof is completed.

Theorem 7 *Equation (1) has no prime period two solutions if l and k are odd when $\gamma \neq a\beta$.*

Theorem 8 *Equation (1) has no prime period two solutions if l is an even and k is an odd when $\alpha + \gamma \neq a\beta$.*

Theorem 9 *Equation (1) has no prime period two solutions if l is an odd and k is an even when $a (\beta + \gamma) \neq 0$.*

5 Special Cases of Equation (1)

5.1 First Equation When $l = k = 1,\ a = 0$ and $\alpha = \beta = \gamma = 1.$

In this subsection we study the following special case of Eq. (1)

$$x_{n+1} = \frac{x_n x_{n-1}}{x_n + x_{n-1}}, \tag{9}$$

where the initial conditions are arbitrary non zero real numbers.

Theorem 10 *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (9). Then for $n = 0,\ 1,\ 2,\ ...$*

$$x_n = \frac{cb}{f_n b + f_{n+1} c},$$

where $x_{-1} = c,\ x_0 = b,\ \{f_n\}_{n=1}^{\infty} = \{1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ ...\}\ f_0 = 0$ and $f_{-1} = 1.$

Proof: For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$ and n. Now, it follows

$$x_{n-2} = \frac{cb}{f_{n-2}b + f_{n-1}c} \text{ and } x_{n-1} = \frac{cb}{f_{n-1}b + f_n c}.$$

Now, it follows from Eq. (9) that

$$\begin{aligned} x_{n+1} &= \frac{x_n x_{n-1}}{x_n + x_{n-1}} = \frac{\left(\frac{cb}{f_n b + f_{n+1} c}\right)\left(\frac{cb}{f_{n-1} b + f_n c}\right)}{\left(\frac{cb}{f_n b + f_{n+1} c}\right) + \left(\frac{cb}{f_{n-1} b + f_n c}\right)} = \frac{\left(\frac{c^2 b^2}{(f_n b + f_{n+1} c)(f_{n-1} b + f_n c)}\right)}{\left(\frac{cb(f_{n-1} b + f_n c) + cb(f_n b + f_{n+1} c)}{(f_n b + f_{n+1} c)(f_{n-1} b + f_n c)}\right)} \\ &= \frac{c^2 b^2}{cb(f_{n-1} b + f_n c) + cb(f_n b + f_{n+1} c)} = \frac{cb}{(f_{n-1} + f_n) b + (f_n + f_{n+1}) c} = \frac{cb}{f_{n+1} b + f_{n+2} c}. \end{aligned}$$

Thus, the proof is completed.

5.2 Second Equation When $l = k = 1,\ a = 0, \quad \alpha = \beta = 1$ and $\gamma = -1.$

In this subsection we study the following special case of Eq. (1)

$$x_{n+1} = \frac{x_n x_{n-1}}{x_n - x_{n-1}}, \tag{10}$$

where the initial conditions are arbitrary non zero real numbers.

Theorem 11 *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (10). Then for $n = 0,\ 1,\ 2,\ ...$*

$$x_n = \frac{(-1)^{n+1} cb}{f_n b - f_{n+1} c},$$

where $x_{-1} = c,\ x_0 = b,$ and $\{f_n\}_{n=-1}^{\infty} = \{1,\ 0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ ...\}.$

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Proof: For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$ and n. Now, it follows

$$x_{n-2} = \frac{(-1)^{n-1}cb}{f_{n-2}b - f_{n-1}c} \text{ and } x_{n-1} = \frac{(-1)^n cb}{f_{n-1}b - f_n c}.$$

Now, it follows from Eq. (10) that

$$\begin{aligned} x_{n+1} &= \frac{x_n x_{n-1}}{x_n - x_{n-1}} = \frac{\left(\frac{(-1)^{n+1}cb}{f_n b - f_{n+1}c}\right)\left(\frac{(-1)^n cb}{f_{n-1}b - f_n c}\right)}{\left(\frac{(-1)^{n+1}cb}{f_n b - f_{n+1}c}\right) - \left(\frac{(-1)^n cb}{f_{n-1}b - f_n c}\right)} = \frac{\left(\frac{(-1)^{2n+1}c^2b^2}{(f_n b - f_{n+1}c)(f_{n-1}b - f_n c)}\right)}{\left(\frac{-cb(f_{n-1}b - f_n c) - cb(f_n b - f_{n+1}c)}{(f_n b - f_{n+1}c)(f_{n-1}b - f_n c)}\right)} \\ &= \frac{(-1)^{2n+2}c^2b^2}{cb(f_{n-1}b - f_n c) + cb(f_n b - f_{n+1}c)} = \frac{(-1)^{n+2}cb}{(f_{n-1} + f_n)b - (f_{n+1} + f_n)c} = \frac{(-1)^{n+2}cb}{f_{n+1}b - f_{n+2}c}. \end{aligned}$$

Thus, the proof is completed.

5.3 Third Equation When $l = k = 1,\ a = 0,\quad \alpha = \gamma = 1$ and $\beta = -1$.

In this subsection we study the following special case of Eq. (1)

$$x_{n+1} = \frac{x_n x_{n-1}}{-x_n + x_{n-1}}, \tag{11}$$

where the initial conditions are arbitrary non zero real numbers.

Theorem 12 *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (11). Then for $n = 0,\ 1,\ 2,\ ...$*

$$x_{3n-1} = (-1)^n c, \quad x_{3n} = (-1)^n b, \text{ and } x_{3n+1} = \frac{(-1)^{n+1}bc}{b-c},$$

where $x_{-1} = c,\ x_0 = b$.

Proof: For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$ and n. Now, it follows

$$x_{3n-4} = (-1)^{n-1} c, \quad x_{3n-3} = (-1)^{n-1} b, \text{ and } x_{3n-2} = \frac{(-1)^n bc}{b-c}.$$

Now, it follows from Eq. (11) that

$$\begin{aligned} x_{3n+2} &= \frac{x_{3n+1}x_{3n}}{-x_{3n+1} + x_{3n}} = \frac{\left(\frac{(-1)^{n+1}bc}{b-c}\right)((-1)^n b)}{-\left(\frac{(-1)^{n+1}bc}{b-c}\right) + (-1)^n b} = \frac{(-1)^{n+1}\left(\frac{b^2c}{b-c}\right)}{\left(\frac{bc}{b-c} + b\right)} = \frac{(-1)^{n+1}b^2c}{b^2} = (-1)^{n+1} c, \\ x_{3n} &= \frac{x_{3n-1}x_{3n-2}}{-x_{3n-1} + x_{3n-2}} = \frac{((-1)^n c)\left(\frac{(-1)^n bc}{b-c}\right)}{-(-1)^n c + \left(\frac{(-1)^n bc}{b-c}\right)} = \frac{(-1)^n\left(\frac{bc^2}{b-c}\right)}{\left(-c + \frac{bc}{b-c}\right)} = \frac{(-1)^n bc^2}{c^2} = (-1)^n b, \end{aligned}$$

and

$$x_{3n+4} = \frac{x_{3n}x_{3n-1}}{-x_{3n} + x_{3n-1}} = \frac{((-1)^n b)((-1)^n c)}{-(-1)^n b + (-1)^n c} = \frac{(-1)^n bc}{-(b-c)} = \frac{(-1)^{n+1}bc}{b-c}.$$

Thus, the proof is completed.

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Theorem 13 *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (11). Then every solution of Eq. (11) is a periodic with period six. Moreover $\{x_n\}_{n=-1}^{\infty}$ takes the form form*

$$\{x_n\} = \left\{c,\ b,\ -\tfrac{bc}{b-c},\ -c,\ -b,\ \tfrac{bc}{b-c},\ c,\ b,\ -\tfrac{bc}{b-c},\ -c,\ -b,\ \tfrac{bc}{b-c},\ ...\right\}.$$

where $x_{-1} = c,\ x_0 = b$.

Example 5. Figure (5) shows the solution of Eq. (11) when the initial conditions $x_{-1} = 0.3$ and $x_0 = 0.6$.

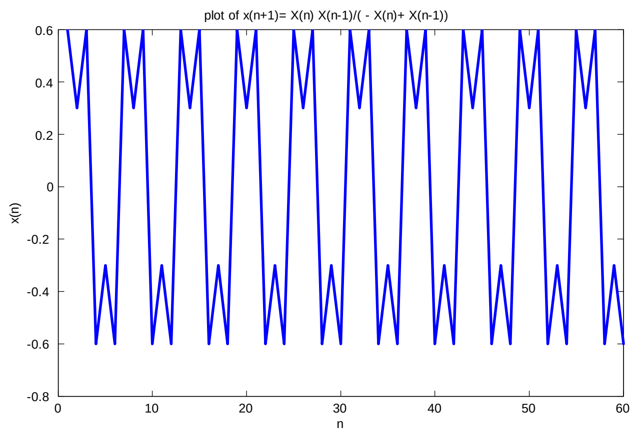

Fig. 5. Draw the solution of equation (11) has a periodic with period six.

5.4 Fourth Equation When $l = k = 1,\ a = 0,\ \ \beta = \gamma = 1$ and $\alpha = -1$.

In this subsection we study the following special case of Eq. (1)

$$x_{n+1} = -\tfrac{x_n x_{n-1}}{x_n + x_{n-1}}, \tag{12}$$

where the initial conditions are arbitrary non zero real numbers.

Theorem 14 *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (12). Then for $n = 0,\ 1,\ 2,\ ...$*

$$x_{3n-1} = c, \quad x_{3n} = b, \text{ and } x_{3n+1} = -\tfrac{bc}{b+c},$$

where $x_{-1} = c,\ x_0 = b$.

Proof: For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1$ and n. Now, it follows

$$x_{3n-4} = c, \quad x_{3n-3} = b, \text{ and } x_{3n-2} = -\tfrac{bc}{b+c}.$$

Now, it follows from Eq. (12) that

$$x_{3n+2} = -\frac{x_{3n+1}x_{3n}}{x_{3n+1}+x_{3n}} = -\frac{\left(-\frac{bc}{b+c}\right)(b)}{\left(-\frac{bc}{b+c}\right)+b} = -\frac{\left(-\frac{b^2c}{b+c}\right)}{\left(\frac{-bc+b^2+bc}{b+c}\right)} = \frac{b^2c}{b^2} = c,$$

$$x_{3n} = -\frac{x_{3n-1}x_{3n-2}}{x_{3n-1}+x_{3n-2}} = -\frac{(c)\left(-\frac{bc}{b-c}\right)}{c+\left(\frac{-bc}{b-c}\right)} = \frac{\left(\frac{bc^2}{b-c}\right)}{\left(\frac{bc+c^2-bc}{b-c}\right)} = \frac{bc^2}{c^2} = b,$$

and

$$x_{3n+4} = \frac{x_{3n}x_{3n-1}}{-x_{3n}+x_{3n-1}} = -\frac{bc}{b+c}.$$

Thus, the proof is completed.

Theorem 15 *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (12). Then every solution of Eq. (12) is a periodic with period three. Moreover $\{x_n\}_{n=-1}^{\infty}$ takes the form form*

$$\{x_n\} = \left\{c,\ b,\ -\frac{bc}{b+c},\ c,\ b,\ -\frac{bc}{b+c}, c,\ b,\ -\frac{bc}{b+c}, \ldots\right\},$$

where $x_{-1} = c,\ x_0 = b$.

Example 6. The solution of Eq. (12) when the initial conditions $x_{-1} = 0.3$ and $x_0 = 0.6$ (See Fig. 6).

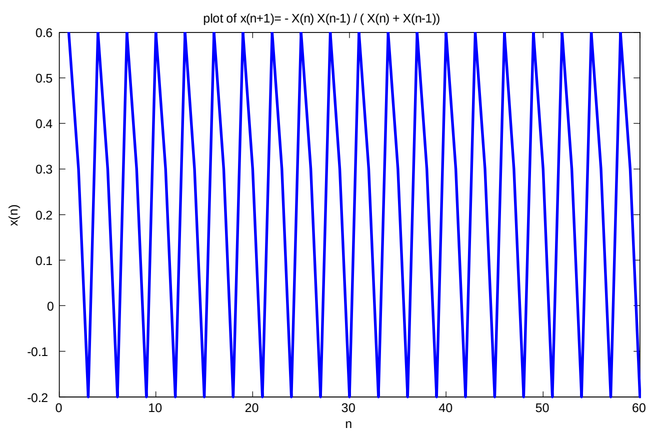

Fig. 6. Polt the solution of equation (12) has a periodic with period three.

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Applications of soft sets in BF-algebras

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Abstract. The aim of this article is to lay a foundation for providing a soft algebraic tool in considering many problems that contain uncertainties. In order to provide these soft algebraic structures, the notion of an intersectional soft subalgebra and an intersectional soft normal subalgebra of a BF-algebra are introduced, and related properties are investigated. A quotient structure of a BF-algebra using an intersectional soft normal subalgebra is constructed. The fundamental homomorphism of a quotient BF-algebra is established.

1. Introduction

The real world is inherently uncertain, imprecise and vague. Various problems in system identification involve characteristics which are essentially non-probabilistic in nature [14]. In response to this situation Zadeh [15] introduced *fuzzy set theory* as an alternative to probability theory. Uncertainty is an attribute of information. In order to suggest a more general framework, the approach to uncertainty is outlined by Zadeh [16]. To solve complicated problem in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [11]. Maji et al. [10] and Molodtsov [11] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [11] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. Worldwide, there has been a rapid growth in interest in soft set theory and its applications in recent years. Evidence of this can be found in the increasing number of high-quality articles on soft sets and related topics that have been published in a variety of international journals, symposia, workshops, and international conferences in recent years.

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Maji et al. [10] described the application of soft set theory to a decision making problem. Maji et al. [9] also studied several operations on the theory of soft sets. Aktaş and Çağman [2] studied the basic concepts of soft set theory, and compared soft sets to fuzzy and rough sets, providing examples to clarify their differences. They also discussed the notion of soft groups. Jun [7] discussed the union soft sets with applications in BCK/BCI-algebras. We refer the reader to the papers [1, 3, 5, 6, 13] for further information regarding algebraic structures/properties of soft set theory.

In this paper, we discuss applications of the an intersectional soft sets in a (normal) subalgebra of a BF-algebra. We introduce the notion of an intersectional (normal) soft subalgebra of a BF-algebra, and investigated related properties. We consider a new construction of a quotient BF-algebra induced by an int-soft normal subalgebra. Also we establish the fundamental homomorphism of a quotient BF-algebra.

2. Preliminaries

We review some definitions and properties that will be useful in our results (see [12]).

By a *BF-algebra* we mean an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions:

(B1) $x * x = 0$,

(B2) $x * 0 = x$,

(B3) $0 * (x * y) = y * x$

for all $x, y \in X$.

A BF-algebra $(X, *, 0)$ is called a BF_1-algebra if it satisfies the following identity:

(BG) $x = (x * y) * (0 * y)$ for all $x, y \in X$.

A BF-algebra $(X, *, 0)$ is called a BF_2-algebra if it satisfies the following identity:

(BH) $x * y = y * x = 0$ imply $x = y$ for all $x, y \in X$.

For brevity, we also call X a BF-algebra. If we can define a binary operation " $\leq$ " by $x \leq y$ if and only if $x * y = 0$. A non-empty subset A of a BF-algebra X is called a *subalgebra* of X if $x * y \in A$ for any $x, y \in A$. A non-empty subset A of a BF-algebra X is said to be *normal* (or *normal subalgebra*) ([8]) of X if $(x * a) * (y * b) \in A$ for any $x * y, a * b \in A$. Note that any normal subalgebra A of a BF-algebra X is a subalgebra of X, but the converse need not be true. A mapping $f : X \to Y$ of BF-algebras is called a *homomorphism* if $f(x*y) = f(x)*f(y)$ for all $x, y \in X$.

Lemma 2.1. *If X is a BF-algebra, then*

(i) $0 * (0 * x) = x$, *for all* $x \in X$.

(ii) $0 * x = 0 * y$ *implied* $x = y$ *for any* $x, y \in X$.

(iii) *if* $x * y = 0$, *then* $y * x = 0$ *for any* $x, y \in X$.

Lemma 2.2. *Let X be a BF-algebra and let N be a subalgebra of X. If $x * y \in N$ for any $x, y \in N$, then $y * x \in N$.*

A BG-algebra $(X;*,0)$ is an algebra of type $(2,0)$ satisfying (B1), (B2) and (BG).

Theorem 2.3 *Let X be a BF_1-algebra. Then*

(i) *X is a BG-algebra.*

(ii) *$x*y=0$ implies $x=y$ for any $x,y \in X$.*

(iii) *The right cancellation law holds in X, i.e., if $x*y=z*y$, then $x=z$ for any $x,y,z \in X$.*

(iv) *The left cancellation law holds in X, i.e., if $y*x=y*z$, then $x=z$ for any $x,y,z \in X$.*

Molodtsov [11] defined the soft set in the following way: Let U be an initial universe set and let E be a set of parameters. We say that the pair (U,E) is a *soft universe*. Let $\mathscr{P}(U)$ denotes the power set of U and $A,B,C,\cdots \subseteq E$.

A fair $(\tilde{f},A)$ is called a *soft set* over U, where $\tilde{f}$ is a mapping given by $\tilde{f}: X \to \mathscr{P}(U)$.

In other words, a soft set over U is parameterized family of subsets of the universe U. For $\varepsilon \in A$, $\tilde{f}(\varepsilon)$ may be considered as the set of ε-approximate elements of the set $(\tilde{f},A)$. A soft set over U can bd represented by the set of ordered pairs:

$$(\tilde{f},A)=\{(x,\tilde{f}(x))|x \in A, \tilde{f}(x) \in \mathscr{P}(U)\},$$

where $\tilde{f}: X \to \mathscr{P}(U)$ such that $\tilde{f}(x)=\emptyset$ if $x \notin A$. Clearly, a soft set is not a set.

For a soft set $(\tilde{f},A)$ of X and a subset γ of U, the γ-*inclusive set* of $(\tilde{f},A)$, defined to be the set

$$i_A(\tilde{f};\gamma):=\{x \in A|\gamma \subseteq \tilde{f}(x)\}.$$

3. Intersectional soft subalgebras

In what follows let X denote a BF-algebra X unless otherwise specified.

Definition 3.1. A soft set $(\tilde{f},X)$ over U is called an *intersectional soft subalgebra* (briefly, *int-soft subalgebra* of X if it satisfies:

(3.1) $\tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x*y)$ for all $x,y \in X$.

Proposition 3.2. *Every int-soft subalgebra $(\tilde{f},X)$ of a BF-algebra X satisfies the following inclusion:*

(3.2) *$\tilde{f}(x) \subseteq \tilde{f}(0)$ for all $x \in X$.*

Proof. Using (3.1) and (B1), we have $\tilde{f}(x)=\tilde{f}(x)\cap\tilde{f}(x) \subseteq \tilde{f}(x*x)=\tilde{f}(0)$ for all $x \in X$. □

Example 3.3. Let $(U=\mathbb{Z},X)$ where $X=\{0,1,2,3\}$ is a BF-algebra ([12]) with the following Cayley table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	1	1
2	2	1	0	1
3	3	1	1	0

Let $(\tilde{f},X)$ be a soft set over U defined as follows:

$$\tilde{f} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \mathbb{Z} & \text{if } x = 0 \\ 2\mathbb{Z} & \text{if } x \in \{1, 2\} \\ 3\mathbb{Z} & \text{if } x = 3. \end{cases}$$

It is easy to check that $(\tilde{f}, X)$ is an int-soft subalgebra over U.

Theorem 3.4. *A soft set $(\tilde{f}, X)$ of a BF-algebra X over U is an int-soft subalgebra of X over U if and only if the γ-inclusive set $i_X(\tilde{f}; \gamma)$ is a subalgebra of X for all $\gamma \in \mathscr{P}(U)$ with $i_X(\tilde{f}; \gamma) \neq \emptyset$.*

The subalgebra $i_X(\tilde{f}; \gamma)$ in Theorem 3.4 is called the *inclusive subalgebra* of X.

Proof. Assume that $(\tilde{f}, X)$ is an int-soft subalgebra over U. Let $x, y \in X$ and $\gamma \in \mathscr{P}(U)$ be such that $x, y \in i_X(\tilde{f}; \gamma)$. Then $\gamma \subseteq \tilde{f}(x)$ and $\gamma \subseteq \tilde{f}(y)$. It follows from (3.1) that $\gamma \subseteq \tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x*y)$ Hence $x*y \in i_X(\tilde{f}; \gamma)$. Thus $i_X(\tilde{f}, X)$ is a subalgebra of X.

Conversely, suppose that $i_X(\tilde{f}; \gamma)$ is a subalgebra X for all $\gamma \in \mathscr{P}(U)$ with $i_X(; \gamma) \neq \emptyset$. Let $x, y \in X$, be such that $\tilde{f}(x) = \gamma_x$ and $\tilde{f}(y) = \gamma_y$. Take $\gamma = \gamma_x \cap \gamma_y$. Then $x, y \in i_X(\tilde{f}; \gamma)$ and so $x * y \in i_X(\tilde{f}; \gamma)$ by assumption. Hence $\tilde{f}(x) \cap \tilde{f}(y) = \gamma_x \cap \gamma_y = \gamma \subseteq \tilde{f}(x * y)$. Thus $(\tilde{f}, X)$ is an int-soft subalgebra over U. □

Theorem 3.5. *Every subalgebra of a BF-algebra can be represented as a γ-inclusive set of an int-soft subalgebra.*

Proof. Let A be a subalgebra of a BF-algebra X. For a subset γ of U, define a soft set $(\tilde{f}, X)$ over U by

$$\tilde{f} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases}$$

Obviously, $A = i_X(\tilde{f}; \gamma)$. We now prove that $(\tilde{f}; \gamma)$ is an int-soft subalgebra over U. Let $x, y \in X$. If $x, y \in A$, then $x * y \in A$ because A is a subalgebra of X. Hence $\tilde{f}(x) = \tilde{f}(y) = \tilde{f}(x * y) = \gamma$, and so $\tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x * y)$. If $x \in A$ and $y \notin A$, then $\tilde{f}(x) = \gamma$ and $\tilde{f}(y) = \emptyset$ which imply that $\tilde{f}(x) \cap \tilde{f}(y) = \gamma \cap \emptyset = \emptyset \subseteq \tilde{f}(x * y)$. Similarly, if $x \notin A$ and $y \in A$, then $\tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x * y)$. Obviously, if $x \notin A$ and $y \notin A$, then $\tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x * y)$. Therefore $(\tilde{f}, X)$ is an int-soft subalgebra over U. □

Any subalgebra of a BF-algebra X may not be represented as a γ-inclusive set of an int-soft subalgebra $(\tilde{f}, X)$ over U in general (see the following example).

Example 3.6. Let $E = X$ be the set of parameters, and let $U = X$ be the initial universe set where where $X = \{0, 1, 2, 3\}$ is a BF-algebra ([12]) with the following Cayley table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	0
2	2	3	0	2
3	3	0	2	0

Consider a soft set $(\tilde{f}, X)$ which is given by

$$\tilde{f} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \{0,3\} & \text{if } x = 0 \\ \{3\} & \text{if } x \in \{1,2,3\} \end{cases}$$

Then $(\tilde{f}, X)$ is an int-soft subalgebra over U. The γ-inclusive set of $(\tilde{f}, X)$ are described as follows:

$$i_X(\tilde{f};\gamma) = \begin{cases} X & \text{if } \gamma \in \{\emptyset, \{3\}\} \\ \{0\} & \text{if } \gamma \in \{\{0\}, \{0,3\}\} \\ \emptyset & \text{otherwise.} \end{cases}$$

The subalgebra $\{0, 2\}$ cannot be a γ-inclusive set $i_X(\tilde{f};\gamma)$ since there is no $\gamma \subseteq U$ such that $i_X(\tilde{f};\gamma) = \{0, 2\}$.

We make a new int-soft subalgebra from old one.

Theorem 3.7. *Let $(\tilde{f}, X)$ be a soft set of a BF-algebra X over U. Define a soft set $(\tilde{f^*}, X)$ of X over U by*

$$\tilde{f^*} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tilde{f}(x) & \text{if } x \in i_X(\tilde{f};\gamma) \\ \emptyset & \text{otherwise} \end{cases}$$

where γ is a non-empty subset subset of U. If $(\tilde{f}, X)$ is an int-soft subalgebra of X, then so is $(\tilde{f^*}, X)$.

Proof. If $(\tilde{f}, X)$ is an int-soft subalgebra over U, then $i_X(\tilde{f};\gamma)$ is a subalgebra of X for all $\gamma \subseteq U$ by Theorem 3.6. Let $x, y \in X$. If $x, y \in i_X(\tilde{f};\gamma)$, then $x * y \in i_X(\tilde{f};\gamma)$. Hence we have

$$\tilde{f^*}(x) \cap \tilde{f^*}(y) = \tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x * y) = \tilde{f^*}(x * y).$$

If $x \notin i_X(\tilde{f};\gamma)$ or $y \notin i_X(\tilde{f};\gamma)$, then $\tilde{f^*}(x) = \emptyset$ or $\tilde{f^*}(y) = \emptyset$. Thus

$$\tilde{f^*}(x) \cap \tilde{f^*}(y) = \emptyset \subseteq \tilde{f^*}(x) * \tilde{f^*}(y).$$

Therefore $(\tilde{f^*}, X)$ is an int-soft subalgebra over U. □

Definition 3.8. A soft set $(\tilde{f}, X)$ over U is called an *intersectional soft normal subalgebra* (briefly, *int-soft normal subalgebra* of X if it satisfies:

(3.3) $\tilde{f}(x * y) \cap \tilde{f}(a * b) \subseteq \tilde{f}((x * a) * (y * b))$ for all $x, y, a, b \in X$.

Proposition 3.9. *Every int-soft subalgebra $(\tilde{f}, X)$ of a BF-algebra X satisfies the following inclusion:*

(3.4) $\tilde{f}(x * y) \subseteq \tilde{f}(y * x)$ *for all* $x, y \in X$.

Proof. Using (B3), (3.1) and (3.2), we have

$$\tilde{f}(y * x) = \tilde{f}(0 * (x * y)) \supseteq \tilde{f}(0) \cap \tilde{f}(x * y) = \tilde{f}(x * y),\ \forall x, y \in X.$$

□

Proposition 3.10. *Every int-soft normal subalgebra $(\tilde{f}, X)$ of a BF-algebra X is an int-soft subalgebra of X.*

Proof. Put $y := 0, b := 0$ and $a := y$ in (3.3). Then $\tilde{f}(x*0) \cap \tilde{f}(y*0) \subseteq \tilde{f}((x*y)*(0*0))$ for any $x, y \in X$. Using (B2) and (B1), we have $\tilde{f}(x) \cap \tilde{f}(y) \subseteq \tilde{f}(x * y)$. Hence $(\tilde{f}, X)$ is an int-soft subalgebra of X. □

The converse of Proposition 3.10 may not be true in general (see Example 3.11).

Example 3.11 Let $E = X$ be the set of parameters where where $X = \{0, 1, 2, 3\}$ is a BF-algebra with the following Cayley table:

$*$	0	1	2	3
0	0	2	1	3
1	1	0	1	1
2	2	2	0	2
3	3	2	1	0

Let $(\tilde{f}, X)$ be a soft set over U defined as follows:

$$\tilde{f} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_3 & \text{if } x = 0 \\ \gamma_2 & \text{if } x = 3 \\ \gamma_1 & \text{if } x \in \{1, 2\}. \end{cases}$$

where γ_1, γ_2 and γ_3 are subsets of U with $\gamma_1 \subsetneq \gamma_2 \subsetneq \gamma_3$. It is easy to check that $(\tilde{f}, X)$ is an int-soft normal subalgebra over U.

Let $(\tilde{g}, X)$ be a soft set over U defined as follows:

$$\tilde{g} : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \alpha_3 & \text{if } x = 0 \\ \alpha_2 & \text{if } x \in \{1, 2\} \\ \alpha_1 & \text{if } x = 3. \end{cases}$$

where α_1, α_2 and α_3 are subsets of U with $\alpha_1 \subsetneq \alpha_2 \subsetneq \alpha_3$. It is easy to check that $(\tilde{f}, X)$ is an int-soft subalgebra over U. But it is not an int-soft normal subalgebra over U since $\tilde{}g(2 * 3) \cap \tilde{g}(2 * 0) = \tilde{g}(2) \cap \tilde{g}(2) = \alpha_2 \nsubseteq \alpha_1 = \tilde{g}(3) = \tilde{g}((2 * 2) * (3 * 0))$.

Theorem 3.12. *A soft set $(\tilde{f}, X)$ of X over U is an int-soft normal subalgebra of X over U if and only if the γ-inclusive set $i_X(\tilde{f}; \gamma)$ is a normal subalgebra of X for all $\gamma \in \mathscr{P}(U)$ with $i_X(\tilde{f}; \gamma) \neq \emptyset$.*

Proof. Similar to Theorem 3.4. □

The normal subalgebra $i_X(\tilde{f}; \gamma)$ in Theorem 3.12 is called the *inclusive normal subalgebra* of X.

4. Quotient BF-algebras induces by soft sets

Let $(\tilde{f}, X)$ be an int-soft normal subalgebra of a BF-algebra X. For any $x, y \in X$, we define a binary operation "$\sim^{\tilde{f}}$" on X as follows:

$$x \sim^{\tilde{f}} y \Leftrightarrow \tilde{f}(x * y) = \tilde{f}(0).$$

Lemma 4.1. *The operation $\sim^{\tilde{f}}$ is an equivalence relation on a BF-algebra X.*

Proof. Obviously, it is reflexive. Let $x \sim^{\tilde{f}} y$. Then $\tilde{f}(x * y) = \tilde{f}(0)$. It follows from (3.4) and (3.2) that $\tilde{f}(0) = \tilde{f}(x * y) \subseteq \tilde{f}(y * x) \subseteq \tilde{f}(0)$. Hence $\tilde{f}(y * x) = \tilde{f}(0)$. Hence $\sim^{\tilde{f}}$ is symmetric. Let $x, y, z \in X$ be such that

$x \sim^{\tilde{f}} y$ and $y \sim^{\tilde{f}} z$. Then $\tilde{f}(x * y) = \tilde{f}(0)$ and $\tilde{f}(y * z) = \tilde{f}(0)$. Using (3.4), (3.3), (B1), (B2) and (3.2), we have

$$\begin{aligned}\tilde{f}(0) = \tilde{f}(x * y) \cap \tilde{f}(y * z) \subseteq \tilde{f}(x * y) \cap \tilde{f}(z * y) \\ &\subseteq \tilde{f}((x * z) * (y * y)) \\ &= \tilde{f}((x * z) * 0) = \tilde{f}(x * z) \subseteq \tilde{f}(0).\end{aligned}$$

Hence $\tilde{f}(x * z) = \tilde{f}(0)$, i.e., $\sim^{\tilde{f}}$ is transitive. Therefore " $\sim^{\tilde{f}}$ " is an equivalence relation on X. □

Lemma 4.2. *For any $x, y, p, q \in X$, if $x \sim^{\tilde{f}} y$ and $p \sim^{\tilde{f}} q$, then $x * p \sim^{\tilde{f}} y * q$.*

Proof. Let $x, y, p, q \in X$ be such that $x \sim^{\tilde{f}} y$ and $p \sim^{\tilde{f}} q$. Then $\tilde{f}(x*y) = \tilde{f}(y*x) = \tilde{f}(0)$ and $\tilde{f}(p*q) = \tilde{f}(q*p) = \tilde{f}(0)$. Using (3.3) and (3.2), we have

$$\begin{aligned}\tilde{f}(0) =& \tilde{f}(x * y) \cap \tilde{f}(p * q) \\ \subseteq& \tilde{f}((x * p) * (y * q)) \subseteq \tilde{f}(0).\end{aligned}$$

Hence $\tilde{f}((x * p) * (y * q)) = \tilde{f}(0)$. By similar way, we get $\tilde{f}((y * q) * (x * p)) = \tilde{f}(0)$. Therefore $x * p \sim^{\tilde{f}} y * q$. Thus " $\sim^{\tilde{f}}$ " is a congruence relation on X. □

Denote $\tilde{f}_x$ and $X/\tilde{f}$ the set of all equivalence classes containing x and the set of all equivalence classes of X, respectively, i.e.,

$$\tilde{f}_x := \{y \in X | y \sim^{\tilde{f}} x\} \text{ and } X/\tilde{f} := \{\tilde{f}_x | x \in X\}.$$

Define a binary relation $\bullet$ on $X/\tilde{f}$ as follows:

$$\tilde{f}_x \bullet \tilde{f}_y = \tilde{f}_{x*y}$$

for all $\tilde{f}_x, \tilde{f}_y \in X/\tilde{f}$. Then this operation is well-defined by Lemma 4.2.

Theorem 4.3. *If $(\tilde{f}, X)$ is an int-soft normal subalgebra of a BF-algebra X, then the quotient $X/\tilde{f} := (X/\tilde{f}, \bullet, \tilde{f}_0)$ is a BF-algebra.*

Proof. Let $\tilde{f}_x, \tilde{f}_y, \tilde{f}_z \in X/\tilde{f}$. Then we have $\tilde{f}_x \bullet \tilde{f}_x = \tilde{f}_{x*x} = \tilde{f}_0$, $\tilde{f}_x \bullet \tilde{f}_0 = \tilde{f}_{x*0} = \tilde{f}_x$, $\tilde{f}_0 \bullet (\tilde{f}_x \bullet \tilde{f}_y) = \tilde{f}_{0*(x*y)} = \tilde{f}_{y*x} = \tilde{f}_y \bullet \tilde{f}_x$. Therefore $X/\tilde{f} = (X/\tilde{f}, \bullet, \tilde{f}_0)$ is a BF-algebra. □

Corollary 4.4. *If $(\tilde{f}, X)$ is an int-soft normal subalgebra of a BF_2-algebra X, then the quotient $X/\tilde{f} := (X/\tilde{f}, \bullet, \tilde{f}_0)$ is a BF_2-algebra.*

Proof. It is enough to show that $X/\tilde{f}$ satisfies (BH). If $\tilde{f}_x \bullet \tilde{f}_y = \tilde{f}_0$ and $\tilde{f}_y \bullet \tilde{f}_x = \tilde{f}_0$ for any $\tilde{f}_x, \tilde{f}_y \in X/\tilde{f}$, then $\tilde{f}_{x*y} = \tilde{f}_0 = \tilde{f}_{y*x}$. Hence $\tilde{f}(x * y) = \tilde{f}(0) = \tilde{f}(y * x)$ and so $x \sim^{\tilde{f}} y$. Hence $\tilde{f}_x = \tilde{f}_y$. Therefore $X/\tilde{f} = (X/\tilde{f}, \bullet, \tilde{f}_0)$ is a BF_2-algebra. □

Proposition 4.5. *Let $\mu : X \to Y$ be a homomorphism of BF-algebras. If $(\tilde{f}, Y)$ is an int-soft normal subalgebra of Y, then $(\tilde{f} \circ \mu, X)$ is an int-soft normal subalgebra of X.*

Proof. For any $x, y, a, b \in X$, we have

$$\begin{aligned}(\tilde{f} \circ \mu)((x * a) * (y * b)) =& \tilde{f}(\mu((x * a) * (y * b)) \\ =& \tilde{f}((\mu(x) * \mu(a)) * (\mu(y) * \mu(b))) \\ \supseteq& \tilde{f}(\mu(x) * \mu(y)) \cap \tilde{f}(\mu(a) * \mu(b)) \\ =& \tilde{f}(\mu(x * y)) \cap \tilde{f}(\mu(a * b)) \\ =& (\tilde{f} \circ \mu)(x * y) \cap (\tilde{f} \circ \mu)(a * b).\end{aligned}$$

Hence $\tilde{f} \circ \mu$ is an int-soft normal subalgebra. Therefore $(\tilde{f} \circ \mu, X)$ is an int-soft normal subalgebra of X. □

Theorem 4.6. *Let $X := (X; *_X, 0_X)$ be a BF-algebra and $Y := (Y; *_Y, 0_Y)$ be a BF_2-algebra and let $\mu : X \to Y$ be an epimorphism. If $(\tilde{f}, Y)$ is an int-soft normal subalgebra of Y, then the quotient algebra $X/(\tilde{f} \circ \mu) := (X/(\tilde{f} \circ \mu), \bullet_X, (\tilde{f} \circ \mu)_{0_X})$ is isomorphic to the quotient algebra $Y/\tilde{f} := (Y/\tilde{f}, \bullet_Y, \tilde{f}_{0_Y})$.*

Proof. By Theorem 4.3, Corollary 4.4, and Proposition 4.5, $X/\tilde{f} \circ \mu : (X/(\tilde{f} \circ \mu), \bullet_X, (\tilde{f} \circ \mu)_{0_X})$ is a BF-algebra and $Y/\tilde{f} := (Y/\tilde{f}, \bullet_Y, \tilde{f}_{0_Y})$ is a BF_2-algebra. Define a map

$$\eta : X/(\tilde{f} \circ \mu) \to Y/\tilde{f}, \ (\tilde{f} \circ \mu)_x \mapsto \tilde{f}_{\mu(x)}$$

for all $x \in X$. Then the function η is well-defined. In fact, assume that $(\tilde{f} \circ \mu)_x = (\tilde{f} \circ \mu)_y$ for all $x, y \in X$. Then we have

$$\begin{aligned}\tilde{f}(\mu(x) *_Y \mu(y)) =& \tilde{f}(\mu(x *_X y)) = (\tilde{f} \circ \mu)(x *_X y) \\ =& (\tilde{f} \circ \mu)(0_X) = \tilde{f}(\mu(0_X)) = \tilde{f}(0_Y)\end{aligned}$$

and

$$\begin{aligned}\tilde{f}(\mu(y) *_Y \mu(x)) =& \tilde{f}(\mu(y *_X x)) = (\tilde{f} \circ \mu)(y *_X x) \\ =& (\tilde{f} \circ \mu)(0_X) = \tilde{f}(\mu(0_X)) = \tilde{f}(0_Y).\end{aligned}$$

Hence $\tilde{f}_{\mu(x)} = \tilde{f}_{\mu(y)}$.

For any $(\tilde{f} \circ \mu)_x, (\tilde{f} \circ \mu)_X \in X/(\tilde{f} \circ \mu)$, we have

$$\begin{aligned}\eta((\tilde{f} \circ \mu)_x \bullet_X (\tilde{f} \circ \mu)_y) =& \eta((\tilde{f} \circ \mu)_{x*y}) = \tilde{f}_{\mu(x *_X y)} \\ =& \tilde{f}_{\mu(x) *_Y \mu(y)} = \tilde{f}_{\mu(x)} \bullet \tilde{f}_{\mu(y)} \\ =& \eta((\tilde{f} \circ \mu)_x) \bullet_Y \eta((\tilde{f} \circ \mu)_y)).\end{aligned}$$

Therefore η is a homomorphism.

Let $\tilde{f}_a \in Y/\tilde{f}$. Then there exists $x \in X$ such that $\mu(x) = a$ since μ is surjective. Hence $\eta((\tilde{f} \circ \mu)_X) = \tilde{f}_{\mu(x)} = \tilde{f}_a$ and so η is surjective.

Let $x, y \in X$ be such that $\tilde{f}_{\mu(x)} = \tilde{f}_{\mu(y)}$. Then we have

$$\begin{aligned}(\tilde{f} \circ \mu)(x *_X y) =& \tilde{f}(\mu(x *_X y)) = \tilde{f}(\mu(x) *_Y \mu(y)) \\ =& \tilde{f}(0_Y) = \tilde{f}(\mu(0_X)) = (\tilde{f} \circ \mu)(0_X)\end{aligned}$$

and

$$\begin{aligned}(\tilde{f}\circ\mu)(y *_X x) =& \tilde{f}(\mu(y *_X x)) = \tilde{f}(\mu(y) *_Y \mu(x))\\ =& \tilde{f}(0_Y) = \tilde{f}(\mu(0_X)) = (\tilde{f}\circ\mu)(0_X).\end{aligned}$$

It follows that $(\tilde{f}\circ\mu)_X = (\tilde{f}\circ\mu)_y$. Thus η is injective. This completes. □

The homomorphism $\pi : X \to X/\tilde{f}$, $x \to \tilde{f}_X$, is called the *natural homomorphism* of X onto $X/\tilde{f}$. In Theorem 4.6, if we define natural homomorphisms $\pi_X : X \to X/\tilde{f}\circ\mu$ and $\pi_Y : Y \to Y/\tilde{f}$ then it is easy to show that $\eta\circ\pi_X = \pi_Y\circ\mu$, i.e., the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\mu} & Y \\ {\scriptstyle \pi_X}\big\downarrow & & {\scriptstyle \pi_Y}\big\downarrow \\ X/(\tilde{f}\circ\mu) & \xrightarrow{\eta} & Y/\tilde{f}. \end{array}$$

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Symmetric solutions for hybrid fractional differential equations

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Abstract

In this paper we introduce a new class of symmetric functions and study the existence of symmetric solutions for hybrid Caputo fractional differential equations. A fixed point theorem in Banach algebra for two operators is used. An example is presented to illustrate our result.

Keywords: Caputo fractional derivative; hybrid fractional differential equation; symmetric solution; fixed point theorem
2010 Mathematics Subject Classifications: 34A08; 34A12.

1 Introduction

The aim of this manuscript is to study the existence at least one symmetric solution for hybrid Caputo fractional differential equation subject to initial and symmetric conditions

$$\begin{cases} D^\alpha \left[\dfrac{x(t)}{f(t,x(t))}\right] + g(t,x(t)) = 0, \quad t \in J := [0,T], \\ x(0) = \beta, \qquad x(t) = x(T-t), \end{cases} \tag{1.1}$$

where D^α denotes the Caputo fractional derivative of order α, $1 < \alpha \leq 2$, $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $g \in C(J \times \mathbb{R}, \mathbb{R})$, $\beta \in \mathbb{R}$. A function $x \in C([0,T], \mathbb{R})$ satisfying the relation $x(t) = x(T-t)$, $t \in [0,T]$, is called *symmetric* on $[0,T]$.

Fractional differential equations have been of great interest recently. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications of such constructions in various science such as physics, mechanics, chemistry, and engineering. There have appeared lots of works, in which fractional derivatives are used for a better description of considered material properties. For details, and some recent results on the subject we refer to [1]-[17] and references cited therein.

Recently, many authors have focused on the existence of symmetric solutions for ordinary differential equation boundary value problems; for example, see [18]-[21] and the references therein. In [22]

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the existence and uniqueness of symmetric solutions for a boundary value problem for nonlinear fractional differential equations with multi-order fractional integral boundary conditions was studied, by using a variety of fixed point theorems (such as Banach contraction principle, nonlinear contractions, Krasnoselskii fixed point theorem and Leray-Schauder nonlinear alternative).

Hybrid fractional differential equations have also been studied by several researchers. This class of equations involves the fractional derivative of an unknown function hybrid with the nonlinearity depending on it. Some recent results on hybrid differential equations can be found in a series of papers ([23]-[28]).

In this paper we prove the existence of symmetric solutions for the hybrid Caputo fractional boundary value problem (1.1). One new result is proved by using a hybrid fixed point theorem for two operators in a Banach algebra due to Dhage [29].

The rest of this paper is organized as follows: In Section 2 we present some preliminary notations, definitions and lemmas that we need in the sequel. Also we introduce a new class of symmetric functions and prove some interesting properties, which are used to establish the Green function. In Section 3 we establish the existence of symmetric solutions for the boundary value problem (1.1). An example illustrating the obtained result is also presented.

2 Preliminaries

In this section, we introduce some notations and definitions of fractional calculus [1, 2] and present preliminary results needed in our proofs later. In addition, a new definition of α-symmetric function is presented and also some properties are proved.

Definition 2.1 *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $g : (0,\infty) \to \mathbb{R}$ is defined by*

$$I^\alpha g(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)ds,$$

provided the right-hand side is point-wise defined on $(0,\infty)$, where Γ is the Gamma function.

Definition 2.2 *The Caputo fractional derivative of order $\alpha > 0$ for an at least n-times differentiable function $g : (0,\infty) \to \mathbb{R}$ is defined by*

$$D^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{g^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds, \quad n-1 < \alpha < n,$$

where $n = [\alpha]+1$, $[\alpha]$ denotes the integer part of real number α.

From the definition of the Caputo fractional derivative, we can obtain the following lemmas.

Lemma 2.3 *(see [1]) Let $\alpha > 0$, the general solution of the fractional differential equation $D^\alpha y(t) = 0$ is given by*

$$y(t) = c_0 + c_1 t + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [\alpha]+1$.

Definition 2.4 *A function $y \in C^2(J,\mathbb{R})$ is called symmetric, if it satisfies the relation $y(t) = y(T-t)$.*

From Definition 2.4 we have $y'(t) = -y'(T-t)$, $y''(t) = y''(T-t)$ and

$$\int_0^{T-t} y(s)ds = \int_0^T y(s)ds - \int_0^t y(s)ds. \tag{2.1}$$

Lemma 2.5 *Let $f \in L^2(J,\mathbb{R})$ be symmetric function. Then we have*

$$I^1 f(T) = \frac{2}{T} I^2 f(T). \tag{2.2}$$

Proof. Since f is symmetric on $[0,T]$, we have

$$\begin{aligned} I^1 f(T) = \int_0^T f(s)ds &= \frac{1}{T}\int_0^T (T-s+s)f(s)ds \\ &= \frac{1}{T}\int_0^T (T-s)f(s)ds + \frac{1}{T}\int_0^T sf(s)ds \\ &= \frac{2}{T}\int_0^T (T-s)f(s)ds \ = \ \frac{2}{T}I^2 f(T). \end{aligned}$$

Therefore, (2.2) holds. □

Now, we define a new class of symmetric functions as follows:

Definition 2.6 *A function $f \in C^1(J,\mathbb{R})$ is called α-symmetric if $D^{2-\alpha}f(t)$ is symmetric function on $[0,T]$, where $1 < \alpha \le 2$.*

Example 2.7 *Let $f : [0,1] \to \mathbb{R}$ be defined as*

$$f(t) = \frac{4}{3\sqrt{\pi}} t^{\frac{3}{2}} \left(1 - \frac{4}{5}t\right).$$

It easy to verify that

$$\begin{aligned} D^{2-\frac{3}{2}} f(t) &= D^{\frac{1}{2}} f(t) \\ &= \frac{4}{3\sqrt{\pi}} D^{\frac{1}{2}} t^{\frac{3}{2}} - \frac{16}{15\sqrt{\pi}} D^{\frac{1}{2}} t^{\frac{5}{2}} \\ &= t(1-t). \end{aligned}$$

Therefore, f is $\frac{3}{2}$-symmetric function.

Remark 2.8 *If $\alpha = 2$, then the class of α-symmetric functions is reduced to the class of usual symmetric functions.*

Lemma 2.9 *Let $z \in C^1(J,\mathbb{R})$ be an α-symmetric function. Then the symmetric solution of linear fractional differential equation*

$$D^\alpha y(t) = z(t), \quad 1 < \alpha \le 2, \ t \in J, \tag{2.3}$$

$$y(t) = y(T-t), \tag{2.4}$$

is given by

$$y(t) = I^\alpha z(t) - \frac{t}{T} I^\alpha z(T) + c_0, \tag{2.5}$$

where $c_0 \in \mathbb{R}$.

Proof. By Lemma 2.3, we have

$$y(t) = I^\alpha z(t) + c_1 t + c_0, \tag{2.6}$$

where $c_0, c_1 \in \mathbb{R}$. We apply symmetric condition to obtain

$$I^\alpha z(t) + c_1 t + c_0 = I^\alpha z(T-t) + c_1(T-t) + c_0. \tag{2.7}$$

Evidently, (2.7) becomes

$$\begin{aligned} c_1(2t-T) &= I^\alpha z(T-t) - I^\alpha z(t) \\ &= \int_0^{T-t} \frac{(T-t-s)^{\alpha-1}}{\Gamma(\alpha)} z(s)ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} z(s)ds. \end{aligned} \tag{2.8}$$

Taking the first-order usual derivative with respect to t in (2.8), we get

$$\begin{aligned} 2c_1 &= -I^{\alpha-1}z(T-t) - I^{\alpha-1}z(t) \\ &= -I^1\left(D^{2-\alpha}z\right)(T-t) - I^1\left(D^{2-\alpha}z\right)(t). \end{aligned}$$

Since $D^{2-\alpha}z(t)$ is symmetric on J, and z is symmetric, by (2.1), we have

$$I^1\left(D^{2-\alpha}z\right)(T-t) = I^1\left(D^{2-\alpha}z\right)(T) - I^1\left(D^{2-\alpha}z\right)(t),$$

which leads to

$$\begin{aligned} 2c_1 &= -I^1\left(D^{2-\alpha}z\right)(T) \\ &= -\frac{2}{T}I^2\left(D^{2-\alpha}z\right)(T), \end{aligned}$$

by using Lemma 2.5.
Therefor, we obtain the constant c_1 as

$$c_1 = -\frac{1}{T}I^\alpha z(T) = -\frac{1}{T}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}z(s)ds.$$

Substituting the constant c_1 in (2.6), we get the result in (2.5) as desired. □

In the following we present the Green function of the hybrid fractional boundary value problem (1.1).

Lemma 2.10 *Let $h \in C^1(J, \mathbb{R})$ be the α-symmetric function and $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$. Then the unique solution of*

$$D^\alpha\left[\frac{x(t)}{f(t, x(t))}\right] + h(t) = 0, \quad t \in J, \tag{2.9}$$

$$x(0) = \beta, \qquad x(t) = x(T-t), \tag{2.10}$$

is given by

$$x(t) = f(t, x(t))\left(\int_0^T G(t,s)h(s)ds + \frac{\beta}{f(0,\beta)}\right), \tag{2.11}$$

where

$$G(t,s) = \begin{cases} \dfrac{t\,(T-s)^{\alpha-1} - T\,(t-s)^{\alpha-1}}{T\Gamma(\alpha)}, & 0 \le s \le t \le T, \\ \dfrac{t\,(T-s)^{\alpha-1}}{T\Gamma(\alpha)}, & 0 \le t \le s \le T. \end{cases} \tag{2.12}$$

Proof. Applying Lemma 2.9, the equation (2.9) can be written as

$$\frac{x(t)}{f(t,x(t))} = -I^\alpha h(t) + \frac{t}{T}I^\alpha h(T) + c_0, \tag{2.13}$$

where $c_0 \in \mathbb{R}$. The condition $x(0) = 0$ implies that

$$c_0 = \frac{\beta}{f(0,\beta)}.$$

Therefore, the unique solution of problem (2.9)-(2.10) is

$$x(t) = f(t, x(t))\Bigg(- \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds$$

$$
\begin{aligned}
&\left. + \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s)ds \right) + \frac{\beta}{f(0,\beta)} f(t,x(t)) \\
= \; & f(t,x(t)) \int_0^T G(t,s)h(s)ds + \frac{\beta}{f(0,\beta)} f(t,x(t)).
\end{aligned}
$$

The proof is completed. □

Remark 2.11 *The Green's function $G(t,s)$ defined by (2.12), is not positive for all $t,s \in J$. For example, if $T=5$, $t=2$, $s=1$ and $\alpha = 3/2$, then we have $G(2,1) = -2/(5\sqrt{\pi})$.*

Lemma 2.12 *The Green's function $G(t,s)$ in (2.12) satisfies the following inequalities*

$$G(t,s) \le G(s,s) \le \frac{((\alpha-1)T)^{\alpha-1}}{\alpha^{\alpha-1}\Gamma(\alpha+1)} \quad \text{for all} \quad s,t \in J. \tag{2.14}$$

Proof. Let us define two functions by

$$g_1(t,s) = t\,(T-s)^{\alpha-1} - T\,(t-s)^{\alpha-1}, \;\; 0 \le s \le t \le T,$$

and

$$g_2(t,s) = t\,(T-s)^{\alpha-1}, \;\; 0 \le t \le s \le T.$$

Obviously, for $0 \le t \le s \le T$, the function $g_2(t,s)$ satisfies

$$g_2(t,s) \le g_2(s,s) = s(T-s)^{\alpha-1}.$$

Let $s \in [0,T)$ be fixed. Differentiating with respect to t the function $g_1(t,s)$, we have

$$\frac{\partial}{\partial t} g_1(t,s) = (T-s)^{\alpha-1} - (\alpha-1)T\,(t-s)^{\alpha-2}, \;\; s < t.$$

We can find that $\partial g_1/\partial t = 0$ if and only if

$$t = t^* = s + \frac{(T-s)^{\frac{\alpha-1}{\alpha-2}}}{((\alpha-1)T)^{\frac{1}{\alpha-2}}}.$$

It follows from $\partial g_1/\partial t > 0$ on $(0,t^*)$ and $\partial g_1/\partial t < 0$ on (t^*,T) that

$$g_1(t,s) \le g_1(t^*,s).$$

Simplifying the above inequality, we get

$$
\begin{aligned}
g_1(t,s) &\le g_1(t^*,s) \\
&= s(T-s)^{\alpha-1} - (2-\alpha)T \cdot \frac{(T-s)^{\frac{(\alpha-1)^2}{\alpha-2}}}{((\alpha-1)T)^{\frac{\alpha-1}{\alpha-2}}} \\
&\le s(T-s)^{\alpha-1} = g_1(s,s),
\end{aligned}
$$

which implies the first inequality.
Next, we will prove the second inequality. Taking the first derivative for $g_2(s,s)$ with respect to s on $[0,T)$, we have

$$g_2'(s,s) = (T-s)^{\alpha-2}(T-\alpha s).$$

Thus $g_2'(s,s)$ has a unique zero at the point $s = s^* = T/\alpha$ such that $s^* \in (0,T)$. Observe that $g_2'(s,s) > 0$ on $(0,s^*)$ and $g_2'(s,s) < 0$ on (s^*,T). Hence

$$g_2(s,s) \le g_2\left(\frac{T}{\alpha}, \frac{T}{\alpha}\right) = \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha} T^\alpha.$$

Then the second inequality is proved. □

Let $E = C([0,T],\mathbb{R})$ be the Banach space endowed with the supremum norm $\|\cdot\|$. Define a multiplication in E by

$$(xy)(t) = x(t)y(t), \quad \forall t \in J.$$

Clearly E is a Banach algebra with respect to above supremum norm and the multiplication in it. The main result is based on the following fixed point theorem for two operators in Banach algebra due to Dhage [29].

Lemma 2.13 *Let S be a non-empty, closed convex and bounded subset of the Banach algebra E, let $A : E \to E$ and $B : S \to E$ be two operators such that:*

- (a) *A is Lipschitzian with a Lipschitz constant δ,*
- (b) *B is completely continuous,*
- (c) *$x = AxBy \Rightarrow x \in S$ for all $y \in S$, and*
- (d) *$M\delta < 1$, where $M = \|B(S)\| = \sup\{\|B(x)\| : x \in S\}$.*

Then the operator equation $x = AxBx$ has a solution in S.

3 Main Result

Now, we are in the position to prove the existence of symmetric solutions for hybrid fractional problem (1.1).

Theorem 3.1 *Assume that the following conditions are satisfied:*

(H_1) *The functions $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C^1(J \times \mathbb{R}, \mathbb{R})$ are symmetric and α-symmetric on J, respectively.*

(H_2) *There exists a bounded function $\phi(t)$, with bound $\|\phi\|$, such that*

$$|f(t,x) - f(t,y)| \leq \|\phi\| \cdot |x - y|$$

for $t \in J$ and $x, y \in \mathbb{R}$.

(H_3) *There exist a function $p \in C(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\Psi : [0,\infty) \to (0,\infty)$ such that*

$$|g(t,x)| \leq p(t)\Psi(|x|), \qquad (t,x) \in J \times \mathbb{R}.$$

(H_4) *There exist a number $r > 0$ such that*

$$r \geq \frac{F_0\left[\dfrac{(\alpha-1)^{\alpha-1}T^\alpha}{\alpha^{\alpha-1}\Gamma(\alpha+1)}\|p\|\Psi(r) + \dfrac{|\beta|}{|f(0,\beta)|}\right]}{1 - \|\phi\|\left[\dfrac{(\alpha-1)^{\alpha-1}T^\alpha}{\alpha^{\alpha-1}\Gamma(\alpha+1)}\|p\|\Psi(r) + \dfrac{|\beta|}{|f(0,\beta)|}\right]} \tag{3.1}$$

where $F_0 = \sup_{t\in J}|f(t,0)|$ and

$$\|\phi\|\left[\frac{(\alpha-1)^{\alpha-1}T^\alpha}{\alpha^{\alpha-1}\Gamma(\alpha+1)}\|p\|\Psi(r) + \frac{|\beta|}{|f(0,\beta)|}\right] < 1. \tag{3.2}$$

Then the problem (1.1) has at least one symmetric solution on J.

Proof. To prove our main result, we first define a subset S of E by

$$S = \{x \in E : \|x\| \leq r\},$$

where r satisfies (3.1). Clearly S is closed, convex and bounded subset of the Banach space E. By Lemma 2.10, we define two operators $\mathcal{A} : E \to E$ by

$$\mathcal{A}x(t) = f(t, x(t)), \qquad t \in J, \tag{3.3}$$

and

$$\mathcal{B}x(t) = \int_0^T G(t,s)g(s,x(s))ds + \frac{\beta}{f(0,\beta)}, \qquad t \in J. \tag{3.4}$$

Hence, the problem (1.1) is transformed into an operator equation as

$$x = \mathcal{A}x\mathcal{B}x. \tag{3.5}$$

Next, we shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Lemma 2.13 under our assumptions. This will be achieved in the series of following steps.

Step 1. *We first show that $\mathcal{A}$ is Lipschitzian on E.*

Let $x, y \in E$. Then by (H_2), for $t \in J$ we have

$$\begin{aligned} |\mathcal{A}x(t) - \mathcal{A}y(t)| &= |f(t,x(t)) - f(t,y(t))| \\ &\leq \phi(t)|x(t) - y(t)| \\ &\leq \|\phi\|\|x - y\|, \end{aligned}$$

which implies that $\|\mathcal{A}x - \mathcal{A}y\| \leq \|\phi\|\|x - y\|$ for all $x, y \in E$. Therefore, $\mathcal{A}$ is a Lipschitzian on E with Lipschitz constant $\delta = \|\phi\|$.

Step 2. *The operator $\mathcal{B}$ is completely continuous on S.*

We first show that the operator $\mathcal{B}$ is continuous on S. Let $\{x_n\}$ be a sequence in S converging to a point $x \in S$. Then by Lebesgue dominated convergence theorem, for all $t \in J$, we have

$$\begin{aligned} \lim_{n\to\infty} \mathcal{B}x_n(t) &= \lim_{n\to\infty} \int_0^T G(t,s)g(s,x_n(s))ds + \frac{\beta}{f(0,\beta)} \\ &= \int_0^T G(t,s) \lim_{n\to\infty} g(s,x_n(s))ds + \frac{\beta}{f(0,\beta)} \\ &= \int_0^T G(t,s)g(s,x(s))ds + \frac{\beta}{f(0,\beta)} \\ &= \mathcal{B}x(t). \end{aligned}$$

This shows that $\{\mathcal{B}x_n\}$ converges to $\mathcal{B}x$ pointwise on J.

Next, we will show that $\{\mathcal{B}x_n\}$ is an equicontinuous sequence of functions in S. Let $\tau_1, \tau_2 \in J$ be arbitrary with $\tau_1 < \tau_2$. Then

$$\begin{aligned} |\mathcal{B}x_n(\tau_2) - \mathcal{B}x_n(\tau_1)| &= \left| \int_0^T G(\tau_2,s)g(s,x_n(s))ds - \int_0^T G(\tau_1,s)g(s,x_n(s))ds \right| \\ &\leq \|p\|\Psi(r) \left| \int_0^T G(\tau_2,s)ds - \int_0^T G(\tau_1,s)ds \right| \\ &\leq \|p\|\Psi(r) \left| \int_0^{\tau_2} \frac{\tau_2(T-s)^{\alpha-1} - T(\tau_2-s)^{\alpha-1}}{T\Gamma(\alpha)} ds \right. \end{aligned}$$

$$
\begin{aligned}
& + \int_{\tau_2}^{T} \frac{\tau_2 (T-s)^{\alpha-1}}{T\Gamma(\alpha)} ds - \int_{\tau_1}^{T} \frac{\tau_1 (T-s)^{\alpha-1}}{T\Gamma(\alpha)} ds \\
& - \int_0^{\tau_1} \frac{\tau_1 (T-s)^{\alpha-1} - T(\tau_1 - s)^{\alpha-1}}{T\Gamma(\alpha)} ds \Bigg| \\
= \;& \|p\|\Psi(r) \Bigg| \int_0^T \frac{\tau_2 (T-s)^{\alpha-1}}{T\Gamma(\alpha)} ds - \int_0^{\tau_2} \frac{(\tau_2 - s)^{\alpha-1}}{\Gamma(\alpha)} ds \\
& - \int_0^T \frac{\tau_1 (T-s)^{\alpha-1}}{T\Gamma(\alpha)} ds + \int_0^{\tau_1} \frac{(\tau_1 - s)^{\alpha-1}}{\Gamma(\alpha)} ds \Bigg| \\
\leq \;& \|p\|\Psi(r) \int_0^T \frac{(\tau_2 - \tau_1)(T-s)^{\alpha-1}}{T\Gamma(\alpha)} ds \\
& + \|p\|\Psi(r) \int_0^{\tau_1} \frac{(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}}{\Gamma(\alpha)} ds \\
& + \|p\|\Psi(r) \int_{\tau_1}^{\tau_2} \frac{(\tau_2 - s)^{\alpha-1}}{\Gamma(\alpha)} ds.
\end{aligned}
$$

Consequently

$$|\mathcal{B}x_n(\tau_2) - \mathcal{B}x_n(\tau_1)| \to 0 \quad \text{as} \quad \tau_2 \to \tau_1$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{B}x_n \to \mathcal{B}x$ is uniformly and hence $\mathcal{B}$ is a continuous operator on S.

Now we will prove that the set $\mathcal{B}(S)$ is a uniformly bounded in S. For any $x \in S$ and using Lemma 2.12, we have

$$
\begin{aligned}
|\mathcal{B}x(t)| &= \left| \int_0^T G(t,s) g(s, x(s)) ds + \frac{\beta}{f(0,\beta)} \right| \\
&\leq \int_0^T |G(t,s)| p(s) \Psi(r) ds + \frac{|\beta|}{|f(0,\beta)|} \\
&\leq \frac{(\alpha-1)^{\alpha-1} T^{\alpha}}{\alpha^{\alpha-1}\Gamma(\alpha+1)} \|p\|\Psi(r) + \frac{|\beta|}{|f(0,\beta)|} := K_1,
\end{aligned}
$$

for all $t \in J$. Therefore, $\|\mathcal{B}x\| \leq K_1$ which shows that $\mathcal{B}$ is uniformly bounded on S.

Next, we will show that $\mathcal{B}(S)$ is an equicontinuous set in E. Let $\tau_1, \tau_2 \in J$ with $\tau_1 < \tau_2$ and $x \in S$. Then, as above, we have

$$
\begin{aligned}
|\mathcal{B}x(\tau_2) - \mathcal{B}x(\tau_1)| &= \left| \int_0^T G(\tau_2, s) g(s, x(s)) ds - \int_0^T G(\tau_1, s) g(s, x(s)) ds \right| \\
&\leq \|p\|\Psi(r) \left| \int_0^T G(\tau_2, s) ds - \int_0^T G(\tau_1, s) ds \right| \\
&\leq \|p\|\Psi(r) \int_0^T \frac{(\tau_2 - \tau_1)(T-s)^{\alpha-1}}{T\Gamma(\alpha)} ds \\
&\quad + \|p\|\Psi(r) \int_0^{\tau_1} \frac{(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}}{\Gamma(\alpha)} ds \\
&\quad + \|p\|\Psi(r) \int_{\tau_1}^{\tau_2} \frac{(\tau_2 - s)^{\alpha-1}}{\Gamma(\alpha)} ds,
\end{aligned}
$$

which is independent of $x \in S$. As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero. Therefore, it follows from the Arzelá-Ascoli theorem that $\mathcal{B}$ is a completely continuous operator on S.

Step 3. *The hypothesis (c) of Lemma 2.13 is satisfied.*

Let $x \in E$ and $y \in S$ be arbitrary elements such that $x = \mathcal{A}x\mathcal{B}y$. Then we have

$$\begin{aligned}
|x(t)| &\leq |\mathcal{A}x(t)||\mathcal{B}y(t)| \\
&= |f(t,x(t))| \left| \int_0^T G(t,s)g(s,y(s))ds + \frac{\beta}{f(0,\beta)} \right| \\
&\leq (|f(t,x(t)) - f(t,0)| + |f(t,0)|) \left(\frac{(\alpha-1)^{\alpha-1}T^\alpha}{\alpha^{\alpha-1}\Gamma(\alpha+1)} \|p\|\Psi(r) + \frac{|\beta|}{|f(0,\beta)|} \right) \\
&\leq (|x(t)| \cdot \|\phi\| + F_0) \left(\frac{(\alpha-1)^{\alpha-1}T^\alpha}{\alpha^{\alpha-1}\Gamma(\alpha+1)} \|p\|\Psi(r) + \frac{|\beta|}{|f(0,\beta)|} \right),
\end{aligned}$$

which leads to

$$\begin{aligned}
\|x\| &\leq (\|x\| \cdot \|\phi\| + F_0) \left(\frac{(\alpha-1)^{\alpha-1}T^\alpha}{\alpha^{\alpha-1}\Gamma(\alpha+1)} \|p\|\Psi(r) + \frac{|\beta|}{|f(0,\beta)|} \right) \\
&\leq r.
\end{aligned}$$

Therefore, $x \in S$.

Step 4. *Finally we show that $\delta M < 1$, that is (d) of Lemma 2.13 holds.*

Since

$$\begin{aligned}
M &= \|\mathcal{B}(S)\| \\
&= \sup_{x\in S} \left\{ \sup_{t\in J} |\mathcal{B}x(t)| \right\} \\
&\leq \frac{(\alpha-1)^{\alpha-1}T^\alpha}{\alpha^{\alpha-1}\Gamma(\alpha+1)} \|p\|\Psi(r) + \frac{|\beta|}{|f(0,\beta)|},
\end{aligned} \tag{3.6}$$

by (3.2) we have

$$\delta\|M \leq \|\phi\| \left(\frac{(\alpha-1)^{\alpha-1}T^\alpha}{\alpha^{\alpha-1}\Gamma(\alpha+1)} \|p\|\Psi(r) + \frac{|\beta|}{|f(0,\beta)|} \right) < 1,$$

with $\delta = \|\phi\|$.

Thus all the conditions of Lemma 2.13 are satisfied and hence the operator equation $x = \mathcal{A}x\mathcal{B}x$ has a solution in S. In consequence, the problem (1.1) has a symmetric solution on J. This completes the proof. □

Next, we present an example to illustrate our result.

Example 3.2 *Consider the following hybrid fractional differential equation with initial and symmetric conditions*

$$\begin{cases}
D^{\frac{3}{2}} \left[\dfrac{x(t)}{\dfrac{x^2(t) + 2|x(t)|}{2(5+(t-1)^2)(|x(t)|+1)} + \dfrac{1}{2}} \right] + \dfrac{1}{24} \left(1 + \sin^2 \left(\dfrac{8t^{\frac{3}{2}}}{3\sqrt{\pi}} \left(1 - \dfrac{2}{5}t \right) \right) \right) \\
\qquad \times \left(\dfrac{x^2(t)}{1+|x(t)|} + 1 \right) = 0, \quad t \in [0,2], \\
x(0) = \dfrac{1}{3}, \qquad x(t) = x(2-t).
\end{cases} \tag{3.7}$$

Here $\alpha = 3/2$, $T = 2$ and $\beta = 1/3$. Since $(t-1)^2$ is symmetric on $[0,2]$ and $(8t^{\frac{3}{2}})(1-(2/5)t)/(3\sqrt{\pi})$ is 3/2-symmetric by

$$D^{\frac{1}{2}} \left(\frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}} \left(1 - \frac{2}{5}t \right) \right) = t(2-t), \quad t \in [0,2],$$

then we get that $f(t,\cdot)$ and $g(t,\cdot)$ are symmetric and 3/2-symmetric functions on $[0,2]$, respectively. With the above information, we find that

$$\begin{aligned} |f(t,x)-f(t,y)| &= \left|\frac{x^2+2|x|}{2(5+(t-1)^2)(|x|+1)} - \frac{y^2+2|y|}{2(5+(t-1)^2)(|y|+1)}\right| \\ &\leq \frac{1}{5+(t-1)^2}|x-y|, \end{aligned}$$

and

$$|g(t,x)| \leq \frac{1}{12}(|x|+1),$$

and $F_0 = \sup_{t\in[0,2]}|f(t,0)| = 1/2$. Choosing $\phi(t) = 1/(5+(t-1)^2)$, $p(t) = 1/12$, we have $\|\phi\| = 1/5$ and $\|p\| = 1/12$. Setting $\Psi(|x|) = |x|+1$, we can find that there exists $0.06962115393 < r < 45.01973321$ which is satisfied (3.1)-(3.2). Thus all the conditions of Theorem 3.1 are satisfied. Therefore, by the conclusion of Theorem 3.1, the problem (3.7) has at least one symmetric solution on $[0,2]$.

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ON THE k-TH DEGENERATION OF THE GENOCCHI POLYNOMIALS

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Abstract. Jeong-Rim-Kim(2015) studied the degenerate Cauchy numbers and polynomials and investigated some properties of these k-times degenerate Cauchy numbers and polynomials. In this paper, we define the degenerate Genocchi polynomials and the k-th degeneration of Genocchi polynomials , and investigate some properties of these polynomials.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will, respectively, denote the rings of p-adic integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|$ is normalized by $|p|_p = \frac{1}{p}$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim as

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x \tag{1}$$

(see [7,8,11,13,14,16,17,20,22]). Then, by (1), we get

$$I_{-1}(f) = -I_{-1}(f) + 2f(0), \tag{2}$$

where $f_1(x) = f(x+1)$.

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From (2), we can derive the following integral equation

$$I_{-1}(f_n) = (-1)^n I_{-1}(f) + 2\sum_{l=0}^{n-1}(-1)^{n-1-l}f(l), \tag{3}$$

where $f_n(x) = f(x+n)$, $(n \in \mathbb{N})$.

As is well known, the Euler polynomials are also defined by the generating function to be

$$\left(\frac{2}{e^t+1}\right)e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \qquad (\text{see } [1,2,4-22]). \tag{4}$$

When $x = 0$, $E_n = E_n(0)$ are called the Euler numbers.

The degenerate Euler polynomials are also defined by the degenerating function to be

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} E_n(\lambda, x)\frac{t^n}{n!} \qquad (\text{see } [1,4,8,11,13,14,16,17,20,22]). \tag{5}$$

When $x = 0, E_n(\lambda) = E_n(\lambda, 0)$ are called the degenerate Euler number.

Note that $\lim_{x\to 0} E_n(\lambda, x) = E_n(x)$. We recall that the Genocchi polynomials are defined by the generating function to be

$$\frac{2}{e^t+1} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} \qquad (\text{see } [18,20,22]). \tag{6}$$

In recent years, many researchers have studied various types of special polynomials, for examples, Barnes-type degenerate Euler polynomials, the degenerate Frobenius-Euler polynomials, the degenerate Frobenius-Genocchi polynomials, and degenerate Bernoulli polynomials (see [2,3,6,9,10,12,13,15]).

In particular, recently, Jeong-Rim-Kim ([5]) studied finite times degenerate Cauchy polynomials and investigated some properties of them.

Thus, our motivation in this paper is to define the degenerate Genocchi polynomials, to define the k-th degeneration of Genocchi polynomials, and to investigate some properties of these k-th degeneration of Genocchi polynomials.

2. The k-th degeneration of Genocchi polynomials

In this section, we define the degenerate Genocchi polynomials which are defined by the generating function to be

$$\frac{2t}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=1}^{\infty} g_n^{(0)}(x|\lambda)\frac{t^n}{n!}. \tag{7}$$

When $x = 0, g_n^{(0)}(\lambda) = g_n^{(0)}(0|\lambda)$ are called the degenerate Genocchi number.

From (2), we easily obtain

$$\frac{2t}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} = t\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x+y}{\lambda}}d\mu_{-1}(y). \tag{8}$$

We note that the Stirling number of the first kind is defined as

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l \qquad (n \geq 0) \tag{9}$$

where $(x)_n = x(x-1)\cdots(x-n+1)$ and $(x)_0 = 1$, and

$$(\log(1+t))^n = n! \sum_{m=n}^{\infty} S_1(m,n)\frac{t^m}{m!} \tag{10}$$

and the Stirling number of the second kind is defined as

$$(e^t - 1)^n = n! \sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!}. \tag{11}$$

From (7), we get

$$\begin{aligned} \sum_{n=1}^{\infty} g_n^{(0)}(x|\lambda)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} g_{n+1}^{(0)}(x|\lambda)\frac{t^{n+1}}{(n+1)!} \\ &= t\sum_{n=0}^{\infty} \frac{g_{n+1}^{(0)}(x|\lambda)}{n+1}\frac{t^n}{n!}. \end{aligned} \tag{12}$$

From (8), we get

$$\begin{aligned} \frac{2t}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} &= t\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x+y}{\lambda}} d\mu_{-1}(y) \\ &= \sum_{n=0}^{\infty} \lambda^{-n} \int_{\mathbb{Z}_p} \left(\frac{x+y}{\lambda}\right)_n d\mu_{-1}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (x+y|\lambda)_n d\mu_{-1}\frac{t^n}{n!}. \end{aligned} \tag{13}$$

Thus, by (7), (12), and (13), we get

$$\frac{g_{n+1}^{(0)}(x|\lambda)}{(n+1)} = \int_{\mathbb{Z}_p} (x+y|\lambda)_n d\mu_{-1}. \tag{14}$$

In the viewpoint of (7), we consider the first degeneration of Genocchi polynomials which are defined by the generating function to be

$$\frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\log(1+\lambda t))^{\frac{1}{\lambda}}+1}(1+\log(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{m=1}^{\infty} g_m^{(1)}(x|\lambda)\frac{t^m}{m!}. \tag{15}$$

By replacing t by $\log(1+\lambda t)^{\frac{1}{\lambda}}$ in (8), we get

$$\begin{aligned} & \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\log(1+\lambda t))^{\frac{1}{\lambda}}+1}(1+\log(1+\lambda t)^{\frac{x}{\lambda}} \\ = & \frac{1}{\lambda}\log(1+\lambda t)\int_{\mathbb{Z}_p} (1+\log(1+\lambda t))^{\frac{x+y}{\lambda}} d\mu_{-1}(y) \\ = & \frac{1}{\lambda}\log(1+\lambda t)\sum_{n=0}^{\infty}\frac{1}{n!}\int_{\mathbb{Z}_p} \lambda^{-n}(x+y|\lambda)_n d\mu_{-1}(\log(1+\lambda t))^n \\ = & \sum_{n=0}^{\infty}\frac{1}{n!}\int_{\mathbb{Z}_p} (x+y|\lambda)_n \lambda^{-n-1} d\mu_{-1}(\log(1+\lambda t))^{n+1} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}(x+y|\lambda)_n\lambda^{-n-1}d\mu_{-1}\frac{(n+1)!}{n!}\sum_{m=n+1}^{\infty}\lambda^m S_1(m,n+1)\frac{t^m}{m!} \\
&= \sum_{m=1}^{\infty}\sum_{n=0}^{m-1}(n+1)\lambda^{m-n-1}S_1(m,n+1)\int_{\mathbb{Z}_p}(x+y|\lambda)_n d\mu_{-1}(y)\frac{t^m}{m!}. \qquad (16)
\end{aligned}
$$

Thus, by (14), (15), and (16), we get

$$
\begin{aligned}
g_m^{(1)}(x|\lambda) &= \sum_{n=0}^{m-1}(n+1)\lambda^{m-n-1}S_1(m,n+1)\int_{\mathbb{Z}_p}(x+y|\lambda)_n d\mu_{-1}(y) \\
&= \sum_{n=0}^{m-1}\lambda^{-m-n-1}S_1(m,n+1)g_{n+1}^{(0)}(x|\lambda). \qquad (17)
\end{aligned}
$$

By (17), we obtain the following theorem.

Theorem 2.1. *Let $m \in \mathbb{N}$. Then we have*

$$
g_m^{(1)}(x|\lambda) = \sum_{n=0}^{m-1}\lambda^{m-n-1}S_1(m,n+1)g_{n+1}^{(0)}(x|\lambda). \qquad (18)
$$

Now, we consider the second degeneration of Genocchi polynomials as follows:

$$
\begin{aligned}
&\frac{2\log(1+\log(1+\lambda t))^{\frac{1}{\lambda}}}{(1+\log(1+\log(1+\lambda t)))^{\frac{1}{\lambda}}+1}(1+\log(1+\log(1+\lambda t)))^{\frac{1}{\lambda}} \\
&= \sum_{m=1}^{\infty}g_m^{(2)}(x|\lambda)\frac{t^m}{m!}. \qquad (19)
\end{aligned}
$$

From (19), we get

$$
\begin{aligned}
&\frac{\frac{2}{\lambda}\log(1+\log(1+\lambda t))}{(1+\log(1+\log(1+\lambda t)))^{\frac{1}{\lambda}}+1}(1+\log(1+\log(1+\lambda t)))^{\frac{x}{\lambda}} \\
&= \frac{1}{\lambda}\log(1+\log(1+\lambda t))\int_{\mathbb{Z}_p}(1+\log(1+\log(1+\lambda t)))^{\frac{x+y}{\lambda}}d\mu_{-1}(y). \qquad (20)
\end{aligned}
$$

From (20), we get

$$
\begin{aligned}
&\frac{1}{\lambda}\log(1+\log(1+\lambda t))\int_{\mathbb{Z}_p}(1+\log(1+\log(1+\lambda t)))^{\frac{x+y}{\lambda}}d\mu_{-1}(y) \\
&= \frac{1}{\lambda}\log(1+\log(1+\lambda t))\sum_{n=0}^{\infty}\frac{1}{n!}\int_{\mathbb{Z}_p}\lambda^{-n}(x+y|\lambda)_n d\mu_{-1}(y)(\log(1+\log(1+\lambda t)))^n \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\int_{\mathbb{Z}_p}(x+y|\lambda)_n\lambda^{-n-1}d\mu_{-1}(\log(1+\log(1+\lambda t)))^n \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\int_{\mathbb{Z}_p}(x+y|\lambda)_n\lambda^{-n-1}d\mu_{-1}(y)(n+1)!\sum_{m=n+1}^{\infty}S_1(m,n+1)\frac{(\log(1+\lambda t))^m}{m!} \\
&= \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}(x+y|\lambda)_n\lambda^{-n-1}d\mu_{-1}(y)(n+1)\sum_{m=n+1}^{\infty}S_1(m,n+1)\sum_{l=m}^{\infty}S_1(l,m)\lambda^l\frac{t^l}{l!} \\
&= \sum_{n_3=0}^{\infty}\sum_{n_2=0}^{n_3}\sum_{n_1=0}^{n_2-1}\lambda^{n_3-n_1-1}S_1(n_3,n_2)S_1(n_2,n_1+1)g_{n_1+1}^{(0)}(x|\lambda)\frac{t^{n_3}}{n_3!}. \qquad (21)
\end{aligned}
$$

From (20) and (21), we obtain the following theorem.

Theorem 2.2. *Let $n_3 \in \mathbb{N}$. Then we have*

$$g_{n_3}^{(2)}(x|\lambda) = \sum_{n_2=0}^{n_3}\sum_{n_1=0}^{n_2-1} \lambda^{n_3-n_1-1} S_1(n_3,n_1)S_1(n_2,n_1+1)g_{n_1+1}^{(0)}(x|\lambda). \tag{22}$$

Inductively, we have the k-th degeneration of Genocchi polynomials as follows:

Theorem 2.3. *Let $k, n_k \in \mathbb{N}$. Then we have*

$$g_{n_k}^{(k-1)}(x|\lambda) = \sum_{n_{k-1}=0} \cdots \sum_{n_1-0}^{n_2-1} \lambda^{n_k-n_1-1} S_1(n_k,n_{k-1})\cdots S_1(n_2,n_1+1)g_{n_1+1}^{(0)}(x|\lambda). \tag{23}$$

By replacing t by $\frac{1}{\lambda}(e^{\lambda t}-1)$ in (19) and (20).
We have

$$\begin{aligned}\sum_{n=1}^{\infty} g_n^{(2)}(x|\lambda)\frac{1}{\lambda^n}\frac{(e^{\lambda t}-1)^n}{n!} &= \sum_{n=1}^{\infty} g_n^{(2)}(x|\lambda)\sum_{l=n}^{\infty}\lambda^{l-n}S_2(l,n)\frac{t^l}{l!} \\ &= \sum_{l=0}^{\infty}(\sum_{n=0}^{l} g_n^{(2)}(x|\lambda)\lambda^{l-n}S_2(l,n))\frac{t^l}{l!}. \end{aligned}\tag{24}$$

By (14) and (23), we obtain the following theorem.

Theorem 2.4. *Let $l \in \mathbb{N}$. Then we have*

$$g_l^{(p)}(x|\lambda) = \sum_{n=0}^{l} g_n^{(2)}(x|\lambda)\lambda^{l-n}S_2(l,n). \tag{25}$$

We note the Daehee polynomials of order r is defined by the generating function to be

$$\left(\frac{\log(1+t)}{t}\right)^r (1+t)^x = \sum_{n=0}^{\infty} D_n^{(r)}(x)\frac{t^n}{n!} \tag{26}$$

When $x=0$, $D_n^{(r)} = D_n^{(r)}(0)$ are called the Daehee numbers of order r.
By replacing t by $\log(1+\lambda t)^{\frac{1}{\lambda}}$ in (7), we get

$$\begin{aligned}\sum_{n=1}^{\infty} g_n^{(0)}(x|\lambda)\frac{(\log(1+\lambda t)^{\frac{1}{\lambda}})^n}{n!} &= \frac{2(\log(1+\lambda t)^{\frac{1}{\lambda}})}{(1+\log(1+\lambda t))^{\frac{1}{\lambda}}+1} - (1+\log(1+\lambda t)^{\frac{x}{\lambda}} \\ &= \sum_{n=1}^{\infty} g_n^{(1)}(\lambda|x)\frac{t^n}{n!}. \end{aligned}\tag{27}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty} g_n^{(0)}(x|\lambda)\left(\frac{\log(1+\lambda t)}{t}\right)^n \lambda^{-n}\frac{t^n}{n!} &= \sum_{n=1}^{\infty} g_n^{(0)}(x|\lambda)\left(\sum_{l=0}^{\infty} D_l^{(n)}\frac{t^l}{l!}\right)\lambda^{-n}\frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty}\sum_{l=0}^{\infty} g_n^{(0)}(x|\lambda)D_l^{(n)}\lambda^{-n}\frac{t^{l+n}}{l!n!} \\ &= \sum_{m=1}^{\infty}\left(\sum_{n=0}^{m} g_n^{(0)}(x|\lambda)D_{m-n}^{(n)}\lambda^{-n}\binom{n}{m}\right)\frac{t^m}{m!}. \end{aligned}\tag{28}$$

Thus, by (27) and (28), we obtain the following theorem.

Theorem 2.5. *Let $m \in \mathbb{N}$. Then we have*

$$g_m^{(1)}(x|\lambda) = \sum_{n=0}^{m}\binom{m}{n}\lambda^{-n}g_n^{(0)}(x|\lambda)D_{m-n}^{(n)}. \tag{29}$$

By replacing t by $\frac{1}{\lambda}(e^{\lambda t}-1)$ in (15), we get

$$\begin{aligned}\sum_{m=1}^{\infty} g_m^{(1)}(x|\lambda)\frac{\left(\frac{e^{\lambda t}-1}{\lambda}\right)^m}{m!} &= \frac{2t}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} \\ &= \sum_{l=1}^{\infty} g_l^{(0)}(x|\lambda)\frac{t^l}{l!}. \qquad (30)\end{aligned}$$

and

$$\begin{aligned}\sum_{m=1}^{\infty} g_m^{(1)}(x|\lambda)\lambda^{-m}\frac{\left(e^{\lambda t}-1\right)^m}{m!} &= \sum_{m=1}^{\infty} g_m^{(1)}(x|\lambda)\lambda^{-m}\sum_{l=m}^{\infty} S_2(l,m)\frac{(\lambda t)^l}{l!} \\ &= \sum_{m=1}^{\infty}\left(\sum_{l=m}^{\infty}\lambda^{l-m}g_m^{(1)}(x|\lambda)S_2(l,m)\right)\frac{t^l}{l!} \\ &= \sum_{l=0}^{\infty}\left(\sum_{m=1}^{l}\lambda^{l-m}g_m^{(1)}(x|\lambda)S_2(l,m)\right)\frac{t^l}{l!}. \qquad (31)\end{aligned}$$

Thus, by (30) and (31), we obtain the following theorem.

Theorem 2.6. *Let $l \in \mathbb{N}$. Then we have*

$$g_l^{(0)}(x|\lambda) = \sum_{m=1}^{l}\lambda^{l-m}g_m^{(1)}(x|\lambda)S_2(l,m). \qquad (32)$$

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Regularization solutions of ill-posed Helmholtz-type equations with fuzzy mixed boundary value†

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Abstract In this study, we discuss the solutions of fuzzy Helmholtz-type equations(FHTEs) and their ill-posedness. A regularization method is required to recover the numerical stability. Moreover, the error estimates and convergence of the method is considered. To support our study, one numerical example is illustrated.
Keywords: Fuzzy numbers; Ill-posed problem; Helmholtz equation; Regularization method; Convergence estimate.

1. Introduction

The study of fuzzy partial differential equations (FPDEs) provides a suitable setting for the mathematical modeling of real-world problems that include uncertainty or vagueness. As a new and powerful mathematical tool, FPDEs have been studied using several approaches. The first definition of an FPDE was presented by Buckley and Feuring in [1]. In [2], the authors considered the application of FPDEs obtained using fuzzy rule-based systems. Furthermore, Oberguggenberger described weak and fuzzy solutions for FPDEs [3] and Chen et al. presented a new inference method with applications to FPDEs [4]. In [5], an interpretation was provided of the use of FPDEs for modeling hydrogeological systems. Studies of heat, wave, and Poisson equations with uncertain parameters were provided in [6]. Fuzzy solutions for heat equations based on generalized Hukuhara differentiability were considered in [7]. Several numerical methods have been proposed to solve FPDEs. Such as Allahviranloo ([8]) proposed a difference method for solving FPDEs. The Adomian decomposition method was studied for finding the approximate solution of the fuzzy heat equation in [9]. Solving FPDEs by the differential transformation method was considered in [10].

In this paper, we proposed a numerical method to solve ill-posed problems for the fuzzy Helmholtz-type equation (FHTEs). The Helmholtz equation arises in many areas, especially in practical physical applications, such as acoustic, wave propagation and scattering, vibration of the structure, electromagnetic scattering and so on, see [11, 12, 13, 14]. The direct problems, i.e. Dirichlet, Neumann or mixed boundary value problems for the Helmholtz equation have been studied extensively in the past century. However, in some practical problems, the boundary data on the whole boundary cannot be obtained. We only know the noisy data on a part of the boundary or at some interior points of the concerned domain, which will lead to some inverse problems and severely ill-posed problems. In 1923, Hadamard [15] introduced the concept of a well-posed problem from philosophy where the mathematical model of a physical problem must have properties where the solution exhibits uniqueness, existence, and stability. If one of the properties fails to hold, the problem is known as ill-posed. Numerical computation is difficult due to the ill-posedness of the problem. That means the solution does not depend continuously on the given Cauchy data and any small perturbation in the given data may cause large change to the solution [15, 16, 17]. The present study addresses two issues. First, we consider the ill-posedness of FHTEs using the decomposition theorem. Second, we use the regularization method to recover the numerical stability.

The remainder of this paper is organized as follows. In Section2, we briefly introduce the necessary notions related to fuzzy numbers and differentiability properties for fuzzy set-valued mappings. In Section3, we define the solution and ill-posedness of FHTEs. The regularization method and convergence

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estimates for the initial-boundary value problems of FHTEs are considered in Section4. In Section5, we present some numerical results and our conclusions are given in Section6.

2. Definitions and preliminaries

Let $P_k(R^n)$ denote the family of all nonempty compact convex subset of R^n and define the addition and scalar multiplication in $P_k(R^n)$ as usual. Let A and B be two nonempty bounded subset of R^n. The distance between A and B is defined by the Hausdorff metric

$$d_H(A,B) = \max\Big\{\sup_{a\in A}\inf_{b\in B}||a-b||, \sup_{b\in B}\inf_{a\in A}||b-a||\Big\}, \tag{2.1}$$

where $||\cdot||$ denotes the usual Euclidean norm in R^n [18]. Then $(P_k(R^n); d_H)$ is a metric space.

Denote

$$E^n = \{u : R^n \to [0,1] | u \text{ satisfies (1)-(4) below}\}$$

is a fuzzy number space, where

(1) u is normal, i.e. there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,

(2) u is fuzzy convex, i.e. $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in R^n$ and $0 \leq \lambda \leq 1$,

(3) u is upper semi-continuous,

(4) $[u]_0 = cl\{x \in R^n | u(x) > 0\}$ is compact.

Here, $cl(X)$ denotes the closure of set X. For $0 < \alpha \leq 1$, the α-level set of u (or simply the α-cut) is defined by $[u]_\alpha = \{x \in R^n | u(x) \geq \alpha\}$. The core of u is the set of elements of R^n having membership grade 1, i.e., $[u]_1 = \{x | x \in R^n, u(x) = 1\}$. Then from above (1)-(4), it follows that the α-level set $[u]_\alpha \in P_k(R^n)$ for all $0 < \alpha \leq 1$. According to Zadeh's extension principle, we have addition and scalar multiplication in fuzzy number space E^n as follows:

$$[u+v]_\alpha = [u]_\alpha + [v]_\alpha = \{x + y | x \in [u]_\alpha, y \in [v]_\alpha\},$$

$$[ku]_\alpha = k[u]_\alpha = \{kx | x \in [u]_\alpha\}, [0]_\alpha = \{0\}.$$

where $u, v \in E^n$ and $0 < \alpha \leq 1$. The distance between two fuzzy numbers u and v is defined by

$$D(u,v) = \sup_{\alpha\in[0,1]} d_H([u]_\alpha, [v]_\alpha). \tag{2.2}$$

We recall some differentiability properties for fuzzy set-valued mappings.

Definition 2.1 (See[19]) Let K_C denote the family of all bounded closed intervals in R, the generalized Hukuhara difference of two intervals $A, B \in K_C$ (gH-difference, for short), denoted by $A \ominus_{gH} B$, is defined by

$$A \ominus_{gH} B = C \Longleftrightarrow \begin{cases} (i) A = B + C; \\ or (ii) B = A + (-C). \end{cases} \tag{2.3}$$

Definition 2.2 (See[20]) The generalized Hukuhara difference of two fuzzy numbers $u, v \in E^1$ (gH-difference, for short) is the fuzzy number ω, if it exists, such that

$$u \ominus_{gH} v = \omega \Longleftrightarrow \begin{cases} (i) u = v + \omega; \\ or (ii) v = u + (-\omega). \end{cases} \tag{2.4}$$

It is easy to show that (i) and (ii) are both valid if and only if w is a crisp number.

It may happen that the gH-difference of two fuzzy numbers does not exist (see, for example, [21]). In order to overcome this shortcoming, in [20, 21], a new difference between fuzzy numbers was proposed, which always exists.

Henceforth, T=]a, b[denotes an open interval in R. A function $F : T \to F_C$ is said to be a fuzzy function. For each $\alpha \in [0, 1]$, associated to F, we define the family of interval-valued functions $F_\alpha : T \to K_C$ given by $F_\alpha(x) = [F(x)]^\alpha$, for $x \in T$. For any $\in [0, 1]$, we denote

$$F_\alpha(x) = [\underline{f}_\alpha(x), \overline{f}_\alpha(x)]. \tag{2.5}$$

Here, for each $\alpha \in [0, 1]$, the endpoint functions $\overline{f}_\alpha, \underline{f}_\alpha : T \to R$ are called upper and lower functions of F, respectively. Next, we present the concept of gH-differentiable fuzzy functions based on the gH-difference of fuzzy intervals.

Definition 2.3 (See [21]) The gH-derivative of a fuzzy function $F : T \to F_C$ at $x_0 \in T$ is defined as

$$F'(x_0) = \lim_{h\to 0} \frac{1}{h}[F(x_0 + h) \ominus_{gH} F(x_0)]. \tag{2.6}$$

If $F(x_0) \in F_C$ satisfying (2.5) exists, we say that F is generalized Hukuhara differentiable (gH-differentiable, for short) at x_0.

Theorem 2.1(See [22]) If $F : T \to F_C$ is gH-differentiable at $x_0 \in T$, then F_α is gH-differentiable at x_0 uniformly in $\alpha \in [0, 1]$ and

$$F'_\alpha(x_0) = [F'(x_0)]^\alpha, \tag{2.7}$$

for all $\alpha \in [0, 1]$.

Theorem 2.2 (See [22]) Let F: $T \to F_C$ be a fuzzy function and $x \in T$. Then F is gH-differentiable at x if and only if one of the following four cases holds:

(a) $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable at x, uniformly in $\alpha \in [0, 1], (\underline{f}_\alpha)'(x)$ is monotonic increasing and $(\overline{f}_\alpha)'(x)$ is monotonic decreasing as functions of α and $(\underline{f}_\alpha)'(x) \le (\overline{f}_\alpha)'(x)$. In this case,

$$F'_\alpha(x) = [(\underline{f}_\alpha)'(x), (\overline{f}_\alpha)'(x)]. \qquad (2.8),$$

for all $\alpha \in [0, 1]$.

(b) $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable at x, uniformly in $\alpha \in [0, 1], (\underline{f}_\alpha)'(x)$ is monotonic increasing and $(\overline{f}_\alpha)'(x)$ is monotonic decreasing as functions of α and $(\overline{f}_\alpha)'(x) \le (\underline{f}_\alpha)'(x)$. In this case,

$$F'_\alpha(x) = [\overline{f}_\alpha)'(x), (\underline{f}_\alpha)'(x)]. \qquad (2.9),$$

for all $\alpha \in [0, 1]$.

(c) $(\underline{f}_\alpha)'_{+/-}(x)$ and $(\overline{f}_\alpha)'_{+/-}(x)$ exist uniformly in $\alpha \in [0, 1], (\underline{f}_\alpha)'_+(x) = (\overline{f}_\alpha)'_-(x)$ is monotonic increasing and $(\overline{f}_\alpha)'_+(x) = (\underline{f}_\alpha)'_-(x)$ is monotonic decreasing as functions of α and $(\underline{f}_\alpha)'_+(x) \le (\overline{f}_\alpha)'_+(x)$. In this case,

$$F'_\alpha(x) = [(\underline{f}_\alpha)'_+(x), \overline{f}_\alpha)'_+(x)] = [\overline{f}_\alpha)'_-(x), (\underline{f}_\alpha)'_-(x)]. \qquad (2.10),$$

for all $\alpha \in [0, 1]$.

(d) $(\underline{f}_\alpha)'_{+/-}(x)$ and $(\overline{f}_\alpha)'_{+/-}(x)$ exist uniformly in $\alpha \in [0, 1], (\underline{f}_\alpha)'_+(x) = (\overline{f}_\alpha)'_-(x)$ is monotonic increasing and $(\overline{f}_\alpha)'_+(x) = (\underline{f}_\alpha)'_-(x)$ is monotonic decreasing as functions of α and $(\overline{f}_\alpha)'_+(x) \le (\underline{f}_\alpha)'_+(x)$. In this case,

$$F'_\alpha(x) = [\overline{f}_\alpha)'_+(x), (\underline{f}_\alpha)'_+(x)] = [(\underline{f}_\alpha)'_-(x), \overline{f}_\alpha)'_-(x)]. \qquad (2.11),$$

for all $\alpha \in [0, 1]$.

Theorem 2.3(**Decomposition Theorem**[23]) If $u \in E^n$, then

$$u = \bigcup_{\alpha\in[0,1]} (\alpha \cdot [u]_\alpha). \tag{2.3}$$

The following well-known characterization theorem makes the connection between a fuzzy interval and its LU-representation.

Theorem 2.4 (See[24]) Let u be a fuzzy number. Then the functions $\underline{u}$, $\overline{u}$: $[0,1] \to R$, defining the endpoints of the α-level sets of u, satisfy the following conditions:

(i) $\underline{u}$ is a bounded, non-decreasing, left-continuous function in $(0,1]$ and it is right-continuous at 0.
(ii) $\overline{u}$ is a bounded, non-increasing, left-continuous function in $(0,1]$ and it is right-continuous at 0.
(iii) $u(1) \leq u(1)$.

Reciprocally, given two functions that satisfy the above conditions, they uniquely determine a fuzzy number.

3. Solutions of FHTEs and Ill-posedness

Now, we consider a Cauchy problem for the Helmholtz-type equation with fuzzy initial-boundary value in a rectangle domain as follows

$$\begin{cases} \dfrac{\partial^2\widetilde{u}}{\partial x^2} + \dfrac{\partial^2\widetilde{u}}{\partial y^2} + k^2\widetilde{u} = \widetilde{0}, & 0 < x < \pi, 0 < y < 1, \\ \widetilde{u}(x,0) = \widetilde{\varphi}(x), & 0 \leq x \leq \pi, \\ \dfrac{\partial\widetilde{u}}{\partial y}(x,0) = \widetilde{0}, & 0 \leq x \leq \pi, \\ \widetilde{u}(0,y) = \widetilde{u}(\pi,y) = \widetilde{0}, & 0 \leq y \leq 1, \end{cases} \tag{3.1}$$

where where constant $k > 0$ is the wave number. $\widetilde{u}$, $\frac{\partial^2\widetilde{u}}{\partial x^2}$, $\frac{\partial^2\widetilde{u}}{\partial y^2}$, $\frac{\partial\widetilde{u}}{\partial y}$, $\widetilde{\varphi}(x)$, $\widetilde{0}$ are fuzzy-number-valued functions and their α-cut sets are shown as follows:

$$\begin{gathered} [\widetilde{u}(x,y)]_\alpha = [\underline{u}(x,y,\alpha),\ \overline{u}(x,y,\alpha)], \\ [\frac{\partial^2\widetilde{u}}{\partial x^2}(x,y)]_\alpha = [\frac{\partial^2\underline{u}}{\partial x^2}(x,y,\alpha),\ \frac{\partial^2\overline{u}}{\partial x^2}(x,y,\alpha)], \\ [\frac{\partial^2\widetilde{u}}{\partial y^2}(x,y)]_\alpha = [\frac{\partial^2\underline{u}}{\partial y^2}(x,y,\alpha),\ \frac{\partial^2\overline{u}}{\partial y^2}(x,y,\alpha)], \\ [\frac{\partial\widetilde{u}}{\partial y}(x,y)]_\alpha = [\frac{\partial\underline{u}}{\partial y}(x,y,\alpha),\ \frac{\partial\overline{u}}{\partial y}(x,y,\alpha)], \\ [\widetilde{\varphi}(x)]_\alpha = [\underline{\varphi}(x,\alpha),\ \overline{\varphi}(x,\alpha)], \quad [\widetilde{0}]_\alpha = [\underline{0}(\alpha),\ \overline{0}(\alpha)]. \end{gathered}$$

From Theorem 2.1 and Theorem 2.2, in order to investigate the solution of (3.1), we consider the following two systems of two partial differential equations

$$\begin{cases} \dfrac{\partial^2\underline{u}}{\partial x^2}(x,y,\alpha) + \dfrac{\partial^2\underline{u}}{\partial y^2}(x,y,\alpha) + k^2\underline{u}(x,y,\alpha) = \underline{0}, & 0 < x < \pi, 0 < y < 1, \\ \underline{u}(x,0) = \underline{\varphi}(x,\alpha), & 0 \leq x \leq \pi, \\ \dfrac{\partial\underline{u}}{\partial y}(x,0,\alpha) = \underline{0}(\alpha), & 0 \leq x \leq \pi, \\ \underline{u}(0,y,\alpha) = \underline{u}(\pi,y) = \underline{0}(\alpha), & 0 \leq y \leq 1, \end{cases} \tag{3.2}$$

$$\begin{cases} \dfrac{\partial^2 \overline{u}}{\partial x^2}(x,y,\alpha) + \dfrac{\partial^2 \overline{u}}{\partial y^2}(x,y,\alpha) + k^2 \overline{u}(x,y,\alpha) = \overline{0}, & 0 < x < \pi, 0 < y < 1, \\ \overline{u}(x,0,\alpha) = \overline{\varphi}(x,\alpha), & 0 \le x \le \pi, \\ \dfrac{\partial \overline{u}}{\partial y}(x,0,\alpha) = \overline{0}(\alpha), & 0 \le x \le \pi, \\ \overline{u}(0,y) = \overline{u}(\pi,y,\alpha) = \overline{0}(\alpha), & 0 \le y \le 1, \end{cases} \tag{3.3}$$

Definition 3.1 (see [1]) Let $\underline{u}(x,y,\alpha)$ and $\overline{u}(x,y,\alpha)$ be solutions of equations (3.2) and (3.3), respectively. If $[\underline{u}(x,y,\alpha), \overline{u}(x,y,\alpha)]$ defines the α-cut of a fuzzy number, for all $(x,y) \in [0,\ \pi] \times [0,\ 1]$, then $\widetilde{u}(x,y)$ is a solution for (3.1).

By the method of separation of variables, it is easy to derive a solution of the direct problem (3.2) and (3.3), respectively as follows:

$$\underline{u}(x,y,\alpha) = \sum_{n=1}^{[k]} \underline{c}_n \sin(nx)\cos(\sqrt{k^2-n^2}y) + \sum_{n=[k]+1}^{\infty} \underline{c}_n \sin(nx)\cosh(\sqrt{n^2-k^2}y) \tag{3.4}$$

where

$$\underline{c}_n = \frac{2}{\pi}\int_0^{\pi} \underline{\varphi}(x,\alpha)\sin(nx)dx \tag{3.5}$$

$$\overline{u}(x,y,\alpha) = \sum_{n=1}^{[k]} \overline{c}_n \sin(nx)\cos(\sqrt{k^2-n^2}y) + \sum_{n=[k]+1}^{\infty} \overline{c}_n \sin(nx)\cosh(\sqrt{n^2-k^2}y) \tag{3.6}$$

where

$$\overline{c}_n = \frac{2}{\pi}\int_0^{\pi} \overline{\varphi}(x,\alpha)\sin(nx)dx \tag{3.7}$$

Obviously, for the solutions $\underline{u}(x,y,\alpha)$ of the equations (3.2) and the solutions $\overline{u}(x,y,\alpha)$ of the equations (3.3), $[\underline{u}(x,y,\alpha), \overline{u}(x,y,\alpha)]$ satisfies the conditions of Theorem 2.2, $[\underline{u}(x,y,\alpha), \overline{u}(x,y,\alpha)]$ determines a solution of problem (3.1) as follows:

$$u = \bigcup_{\alpha\in[0,1]} (\alpha \cdot [\underline{u}(x,y,\alpha), \overline{u}(x,y,\alpha)]). \tag{3.8}$$

Remark 3.1 If $0 < k < 1$, the first term in Equations (3.4) and (3.6) is vanished.

In the following, we discuss the ill-posedness of problem (3.1).

Definition 3.2 (Hadamard's definition of well-posedness [15]) If a deterministic solution problem of FPDE satisfies the following properties (3.9-3.11), then it is well-posed.

For all admissible date, a solution exists. (3.9)

For all admissible date, the solution is unique. (3.10)

The solution depends continuously on the date. (3.11)

Conversely, if one of the properties (3.9-3.11) does not satisfy for a deterministic solution problem of FPDE, then it is ill-posed.

Next, we are always suppose that (3.9) and (3.10) hold for the convenience of discussion, (3.11) does not hold.

Definition 3.3 Problem of FHTEs (3.1) is said to be ill-posed if both problems of PDE (3.2) and PDE (3.3) are ill-posed.

The the systems of PDE (3.2) and (3.3) are highly ill-posed, see[16]. Thus, the systems (3.1) is ill-posed.

Ill-posed problem means the solution does not depend continuously on the given Cauchy data and any small perturbation in the given data may cause large change to the solution. Thus regularization techniques are required to stabilize numerical computations. In general terms, regularization is the approximation of an ill-posed problem by a family neighbouring well-posed problems.

4. Regularization and Convergence estimates

In this section, we use the solution of perturbation problems to approach the solution of problems (3.2) and (3.3). Thus the regularization solution of problems (3.1) be derived by (3.4).

For $0 < k < 1$, we consider the following problem

$$\begin{cases} \Delta \underline{v}(x,y) + k^2 \underline{v}(x,y) = \underline{0}, & 0 < x < \pi, 0 < y < 1, \\ \underline{v}(x,0) + \beta \underline{v}(x,1) = \underline{\varphi}^{\delta_1}(x,\alpha), & 0 \le x \le \pi, \\ \underline{v}_y(x,0) = 0, & 0 \le x \le \pi, \\ \underline{v}(0,y) = \underline{v}(\pi,y) = \underline{0}, & 0 \le y \le 1, \end{cases} \tag{4.1}$$

$$\begin{cases} \Delta \overline{v}(x,y) + k^2 \overline{v}(x,y) = \overline{0}, & 0 < x < \pi, 0 < y < 1, \\ \overline{v}(x,0) + \beta \overline{v}(x,1) = \overline{\varphi}^{\delta_2}(x,\alpha), & 0 \le x \le \pi, \\ \overline{v}_y(x,0) = 0, & 0 \le x \le \pi, \\ \overline{v}(0,y) = \overline{v}(\pi,y) = \overline{0}, & 0 \le y \le 1, \end{cases} \tag{4.2}$$

where $0 < \alpha \le 1$ is α-level set parameter, and $\beta > 0$ is a regularization parameter. The measured data of equations (3.1) is fuzzy-number-valued function $\widetilde{\varphi}(x)$, and its α-level set is defined as $[\widetilde{\varphi}(x)]_\alpha = [\underline{\varphi}(x,\alpha),\ \overline{\varphi}(x,\alpha)]$. $\underline{\varphi}^{\delta_1} \in L^2(0,\pi)$, $\overline{\varphi}^{\delta_2} \in L^2(0,\pi)$ satisfies

$$\|\underline{\varphi}^{\delta_1} - \underline{\varphi}\| \le \delta_1, \tag{4.3}$$

$$\|\overline{\varphi}^{\delta_2} - \overline{\varphi}\| \le \delta_2, \tag{4.4}$$

in which the constant $\delta_1 > 0$ and $\delta_2 > 0$ is called an error level and $\|\cdot\|$ denotes the L^2-norm. Further assume that there exists a constant $E > 0$ such that the following a priori bound exists

$$\|u(\cdot,1)\| \le E. \tag{4.5}$$

By the method of separation of variables, it is easy to derive a solution of direct problem (4.1) and (4.2) as follows, respectively

$$\underline{v}(x,y,\alpha) = \sum_{n=1}^{\infty} \underline{c}_n^{\delta_1} \frac{\cosh(\sqrt{n^2-k^2}y)}{1+\beta\cosh(\sqrt{n^2-k^2})} \sin(nx), \tag{4.6}$$

where

$$\underline{c}_n^{\delta_1} = \frac{2}{\pi}\int_0^\pi \underline{\varphi}^{\delta_1}(x,\alpha)\sin(nx)dx. \tag{4.7}$$

$$\overline{v}(x,y,\alpha) = \sum_{n=1}^{\infty} \overline{c}_n^{\delta_2} \frac{\cosh(\sqrt{n^2-k^2}y)}{1+\beta\cosh(\sqrt{n^2-k^2})} \sin(nx), \tag{4.8}$$

where

$$\overline{c}_n^{\delta_2} = \frac{2}{\pi}\int_0^\pi \overline{\varphi}^{\delta_2}(x,\alpha)\sin(nx)dx. \tag{4.9}$$

For $k \geq 1$, we define a regularized solution v as follows:

$$\underline{v}(x,y,\alpha) = \sum_{n=1}^{[k]} \underline{c}_n^{\delta_1} \cosh(\sqrt{n^2-k^2}y) + \sum_{n=[k]+1}^{\infty} \underline{c}_n^{\delta_1} \frac{\cosh(\sqrt{n^2-k^2}y)}{1+\beta\cosh(\sqrt{n^2-k^2})} \sin(nx), \tag{4.10}$$

where $\underline{c}_n^{\delta_1}$ is defined by Equation (4.7).

$$\overline{v}(x,y,\alpha) = \sum_{n=1}^{[k]} \overline{c}_n^{\delta_2} \cosh(\sqrt{n^2-k^2}y) + \sum_{n=[k]+1}^{\infty} \overline{c}_n^{\delta_2} \frac{\cosh(\sqrt{n^2-k^2}y)}{1+\beta\cosh(\sqrt{n^2-k^2})} \sin(nx), \tag{4.11}$$

where $\overline{c}_n^{\delta_2}$ is defined by Equation (4.9).

Remark 4.1 (see[25]) For $k \geq 1$, the regularized solution (4.10) and (4.11) be not an exact solution of the problem (4.1) and (4.2), respectively, but a modified solution. This is done to avoid the case $1+\beta\cos(\sqrt{n^2-k^2}) = 0$ for $k \geq 1$ and $n < k$ and prove a convergence result.

In the following results shall show that the regularization solution $\underline{v}$ given by Equation (4.6)and(4.10), and $\overline{v}$ given by Equation (4.8) and (4.11) are a stable approximation to the exact solution $\underline{u}$ and $\overline{u}$ given by Equation (3.4) and (3.6),respectively. The regularization solution $\underline{v}$ and $\overline{v}$ depends continuously on the measured data $\underline{\varphi}^{\delta_1}$ and $\overline{\varphi}^{\delta_2}$ for a fixed parameter $\beta > 0$, respectively.

Theorem 4.1 (see[25]) Suppose that $\underline{u}$ and $\overline{u}$ is defined by Equation (3.4) and (3.6) with the exact data $\underline{\varphi}$ and $\overline{\varphi}$, respectively. Suppose that $\underline{v}$ is defined by Equation (4.6) for the case $0 < k < 1$ or Equation (4.10)for the case $k \geq 1$ with the measured data $\underline{\varphi}^{\delta_1}$, $\overline{v}$ is defined by Equation (4.8) for the case $0 < k < 1$ or Equation (4.11) for the case $k \geq 1$ with the measured data $\overline{\varphi}^{\delta_2}$. Let the measured data $\underline{\varphi}^{\delta_1}$ and $\overline{\varphi}^{\delta_2}$ satisfy Equation (4.3) and (4.4), respectively. Let the exact solution u at $y = 1$ satisfy Equation (4.5). If the regularization parameter β is chosen as, respectively

$$\beta = \frac{\delta_1}{E}, \tag{4.11}$$

$$\beta = \frac{\delta_2}{E}, \tag{4.12}$$

then for fixed $0 < y < 1$, we have the following convergence estimate

$$\|\underline{v}(\cdot,y) - \underline{u}(\cdot,y)\| \leq \delta_1 + 2C_y E^y \delta_1^{1-y}. \tag{4.13}$$

$$\|\overline{v}(\cdot,y) - \overline{u}(\cdot,y)\| \leq \delta_2 + 2C_y E^y \delta_2^{1-y}. \tag{4.14}$$

where $C_y = \frac{1-y}{(\frac{2y}{(1-y)})^y}$.

However, the convergence estimate in Equation (4.13) and (4.14) is not useful for $y = 1$. In order to obtain a convergence rate at $y = 1$, we need a stronger a priori assumption

$$\|\frac{\partial^p u(\cdot,1)}{\partial y^p}\| \leq E, \tag{4.15}$$

where $p \geq 1$ is an integer. We have the following convergence estimate.

Theorem 4.2 (see[25]) Suppose that $\underline{u}$ and $\overline{u}$ is defined by Equation (3.4) and (3.6) with the exact data $\underline{\varphi}$ and $\overline{\varphi}$, respectively. Suppose that $\underline{v}$ is defined by Equation (4.6) for the case $0 < k < 1$ or Equation (4.10) for the case $k \geq 1$ with the measured data $\underline{\varphi}^{\delta_1}$, $\overline{v}$ is defined by Equation (4.8) for the case $0 < k < 1$ or Equation (4.11) for the case $k \geq 1$ with the measured data $\overline{\varphi}^{\delta_2}$. Let the measured data $\underline{\varphi}^{\delta_1}$ and $\overline{\varphi}^{\delta_2}$

satisfy Equation (4.3) and (4.4), respectively. Let the exact solution u at $y = 1$ satisfy Equation (4.15). If the regularization parameter β is chosen as, respectively

$$\beta = \frac{\delta_1}{E}, \tag{4.16}$$

$$\beta = \frac{\delta_2}{E}, \tag{4.17}$$

then we have the following convergence estimate at $y = 1$,

$$\|\underline{v}(\cdot, 1) - \underline{u}(\cdot, 1)\| \leq \delta_1 + \sqrt{\delta_1 E} + \frac{2E}{1 - e^{-2k}} \max\{K^{-p}(\frac{\delta_1}{E})^{\frac{1}{3}}, 2(\frac{1}{6}\ln\frac{E}{\delta_1})^{-p}\}. \tag{4.18}$$

$$\|\overline{v}(\cdot, 1) - \overline{u}(\cdot, 1)\| \leq \delta_2 + \sqrt{\delta_2 E} + \frac{2E}{1 - e^{-2k}} \max\{K^{-p}(\frac{\delta_2}{E})^{\frac{1}{3}}, 2(\frac{1}{6}\ln\frac{E}{\delta_2})^{-p}\}. \tag{4.19}$$

where $K = \sqrt{([k] + 1)^2 - k^2}$ and $[\cdot]$ denotes the integer part of a real number.

Theorem 4.3 Suppose that $\widetilde{u}$ defined by Equation (3.8) is a solution of problem (3.1) and $\widetilde{v}$ is its regularization solution. If $\underline{u}$ is defined by Equation (3.4) and $\underline{v}$ is its regularization solution defined by Equation (4.6) for the case $0 < k < 1$ or Equation (4.10) for the case $k \geq 1$, while $\overline{u}$ is defined by Equation (3.6) and $\overline{v}$ is its regularization solution defined by Equation (4.8) for the case $0 < k < 1$ or Equation (4.11) for the case $k \geq 1$. then $\widetilde{v}$ is a stable approximation to $\widetilde{u}$, where

$$\widetilde{v} = \bigcup_{\alpha\in[0,1]} (\alpha \cdot [\underline{v}(x, y, \alpha), \overline{v}(x, y, \alpha)]). \tag{4.20}$$

Proof By Equation(2.2), since

$$\begin{aligned} D(\widetilde{u}, \widetilde{v}) &= \sup_{\alpha\in[0,1]} d_H([\widetilde{u}]_\alpha, [\widetilde{v}]_\alpha) \\ &= \sup_{\alpha\in[0,1]} \max\{|\underline{u}(\alpha) - \underline{v}(\alpha)|, |\overline{u}(\alpha) - \overline{v}(\alpha)|\}, \end{aligned} \tag{4.21}$$

from Theorem 4.1 and 4.2, $\underline{v}(\alpha)$ is a stable approximation to $\underline{u}(\alpha)$ and $\overline{v}(\alpha)$ is a stable approximation to $\overline{u}(\alpha)$. Hence, From (4.21) we have that $\widetilde{v}$ is a stable approximation to $\widetilde{u}$. The proof is complete.

5. Numerical examples

Consider the following direct problem for the Helmholtz equation with fuzzy mixed boundary value

$$\begin{cases} \dfrac{\partial^2\widetilde{u}}{\partial x^2} + \dfrac{\partial^2\widetilde{u}}{\partial y^2} + k^2\widetilde{u} = \widetilde{0}, & 0 < x < \pi, 0 < y < 1, \\ \widetilde{u}(x, 1) = \widetilde{f}(x), & 0 \leq x \leq \pi, \\ \dfrac{\partial\widetilde{u}}{\partial y}(x, 0) = \widetilde{0}, & 0 \leq x \leq \pi, \\ \widetilde{u}(0, y) = \widetilde{u}(\pi, y) = \widetilde{0}, & 0 \leq y \leq 1, \end{cases} \tag{5.1}$$

in which $\widetilde{f} : [0, \pi] \to E^1$.

$$\widetilde{f} = \widetilde{v} \cdot 2x(\pi - x), x \in [0, \pi]. \tag{5.2}$$

where $\widetilde{v} \in E^1$ is given by a triangular fuzzy number

$$\widetilde{v}(t) = \begin{cases} t + 1, & t \in (-1, 0), \\ -t + 1, & t \in (0, 1), \\ 0, & t \in (-\infty, -1] \cup [1, +\infty). \end{cases} \tag{5.3}$$

The α-cut set of $\widetilde{f}(x)$ is given by

$$\begin{aligned}[\widetilde{f}(x)]_\alpha &= [2x(\pi-x)\underline{v}(t,\alpha), 2x(\pi-x)\overline{v}(t,\alpha)] \\ &= [2x(\pi-x)(\alpha-1), 2x(\pi-x)(1-\alpha)]. \end{aligned} \tag{5.4}$$

In order to investigate the numerical solution of (5.1), we consider the following two systems of two partial differential equations

$$\begin{cases} \dfrac{\partial^2 \underline{u}}{\partial x^2} + \dfrac{\partial^2 \underline{u}}{\partial y^2} + k^2 \underline{u} = \underline{0}, & 0 < x < \pi, 0 < y < 1, \\ \underline{u}(x,1) = 2x(\pi-x)(\alpha-1), & 0 \le x \le \pi, \\ \dfrac{\partial \underline{u}}{\partial y}(x,0) = \underline{0}, & 0 \le x \le \pi, \\ \underline{u}(0,y) = \underline{u}(\pi,y) = \underline{0}, & 0 \le y \le 1, \end{cases} \tag{5.5}$$

$$\begin{cases} \dfrac{\partial^2 \overline{u}}{\partial x^2} + \dfrac{\partial^2 \overline{u}}{\partial y^2} + k^2 \overline{u} = \overline{0}, & 0 < x < \pi, 0 < y < 1, \\ \overline{u}(x,1) = 2x(\pi-x)(1-\alpha), & 0 \le x \le \pi, \\ \dfrac{\partial \overline{u}}{\partial y}(x,0) = \overline{0}, & 0 \le x \le \pi, \\ \overline{u}(0,y) = \overline{u}(\pi,y) = \overline{0}, & 0 \le y \le 1, \end{cases} \tag{5.6}$$

By the method of separation of variables, the solution of the direct problem (5.5) and (5.6) can be obtained as follows, respectively.

$$\underline{u}(x,y,\alpha) = \sum_{n=1}^{[k]} \underline{c}_n \sin(nx)\cos(\sqrt{k^2-n^2}y) + \sum_{n=[k]+1}^{\infty} \underline{c}_n \sin(nx)\cosh(\sqrt{n^2-k^2}y), \tag{5.7}$$

$$\overline{u}(x,y,\alpha) = \sum_{n=1}^{[k]} \overline{c}_n \sin(nx)\cos(\sqrt{k^2-n^2}y) + \sum_{n=[k]+1}^{\infty} \overline{c}_n \sin(nx)\cosh(\sqrt{n^2-k^2}y), \tag{5.8}$$

where $\underline{\varphi}_n = \frac{2}{\pi\cosh(n)} d_n, d_n = \int_0^\pi 2x(\pi-x)(\alpha-1)\sin(nx)dx$, $\overline{\varphi}_n = \frac{2}{\pi\cosh(n)} d_n, d_n = \int_0^\pi 2x(\pi-x)(1-\alpha)\sin(nx)dx$, and they can be computed by the Simpson formulation, respectively.

Remark 5.1 If $0 < k < 1$, the first term in Equations (5.7) and (5.8) is vanished.

Then we choose the initial data $\underline{\varphi}(x)$ for equation (3.2) and $\overline{\varphi}(x)$ for equation (3.3) as follows,

$$\underline{\varphi}(x) = u(x,0) \approx \sum_{n=1}^{25} \underline{\varphi}_n \sin(nx). \tag{5.9}$$

$$\overline{\varphi}(x) = u(x,0) \approx \sum_{n=1}^{25} \overline{\varphi}_n \sin(nx). \tag{5.10}$$

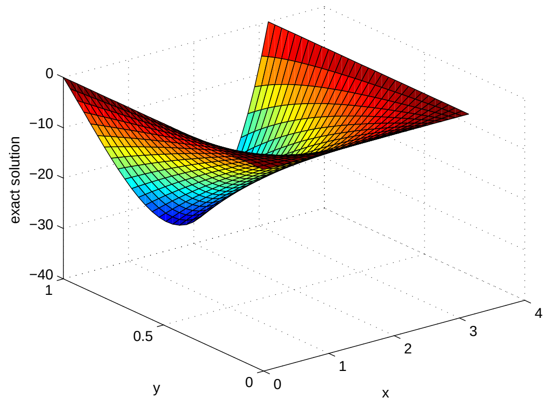
The measured data $\underline{\varphi}_{\delta_1}$ and $\overline{\varphi}_{\delta_2}$is given by $\underline{\varphi}^{\delta_1}(x_i) = \varphi(x_i) + \varepsilon \cdot \mathrm{rand}(i)$, and $\overline{\varphi}^{\delta_2}(x_i) = \varphi(x_i) + \varepsilon \cdot \mathrm{rand}(i)$, respectively, where ε is an error level,

$$\delta_1 := \|\underline{\varphi}_{\delta_1} - \underline{\varphi}\|_{l_2} = \left(\frac{1}{N_1}\sum_{i=1}^{N_1} \left|\underline{\varphi}_{\delta_1}(x_i) - \underline{\varphi}(x_i)\right|^2\right)^{1/2}. \tag{5.11}$$

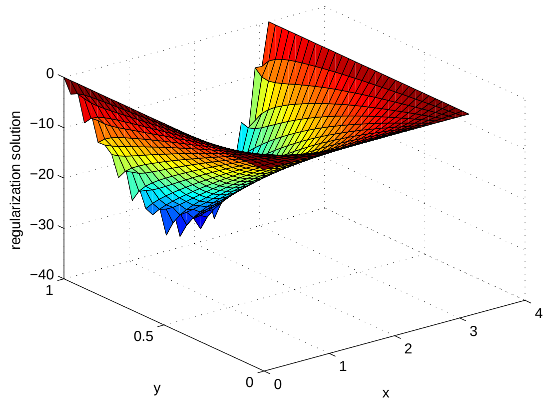
$$\delta_2 := \|\overline{\varphi}_{\delta_2} - \overline{\varphi}\|_{l_2} = \left(\frac{1}{N_1} \sum_{i=1}^{N_1} \left| \overline{\varphi}_{\delta_2}(x_i) - \overline{\varphi}(x_i) \right|^2 \right)^{1/2}. \tag{5.12}$$

The function rand$(\cdot)$ denotes a random number uniformly distributed in the interval $[0, 1]$. In our numerical computations, we always take $N_1 = 31$. The regularization parameter β is chosen by (4.10),(4.11) and (4.15),(4.16) respectively.

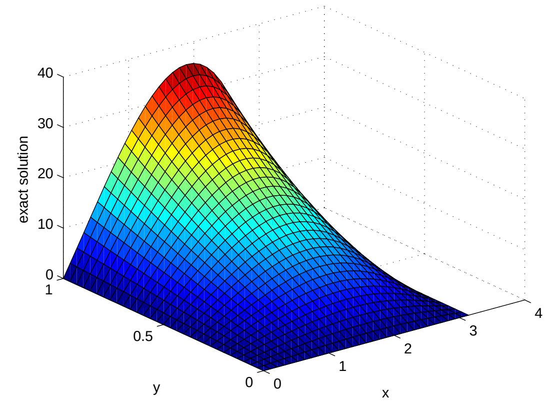
The numerical results for $u(\cdot, y)$ and $u_\beta^\delta(\cdot, y)$ with $k = \frac{1}{2}, \varepsilon = 0.0001, \alpha = \frac{1}{2}$ are shown in Figure1.



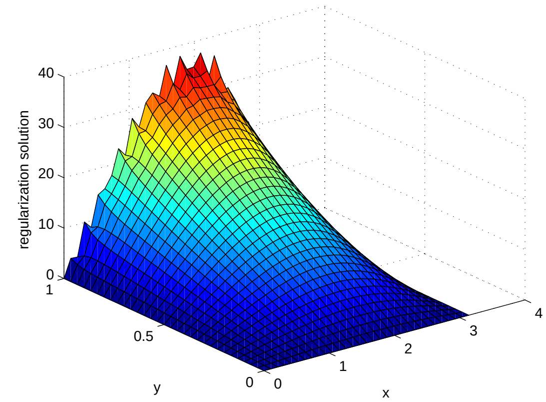
(a) ***exact solution*** $\underline{u}$.



(b) ***regularization solution*** $\underline{u}_\beta^\delta$.



(c) ***exact solution*** $\overline{u}$.



(d) ***regularization solution*** $\overline{u}_\beta^\delta$.

Figure 1: $\varepsilon = 1 \times 10^{-4}, \alpha = \frac{1}{2}, k = \frac{1}{2}$.

6. Conclusion

In this paper, we investigate a new numerical method of solution for inverse problem of FHTEs. We defined the ill-posedness for deterministic solution problem of FHTEs and the regularization method is proposed to solve a Cauchy problem for the ill-posed FHTEs. The convergence and stability estimates for $0 < y < 1$, $y = 1$ have been obtained under a-priori bound assumptions for the exact solution. Finally, one example shows that our proposed numerical method is effective.

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Behavior of the Difference Equations $x_{n+1} = x_n\, x_{n-1} - 1$

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Abstract

In this paper, the behavior of solutions of a kind of nonlinear difference equations was studied. According to the first initial value, the regions of the second initial values was partitioned by zeroes of auxiliary functions such that the asymptotical behavior of the equation was determined, which was convergent or unbounded.

Key words: Nonlinear difference equations; Convergent; Unbounded

AMS 2000 Subject Classification: 39A10, 39A11

1 Introduction

In 2011, Kosmala[1] proposed a kind of nonlinear difference equations

$$x_{n+1} = x_{n-k}\, x_{n-l} - 1, \qquad n = 1, 2, \ldots \tag{1}$$

with k, $l \in N$ and the initial values being real numbers. It stems from investigating periodic difference equations.

Stević and Iričanin [2] presented the first general result on the behavior of solutions of (1), by describing the long-term behavior of the solutions of (1) for all values of parameters k and l, where the initial values satisfy a special condition.

Moreover, some particular cases of (1) were investigated in [3–7]. Paper [3] investigated the case where $k = 1$, $l = 2$; paper [4] and [7] investigated the case where $k = 0$, $l = 1$; paper [5] investigated the case where $k = 0$, $l = 2$; paper [6] investigated the case where $k = 0$, $l = 3$.

The relatively simple appearance of (1) is deceiving in that its behavior changes significantly for different choices of k and l. These results of (1) were mainly about the periodicity, unboundedness and stability for particular choices of k and l.

In this paper, we consider a special case of (1), which was investigated in [4] and [7],

$$x_{n+1} = x_n\, x_{n-1} - 1, \qquad n = 0, 1, 2, \ldots \tag{2}$$

with the initial values x_{-1} and x_0 being real numbers. Note that the equilibria $\bar{x}$ of (2) are

$$\bar{x}_1 = \frac{1-\sqrt{5}}{2}, \qquad \bar{x}_2 = \frac{1+\sqrt{5}}{2}.$$

Furthermore, $\bar{x}_1$ was locally asymptotically stable and $\bar{x}_2$ is unstable[4].

We first summarize the main results[4, 7] on the solutions of (2).

(1) (C) If $-1 < x_{-1},\ x_0 < 0$, then every solution of (2) converges to $\bar{x}_1$.

(2) (UB) If one of the following holds, then the solution of (2) is unbounded.

$$\begin{array}{lll} (i) & x_{-1} > \bar{x}_2, & x_0 > \bar{x}_2; \\ (ii) & x_{-1} < -1, & x_0 < -1; \\ (iii) & x_{-1} < 0, & x_0 > 0; \\ (iv) & 0 < x_{-1} < 1, & 0 < x_0 < 1, \\ & x_0^2 x_{-1}^2 - 2x_0 x_{-1} + 1 - x_{-1} > 0. & \end{array}$$

(3) (UB or C)

- If $1 < x_{-1},\ x_0 < \bar{x}_2$, then one of the following occurs.
  - (i) The solution of (2) is unbounded.
  - (ii) There exists $n_0 \geq 1$ such that $x_n \in (-1, 0)$ for all $n \geq n_0$.
- If $x_{-1} > 0,\ x_0 < 0$, then the solution of (2) in certain cases is bounded and in other cases is unbounded.
- If $0 < x_{-1},\ x_0 < \bar{x}_2$, then the solutions of (2) exhibit somewhat chaotic behavior relative to the initial values. A little change in the initial conditions can cause a drastic difference in the long-term behavior of the solutions.

For simplicity, we show them in Figure 1. For the initial values $(x_{-1},\ x_0)$ in different colored regions, the solution of (2) is of three kinds: being convergent(C) and being unbounded(UB), being unbounded or convergent(UB or C).

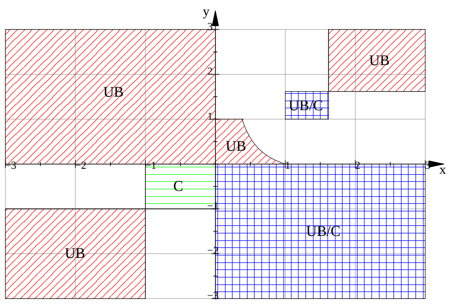

Figure 1: Different regions of the initial values of (2)

From the above results, one can see that these regions were presented from the perspective of the relation of two initial values of (2). For the initial values in the green regions, the corresponding solution is bounded and convergent. For the initial values in the red regions, the solution is unbounded. As far as the initial values in the blue regions is concerned, the solution was either unbounded or convergent and such a conclusion was not concise.

Specially, for the initial values in the blank regions, the behavior of the solution is unknown.

It is interesting to investigate the evolution of the solution according to the initial values in the plane. In the following, we try to use a new method to consider the behavior of (2). Different from the method in [4], we construct auxiliary functions and then use the zeroes of them to create new partitions of the second initial value. In this way, for the first initial value which is arbitrarily chosen, the corresponding solution is convergent only for the second initial value in some intervals which are determined by the zeroes of auxiliary functions. And the lengths of these intervals are decreasing to zero.

2 Main Results

In this section, we present the main result by investigating the behavior of solutions of (2).

First of all, from the results in [4], we made a little generalization.

Theorem 2.1.

(I) If there is an $N \geq 0$ *such that* $-1 < x_{N-1},\, x_N < 0$*, then* $\{x_n\}$ *of* (2) *converges to* $\bar{x}_1$.

(II) If there is an $N \geq 0$ *such that one of the following five conditions holds, then the solution of* (2) *is unbounded.*

1) $x_{N-1} > \bar{x}_2, \quad x_N > \bar{x}_2;$

2) $x_{N-1} < -1, \quad x_N < -1;$

3) $x_{N-1} < 0, \quad x_N > 0;$

4) $0 < x_{N-1} < 1, \quad 0 < x_N < 1,$
$x_N^2 x_{N-1}^2 - 2x_N x_{N-1} + 1 - x_{N-1} > 0;$

5) $x_{N-1} > 0, \quad x_N < -1.$

It is worth pointing out that the last case *5)* is a direct result of the case *2)* and it is crucial for our main result.

Thus, the behavior of solutions of (2) depends on the location of its two consecutive terms of x_{N-1} and x_N being less than -1, greater than $\bar{x}_2$ or in the interval $(-1, 0)$. However, it is still complicated in terms of the boundedness of solutions of (2) for other cases.

By Remark 2.6 in [4], if the solution of (2) is not periodic or eventually periodic with minimal period three, then the solution is either bounded, while inside $(-1, 0)$, or unbounded.

Now, we present a necessary and sufficient condition on the existence of eventually prime period-three solutions of (2).

Lemma 2.1. *The solution* $\{x_n\}$ *of (2) is an eventually prime period-three solution if and only if there is an* $N \geq 1$ *such that* $x_N = 0$.

Proof. By Theorem 2.1 in [4], if the solution $\{x_n\}$ is an eventually prime period-three solution, then there is an $N \geq 1$ such that $x_N = 0$.

On the other hand, if there is an $N \geq 1$ such that $x_N = 0$, then we have $x_{N+1} = -1$ and $x_{N+2} = -1$ from (2). Thus, it is an eventually prime period-three solution. □

In the following, letting the first initial value x_{-1} being fixed, we consider the behavior of the solution for the second initial value x_0, mainly on the convergence and unboundedness of the corresponding solution of (2).

For simplicity, we could assume that $x_{-1} = a$ and $x_0 = b$, where a and b are real numbers.

Now, we introduce auxiliary functions $F_i(b) = x_i$ for $i \geq 1$, from (2), which are

$$\begin{aligned}
F_1(b) &= ab - 1, && (3)\\
F_2(b) &= b\,F_1(b) - 1 \;=\; ab^2 - b - 1, && (4)\\
F_3(b) &= F_2(b)\,F_1(b) - 1 \;=\; b\,(F_1^2(b) - a), && (5)\\
F_4(b) &= F_3(b)\,F_2(b) - 1 \;=\; F_1(b)\,(F_2^2(b) - b), && (6)\\
F_5(b) &= F_4(b)\,F_3(b) - 1 \;=\; F_2(b)\,(F_3^2(b) - F_1(b)), && (7)
\end{aligned}$$

and by induction, for $i \geq 5$, we have

$$F_{i+1}(b) = F_i(b)\,F_{i-1}(b) - 1 \;=\; F_{i-2}(b)(F_{i-1}^2(b) - F_{i-3}(b)), \tag{8}$$

from which we know that $F_i(b)$ is a higher-degree polynomial of b.

By listing the roots of $F_i(b) = 0$ for each $i \geq 1$, we consider the behavior of $F_i(b)$ with b in the intervals between these adjacent roots, which describes the long term behavior of the solution of (2) with the second initial value x_0 in such intervals for the first one x_{-1} being fixed.

In the following, we investigate the roots of $F_i(b) = 0$ step by step.

It is obvious that $r_{11} = 1/a$ is the root of $F_1(b) = 0$ if $a \neq 0$.

If $a \geq -0.25$ and $a \neq 0$, then $F_2(b)=0$ has two roots which are

$$r_{21} = \frac{1 - \sqrt{1+4a}}{2a}, \qquad r_{22} = \frac{1 + \sqrt{1+4a}}{2a}$$

and they satisfy $r_{21} < r_{11} < r_{22}$ for $a > 0$.

It is noted that 0 is always a root of $F_3(b)=0$ (for convenience, denoted by itself) and for $a > 0$, the other two roots are

$$r_{31} = \frac{1 - \sqrt{a}}{a}, \qquad r_{32} = \frac{1 + \sqrt{a}}{a}$$

satisfying $0 < r_{31} < r_{11} < r_{22} < r_{32}$ for $0 < a < 1$ and $r_{31} < 0 < r_{11} < r_{22} < r_{32}$ for $a > 1$.

From (6), we know that $F_4(b)=0$ is equivalent to $F_1(b)=0$ or $F_2^2(b)=b$. Thus r_{11} is always a root of $F_4(b)=0$. From $F_2^2(b) = b$, in view of the strict monotonicity of $F_2(b)$ for $b>r_{11}$, there are only two roots of $F_4(b)=0$, satisfying $r_{41} < r_{22} < r_{42}$ for $a>0$ and $b>0$.

Similarly, the other two roots of $F_5(b) = 0$ satisfy $r_{51} < r_{32} < r_{52}$ for $a > 0$ and $b > r_{11}$, which are different from r_{21} and r_{22}.

Here and after, we only focus on these "new" roots of $F_i(b) = 0$, which have not been labeled by other smaller indices.

Now, we conclude the existence of two roots of $F_{i+1}(b)=0$ for $i\geq 5$.

Lemma 2.2. *$F_{i+1}(b)=0$ has only two roots for $a>0$ and $b>r_{(i-3)2}$ for $i\geq 5$.*

Proof. Letting r_{ij} be the roots of $F_i(b)=0$ for $i>1$ and $j=1$, 2, from (8), we have

$$F'_{i+1}(b)=F'_i(b)F_{i-1}(b) + F_i(b)F'_{i-1}(b) > 0 \tag{9}$$

for $b>r_{i2}$ and thus $F_{i+1}(b)$ is strictly increasing for $b>r_{i2}$.

From (8), we have $F_{i-1}^2(b)=F_{i-3}(b)$ for $i\geq 5$. Hence, in view of the monotonicity of $F_{i-1}(b)$ for $b>r_{(i-2)2}$ and the positivity of $F_{i-3}(b)$ for $b>r_{(i-3)2}$, by induction, there are only two roots of $F_{i+1}(b)=0$ satisfying $r_{(i+1)1} < r_{(i-1)2} < r_{(i+1)2}$ for $a>0$ and $b>r_{(i-3)2}$. □

Furthermore, we could conclude that $\{r_{i1}\}_{i=2}^{+\infty}$ and $\{r_{i2}\}_{i=2}^{+\infty}$ are convergent.

Lemma 2.3.

$$\lim_{i\to+\infty} r_{i1} = \lim_{i\to+\infty} r_{i2}. \tag{10}$$

Proof. First, from the strict monotonicity of $F_{i+1}(b)$ for $b > r_{i2}$, we have $r_{i2} < r_{(i+1)2}$ and thus $r_{22} < r_{32} < r_{42} < \ldots$. The convergence of $\{r_{i2}\}_{i=2}^{+\infty}$ is guaranteed by $F_i(b)$ being a higher-degree polynomial of b. Similarly, $\{r_{i1}\}_{i=2}^{+\infty}$ is convergent.

Denote

$$\lim_{i\to+\infty} r_{i1} = \underline{b} = \underline{b}(a), \qquad \lim_{i\to+\infty} r_{i2} = \hat{b} = \hat{b}(a).$$

In order to prove (10), we suppose that $\underline{b} < \hat{b}$. Then there exists N such that $r_{N2} > \underline{b}$. From the above, there exists a root $r_{(N+2)1}$ of $F_{N+2}(b) = 0$ near r_{N2}. In view of $\{r_{i2}\}_{i=2}^{+\infty}$ being increasing, it is enough to choose such a r_{N2} that the corresponding $r_{(N+2)1} > \underline{b}$, which contradicts the convergence of $\{r_{i1}\}_{i=2}^{+\infty}$. The case of $\hat{b} < \underline{b}$ is similar. In fact, we can find such an M that $\hat{b} < r_{M1} < r_{M2}$ and they are roots of $F_M(b) = 0$, which contradicts the convergence of $\{r_{i2}\}_{i=2}^{+\infty}$. Hence, (10) is true. $\square$

From the above, for the first initial value $x_{-1} = a$ being fixed, we have obtained that $F_i(b) = 0$ has only two "new" roots for $i > 3$ and the sequences $\{r_{i1}\}_{i=2}^{+\infty}$ and $\{r_{i2}\}_{i=2}^{+\infty}$ converge to a same number.

To investigate the behavior of the solution of (2) with initial values in these intervals which are partitioned by the adjacent roots r_{ij}, we consider three cases.

Case 1 $\quad a = 0$

If $-1 < b < 0$, it follows that both $F_2(b) = -b - 1$ and $F_3(b) = b$ are in the interval $(-1,\ 0)$. Thus $\{x_n\}$ of (2) converges to $\bar{x}_1$ by Theorem 2.1.

Case 2 $\quad a < 0$

From $r_{11} = 1/a < 0$ and $F_2(b) = a\,b^2 - b - 1$, we have that $F_2(r_{11}) = F_2(0) = -1$. Hence, only two cases in the following are needed.

(i) If $a < -0.25$ and $b \in (r_{11},\ 0)$, then $-1 < F_1(b),\ F_2(b) < 0$. Thus, by Theorem 2.1, $\{x_n\}$ of (2) converges to $\bar{x}_1$.

(ii) If $-0.25 \le a < 0$, then $F_2(b) = 0$ has two roots satisfying $r_{11} < r_{21} \le r_{22} < 0$. For $b \in (r_{11},\ r_{21}) \cup (r_{22},\ 0)$, we have $-1 < F_1(b),\ F_2(b) < 0$ and $\{x_n\}$ of (2) converges to $\bar{x}_1$ by Theorem 2.1. For $b \in (-\infty,\ r_{11})$, we have $F_2(b) < -1$ and $F_3(b) < -1$ and thus $\{x_n\}$ of (2) is unbounded by Theorem 2.1. For $b \in (r_{21},\ r_{22})$, we have $F_1(b) < 0$ and $F_2(b) > 0$ and thus $\{x_n\}$ of (2) is unbounded by Theorem 2.1.

Generally speaking, for $a < 0$, the solution $\{x_n\}$ of (2) converges to $\bar{x}_1$ only for two cases: one is $a < -0.25$ and $b \in (r_{11},\ 0)$, the other is $-0.25 \le a < 0$ and $b \in (r_{11},\ r_{21}) \cup (r_{22},\ 0)$. Hence , for $a \le 0$ and $b > 0$, $\{x_n\}$ of (2) is unbounded. Thus, the dynamics of (2) is clear.

Case 3 $\quad a > 0$

For this case, it is complicated to arrange these roots r_{ij}. We divide it into three cases.

3.1 $\quad 0 < a < 1$

In this case, we prove that

$$r_{21} < 0 < r_{31} < r_{41} < r_{11} < r_{22} < r_{51} < r_{61} < r_{32} < r_{42} < r_{52} < r_{62}. \tag{11}$$

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From the above, we only need to show $r_{31} < r_{41} < r_{11}$ and $r_{22} < r_{51} < r_{61} < r_{32}$.
First, from $r_{41} < r_{22} < r_{42}$, in view of $F_2(r_{31}) < 0$ and $F_3'(r_{31}) < 0$, it follows that

$$F_4(r_{31}) = -1, \quad F_4(r_{11}) = 0, \quad F_4'(r_{31}) > 0, \quad F_4'(r_{11}) = a-1 < 0 \tag{12}$$

for $0 < a < 1$. Thus, $r_{31} < r_{41} < r_{11}$.
Second, in order to prove $r_{22} < r_{51} < r_{61} < r_{32}$, we only need to show $r_{22} < r_{51}$ and $r_{61} < r_{32}$. Thus, the key is to compare r_{i2} with r_{j1} for $i \geq 2$ and $j = i + 3$.
From $r_{51} < r_{32} < r_{52}$, we could conclude that $r_{22} < r_{51}$.
In fact, in view of $F_5(r_{22}) = 0$ and $F_5(r_{32}) = -1$, we have

$$\begin{aligned} F_5'(r_{32}) &= -F_3'(r_{32}) < 0, & (13)\\ F_5'(r_{22}) &= F_2'(r_{22})\,(1 - F_1(r_{22})) > 0 & (14)\end{aligned}$$

which is guaranteed by

$$1 - F_1(r_{22}) = \frac{2(2-a)}{3 + \sqrt{1+4a}} > 0 \tag{15}$$

for $0 < a < 2$. Thus, the conclusion is true.
In a similar way, from $r_{61} < r_{42} < r_{62}$, we conclude $r_{51} < r_{61} < r_{32}$.
In fact, in view of $F_6(r_{32}) = 0$ and $F_6(r_{51}) = -1$, we have

$$\begin{aligned} F_6'(r_{51}) &= F_5'(r_{51})F_4(r_{51}) > 0,\\ F_6'(r_{32}) &= F_3'(r_{22})\,(1 - F_2(r_{32})) < 0 \end{aligned} \tag{16}$$

which is guaranteed by

$$1 - F_2(r_{32}) = \frac{a-1}{a+\sqrt{a}} < 0 \tag{17}$$

for $0 < a < 1$. Thus, $r_{51} < r_{61} < r_{32}$ holds.
Hence, from the above, it is proved that (11) holds for $0 < a < 1$.
Third, we analyze the behavior of $\{x_n\}$ of (2) with x_0 being in the intervals partitioned by these adjacent roots.

(1) For $b \in (-\infty, r_{21})$, we have $F_1(b) < 0$, $F_2(b) > 0$. Thus, $\{x_n\}$ is unbounded by Theorem 2.1. It is also true for $b \in (0, r_{31}) \cup (r_{41}, r_{11}) \cup (r_{22}, r_{51}) \cup (r_{61}, r_{32})$ which are listed in Table 1.
(2) For $b \in (r_{21}, 0)$, we have $-1 < F_2(b)$, $F_3(b) < 0$. Thus, $\{x_n\}$ converges to $\bar{x}_1$ by Theorem 2.1. It is also true for $b \in (r_{31}, r_{41}) \cup (r_{11}, r_{22}) \cup (r_{51}, r_{61})$ which are listed in Table 2.

Table 1: Intervals of x_0 such that $\{x_n\}$ is unbounded for $0 < x_{-1} < 1$

Intervals of x_0	Reasons	$\{x_n\}$ is
$(-\infty,\ r_{21})$	$F_1(b) < 0,\ \ F_2(b) > 0$	unbounded
$(0,\ \ r_{31})$	$F_2(b) < 0,\ \ F_3(b) > 0$	unbounded
$(r_{41},\ r_{11})$	$F_3(b) < 0,\ \ F_4(b) > 0$	unbounded
$(r_{22},\ r_{51})$	$F_4(b) < 0,\ \ F_5(b) > 0$	unbounded
$(r_{61},\ r_{32})$	$F_5(b) < 0,\ \ F_6(b) > 0$	unbounded

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Table 2: Intervals of x_0 such that $\{x_n\}$ is convergent for $0 < x_{-1} < 1$

Intervals of x_0	Reasons	$\{x_n\}$
$(r_{21},\ 0)$	$-1<F_2(b),\ F_3(b)<0$	converges to $\bar{x}_1$
$(r_{31},\ r_{41})$	$-1<F_3(b),\ F_4(b)<0$	converges to $\bar{x}_1$
$(r_{11},\ r_{22})$	$-1<F_4(b),\ F_5(b)<0$	converges to $\bar{x}_1$
$(r_{51},\ r_{61})$	$-1<F_5(b),\ F_6(b)<0$	converges to $\bar{x}_1$

3.2 $1 \le a < 2$

In this case, we prove that

$$r_{21} < r_{31} \le 0 < r_{11} \le r_{41} < r_{22} < r_{51} < r_{32} \le r_{61} < r_{42} < r_{52} < r_{62}. \tag{18}$$

In fact, for $a = 1$, in view of their expressions and $F_4(b) = (b-1)^2\,(b^3 - b^3 - 2b - 1)$, we have $r_{31}=0,\ \ r_{32}=2=r_{61},\, r_{11}=1=r_{41}$.

For $1 < a < 2$, it is apparent that $r_{21} < r_{31} < 0 < r_{11} < r_{22} < r_{32}$. From (12), (16) and (17), we have

$$\begin{aligned} F_4'(r_{11}) &= a-1 > 0, \\ F_6'(r_{32}) &= F_3'(r_{22})\tfrac{a-1}{a+\sqrt{a}} > 0. \end{aligned} \tag{19}$$

It follows that $r_{11} < r_{41}$ and $r_{32} < r_{61}$.

And $r_{22} < r_{51}$ follows from (14) and (15). Thus, (18) holds for $1 \le x_{-1} < 2$.

It is worth pointing out that $\{x_n\}$ of (2) converges to $\bar{x}_1$ for $1 \le x_{-1} < 2$ and $x_0 \in (r_{21}, r_{31}) \cup (0, r_{11}) \cup (r_{41}, r_{22}) \cup (r_{51}, r_{32})$ which are listed in Table 3.

Table 3: Intervals of x_0 such that $\{x_n\}$ is convergent for $1 \le x_{-1} < 2$

Intervals of x_0	Reasons	$\{x_n\}$
$(r_{21},\ r_{31})$	$-1<F_2(b),\ F_3(b)<0$	converges to $\bar{x}_1$
$(0,\ \ r_{11})$	$-1<F_3(b),\ F_4(b)<0$	converges to $\bar{x}_1$
$(r_{41},\ r_{22})$	$-1<F_4(b),\ F_5(b)<0$	converges to $\bar{x}_1$
$(r_{51},\ r_{32})$	$-1<F_5(b),\ F_6(b)<0$	converges to $\bar{x}_1$

3.3 $a \ge 2$

In this case, we prove that

$$r_{21} < r_{31} < 0 < r_{11} < r_{41} < r_{51} \le r_{22} < r_{32} < r_{61} < r_{42} < r_{52} < r_{62}. \tag{20}$$

Compared with (18), we only need to prove $r_{22} \ge r_{51}$ for $a \ge 2$. In fact, from (14) and (15), for $a \ge 2$, we have that $r_{22} \ge r_{51}$.

It is worth pointing out that $\{x_n\}$ of (2) converges to $\bar{x}_1$ for $x_{-1} \ge 2$ and $x_0 \in (r_{21}, r_{31}) \cup (0, r_{11}) \cup (r_{41}, r_{51}) \cup (r_{22}, r_{32})$ which are listed in Table 4.

From the above, we derive such intervals of x_0 for x_{-1} such that $\{x_n\}$ of (2) is convergent. It is worth pointing out that we couldn't continue such a procedure because there are no explicit expressions of r_{42} and so on. From the above procedures, we know that the key is how to compare r_{i2} with r_{j1} where $j = i+3$ for $i \ge 4$.

In fact, for such an interval $I_i = (r_{i2},\ r_{j1})$ (or $(r_{j1},\ r_{i2})$) where $j = i+3$ for $i \ge 4$, in view of auxiliary functions $F_j(b)$, we have $F_{j-1}(b) < 0$ and $F_j(b) > 0$. Thus, for x_{-1} being fixed and $x_0 \in \bigcup I_i$ (the union of I_i for $i \ge 4$), $\{x_n\}$ of (2) is unbounded by Theorem 2.1.

Table 4: Intervals of x_0 such that $\{x_n\}$ is convergent for $x_{-1} \geq 2$

Intervals of x_0	Reasons	$\{x_n\}$
$(r_{21},\ r_{31})$	$-1 < F_2(b),\ F_3(b) < 0$	converges to $\bar{x}_1$
$(0,\quad r_{11})$	$-1 < F_3(b),\ F_4(b) < 0$	converges to $\bar{x}_1$
$(r_{41},\ r_{51})$	$-1 < F_4(b),\ F_5(b) < 0$	converges to $\bar{x}_1$
$(r_{22},\ r_{32})$	$-1 < F_5(b),\ F_6(b) < 0$	converges to $\bar{x}_1$

In view of Lemma 2.3, we obtain that the lengths of these open intervals I_i for $i \geq 4$ tend to zero as i tends to $+\infty$. For $x_0 > \hat{b}$, the increasing property of $\{x_n\}$ of (2) leads to its divergence.

Therefore, we generalize the above results into the following theorem.

Theorem 2.2. *The solution $\{x_n\}$ of (2) is unbounded only for its second initial value x_0 in such open intervals depending on the first initial valuex_0, which are listed in Table 5, where the endpoints r_{ij} are the roots of auxiliary functions $F_i(b) = x_i = 0$ with $x_0 = b$ and $x_{-1} = a$ for $i \geq 1$. And $\{x_n\}$ of (2) is an eventually prime period-three solution just at $x_0 = r_{ij}$ or $x_0 = 0$. For x_0 belongs to the complementary set of such intervals except those endpoints, $\{x_n\}$ of (2) is convergent to the negative equilibrium $\bar{x}_1$.*

Table 5: Intervals of x_0 for x_{-1} such that $\{x_n\}$ is unbounded

x_{-1}	Intervals of x_0
$(-\infty,\ -0.25)$	$(-\infty,\ r_{11}) \cup (0,\ +\infty)$
$[-0.25,\ 0)$	$(-\infty,\ r_{11}) \cup (r_{21}, r_{22}) \cup (0,\ +\infty)$
0	$(-\infty,\ -1) \cup (0,\ +\infty)$
$(0,\ 1)$	$(-\infty,\ r_{21}) \cup (0, r_{31}) \cup (r_{41},\ r_{11}) \cup (\hat{b},\ +\infty) \cup (\bigcup I_i)$
$[1,\ +\infty)$	$(-\infty,\ r_{21}) \cup (0, r_{31}) \cup (r_{41},\ r_{11}) \cup (\hat{b},\ +\infty) \cup (\bigcup I_i)$

From Theorem 2.2 and Table 5, for x_{-1} and x_0 greater than zero, solutions of (2) would exhibit somewhat chaotic behavior[4], that is, $\{x_n\}$ is either unbounded or convergent alternately for x_0 depending on x_{-1}, which is more concise from Table 5.

Now, we give some examples for particular x_{-1} which are listed in Table 6. Here, we only present the former six intervals of x_0 such that the solution $\{x_n\}$ of (2) is convergent. It is noted that the numerical values of these endpoints of these intervals are approximated to the values of the solutions of the auxiliary equations $F_i(b) = 0$.

From Table 6, for $x_{-1} = 1.5$, it is shown the former six intervals of x_0 such that the solution $\{x_n\}$ of (2) is convergent, which are on both sides of zero. If $x_0 = 1.6$ in $(1.4975,\ 1.6073)$, then the solution of (2) enters and then remains in the interval $(-1,\ 0)$, and hence is bounded and convergent. Whereas if $x_0 = 1.61$, then the solution is unbounded. It is clear for the third case that the solution is UB or C.

3 Conclusion

The existence of prime period-three solutions of (2) is proved in [4] and the convergence of (2) in its invariant interval $(-1,\ 0)$ is proved in [7]. In this paper, we present a new method to partition the intervals of x_0 depending on x_{-1} to describe the behavior of solutions of (2) and explain in detail that the solution of (2) exhibits somewhat chaotic behavior relative to the

Table 6: Intervals of x_0 for $x_{-1} > 0$ such that $\{x_n\}$ is convergent

x_{-1}	Intervals of x_0
0.1	$(-0.9161,\ 0)$, $(6.8377,\ 7.6946)$, $(10,\ 10.9161)$, $(12.4540,\ 12.8553)$, $(13.1623,\ 13.4675)$, $(13.6396,\ 13.7755)$
0.618	$(-0.6985,\ 0)$, $(0.3461,\ 1.4048)$, $(1.6181,\ 2.3166)$, $(2.5350,\ 2.8614)$, $(2.8902,\ 3.0690)$, $(3.0996,\ 3.1756)$
1	$(-0.618,\ 0)$, $(0,\ 1)$, $(1,\ 1.6180)$, $(1.7121,\ 2)$, $(2,\ 2.1479)$, $(2.1637,\ 2.2237)$
1.5	$(-0.5486,\ -0.1498)$, $(0,\ 0.6667)$, $(0.7717,\ 1.2153)$, $(1.2447,\ 1.4832)$, $(1.4975,\ 1.6073)$, $(1.6149,\ 1.6633)$
2.5	$(-0.4633,\ -0.2325)$, $(0,\ 0.4)$, $(0.5711,\ 0.8476)$, $(0.8633,\ 1.0325)$, $(1.0558,\ 1.1302)$, $(1.1316,\ 1.1680)$
10	$(-0.2702,\ -0.2162)$, $(0,\ 0.1)$, $(0.2740,\ 0.3327)$, $(0.3702,\ 0.4162)$, $(0.4383,\ 0.4596)$, $(0.4630,\ 0.4752)$

initial values. Compared with the known results[4], our results are much more accurate and easy to obtain by computers to describe the evolution of (2) for the initial values in the plane.

We conclude that the solution of (2) is bounded and convergent only for x_0 in particular intervals depending on x_{-1}, which are partitioned by the zeroes of auxiliary functions presented in this paper. Specially, it is unbounded only for x_0 in such open intervals listed in Table 5 which depend on x_{-1}.

It is of great interest to continue the investigation of the monotonicity, periodicity, and boundedness nature of solutions of (1) for different choices of parameters k and l and other equations presented in [4]. We believe that prime-period solutions and the negative equilibrium are crucial for the dynamics of difference equations (1). The future work is to extend our study to a more generalized equation (1).

Conflict of Interests

The authors declare that they have no competing interests.

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